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DEPARTMENT OF MECHANICAL ENGINEERING

SUBJECT NOTES

SUB NAME: OPERATION RESEARCH

SUBCODE : MET61 **YEAR & SEM : III/VI**

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OPERATION RESEARCH / MET 61

UNIT-I
Linear Programming Problems - Formulation and Duality concepts. Methods of solving LPP - Graphical Method, Simplex method (Computational Procedure) – Two Phase, Dual Simplex - Sensitivity analysis. Integer Programming: Introduction -Cutting plane method.

UNIT – II
Revised Simplex method - Transportation problem - optimal solution - MODI method - Transshipment problem. Assignment problem - various types. Dynamic programming - Solving General allocation, Investment, Stagecoach, Equipment replacement problems.

UNIT – III
Inventory Control Fundamentals-Inventory concepts and costs, DeterministicInventory models - Single item models-Classic EOQ and gradual replacement /manufacturing models with and without shortages, EOQ with price breaks, Introduction to inventory control applications. Game theory- Two persons zero sum games- Pure strategies, Mixed strategies, Dominance property, Graphical solution of (2xn) and (mx2) games.

UNIT – IV
PERT and CPM - Network Diagram , Crictical Path, Crashing, Probability Considaration, Resource leveling and allocation.

UNIT – V
Waiting line problems - Poisson arrivals and exponential service times, single channel and single stage problems. Logical flow charts for single server and Parallel server Queuing Models.

<u>TEXT BOOKS</u>

1. R Pannerselvam, Operations Research, PHI Learning Private Ltd., New Delhi, 2008
2. Hamdy A.Taha, Operations Research - An Introduction, Prentice Hall of India, 1995.
3. P.K.Gupta and D.S.Hira, Operations Research, S.Chand & Sons Ltd., New Delhi, 2007.

<u>REFERENCE BOOK</u>

1. Harvey M.Wagner, Principles of Operations Research with applications to managerial decisions, Prentice Hall of India, 2001.

Historical Development

INTRODUCTION

The subject **OPERATIONS RESEARCH** is a branch of mathematics - specially applied mathematics, used to provide a scientific base for management to take timely and effective decisions to their problems. It tries to avoid the dangers from taking decisions merely by guessing or by using thumb rules. Management is the multidimensional and dynamic concept. It is multidimensional, because management problems and their solutions have consequences in several dimensions, such as human, economic social and political fields. As the manager operates his system in an environment, which will never remain static, hence is dynamic in nature. Hence any manager, while making decisions, consider all aspects in addition to economic aspect, so that his solution should be useful in all aspects. The general approach is to **analyse the problem in economic terms and then implement the solution if it does not aggressive or violent to other aspects like human, social and political constraints**.

Management may be considered as the process of integrating the efforts of a purposeful group, or organisation, whose members have at least one common goal. You have studied various schools of management in your management science. Most important among them which uses scientific basis for decision making are:

(i) The Decision theory or Decisional Management School and

(ii) The Mathematical or Quantitative Measurement School.

The above-mentioned schools of management thought advocate the use of **mathematical methods** or **quantitative methods** for making decisions. Quantitative approach to management problems requires that decision problems be defined, analyzed, and solved in a conscious, rational, logical and systematic and scientific manner - based on data, facts, information and logic, and not on mere guess work or thumb rules. Here we use objectively measured decision criteria. Operations research is the body of knowledge, which uses mathematical techniques to solve management problems and make timely optimal decisions. Operations Research is concerned with helping managers and executives to make better decisions. Today's manager is working in a highly competitive and dynamic environment. In present environment, the manager has to deal with systems with complex interrelationship of various factors among them as well as equally complicated dependence of the criterion of effective performance of the system on these factors, conventional methods of decision-making is found very much inadequate. Though the common sense, experience, and commitment of the manager is essential in making decision, we cannot deny the role-played by scientific methods in making optimal decisions. Operations Research uses logical analysis and analytical techniques to study the behaviour of a system in relation to its overall working as resulting from its functionally interconnected constraints, whose parameters are recognized, quantified wherever possible relationships identified to the extent possible and alterative decisions are derived.

Conventional managers were very much worried about that an Operations Research analyst replace them as a decision maker, but immediately they could appreciated him due to his mathematical and logical knowledge, which he applies while making decisions. But operations research analyst list out alternative solutions and their consequences to ease manager's work of decision making. Operations research gives rationality to decision-making with clear view of possible consequences.

The scope of quantitative methods is very broad. They are applied in defining the problems and getting solutions of various organisatons like, business, Government organisations, profit making units and non-profit units and service units. They can be applied to variety of problems like deciding plant location, Inventory control, Replacement problems, Production scheduling, return on investment analysis (ROI), Portfolio selection, marketing research and so on. This book, deals with basic models of Operations research and quantitative methods. The students have to go through advanced Operations Research books, to understand the scope of the subject.

Two important aspects of quantitative methods are:

(a) Availability of well-structured models and methods in solving the problems,

(b) The attitude of search, conducted on a scientific basis, for increased knowledge in the management of organisations.

Therefore, the attitude encompassed in the quantitative approaches is perhaps more important than the specific methods or techniques. It is only by adopting this attitude that the boundaries and application of the quantitative approach can be advanced to include those areas where, at first glance, quantitative data and facts are hard to come by. Hence, quantitative approach has found place in traditional business and as well in social problems, public policy, national, international problems and interpersonal problems. In fact we can say that the application of quantitative techniques is not limited to any area and can be conveniently applied to all walks of life as far as decision-making is concerned. **The quantitative approach does not preclude the qualitative or judgemental elements that almost always exert a substantial influence on managerial decision-making. Quite the contrary. In actual practice, the quantitative approach must build upon, be modified by, and continually benefit from the experiences and creative insight of business managers.** In fact quantitative approach imposes a special responsibility on the manager. It makes modern manager to cultivate a managerial style that demand conscious, systematic and scientific analysis - and resolution - of decision problems.

In real world problems, we can notice that there exists a relationship among *intuition, judgement, science, quantitative attitudes, practices, methods and models*, as shown in figure 1.1.

The figure depicts that higher the degree of complexity and the degree of turbulence in the environment, the greater is the importance of the qualitative approach to management. On the other hand, the lower the degree of complexity *i.e.*, simple and well-structured problems, and lesser degree of turbulence in the environment, the greater is the potential of quantitative models. The advancement in quantitative approach to management problems is due to two facts. They are:

(a) Research efforts have been and are being directed to discover and develop more efficient tools and techniques to solve decision problems of all types.

(b) Through a continuous process of testing new frontiers, attempts have been made to expand the boundaries and application potential of the available techniques.

Quantitative approach is assuming an increasing degree of importance in the theory and practice of management because of the following reasons.

(a) Decision problems of modern management are so complex that only a conscious, systematic and scientifically based analysis can yield a realistic fruitful solution.

(b) Availability of list of more potential models in solving complex managerial problems.

(c) The most important one is that availability of high speed computers to solve large and complex real world problems in less time and at least cost and which help the managers to take timely decision.

One thing we have to remember here is that if managers are to fully utilize the potentials of management science models and computers, then problems will have to be stated in quantitative terms.

As far as the title of the subject is concerned, the terms **'quantitative approach', 'operations research', 'management science', 'systems analysis' and 'systems science'** are often used interchangeably. What ever be the name of the subject, the syllabi and subject matter dealt which will be same. This analog to 'god is one but the names are different'.

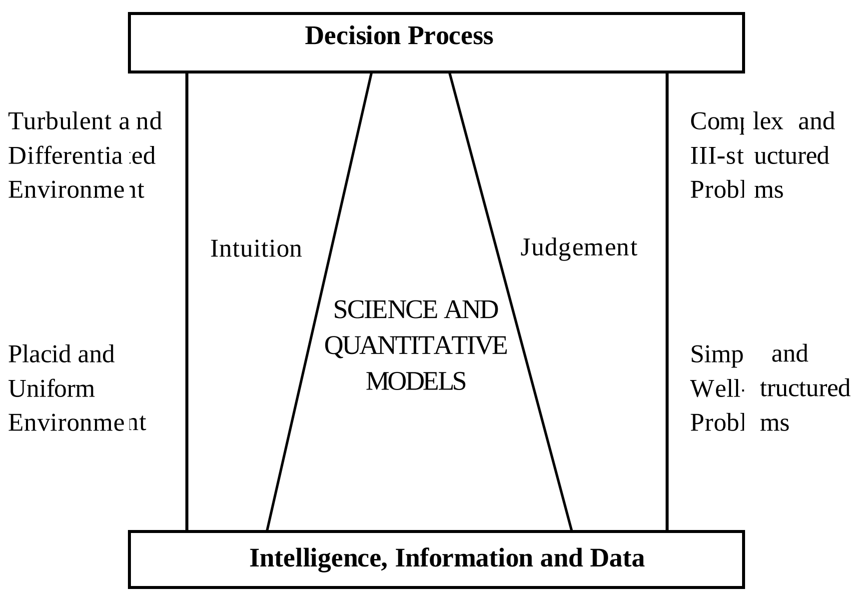


Figure. 1.1. Qualitative Thinking and Quantitative models.

HISTORY OF OPERATIONS RESEARCH

Operations Research is a **'war baby'**. It is because, the first problem attempted to solve in a systematic way was concerned with *how to set the time fuse bomb to be dropped from an aircraft on to a submarine*. In fact the main origin of Operations Research was during the **Second World War**. At the time of Second World War, the military management in England invited a team of scientists to study the strategic and tactical problems related to air and land defense of the country. The problem attained importance because at that time the resources available with England was very limited and the objective was to win the war with available meager resources. The resources such as food, medicines, ammunition, manpower etc., were required to manage war and for the use of the population of the country. It was necessary to decide upon the most effective utilization of the available resources to achieve the objective. It was also necessary to utilize the military resources cautiously. Hence, the Generals of military, invited a team of experts in various walks of life such as scientists, doctors, mathematicians, business people, professors, engineers etc., and the problem of resource utilization is given to them to discuss and come out with a feasible solution. These specialists had a brain storming session and came out with a method of solving the problem, which they coined the name **"Linear Programming".** This method worked out well in solving the war problem. As the name indicates, the word **Operations** is used to refer to the problems of military and the word **Research** is use for inventing new method. As this method of solving the problem was invented during the war period, the subject is given the name **'OPERATIONS RESEARCH'** and abbreviated as **'O.R.'** After the World War there was a scarcity of industrial material and industrial productivity reached the lowest level. Industrial recession was there and to solve the industrial problem the method **linear programming** was used to get optimal solution. From then on words, lot of work done in the field and today the subject of O.R. have numerous methods to solve different types of problems. After seeing the success of British military, the United States military management started applying the techniques to various activities to solve military, civil and industrial problems. They have given various names to this discipline. Some of them are Operational Analysis, Operations Evaluation, Operations Research, System Analysis, System Evaluation, Systems Research, Quantitative methods, Optimisation Techniques and Management Science etc. But most widely used one is **OPERATIONS RESEARCH.** In industrial world, most important problem for which these techniques used is how to *optimise the profit or how to reduce the costs*. The introduction of Linear Programming and Simplex method of solution developed by American Mathematician George B. Dontzig in 1947 given an opening to go for new techniques and applications through the efforts and co-operation of interested individuals in academic field and industrial field. Today the scenario is totally different. A large number of Operations Research consultants are available to deal with different types of problems. In India also, we have O.R. Society of India (1959) to help in solving various problems. Today the Operations Research techniques are taught at High School levels. To quote some Indian industries, which uses Operations Research for problem solving are: M/S Delhi Cloth Mills, Indian Railways, Indian Airline, Hindustan Lever, Tata Iron and Steel Company, Fertilizers Corporation of India and Defense Organizations. In all the above organizations, Operations Research people act as staff to support line managers in taking decisions.

In one word we can say that Operations Research play a vital role in every organization, especially in decision-making process.

DECISION MAKING AND SOME ASPECTS OF DECISION

Many a time we speak of the word **decision**, as if we know much about decision. But what is decision? What it consists of? What are its characteristics? Let us have brief discussion about the word decision, as much of our time we deal with decision-making process in Operations Research.

A **decision** is the conclusion of a process designed to weigh the relative uses or utilities of a set of alternatives on hand, so that decision maker selects the best alternative which is best to his problem or situation and implement it. **Decision Making** involves all activities and thinking that are necessary to identify the most optimal or preferred choice among the available alternatives. The basic requirements of decision-making are (*i*) A set of goals or objectives, (*ii*) Methods of evaluating alternatives in an objective manner, (*iii*) A system of choice criteria and a method of projecting the repercussions of alternative choices of courses of action. The evaluation of consequences of each course of action is important due to sequential nature of decisions.

The necessity of making decisions arises because of our existence in the world with various needs and ambitions and goals, whose resources are limited and some times scarce. Every one of us competes to use these resources to fulfill our goals. Our needs can be biological, physical, financial, social, ego or higher-level self-actualisation needs. One peculiar characteristics of decision-making is the inherent conflict that desists among various goals relevant to any decision situation (for example, a student thinking of study and get first division and at the same time have youth hood enjoyment without attending classes, OR a man wants to have lot of leisure in his life at the same time earn more etc.). The process of decision-making consists of two phases. The first phase consists of formulation of goals and objectives, enumeration of environmental constraints, identification and evaluation of alternatives. The second stage deals with selection of optimal course of action for a given set of constraints. **In Operations Research, we are concerned with *how to choose optimal strategy under specified set of assumptions*, including all available strategies and their associated payoffs.**

Decisions may be classified in different ways, depending upon the criterion or the purpose of classification. Some of them are shown below:

I Decisions (depending on the purpose)

PURPOSE:

- Strategic
  Related to external Environment
- Administrative
  Resource utilisation of an orgnisation.
- Operational
  Related to day –to-day Repetitive work.

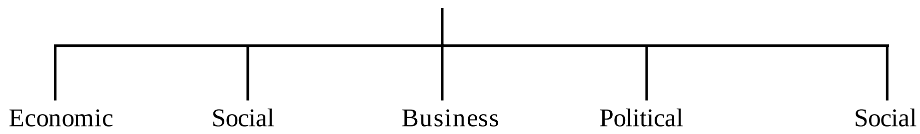
II Decision (Depending on the nature)

- Programmed decisions
  Meant for repetitive and well-structured problems. Inventory Problems, Product Mix Problems, etc.
- Non Programmed decisions
  Meant for non-routine, novel, ill-structured problems. Policy matters, Product market mix, plant location Etc.

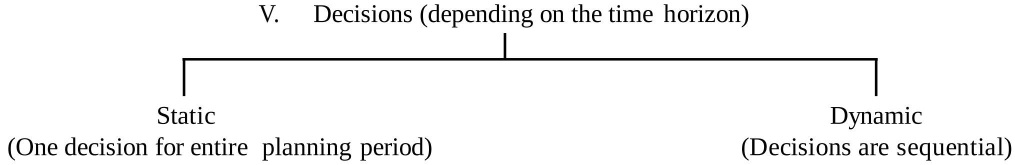
III Decisions (Depending on the persons involved)

- Individual
- Managerial

IV. Decisions (Depending on the Sphere of interest)



V. Decisions (depending on the time horizon)



Decisions may also be classified depending on the situations such as **degree of certainty**. For example, (*i*) Decision making under certainty (*ii*) Decision making under Uncertainty and (*iii*) Decision making under risk. The first two are two extremes and the third one is falls between these two with certain probability distribution.

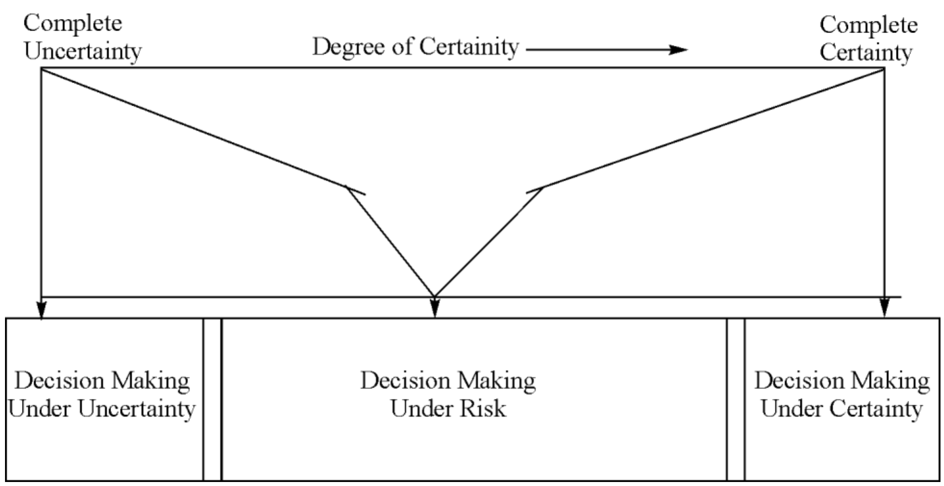


Figure 1.2. Decision based on degree of certainty.

OBJECTIVE OF OPERATIONS RESEARCH

Today's situation in which a manager has to work is very complicated due to complexity in business organizations. Today's business unit have number of departments and each department work for fulfilling the objectives of the organization. While doing so the individual objective of one of the department may be conflicting with the objective of the other department, though both working for achieving the common goal in the interest of the organization. In such situations, it will become a very complicated issue for the general manager to get harmony among the departments and to allocate the available resources of all sorts to the departments to achieve the goal of the organization. At the same time the environment in which the organization is operating is very dynamic in nature and the manager has to take decisions without delay to stand competitive in the market. At the same time a wrong decision or an untimely decision may be very costly. Hence the decision making process has become very complicated at the same time very important in the environment of conflicting interests and competitive strategies. Hence it is desirable for modern manager to use scientific methods with mathematical base while making decisions instead of depending on guesswork and thumb rule methods. Hence the knowledge of Operations Research is an essential tool for a manager who is involved in decision-making process. He must have support of knowledge of mathematics, statistics, economics etc., so that the decision he takes will be an optimal decision for his organisaton. Operation Research provides him this knowledge and helps him to take quick, timely, decisions, which are optimal for the organisaton. Hence the **objective** of operations research is:

"The objective of Operations Research is to provide a scientific basis to the decision maker for solving the problems involving the interaction of various components of an organization by employing a team of scientists from various disciplines, all working together for finding a solution which is in the best interest of the organisaton as a whole. The best solution thus obtained is known as optimal decision".

DEFINITION OF OPERATIONS RESEARCH

Any subject matter when defined to explain what exactly it is, we may find one definition. Always a definition explains what that particular subject matter is. Say for example, if a question is asked what is Boyel's law, we have a single definition to explain the same, irrespective of the language in which it is defined. But if you ask, what Operations research is? The answer depends on individual objective. Say for example a student may say that the Operations research is technique used to obtain first class marks in the examination. If you ask a businessman the same question, he may say that it is the technique used for getting higher profits. Another businessman may say it is the technique used to capture higher market share and so on. Like this each individual may define in his own way depending on his objective. Each and every definition may explain one or another characteristic of Operations Research but none of them explain or give a complete picture of Operations research. But in the academic interest some of the important definitions are discussed below.

(a) Operations Research is the art of winning wars without actually fighting. - Aurther Clarke.

This definition does not throw any light on the subject matter, but it is oriented towards warfare. It means to say that the directions for fighting are planned and guidance is given from remote area, according to which the war is fought and won. Perhaps you might have read in Mahabharatha or you might have seen some old pictures, where two armies are fighting, for whom the guidance is given by the chief minister and the king with a chessboard in front of them. Accordingly war is fought in the warfront. Actually the chessboard is a model of war field.

(b) Operations Research is the art of giving bad answers to problems where otherwise worse answers are given. - T.L. Satty.

This definition covers one aspect of decision-making, *i.e.*, choosing the best alternative among the list of available alternatives. It says that if the decisions are made on guesswork, we may face the worse situation. But if the decisions are made on scientific basis, it will help us to make better decisions. Hence this definition deals with one aspect of decision-making and not clearly tells what is operations research.

(c) **Operations Research is Research into Operations. - J. Steinhardt.**

This definition does not give anything in clear about the subject of Operations Research and simply says that it is research in to operations. Operations may here be referred as military activities or simply the operations that an executive performs in his organisations while taking decisions. Research in the word means that finding a new approach. That is when an executive is involved in performing his operations for taking decisions he has to go for newer ways so that he can make a better decision for the benefit of his organisation.

(d) **Operations Research is defined as Scientific method for providing executive departments a quantitative basis for decisions regarding the operations under their control. - P.M. Morse and G.E. Kimball.**

This definition suggests that the Operations Research provides scientific methods for an executive to make optimal decisions. But does not give any information about various models or methods. But this suggests that executives can use scientific methods for decision-making.

(e) **Operations Research is th study of administrative system pursued in the same scientific manner in which system in Physics, Chemistry and Biology are studied in natural sciences.**

This definition is more elaborate than the above given definitions. It compares the subject Operations Research with that of natural science subjects such as Physics, Chemistry and Biology, where while deciding any thing experiments are conducted and results are verified and then the course of action is decided. It clearly directs that Operations Research can also be considered as applied science and before the course of action is decided, the alternatives available are subjected to scientific analysis and optimal alternative is selected. But the difference between the experiments we conduct in natural sciences and operations research is: in natural sciences the research is rigorous and exact in nature, whereas in operations research, because of involvement of human element and uncertainty the approach will be totally different.

(f) **Operations Research is the application of scientific methods, techniques and tools to operation of a system with optimum solution to the problem. - Churchman, Ackoff and Arnoff.**

This definition clearly states that the operations research applies scientific methods to find an optimum solution to the problem of a system. A system may be a production system or information system or any system, which involves men, machine and other resources. We can clearly identify that this definition tackles three important aspects of operations research *i.e.* application of scientific methods, study of a system and optimal solution. This definition too does not give any idea about the characteristics of operations research.

(g) **Operations Research is the application of the theories of Probability, Statistics, Queuing, Games, Linear Programming etc., to the problems of War, Government and Industry.**

This definition gives a list of various techniques used in Operations Research by various managers to solve the problems under their control. A manager has to study the problem, formulate the problem, identify the variables and formulate a model and select an appropriate technique to get optimal solution. We can say that operations research is a bunch of mathematical techniques to solve problems of a system.

(h) **Operations Research is the use of Scientific Methods to provide criteria or decisions regarding man-machine systems involving repetitive operations.**

This definition talks about man- machine system and use of scientific methods and decision-making. It is more general and comprehensive and exhaustive than other definitions. Wherever a study of system involving man and machine, the person in charge of the system and involved in decision-making will use scientific methods to make optimal decisions.

(i) Operations Research is applied decision theory. It uses any scientific, mathematical or logical means to attempt to cope with problems that confront the executive, when he tries to achieve a thorough going rationally in dealing with his decision problem. - D.W. Miller and M.K. Starr.

This definition also explains that operations research uses scientific methods or logical means for getting solutions to the executive problems. It too does not give the characteristics of Operations Research.

(j) Operations Research is the application of the methods of science to complex problems arising in the direction and management of large systems of men, materials and money in industry, business, Government and defense. The distinctive approach is to develop a scientific model of the system, incorporating measurements of factors such as chance and risk, with which to predict and compare the outcome of alternative decisions, strategies or controls. The purpose is to help management to determine its policy and actions scientifically. - Operations Society of Great Britain.

The above definition is more elaborate and says that operations research applies scientific methods to deal with the problems of a system where men, material and other resources are involved and the system under study may be industry, defense or business etc, gives this definition. It also say that the manager has to build a scientific model to study the system which must be provided with facility to measure the outcomes of various alternatives under various degrees of risk, which helps the managers to take optimal decisions.

In addition to the above there are hundreds of definitions available to explain what Operations Research is? But many of them are not satisfactory because of the following reasons.

(i) Operations Research is not a well-defined science like Physics, Chemistry etc. All these sciences are having well defined theory about the subject matter, where as operations research do not claim to know or have theories about operations. Moreover, Operations Research is not a scientific research into the control of operations. It is only the application of mathematical models or logical analysis to the problem solving. Hence none of the definitions given above defines operations research precisely.

(ii) The objective of operations research says that the decisions are made by brain storming of people from various walks of life. This indicates that operations research approach is inter- disciplinary approach, which is an important character of operations research. This aspect is not included in any of the definitions hence they are not satisfactory.

(iii) The above-discussed definitions are given by various people at different times and stages of development of operations research as such they have considered the field in which they are involved hence each definition is concentrating on one or two aspects. No definition is having universal approach.

But salient features of above said definitions are:

* Operations Research uses Scientific Methods for making decisions.
* It is interdisciplinary approach for solving problems and it uses the knowledge and experience of experts in various fields.

* While analyzing the problems all aspects are considered and examined and analyzed scientifically for finding the optimal solution for the problem on hand.
* As operations research has scientific approach, it improves the quality of answers to the problems.
* Operations research provides scientific base for decision-making and provide scientific substitute for judgement and intuition.

CHARACTERISTICS OF OPERATIONS RESEARCH

After considering the objective and definitions of Operations Research, now let us try to understand what are the characteristics of Operations Research.

(a) Operations Research is an interdisciplinary team approach.

The problems an operations research analyst face is heterogeneous in nature, involving the number of variables and constraints, which are beyond the analytical ability of one person. Hence people from various disciplines are required to understand the operations research problem, who applies their special knowledge acquired through experience to get a better view of cause and effects of the events in the problem and to get a better solution to the problem on hand. This type of team approach will reduce the risk of making wrong decisions.

(b) **Operations Research increases the creative ability of the decision maker.**

Operations Research provides manager mathematical tools, techniques and various models to analyse the problem on hand and to evaluate the outcomes of various alternatives and make an optimal choice. This will definitely helps him in making better and quick decisions. A manager, without the knowledge of these techniques has to make decisions by thumb rules or by guess work, which may click some times and many a time put him in trouble. Hence, a manager who uses Operations Research techniques will have a better creative ability than a manager who does not use the techniques.

(c) Operations Research is a systems approach.

A business or a Government organization or a defense organization may be considered as a system having various sub-systems. The decision made by any sub-system will have its effect on other sub-systems. Say for example, a decision taken by marketing department will have its effect on production department. When dealing with Operations Research problems, one has to consider the entire system, and characteristics or sub- systems, the inter-relationship between sub-systems and then analyse the problem, search for a suitable model and get the solution for the problem. Hence we say Operations Research is a Systems Approach.

SCOPE OF OPERATIONS RESEARCH

The scope aspect of any subject indicates, the limit of application of the subject matter/techniques of the subject to the various fields to solve the variety of the problems. But we have studied in the objective, that the subject Operations Research will give scientific base for the executives to take decisions or to solve the problems of the systems under their control. The system may be business, industry, government or defense. Not only this, but the definitions discussed also gives different versions. This indicates that the techniques of Operations Research may be used to solve any type of problems. The problems may pertain to an individual, group of individuals, business, agriculture, government or defense. Hence, we can say that there is no limit for the application of Operations Research methods and techniques; they may be applied to any type of problems. Let us now discuss some of the fields where Operations Research techniques can be applied to understand how the techniques are useful to solve the problems. In general we can state that whenever there is a problem, simple or complicated, we can use operations research techniques to get best solution.

(i) In Defense Operations

In fact, the subject Operations research is the baby of World War II. To solve war problems, they have applied team approach, and come out with various models such as resource allocation model, transportation model etc.In any war field two or more parties are involved, each having different resources (manpower, ammunition, etc.), different courses of actions (strategies) for application. Every opponent has to guess the resources with the enemy, and his courses of action and accordingly he has to attack the enemy. For this he needs scientific, logical analysis of the problem to get fruitful results. Here one can apply the techniques like *Linear Programming, Game theory, and inventory models etc.* to win the game. In fact in war filed every situation is a competitive situation. More over each party may have different bases, such as Air force, Navy and Army. The decision taken by one will have its effect on the other. Hence proper co-ordination of the three bases and smooth flow of information is necessary. Here operations research techniques will help the departmental heads to take appropriate decisions.

(ii) In Industry

After the II World War, the, Industrial world faced a depression and to solve the various industrial problems, industrialist tried the models, which were successful in solving their problems. Industrialist learnt that the techniques of operations research can conveniently applied to solve industrial problems. Then onwards, various models have been developed to solve industrial problems. Today the managers have on their hand numerous techniques to solve different types of industrial problems. In fact *decision trees, inventory model, Linear Programming model, Transportation model, Sequencing model, Assignment model and replacement models* are helpful to the managers to solve various problems, they face in their day to day work. These models are used to minimize the cost of production, increase the productivity and use the available resources carefully and for healthy industrial growth. An industrial manager, with these various models on his hand and a computer to workout the solutions (today various packages are available to solve different industrial problems) quickly and preciously.

(iii) In Planning For Economic Growth

In India we have five year planning for steady economic growth. Every state government has to prepare plans for balanced growth of the state. Various secretaries belonging to different departments has to co-ordinate and plan for steady economic growth. For this all departments can use Operations research techniques for planning purpose. The question like how many engineers, doctors, software people etc. are required in future and what should be their quality to face the then problems etc. can be easily solved.

(iv) In Agriculture

The demand for food products is increasing day by day due to population explosion. But the land available for agriculture is limited. We must find newer ways of increasing agriculture yield. So the selection of land area for agriculture and the seed of food grains for sowing must be meticulously done so that the farmer will not get loss at the same time the users will get what they desire at the desired time and desired cost.

(v) In Traffic control

Due to population explosion, the increase in the number and verities of vehicles, road density is continuously increasing. Especially in peak hours, it will be a headache to control the traffic. Hence proper timing of traffic signaling is necessary. Depending on the flow of commuters, proper signaling time is to be worked out. This can be easily done by the application of *queuing theory*.

(vi) In Hospitals

Many a time we see very lengthy queues of patient near hospitals and few of them get treatment and rest of them have to go without treatment because of time factor. Some times we have problems non-availability of essential drugs, shortage of ambulances, shortage of beds etc. These problems can be conveniently solved by the application of operations research techniques.

The above-discussed problems are few among many problems that can be solved by the application of operation research techniques. This shows that Operations Research has no limit on its scope of application.

PHASES IN SOLVING OPERATIONS RESEARCH PROBLEMS OR STEPS IN SOLVING OPERATIONS RESEARCH PROBLEMS

Any Operations Research analyst has to follow certain sequential steps to solve the problem on hand. The steps he has to follow are discussed below:

First he has to study the situation and collect all information and formulate the statement of the problem. Hence the first step is the *Formulation of the problem*. The figure 1.3 shows the various steps to be followed.

Formulation of the Problem

The Operations Research analyst or team of experts first have to examine the situation and clearly define what exactly happening there and identify the variables and constraints. Similarly identify what is the objective and put them all in the form of statement. The statement must include a) a precise description goals or objectives of the study, b) identification of controllable and uncontrollable variables and c) restrictions of the problem. The team should consult the personals at the spot and collect information, if something is beyond their reach, they have to consult duty engineers available and understand the facts and formulate the problem. Let us consider the following statement:

Statement: A company manufactures **two products *X* and *Y***, by using the **three machines *A*, *B*, and *C*.** Each unit of *X* takes **1 hour** on machine A, **3 hours** on machine B and **10 hours** on machine *C*. Similarly, product *Y* takes **one hour, 8 hours and 7 hours** on Machine *A*, *B*, and *C* respectively. In the coming planning period, **40 hours of machine *A*, 240 hours of machine *B* and 350 hours of machine *C*** is available for production. Each unit of ***X* brings a profit of Rs 5/-** and ***Y* brings Rs. 7 per unit. How much of *X* and *Y* are to be manufactured** by the company **for maximizing the profit**?

The team of specialists prepares this statement after studying the system. As per requirement this must include the variables, constraints, and objective function.

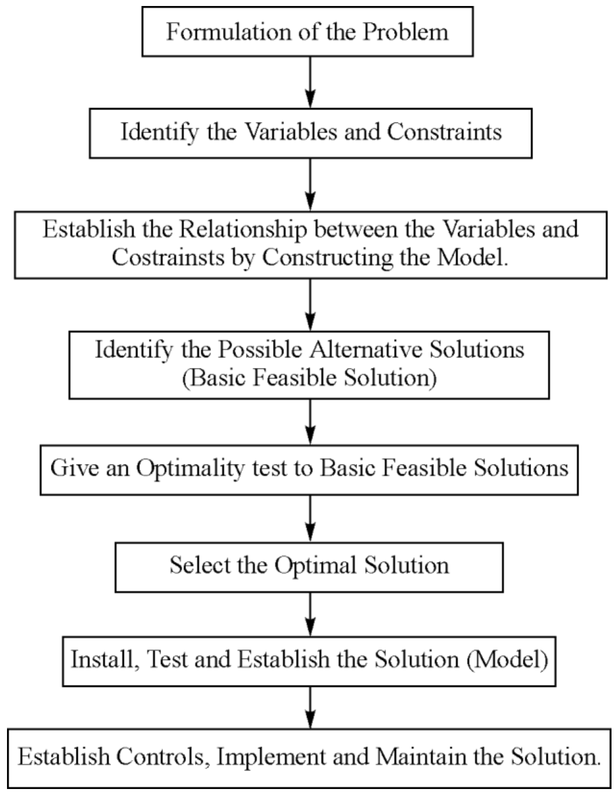


Figure 1.3. Phases of Solving Operations Research Problems.

Variables

The Company is manufacturing two products *X* and *Y*. These are the two variables in the problem. When they are in the problem statement they are written in *capital letters*. Once they are entered in the model small letters (lower case) letters are used (*i.e*,. *x* and *y*). We have to find out how much of *X* and how much of *Y* are to be manufactured. Hence they are variables. In linear programming language, these are known as c**ompeting candidates**. Because they compete to use or consume available resources.

Resources and Constraints

There are three machines *A*, *B*, and C on which the products are manufactured. These are known as resources. The capacity of machines in terms of machine hours available is the available resources. The competing candidates have to use these available resources, which are limited in nature. Now in the above statement, machine *A* has got available 40 hours and machine B has available a capacity of 240 hours and that of machine *C* is 350 hours. The products have to use these machine hours in required proportion. That is one unit of product *X* consumes one hour of machine *A*, 3 hours of machine *B* and 10 hours of machine *C*. Similarly, one unit of *Y* consumes one hour of machine *B*, 8 hours of machine *B* and 7 hours of machine *C*. These machine hours given are the available resources and they are limited in nature and hence they are **constraints** given in the statement.

Objective of the Problem

To maximise the profit how much of *X* and *Y* are to be manufactured? That is **maximization of the profit or maximization of the returns** is the objective of the problem. For this in the statement it is given that the profit contribution of X is Rs 5/- per unit and that of product Y is Rs. 7/- per unit.

To establish relationship between variables and constraints and build up a model

Let us say that company manufactures x units of *X* and y units of *Y*. Then as one unit of *x* consumes one hour on machine A and one unit of y consumes one hour on machine A, the total consumption by manufacturing *x* units of *X* and y units of *Y* is, $1x + 1y$ and this should not exceed available capacity of 40 hours. Hence the **mathematical relationship in the form of mathematical model is $1x + 1y \leq 40$**. This is for resource machine *A*. Similarly for machine *B* and machine *C* we can formulate the mathematical models. They appear as shown below:
$3x + 8y \leq 240$ for machine *B* **and $10x + 7y \leq 350$** for machine *C*. Therefore, the mathematical model for these resources are:

$$1x + 1y \leq 40$$
$$3x + 8y \leq 240 \text{ and}$$
$$10x + 7y \leq 350.$$

Similarly for objective function as the company manufacturing x units of *X* and y units of *Y* and the profit contribution of *X* and Y are Rs.5/- and Rs 7/- per unit of *X* and *Y* respectively, the total profit earned by the company by manufacturing *x* and *y* units is **$5x + 7y$.** This we have to maximise. Therefore **objective function is Maximise $5x + 7y$.** At the same time, we have to remember one thing that the company can manufacture any number of units or it may not manufacture a particular product, for example say $x = 0$. But it cannot manufacture negative units of *x* and *y*. Hence one more constraint is to be introduced in the model *i.e.* **a non - negativity constraint.** Hence the mathematical representation of the contents of the statement is as given below:

Maximise $Z = 5x + 7y$ Subject to a condition (written as s.t.) ⟶ **OBJECTIVE FUNCTION.**

$$\left.\begin{array}{l} 1x + 1y \leq 40 \\ 3x + 8y \leq 240 \\ 10x + 7y \leq 350 \text{ and} \end{array}\right\| \longrightarrow \textbf{STRUCTURAL CONSTRAINTS.}$$

Both *x* and *y* are ≥ 0 ⟶ **NON-NEGATIVITY CONSTRAINT.**

Identify the possible alternative solutions (or known as Basic Feasible Solutions or simply BFS)

There are various methods of getting solutions. These methods will be discussed later. For example we go on giving various values (positive numbers only), and find various values of objective function. All these are various Basic Feasible Solutions. For example $x = 0,1,2,3$, etc. and $y = 0,1,2,3$ etc are all feasible values as far as the given condition is concerned. Once we have feasible solutions on hand go on asking is it maximum? Once we get maximum value, those values of x and y are optimal values. And the value of objective function is **optimal value of the objective function**. These two steps we shall discuss in detail in the next chapter.

Install and Maintain the Solution

Once we get the optimal values of x and y and objective function instructions are given to the concerned personal to manufacture the products as per the optimal solution, and maintain the same until further instructions.

MEANING AND NECESSITY OF OPERATIONS RESEARCH MODELS

Management deals with reality that is at once complex, dynamic, and multifacet. It is neither possible nor desirable, to consider each and every element of reality before deciding the courses of action. It is impossible because of time available to decide the courses of action and the resources, which are limited in nature. More over in many cases, it will be impossible for a manager to conduct experiment in real environment. For example, if an engineer wants to measure the inflow of water in to a reservoir through a canal, he cannot sit on the banks of canal and conduct experiment to measure flow. He constructs a similar model in laboratory and studies the problem and decides the inflow of water. Hence for many practical problems, a model is necessary. We can **define an operations research model as some sort of mathematical or theoretical description of various variables of a system representing some aspects of a problem on some subject of interest or inquiry. The model enables to conduct a number of experiment involving theoretical subjective manipulations to find some optimum solution to the problem on hand.**

Let us take a very simple example. Say you have a small child in your house. You want to explain to it what is an elephant. You can say a story about the elephant saying that it has a trunk, large ears, small eyes etc. The child cannot understand or remember anything. But if you draw small drawing of elephant on a paper and show the trunk, ears, eyes and it will grasp very easily the details of elephant. When a circus company comes to your city and take elephants in procession, then the child if observe the procession, it will immediately recognize the elephant. This is the exact use of a model. In your classrooms your teacher will explain various aspects of the subject by drawing neat sketches on the black board. You will understand very easily and when you come across real world system, you can apply what all you learnt in your classroom. Hence through a model, we can explain the aspect of the subject / problem / system. The inequalities given in section 1.8.5 is a mathematical model, which explains clearly the manufacturing system, given in section 1.8.1. **(*Here we can say a model is a relationship among specified variables and parameters of the system*).**

Classification of Models

The models we use in operations research may broadly classified as:

(i) Mathematical and Descriptive models, and (*ii*) Static and Dynamic Models.

Mathematical and Descriptive Models

(i) Descriptive Model

A descriptive model explains or gives a description of the system giving various variables, constraints and objective of the system or problem. In article 1.8.1 gives the statement of the problem, which is exactly a descriptive model. The drawback of this model is as we go on reading and proceed; it is very difficult to remember about the variables and constraints, in case the problem or description of the system is lengthy one. It is practically impossible to keep on reading, as the manager has to decide the course of action to be taken timely.

Hence these models, though necessary to understand the system, have limited use as far as operations research is concerned.

(ii) Mathematical Model

In article, 1.8.2 we have identified the variables and constraints and objective in the problem statement and given them mathematical symbols *x* and *y* and a model is built in the form of an inequality of ≤ type. Objective function is also given. This is exactly a mathematical model, which explains the entire system in mathematical language, and enables the operations research person to proceed towards solution.

Types of Models

Models are also categorized depending on the *structure, purpose, nature of environment, behaviour, by method of solution and by use of digital computers.*

(a) Classification by Structure

(i) *Iconic Models:* These models are scaled version of the actual object. For example a toy of a car is an iconic model of a real car. In your laboratory, you might have seen Internal Combustion Engine models, Boiler models etc. All these are iconic models of actual engine and boiler etc. They explain all the features of the actual object. In fact a globe is an iconic model of the earth. These models may be of enlarged version or reduced version. Big objects are scaled down (reduced version) and small objects, when we want to show the features, it is scaled up to a bigger version. In fact it is a descriptive model giving the description of various aspects of real object. As far as operations research is concerned, is of less use. The advantages of these models: are It is easy to work with an iconic model in some cases, these are easy to construct and these are useful in describing static or dynamic phenomenon at some definite time. The limitations are, we cannot study the changes in the operation of the system. For some type of systems, the model building is very costly. It will be sometimes very difficult to carry out experimental analysis on these models.

(ii) *Analogue Model:* In this model one set of properties are used to represent another set of properties. Say for example, blue colour generally represents water. Whenever we want to show water source on a map it is represented by blue colour. Contour lines on the map is also analog model. Many a time we represent various aspects on graph by defferent colours or different lines all these are analog models. These are also not much used in operations research. The best examples are warehousing problems and layout problems.

(iii) *Symbolic Models or Mathematical Models*: In these models the variables of a problem is represented by mathematical symbols, letters etc. To show the relationships between variables and constraints we use mathematical symbols. Hence these are known as symbolic models or mathematical models. These models are used very much in operations research. Examples of such models are Resource allocation model, Newspaper boy problem, transportation model etc.

(b) Classification by utility

Depending on the use of the model or purpose of the model, the models are classified as D*escriptive, Predictive and Prescriptive models.*

(i) *Descriptive model*: The descriptive model simply explains certain aspects of the problem or situation or a system so that the user can make use for his analysis. It will not give full details and clear picture of the problem for the sake of scientific analysis.

(ii) *Predictive model*: These models basing on the data collected, can predict the approximate results of the situation under question. For example, basing on your performance in the examination and the discussions you have with your friends after the examination and by verification of answers of numerical examples, you can predict your score or results. This is one type of predictive model.

(iii) *Prescriptive models*: We have seen that predictive models predict the approximate results. But if the predictions of these models are successful, then it can be used conveniently to prescribe the courses of action to be taken. In such case we call it as Prescriptive model. Prescriptive models prescribe the courses of action to be taken by the manager to achieve the desired goal.

(c) Classification by nature of environment

Depending on the environment in which the problem exists and the decisions are made, and depending on the conditions of variables, the models may be categorized as *Deterministic models* and *Probabilistic models.*

(i) *Deterministic Models*: In this model the operations research analyst assumes complete certainty about the values of the variables and the available resources and expects that they do not change during the planning horizon. All these are deterministic models and do not contain the element of uncertainty or probability. The problems we see in Linear Programming, assumes certainty regarding the values of variables and constraints hence the Linear Programming model is a Deterministic model.

(ii) *Probabilistic or Stochastic Models*: In these models, the values of variables, the pay offs of a certain course of action cannot be predicted accurately because of element of probability. It takes into consideration element of risk into consideration. The degree of certainty varies from situation to situation. A good example of this is the sale of insurance policies by Life Insurance Companies to its customers. Here the failure of life is highly probabilistic in nature. The models in which the pattern of events has been compiled in the form of probability distributions are known as Probabilistic or Stochastic Models.

(d) Classification depending on the behaviour of the problem variables

Depending on the behaviour of the variables and constraints of the problem they may be classified as *Static Models* or *Dynamic models*.

(i) *Static Models*: These models assumes that no changes in the values of variables given in the problem for the given planning horizon due to any change in the environment or conditions of the system. All the values given are independent of the time. Mostly, in static models, one decision is desirable for the given planning period.

(ii) *Dynamic Models*: In these models the values of given variables goes on changing with time or change in environment or change in the conditions of the given system. Generally, the dynamic models then exist a series of interdependent decisions during the planning period.

(e) Classification depending on the method of getting the solution

We may use different methods for getting the solution for a given model. Depending on these methods, the models are classified as A*nalytical Models* and *Simulation Models.*

(i) *Analytical Models:* The given model will have a well-defined mathematical structure and can be solved by the application of mathematical techniques. We see in our discussion that the Resource allocation model, Transportation model, Assignment model, Sequencing model etc. have well defined mathematical structure and can be solved by different mathematical techniques. For example, Resource allocation model can be solved by Graphical method or by Simplex method depending on the number of variables involved in the problem. All models having mathematical structure and can be solved by mathematical methods are known as Analytical Models.

(ii) *Simulation Models:* The meaning of simulation is *imitation*. These models have mathematical structure but cannot be solved by using mathematical techniques. It needs certain experimental analysis. To study the behaviour of the system, we use random numbers. More complex systems can be studied by simulation. Studying the behaviour of laboratory model, we can evaluate the required values in the system. Only disadvantage of this method is that it does not have general solution method.

Some of the Points to be Remembered while Building a Model

* When we can solve the situation with a simple model, do not try to build a complicated model.
* Build a model that can be easily fit in the techniques available. Do not try to search for a technique, which suit your model.
* In order to avoid complications while solving the problem, the fabrication stage of modeling must be conducted rigorously.
* Before implementing the model, it should be validated / tested properly.
* Use the model for which it is deduced. Do not use the model for the purpose for which it is not meant.
* Without having a clear idea for which the model is built do not use it. It is better before using the model; you consult an operations research analyst and take his guidance.
* Models cannot replace decision makers. It can guide them but it cannot make decisions. Do not be under the impression, that a model solves every type of problem.
* The model should be as accurate as possible.
* A model should be as simple as possible.
* Benefits of model are always associated with the process by which it is developed.

Advantages of a Good Model

(i) A model provides logical and systematic approach to the problem.
(ii) It provides the analyst a base for understanding the problem and think of methods of solving.
(iii) The model will avoid the duplication work in solving the problem.
(iv) Models fix the limitation and scope of an activity.
(v) Models help the analyst to find newer ways of solving the problem.
(vi) Models saves resources like money, time etc.
(vii) Model helps analyst to make complexities of a real environment simple.

(viii) Risk of tampering the real object is reduced, when a model of the real system is subjected to experimental analysis.

(ix) Models provide distilled economic descriptions and explanations of the operation of the system they represent.

Limitations of a Model

(i) Models are constructed only to understand the problem and attempt to solve the problem; they are not to be considered as real problem or system.

(ii) The validity of any model can be verified by conducting the experimental analysis and with relevant data characteristics.

Characteristics of a Good Model

(i) The number of parameters considered in a model should be less to understand the problem easily.

(ii) A good model should be flexible to accommodate any necessary information during the stages of building the model.

(iii) A model must take less time to construct.

(iv) A model may be accompanied by lower and upper bounds of parametric values.

Steps in Constructing a Model

(i) *Problem environment analysis and formulation*: One has to study the system in all aspects, if necessary make relevant assumptions, have the decision for which he is constructing the model in mind and formulate the model.

(ii) *Model construction and assumptions*: Identify the main variables and constraints and relate them logically to arrive at a model.

(iii) *Testing the model*: After the formulation, before using check the model for its validity.

Methods of Solving Operations Research Problems

There are three methods of solving an operations research problem. They are:

(i) Analytical method, (*ii*) Iterative Method, (*iii*) The Monte-Carlo Technique.

(i) *Analytical Method*: When we use mathematical techniques such as differential calculus, probability theory etc. to find the solution of a given operations research model, the method of solving is known as analytical method and the solution is known as analytical solution. Examples are problems of inventory models. This method evaluates alternative policies efficiently.

(ii) *Iterative Method (Numerical Methods)*: This is trial and error method. When we have large number of variables, and we cannot use classical methods successfully, we use iterative process. First, we set a trial solution and then go on changing the solution under a given set of conditions, until no more modification is possible. The characteristics of this method is that the trial and error method used is laborious, tedious, time consuming and costly. The solution we get may not be accurate one and is approximate one. Many a time we find that after certain number of iterations, the solution cannot be improved and we have to accept it as the expected optimal solution.

(iii) *Monte-Carlo Method*: This method is based on random sampling of variable's values from a distribution of the variable. This uses sampling technique. A table of random numbers must be available to solve the problems. In fact it is a simulation process.

SOME IMPORTANT MODELS (PROBLEMS) WE COME ACROSS IN THE STUDY OF OPERATIONS RESEARCH

1. Linear Programming Model

This model is used for resource allocation when the resources are limited and there are number of competing candidates for the use of resources. The model may be used to maximise the returns or minimise the costs. Consider the following two situations:

(a) A company which is manufacturing variety of products by using available resources, want to use resources optimally and manufacture different quantities of each type of product, which yield different returns, so as to maximise the returns.

(b) A company manufactures different types of alloys by purchasing the three basic materials and it want to maintain a definite percentage of basic materials in each alloy. The basic materials are to be purchased from the sellers and mix them to produce the desired alloy. This is to be done at minimum cost.

Both of them are resource allocation models, the case (*a*) is maximisation problem and the case (*b*) is minimisation problem.

(c) Number of factories are manufacturing the same commodities in different capacities and the commodity is sent to various markets for meeting the demands of the consumers, when the cost of transportation is known, the linear programming helps us to formulate a programme to distribute the commodity from factories to markets at minimum cost. The model used is transportation model.

(d) When a company has number of orders on its schedule, which are to be processed on same machines and the processing time, is known, then we have to allocate the jobs or orders to the machines, so as to complete all the jobs in minimum time. This we can solve by using Assignment model.

All the above-discussed models are Linear Programming Models. They can be solved by application of appropriate models, which are linear programming models.

2. Sequencing Model

When a manufacturing firm has some job orders, which can be processed on two or three machines and the processing times of each job on each machine is known, then the problem of processing in a sequence to minimise the cost or time is known as Sequencing model.

3. Waiting Line Model or Queuing Model

A model used for solving a problem where certain service facilities have to provide service to its customers, so as to avoid lengthy waiting line or queue, so that customers will get satisfaction from effective service and idle time of service facilities are minimised is waiting line model or queuing model.

4. Replacement Model

Any capital item, which is continuously used for providing service or for producing the product is subjected to wear and tear due to usage, and its efficiency goes on reducing. This reduction in efficiency can be predicted by the increasing number of breakdowns or reduced productivity. The worn out parts or components are to be replaced to bring the machine back to work. This action is known as maintenance. A time is reached when the maintenance cost becomes very high and the manager feels to replace the old machine by new one. This type of problems known as replacement problems and can be solved by replacement models.

5. Inventory Models

Any manufacturing firm has to maintain stock of materials for its use. This stock of materials, which are maintained in stores, is known as inventory. Inventory is one form of capital or money. The company has to maintain inventory at optimal cost. There are different types of inventory problems, depending the availability and demand pattern of the materials. These can be solved by the application of inventory models.

In fact depending on the number of variables, characteristics of variables, and the nature of constraints different models are available. These models, we study when we go chapter wise.

QUESTIONS

1. Trace the history of Operations Research.
2. Give a brief account of history of Operations Research.
3. Discuss the objective of Operations Research.
4. "Operations Research is a bunch of mathematical techniques to break industrial problems". Critically comment.
5. What is a Operations Research model? Discuss the advantages of limitation of good Operations Research model.
6. Discuss three Operations Research models.
7. What is a decision and what are its characteristics.
8. Briefly explain the characteristics of Operations Research.
9. Discuss the various steps used in solving Operations Research problems.
10. Discuss the scope of Operations Research.

CHAPTER – 2

Linear Programming Models

(Resource Allocation Models)

INTRODUCTION

A model, which is used for optimum allocation of scarce or limited resources to competing products or activities under such assumptions as certainty, linearity, fixed technology, and constant profit per unit, is *linear programming*.

Linear Programming is one of the most versatile, powerful and useful techniques for making managerial decisions. Linear programming technique may be used for solving broad range of problems arising in business, government, industry, hospitals, libraries, etc. Whenever we want to allocate the available limited resources for various competing activities for achieving our desired objective, the technique that helps us is **LINEAR PROGRAMMING**. As a decision making tool, it has demonstrated its value in various fields such as production, finance, marketing, research and development and personnel management. Determination of optimal product mix (a combination of products, which gives maximum profit), transportation schedules, Assignment problem and many more. In this chapter, let us discuss about various types of linear programming models.

PROPERTIES OF LINEAR PROGRAMMING MODEL

Any linear programming model (problem) must have the following properties:

(a) **The relationship between variables and constraints must be linear.**

(b) **The model must have an objective function.**

(c) **The model must have structural constraints.**

(d) **The model must have non-negativity constraint.**

Let us consider a product mix problem and see the applicability of the above properties.

Example 2.1. A company manufactures two products *X* and *Y*, which require, the following resources. The resources are the capacities machine M_1, M_2, and M_3. The available capacities are 50,25,and 15 hours respectively in the planning period. Product *X* requires 1 hour of machine M_2 and 1 hour of machine M_3. Product *Y* requires 2 hours of machine M_1, 2 hours of machine M_2 and 1 hour of machine M_3. The profit contribution of products *X* and *Y* are Rs.5/- and Rs.4/- respectively.

The contents of the statement of the problem can be summarized as follows:

Machines	*Products*		*Availability in hours*
	X	*Y*	
M_1	0	2	50
M_2	1	2	25
M_3	1	1	15
Profit in Rs. Per unit	5	4	

In the above problem, Products *X* and *Y* are competing candidates or variables.

Machine capacities are available resources. Profit contribution of products *X* and *Y* are given. Now let us formulate the model.

Let the company manufactures *x* units of *X* and y units of *Y*. As the profit contributions of *X* and *Y* are Rs.5/- and Rs. 4/- respectively. The objective of the problem is to maximize the profit *Z*, hence objective function is:

Maximize $Z = 5x + 4y$ $\longrightarrow$ OBJECTIVE FUNCTION.

This should be done so that the utilization of machine hours by products x and y should not exceed the available capacity. This can be shown as follows:

For Machine M_1 $0x + 2y \leq 50$
For Machine M_2 $1x + 2y \leq 25$ and $\longrightarrow$ **LINEAR STRUCTURAL CONSTRAINTS.**
For machine M_3 $1x + 1y \leq 15$

But the company can stop production of *x* and *y* or can manufacture any amount of *x* and *y*. It cannot manufacture negative quantities of *x* and *y*. Hence we have write,

Both x and y are ≥ 0 . $\longrightarrow$ **NON -NEGATIVITY CONSTRAINT.**

As the problem has got objective function, structural constraints, and non-negativity constraints and there exist a linear relationship between the variables and the constraints in the form of inequalities, the problem satisfies the properties of the Linear Programming Problem.

Basic Assumptions

The following are some important assumptions made in formulating a linear programming model:

1. It is assumed that the decision maker here is *completely certain* (*i.e.*, deterministic conditions) regarding all aspects of the situation, *i.e.*, availability of resources, profit contribution of the products, technology, courses of action and their consequences etc.
2. It is assumed that the relationship between variables in the problem and the resources available *i.e.*, constraints of the problem exhibits *linearity*. Here the term linearity implies proportionality and additivity. This assumption is very useful as it simplifies modeling of the problem.
3. We assume here *fixed technology*. Fixed technology refers to the fact that the production requirements are fixed during the planning period and will not change in the period.
4. It is assumed that the *profit contribution of a product remains constant*, irrespective of level of production and sales.

5. It is assumed that the decision variables are *continuous*. It means that the companies manufacture products in fractional units. For example, company manufacture 2.5 vehicles, 3.2 barrels of oil etc. This is referred too as the assumption of *divisibility*.

6. It is assumed that *only one decision* is required for the planning period. This condition shows that the linear programming model is a static model, which implies that the linear programming problem is a *single stage decision problem*. (Note: Dynamic Programming problem is a multistage decision problem).

7. All variables are restricted to *nonnegative values* (*i.e.*, their numerical value will be ≥ 0).

Terms Used in Linear Programming Problem

Linear programming is a method of obtaining an optimal solution or programme (say, product mix in a production problem), when we have limited resources and a good number of *competing candidates to consume* the limited resources in *certain proportion*. The term linear implies the condition of proportionality and additivity. The *programme* is referred as a course of action covering a specified period of time, say planning period. The manager has to find out the best course of action in the interest of the organization. This best course of action is termed as *optimal course of action or optimal solution* to the problem. A programme is optimal, when it *maximizes or minimizes* some measure or criterion of effectiveness, such as profit, sales or costs.

The term *programming* refers to a systematic procedure by which a particular program or plan of action is designed. Programming consists of a series of instructions and computational rules for solving a problem that can be worked out manually or can fed into the computer. In solving linear programming problem, we use a systematic method known as *simplex method* developed by American mathematician George B. Dantzig in the year 1947.

The candidates or activity here refers to number of products or any such items, which need the utilization of available resources in a certain required proportion. The available resources may be of any nature, such as money, area of land, machine hours, and man-hours or materials. But they are *limited in availability* and which are desired by the activities / products for consumption.

General Linear Programming Problem

A general mathematical way of representing a Linear Programming Problem (L.P.P.) is as given below:

$Z = c_1x_1 + c_2x_2 + \ldots c_nx_n$ **subjects to the conditions,** ⟶ **OBJECTIVE FUNCTION**

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1j}x_j + \ldots + ..a_{1n}x_n\ (\geq, =, \leq)\ b_1$
$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots\ldots\ldots\ldots + a_{2j}x_j + \ldots\ldots + a_{2n}x_n\ (\geq, =, \leq)\ b_2$
....
....
$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mj}x_j \ldots + a_{mn}x_n\ (\geq, =, \leq)\ b_m$ **Structural Constraints**

and all x_j **are = 0** ⟶ **NON NEGETIVITY CONSTRINT.**
Where $j = 1, 2, 3, \ldots n$

Where all $c_j s$, b_i s and $a_{ij} s$ are constants and $x_j s$ are decision variables.

To show the relationship between left hand side and right hand side the symbols ≤ , = , ≥ are used. Any one of the signs may appear in real problems. Generally ≤ sign is used for maximization problems and ≥ sign is used for minimization problems and in some problems, which are known as mixed problems we may have all the three signs. The word optimize in the above model indicates either maximise or minimize. The linear function, which is to be optimized, is the objective function. The inequality conditions shown are constraints of the problem. Finally all x_i *s* should be positive, hence the non-negativity function.

The steps for formulating the linear programming are:

1. ***Identify the unknown decision variables to be determined and assign symbols to them.***
2. ***Identify all the restrictions or constraints in the problem and express them as linear equations or inequalities of decision variables.***
3. ***Identify the objective or aim and represent it also as a linear function of decision variables.***

Construct linear programming model for the following *problems*:

MAXIMIZATION MODELS

Example 2.2. A retail store stocks two types of shirts *A* and *B*. These are packed in attractive cardboard boxes. During a week the store can sell a maximum of 400 shirts of type *A* and a maximum of 300 shirts of type *B*. The storage capacity, however, is limited to a maximum of 600 of both types combined. Type *A* shirt fetches a profit of Rs. 2/- per unit and type *B* a profit of Rs. 5/- per unit. How many of each type the store should stock per week to maximize the total profit? Formulate a mathematical model of the problem.

Solution: Here shirts *A* and *B* are problem variables. Let the store stock '*a*' units of *A* and '*b*' units of *B*. As the profit contribution of *A* and *B* are Rs.2/- and Rs.5/- respectively, objective function is:

Maximize $Z = 2a + 5b$ subjected to condition (s.t.)

Structural constraints are, stores can sell 400 units of shirt *A* and 300 units of shirt *B* and the storage capacity of both put together is 600 units. Hence the structural constraints are:

$1a + 0b \geq 400$ and $0a + 1b \leq 300$ for sales capacity and $1a + 1b \leq 600$ for storage capacity.

And non-negativity constraint is both *a* and *b* are ≥ 0. Hence the model is:

Maximize $Z = 2a + 5b$ s.t.

$1a + 0b \leq 400$

$0a + 1b \leq 300$

$1a + 1b \leq 600$ and

Both *a* and *b* are ≥ 0.

Problem 2.3. A ship has three cargo holds, forward, aft and center. The capacity limits are:

Forward 2000 tons, 100,000 cubic meters

Center 3000 tons, 135,000 cubic meters

Aft 1500 tons, 30,000 cubic meters.

The following cargoes are offered, the ship owners may accept all or any part of each commodity:

Commodity	*Amount in tons.*	*Volume/ton in cubic meters*	*Profit per ton in Rs.*
A	6000	60	60
B	4000	50	80
C	2000	25	50

In order to preserve the trim of the ship the weight in each hold must be proportional to the capacity in tons. How should the cargo be distributed so as to maximize profit? Formulate this as linear programming problem.

Solution: Problem variables are commodities, *A*, *B*, and *C*. Let the shipping company ship '*a*' units of *A* and '*b*' units of *B* and '*c*' units of *C*. Then Objective function is:

Maximize $Z = 60a + 80b + 50c$ s.t.

Constraints are:

Weight constraint: $6000a + 4000b + 2000c \leq 6{,}500$ (= 2000+3000+1500)

The tonnage of commodity is 6000 and each ton occupies 60 cubic meters, hence there are 100 cubic meters capacity is available.

Similarly, availability of commodities *B* and *C*, which are having 80 cubic meter capacities each. Hence capacity inequality will be:

$100a + 80b + 80c \leq 2{,}65{,}000$ (= 100,000+135,000+30,000). Hence the l.p.p. Model is:

Maximise $Z = 60a+80b+50c$ s.t. $100a = 6000/60 = 100$

$6000a + 4000b + 2000c \leq 6{,}500$ $80b = 4000/50 = 80$

$100a+80b+80c \leq 2{,}65{,}000$ and $80c = 2000/25 = 80$ etc.

a, b, c all ≥ 0

MINIMIZATION MODELS

Problem 2.4. A patient consult a doctor to check up his ill health. Doctor examines him and advises him that he is having deficiency of two vitamins, vitamin *A* and vitamin *D*. Doctor advises him to consume vitamin *A* and *D* regularly for a period of time so that he can regain his health. Doctor prescribes tonic *X* and tonic *Y*, which are having vitamin *A*, and *D* in certain proportion. Also advises the patient to consume **at least** 40 units of vitamin *A* and 50 units of vitamin Daily. The cost of tonics *X* and *Y* and the proportion of vitamin *A* and *D* that present in *X* and *Y* are given in the table below. Formulate l.p.p. to minimize the cost of tonics.

Vitamins	*Tonics*		*Daily requirement in units.*
	X	*Y*	
A	2	4	40
D	3	2	50
Cost in Rs. per unit.	5	3	

Solution: Let patient purchase *x* units of *X* and y units of *Y*.

Objective function: Minimize $Z = 5x + 3y$

Inequality for vitamin *A* is $2x + 4y \geq 40$ (Here **at least** word indicates that the patient can consume more than 40 units but not less than 40 units of vitamin A daily).

Similarly the inequality for vitamin D is $3x + 2y \geq 50$.

For non–negativity constraint the patient cannot consume negative units. Hence both *x* and *y* must be ≥ 0.

Now the l.p.p. model for the problem is:

Minimize $Z = 5x + 3y$ s.t.

$2x + 4y \geq 40$

$3x + 2y \geq 50$ and

Both x and y are ≥ 0.

Problem 2.5. A machine tool company conducts a job-training programme at a ratio of one for every ten trainees. The training programme lasts for one month. From past experience it has been found that out of 10 trainees hired, only seven complete the programme successfully. (The unsuccessful trainees are released). Trained machinists are also needed for machining. The company's requirement for the next three months is as follows:

January: 100 machinists, February: 150 machinists and March: 200 machinists.

In addition, the company requires 250 trained machinists by April. There are 130 trained machinists available at the beginning of the year. Pay roll cost per month is:

Each trainee Rs. 400/- per month.

Each trained machinist (machining or teaching): Rs. 700/- p.m.

Each trained machinist who is idle: Rs.500/- p.m.

(Labour union forbids ousting trained machinists). Build a l.p.p. for produce the minimum cost hiring and training schedule and meet the company's requirement.

Solution: There are three options for trained machinists as per the data given. (*i*) A trained machinist can work on machine, (*ii*) he can teach or (*iii*) he can remain idle. It is given that the number of trained machinists available for machining is fixed. Hence the unknown decision variables are the number of machinists goes for teaching and those who remain idle for each month. Let,

'a' be the trained machinists teaching in the month of January.

'b' be the trained machinists idle in the month of January.

'c' be the trained machinists for teaching in the month of February.

'd' be the trained machinists remain idle in February.

'e' be the trained machinists for teaching in March.

'f' be the trained machinists remain idle in the month of March.

The constraints can be formulated by the rule that the number of machinists used for (machining + teaching + idle) = Number of trained machinists available at the beginning of the month.

For January $100 + 1a + 1b \geq 130$

For February, $150 + 1c + 1d = 130 + 7a$ (Here $7a$ indicates that the number of machinist trained is $10 \times a = 10a$. But only 7 of them are successfully completed the training *i.e.* $7a$).

For the month of March, $200 + 1e + 1f \geq 130 + 7a + 7c$

The requirement of trained machinists in the month of April is 250, the constraints for this will be

$130 + 7a + 7c + 7e \geq 250$ and the objective function is

Minimize $Z = 400\ (10a + 10c + 10e) + 700\ (1a + 1c + 1e) + 400\ (1b + 1d + 1f)$ and the non-negativity constraint is a, b, c, d, e, f all ≥ 0. The required model is:

Minimize $Z = 400\ (10a + 10c + 10e) + 700\ (1a + 1c + 1e) + 400\ (1b + 1d + 1f)$ s.t.

$100 + 1a + 1b \geq 130$

$150 + 1c + 1d \geq 130 + 7a$

$200 + 1e + 1f \geq 130 + 7a + 7c$

$130 + 7a + 7c + 7e \geq 250$ and

a, b, c, d, e, f all ≥ 0.

METHODS FOR THE SOLUTION OF A LINEAR PROGRAMMING PROBLEM

Linear Programming, is a method of solving the type of problem in which two or more **candidates** or **activities** are competing to utilize the available limited resources, with a view to **optimize** the **objective function** of the problem. The objective may be to maximize the **returns** or to minimize the **costs**. The various methods available to solve the problem are:

1. The Graphical Method when we have two decision variables in the problem. (To deal with more decision variables by graphical method will become complicated, because we have to deal with planes instead of straight lines. Hence in graphical method let us limit ourselves to two variable problems.
2. The Systematic Trial and Error method, where we go on giving various values to variables until we get optimal solution. This method takes too much of time and laborious, hence this method is not discussed here.
3. The Vector method. In this method each decision variable is considered as a vector and principles of vector algebra is used to get the optimal solution. This method is also time consuming, hence it is not discussed here.
4. The Simplex method. When the problem is having more than two decision variables, simplex method is the most powerful method to solve the problem. It has a systematic programme, which can be used to solve the problem.

One problem with two variables is solved by using both graphical and simplex method, so as to enable the reader to understand the relationship between the two.

Graphical Method

In graphical method, the inequalities (structural constraints) are considered to be equations. This is because; one cannot draw a graph for inequality. Only two variable problems are considered, because we can draw straight lines in two-dimensional plane (*X*- axis and *Y*-axis). More over as we have non-negativity constraint in the problem that is all the decision variables must have positive values always the solution to the problem lies in first quadrant of the graph. Some times the value of variables may fall in quadrants other than the first quadrant. In such cases, the line joining the values of the variables must be extended in to the first quadrant. The procedure of the method will be explained in detail while solving a numerical problem. The characteristics of Graphical method are:

(i) Generally the method is used to solve the problem, when it involves two decision variables.

(ii) For three or more decision variables, the graph deals with planes and requires high imagination to identify the solution area.

(iii) Always, the solution to the problem lies in first quadrant.

(iv) This method provides a basis for understanding the other methods of solution.

Problem 2.6. A company manufactures two products, *X* and *Y* by using three machines *A*, *B*, and *C*. Machine *A* has 4 hours of capacity available during the coming week. Similarly, the available capacity of machines *B* and *C* during the coming week is 24 hours and 35 hours respectively. One unit of product *X* requires one hour of Machine *A*, 3 hours of machine *B* and 10 hours of machine *C*. Similarly one unit of product *Y* requires 1 hour, 8 hour and 7 hours of machine *A*, *B* and *C* respectively. When one unit of *X* is sold in the market, it yields a profit of Rs. 5/- per product and that of *Y* is Rs. 7/- per unit. Solve the problem by using graphical method to find the optimal product mix.

Solution: The details given in the problem is given in the table below:

Machines	*Products (Time required in hours).*		*Available capacity in hours.*
	X	*Y*	
A	1	1	4
B	3	8	24
C	10	7	35
Profit per unit in Rs.	5	7	

Let the company manufactures x units of *X* and y units of *Y*, and then the L.P. model is:

Maximise $Z = 5x + 7y$ s.t.

$1x + 1y \leq 4$

$3x + 8y \leq 24$

$10x + 7y \leq 35$ and

Both x and y are ≥ 0.

As we cannot draw graph for inequalities, let us consider them as equations.

Maximise $Z = 5x + 7y$ s.t.

$1x + 1y = 4$

$3x + 8y = 24$

$10x + 7y = 35$ and both x and y are ≥ 0

Let us take machine *A*. and find the boundary conditions. If $x = 0$, machine *A* can manufacture $4/1 = 4$ units of y.

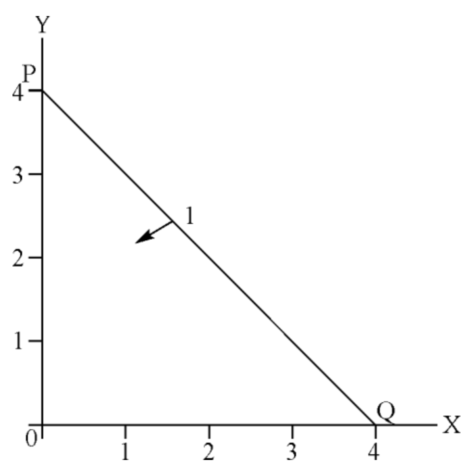


Figure 2.1 Graph for machine A

Similarly, if $y = 0$, machine *A* can manufacture $4/1 = 4$ units of x. For other machines:

Machine *B* When $x = 0$, $y = 24/8 = 3$ and when $y = 0$ $x = 24/3 = 8$

Machine *C* When $x = 0$, $y = 35/10 = 3.5$ and when $y = 0$, $x = 35 / 7 = 5$.

These values we can plot on a graph, taking product *X* on *x*-axis and product *Y* on *y*- axis.

First let us draw the graph for machine *A*. In figure 2. 1 we get line 1 which represents $1x + 1y = 4$. The point *P* on *Y* axis shows that the company can manufacture 4 units of *Y* only when does not want to manufacture *X*. Similarly the point *Q* on *X* axis shows that the company can manufacture 4 units of *X*, when does not want to manufacture *Y*. In fact triangle *POQ* is the capacity of machine *A* and the line *PQ* is the boundary line for capacity of machine *A*.

Similarly figure 2.2 show the Capacity line RS for machine *B*. and the triangle *ROS* shows the capacity of machine *B i.e.*, the machine *B* can manufacture 3 units of product *Y* alone or 8 units of product *X* alone.

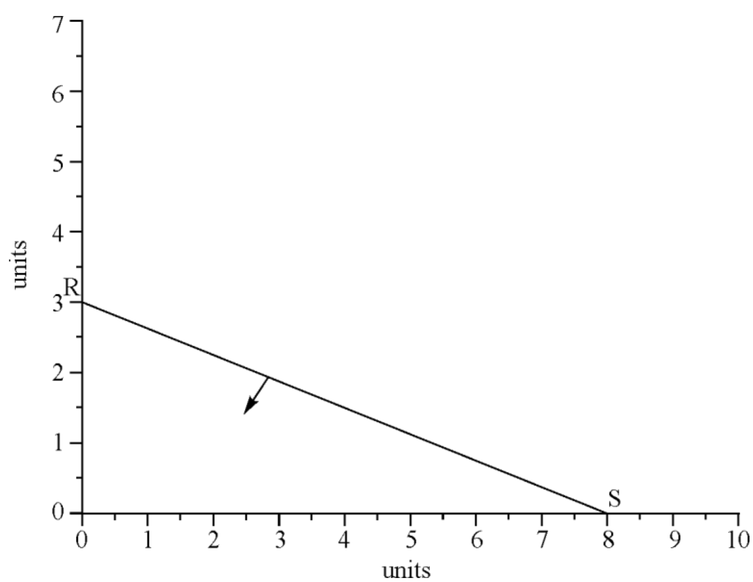


Figure 2.2. Graph for machine B

The graph 2.3 shows that the machine *C* has a capacity to manufacture 5 units of *Y* alone or 3.5 units of *X* alone. Line *TU* is the boundary line and the triangle *TOU* is the capacity of machine *C*.

The graph is the combined graph for machine *A* and machine *B*. Lines *PQ* and RS intersect at *M*. The area covered by both the lines indicates the products (*X* and *Y*) that can be manufactured by using both machines. This area is the feasible area, which satisfies the conditions of inequalities of machine *A* and machine *B*. As *X* and *Y* are processed on *A* and *B* the number of units that can be manufactured will vary and the there will be some idle capacities on both machines. The idle capacities of machine *A* and machine *B* are shown in the figure 2.4.

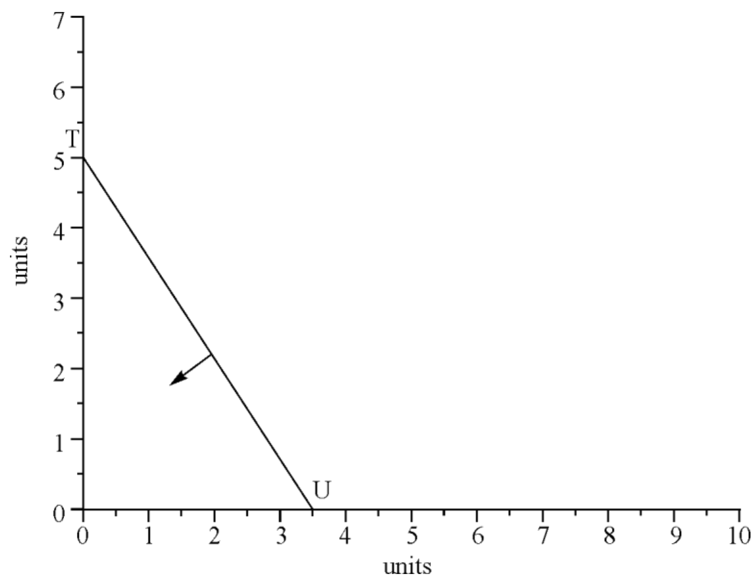


Figure 2.3. Graph for machine C

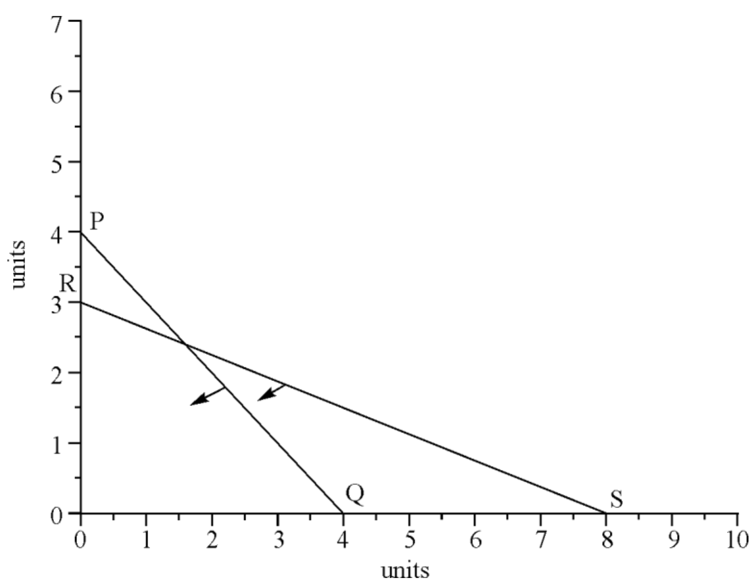


Figure 2.4. Graph of Machines A and B

Figure 2.5 shows the feasible area for all the three machines combined. This is the fact because a products *X* and *Y* are complete when they are processed on machine *A*, *B*, and *C*. The area covered by all the three lines *PQ*. *RS*, and *TU* form a closed polygon *ROUVW*. This polygon is the feasible area for the three machines. This means that all the points on the lines of polygon and any point within the polygon satisfies the inequality conditions of all the three machines. To find the optimal solution, we have two methods.

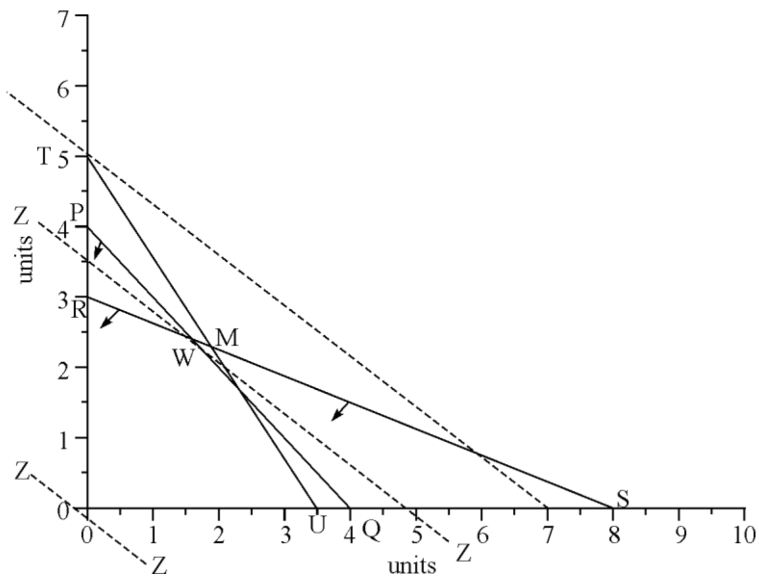


Figure 2.5. Graph for machine A, B and C combined

Method 1. Here we find the co-ordinates of corners of the closed polygon *ROUVW* and substitute the values in the objective function. In maximisaton problem, we select the co-ordinates giving maximum value. And in minimisaton problem, we select the co-ordinates, which gives minimum value.
In the problem the co-ordinates of the corners are:

R = (0, 3.5), *O* = (0,0), *U* = (3.5,0), *V* = (2.5, 1.5) and *W* = (1.6,2.4). Substituting these values in objective function:

$$Z_{(0,3.5)} = 5 \times 0 + 7 \times 3.5 = \text{Rs. } 24.50, \text{ at point } R$$
$$Z_{(0,0)} = 5 \times 0 + 7 \times 0 = \text{Rs. } 00.00, \text{ at point } O$$
$$Z_{(3.5,0)} = 5 \times 3.5 + 7 \times 0 = \text{Rs. } 17.5 \text{ at point } U$$
$$Z_{(2.5,\ 1.5)} = 5 \times 2.5 + 7 \times 1.5 = \text{Rs. } 23.00 \text{ at point } V$$
$$\boldsymbol{Z_{(1.6,\ 2.4)} = 5 \times 1.6 + 7 \times 2.4 = \textbf{Rs. 24.80 at point } W}$$

Hence the optimal solution for the problem is company has to manufacture 1.6 units of product *X* and 2.4 units of product *Y*, so that it can earn a maximum profit of Rs. 24.80 in the planning period.

Method 2. Isoprofit Line Method: Isoprofit line, a line on the graph drawn as per the objective function, assuming certain profit. On this line any point showing the values of x and y will yield same profit. For example in the given problem, the objective function is Maximise $Z = 5x + 7y$. If we assume a profit of Rs. 35/-, to get Rs. 35, the company has to manufacture either 7 units of *X* or 5 units of *Y*.

Hence, we draw line *ZZ* (preferably dotted line) for $5x + 7y = 35$. Then draw parallel line to this line *ZZ* at origin. The line at origin indicates zero rupees profit. No company will be willing to earn zero rupees profit. Hence slowly move this line away from origin. Each movement shows a certain profit, which is greater than Rs.0.00. While moving it touches corners of the polygon showing certain higher profit. Finally, it touches the farthermost corner covering all the area of the closed polygon. This point where the line passes (farthermost point) is the **OPTIMAL SOLUTION** of the problem. In the figure 2.6. the line *ZZ* passing through point *W* covers the entire area of the polygon, hence it is the point that yields highest profit. Now point *W* has co-ordinates (1.6, 2.4). **Now Optimal profit $Z = 5 \times 1.6 + 7 \times 2.4 =$ Rs. 24.80.**

Points to be Noted:

(i) In case lsoprofit line passes through more than one point, then it means that the problem has more than one optimal solution, *i.e.*, alternate solutions all giving the same profit. This helps the manager to take a particular solution depending on the demand position in the market. He has options.

(ii) If the lsoprofit line passes through single point, it means to say that the problem has unique solution.

(iii) If the Isoprofit line coincides any one line of the polygon, then all the points on the line are solutions, yielding the same profit. Hence the problem has innumerable solutions.

(iv) If the line do not pass through any point (in case of open polygons), then the problem do not have solution, and we say that the problem is UNBOUND.

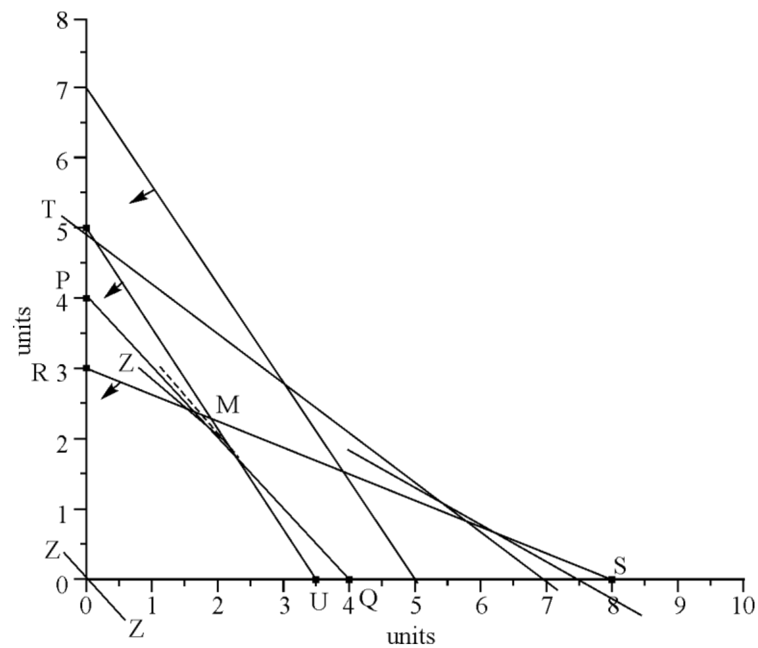


Figure 2.6. ISO profit line method.

Now let us consider some problems, which are of mathematical interest. Such problems may not exist in real world situation, but they are of mathematical interest and the student can understand the mechanism of graphical solution.

Problem 2.7. Solve graphically the given linear programming problem. (Minimization Problem).

Minimize $Z = 3a + 5b$ S.T

$-3a + 4b \leq 12$

$2a - 1b \geq -2$

$2a + 3b \geq 12$

$1a + 0b \geq 4$

$0a + 1b \geq 2$

And both a and b are ≥ 0.

Points to be Noted:

(i) In inequality $-3a + 4b \leq 12$, product/the candidate/activity requires –3 units of the resource. It does not give any meaning (or by manufacturing the product A the manufacturer can save 3 units of resource No.1 or one has to consume –3 units of A. (All these do not give any meaning as far as the practical problems or real world problems are concerned).

(ii) In the second inequality, on the right hand side we have –2. This means that –2 units of resource is available. It is absolutely wrong. Hence in solving a l.p.p. problem, one must see that the right hand side we must have always a positive integer. Hence the inequality is to be multiplied by –1 so that the inequality sign also changes. In the present case it becomes: $-2a + 1b \leq 2$.

Solution: Now the problem can be written as:

Minimize $Z = 3a + 5b$ S.T.

When converted into equations they can be written as Min. $Z = 3a + 5b$ *S.T.*

$-3a + 4b \leq 12$	$-3a + 4b = 12$
$-2a + 1b \leq 2$	$-2a + 1b = 2$
$2a - 3b \geq 12$	$2a - 3b = 12$
$1a + 0b \leq 4$	$1a + 0b = 4$

$0a + 1b \geq 2$ and both a and b are $\geq = 0.0a + 1b \geq 2$ and both a and b are ≥ 0.

The lines for inequalities $-3a + 4b \leq 12$ and $-2a + 1b \leq 2$ starts from quadrant 2 and they are to be extended into quadrant 1. Figure 2.7 shows the graph, with Isocost line.

Isocost line is a line, the points on the line gives the same cost in Rupees. We write Isocost line at a far off place, away from the origin by assuming very high cost in objective function. Then we move line parallel towards the origin (in search of least cost) until it passes through a single corner of the closed polygon, which is nearer to the origin, (Unique Solution), or passes through more than one point, which are nearer to the origin (more than one solution) or coincides with a line nearer to the origin and the side of the polygon (innumerable solution). The solution for the problem is the point P (3,2,) and the **Minimum cost is Rs. 3 × 3 + 2 × 5 = Rs. 19/-**

Problem 2.8. The cost of materials *A* and *B* is Re.1/- per unit respectively. We have to manufacture an alloy by mixing these to materials. The process of preparing the alloy is carried out on three facilities *X*, *Y* and *Z*. Facilities *X* and *Z* are machines, whose capacities are limited. *Y* is a furnace, where heat treatment takes place and the material must use a minimum given time (even if it uses more than the required, there is no harm). Material *A* requires 5 hours of machine *X* and it does not require processing on machine *Z*. Material *B* requires 10 hours of machine *X* and 1 hour of machine *Z*. Both *A* and *B* are to be heat treated at last one hour in furnace *Y*. The available capacities of *X*, *Y* and *Z* are 50 hours, 1 hour and 4 hours respectively. Find how much of *A* and *B* are mixed so as to minimize the cost.

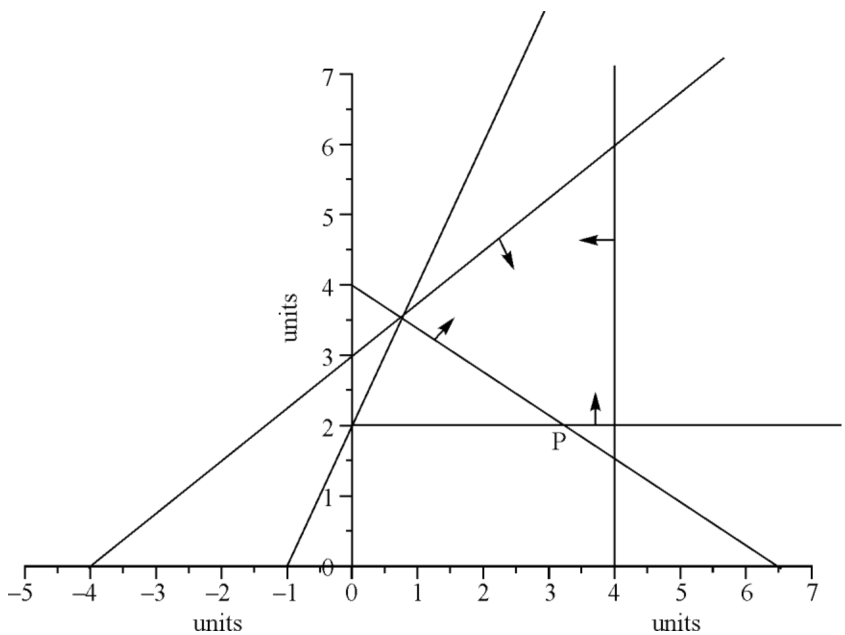


Figure 2.7. Graph for the problem 2.7

Solution: The l.p.p. model is:

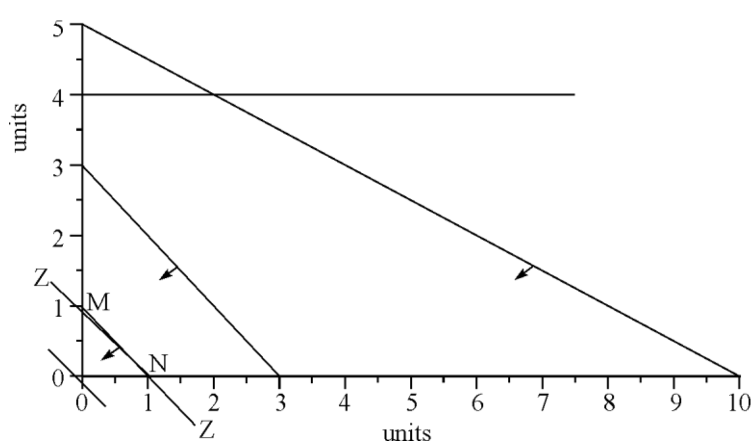


Figure 2.8. Graph for the problem 2.8

Minimize $Z = 1a + 1b$ *S.T.* Equations are: Minimise $Z = 1a + 1b$ *S.T*

$5a + 10b \leq 50$, $5a + 10b = 50$

$1a + 1b \geq 1$ $1a + 1b = 1$

$0a + 1b \leq 4$ and both a and b are ≥ 0. $0a + 1b = 4$ and both a and b are ≥ 0.

Figure 2.8 shows the graph. Here Isocost line coincides with side of the polygon, *i.e.*, the line MN. Hence the problem has innumerable solutions. Any value on line (1,1) will give same cost. **Optimal cost is Re.1/-**

Problem 2.9. Maximise $Z = 0.75\ a + 1b$ *S.T.*

$1a + 1b \geq 0$

$-0.5\ a + 1b \leq 1$ and both a and b are ≥ 0.

Solution: Writing the inequalities as equations,

$1a + 1b = 0$ *i.e.*, $a = b = 1$ which is a line passing through origin at 45°

$0.5\ a + 1\ b = 1$ and both a and b are ≥ 0. Referring to figure 2.9.

The polygon is not closed one *i.e.*, the feasible area is unbound. When Isoprofit line is drawn, it passes through open side of the polygon and it does not coincide with any corner or any line. Hence the line can be moved indefinitely, still containing a part of the feasible area. Thus there is no finite maximum value of Z. That the value of Z can be increased indefinitely. **When the value of Z can be increased indefinitely, the problem is said to have an UNBOUND solution.**

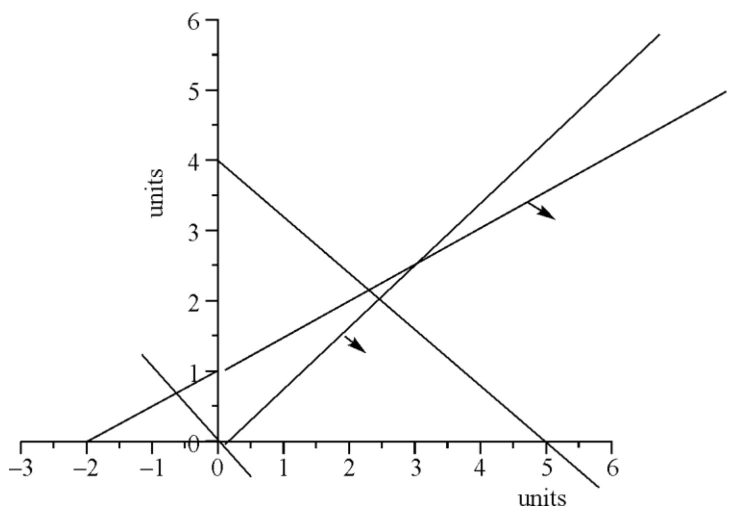


Figure 2.9. Graph for the problem 2.9

Problem 2.10. A company manufactures two products X and Y on two facilities A and B. The data collected by the analyst is presented in the form of inequalities. Find the optimal product mix for maximising the profit.

Maximise $Z = 6x - 2y$ S.T. Writing in the equation form: Maximise $Z = 6x - 2y$ S.T.

$2x - 1y \leq 2$ $2x - 1y = 2$

$1x + 0y \leq 3$ and both x and y are ≥ 0 $1x + 0y = 3$ and both x and y are ≥ 0

Solution: The straight line for $2x - 1y = 2$ starts in 4^{th} quadrant and is to be extended into first quadrant. The polygon is not a closed one and the feasible area is unbound. But when an Isoprofit line is drawn it passes through a corner of the feasible area that is the corner M of the open polygon. The (figure 2.10) coordinates of M are (3, 4) and the **maximum Z = Rs. 10/-**

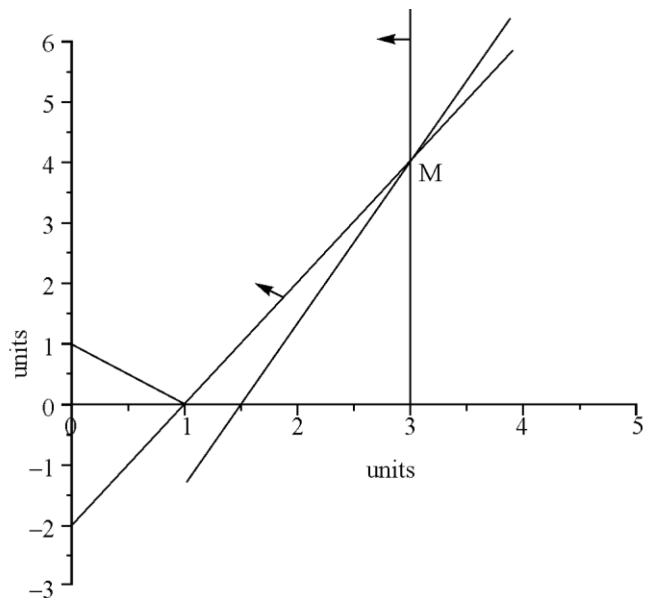


Figure 2.10. Graph for the problem 2.10

Problem 2.11. A company manufactures two products *X* and *Y*. The profit contribution of *X* and *Y* are Rs.3/- and Rs. 4/- respectively. The products *X* and *Y* require the services of four facilities. The capacities of the four facilities *A*, *B*, *C*, and *D* are limited and the available capacities in hours are 200 Hrs, 150 Hrs, and 100 Hrs. and 80 hours respectively. Product *X* requires 5, 3, 5 and 8 hours of facilities *A*, *B*, *C* and *D* respectively. Similarly the requirement of product *Y* is 4, 5, 5, and 4 hours respectively on *A*, *B*, *C* and *D*. Find the optimal product mix to maximise the profit.

Solution: Enter the given data in the table below:

	products		
Machines	*X* (Time in hours)	*Y*	*Availability in hours.*
A	5	4	200
B	3	5	150
C	5	4	100
D	8	4	80
Profit in Rs. Per unit:	3	4	

The inequalities and equations for the above data will be as follows. Let the company manufactures *x* units of *X* and *y* units of *Y*. (Refer figure 2.11)

Maximise $Z\ 3x + 4y$ S.T.	Maximise $Z = 3x + 4y$ S.T.
$5x + 4y \leq 200$	$5x + 4y = 200$
$3x + 5y \leq 150$	$3x + 5y = 150$
$5x + 4y \leq 100$	$5x + 4y = 100$
$8x + 4y \leq 80$	$8x + 4y = 80$
And both x and y are ≥ 0	And both x and y are ≥ 0

In the graph the line representing the equation $8x + 4y$ is out side the feasible area and hence it is a redundant equation. It does not affect the solution. The Isoprofit line passes through corner T of the polygon and is the point of maximum profit. Therefore $Z_T = Z_{(32,10)} = 3 \times 32 + 4 \times 10 =$ Rs. 136/.

Problem 2.12. This problem is of mathematical interest.

Maximise $Z = 3a + 4b$ S.T. Converting and writing in the form of equations,

$1a - 1b \leq -1.$	Maximise $Z = 3a + 4b$ S.T
$-1a + 1b \leq 0$	$1a - 1b = 0$
And both a and b are ≥ 0	$-1a + 1b = 0$
	And both a and b are ≥ 0

Referring to figure 2.11, the straight line for equation 1 starts in second quadrant and extended in to first quadrant. The line for equation 2 passes through the origin. We see that there is no point, which satisfies both the constraints simultaneously. Hence there is no feasible solution. **Given l.p.p. has no feasible solution.**

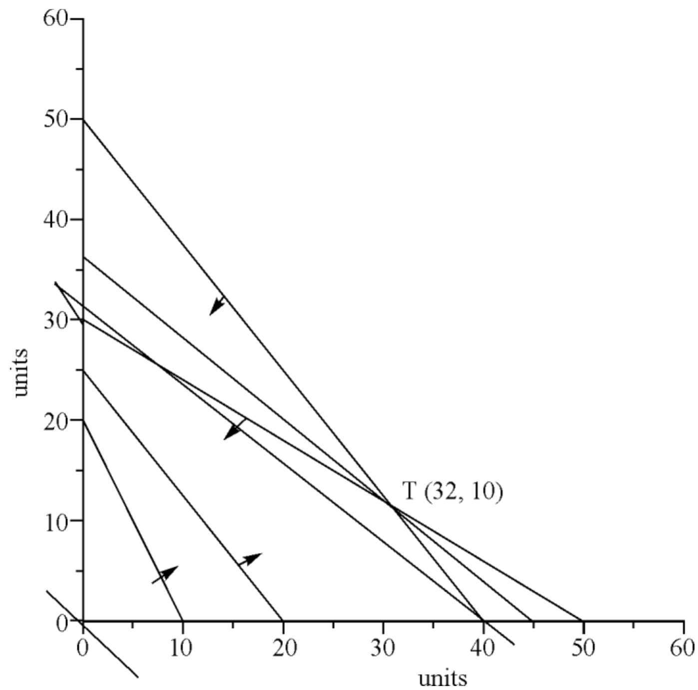


Figure 2.11. Graph for the problem 2.11

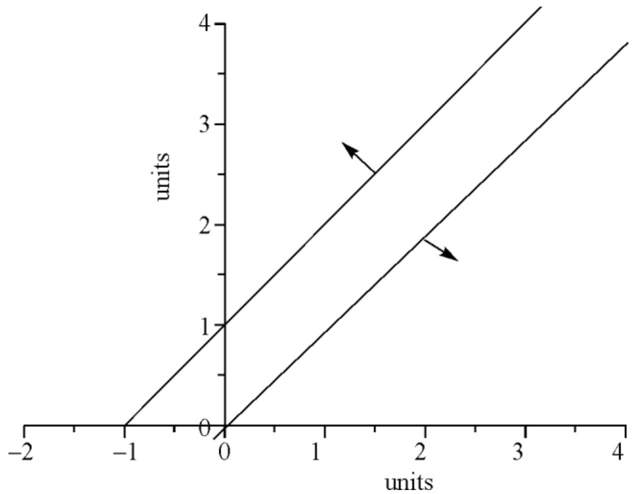


Figure 2.12. Graph for the problem 2.12

Problem 2.13. Solve the l.p.p. by graphical method.

Maximise $Z = 3a + 2b$ S.T.

$1a + 1b \leq 4$

$1a - 1b \leq 2$ and both a and b are ≥ 0.

Solution: The figure 2.13 shows the graph of the equations.

Equations are: Maximise $Z = 3a + 2b$ S.T.

$1a + 1b = 4$

$1a - 1b = 2$ and both a and b are ≥ 0.

In the figure the Isoprofit line passes through the point N (3,1). Hence **optimal Profit $Z = 3 \times 3 + 2 \times 1$ = Rs.11 /-**

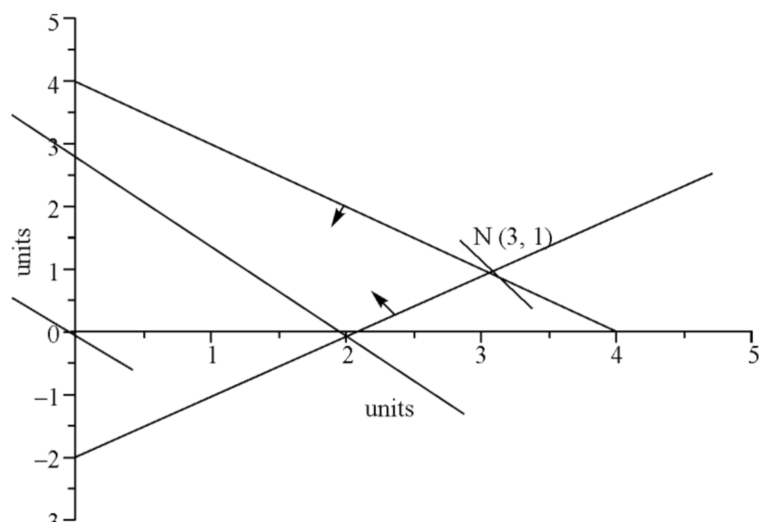


Figure 2.13. Graph for the problem 2.13

Problem 2.14: Formulate the l.p.p. and solve the below given problem graphically.
Old hens can be bought for Rs.2.00 each but young ones costs Rs. 5.00 each. The old hens lay 3 eggs per week and the young ones lay 5 eggs per week. Each egg costs Rs. 0.30. A hen costs Rs.1.00 per week to feed. If the financial constraint is to spend Rs.80.00 per week for hens and the capacity constraint is that total number of hens cannot exceed 20 hens and the objective is to earn a profit more than Rs.6.00 per week, find the optimal combination of hens.

Solution: Let x be the number of old hens and y be the number of young hens to be bought. Now the old hens lay 3 eggs and the young one lays 5 eggs per week. Hence total number of eggs one get is $3x + 5y$.

Total revenues from the sale of eggs per week is Rs. 0.30 $(3x + 5y)$ *i.e.*, $0.90\, x + 1.5\, y$

Now the total expenses per week for feeding hens is Re.1 $(1x + 1y)$ *i.e.*, $1x + 1y$.

Hence the net income = Revenue – Cost = $(0.90\, x + 1.5\, y) - (1\, x + 1\, y) = -\,0.1\, x + 0.5\, y$ or $0.5y - 0.1\, x$. Hence the desired l.p.p. is

Maximise $Z = 0.5\, y - 0.1 \times$ S.T.

$2\, x + 5\, y \leq 80$

$1x + 1y \leq 20$ and both x and y are ≥ 0

The equations are:

Maximise $Z = 0.5\, y - 0.1 \times$ S.T.

$2\, x + 5\, y = 80$

$1x + 1y = 20$ and both x and y are ≥ 0

In the figure 2.13, which shows the graph for the problem, the isoprofit line passes through the point C. Hence $Zc = Z\,(0{,}16)$ = Rs.8.00. Hence, one has to buy 16 young hens and his weekly profit will be Rs.8.00

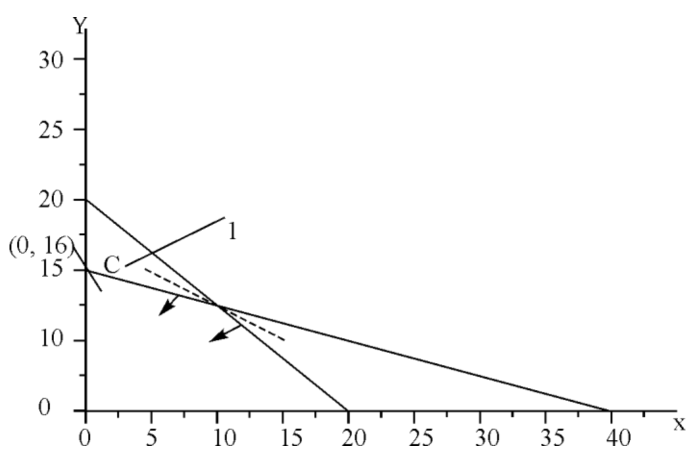


Figure 2.14. Graph for the problem 2.14

***Point to Note*: In case in a graphical solution, after getting the optimal solution, one more constraint is added, we may come across following situations.**

(i) The feasible area may reduce or increase and the optimal solution point may be shifted depending the shape of the polygon leading to decrease or increase in optimal value of the objective function.

(ii) Some times the new line for the new constraint may remain as redundant and imposes no extra restrictions on the feasible area and hence the optimal value will not change.

(iii) Depending on the position of line for the new constraint, there may not be any point in the feasible area and hence there may not be a solution. OR the isoprofit line may coincide with a line and the problem may have innumerable number of solutions.

SUMMARY

1. The graphical method for solution is used when the problem deals with 2 variables.
2. The inequalities are assumed to be equations. As the problem deals with 2 variables, it is easy to write straight lines as the relationship between the variables and constraints are linear. In case the problem deals with three variables, then instead of lines we have to draw planes and it will become very difficult to visualize the feasible area.
3. If at all there is a feasible solution (feasible area of polygon) exists then, the feasible area region has an important property known as **convexity Property** in geometry. (**Convexity** means: Convex polygon consists of a set points having the property that the segment joining any two points in the set is *entirely* in the convex set. There is a mathematical theorem, which states, "The points which are simulations solutions of a system of inequalities of the $\leq$ type form a polygonal convex set".

   The regions will not have any holes in them, *i.e.*, they are solids and the boundary will not have any breaks. This can be clearly stated that **joining any two points in the region also lies in the region.**
4. The boundaries of the regions are lines or planes.
5. There will be corners or extreme points on the boundary and there will be edges joining the various corners. The closed figure is known as **polygon**.
6. The objective function of a maximisation is represented by assuming suitable outcome (revenue) and is known as **Isoprofit line**. In case of minimisation problem, assuming suitable cost, a line, known as **Isocost line**, represents the objective function.
7. If isoprofit or isocost line coincides with one corner, then the problem has **unique** solution. In case it coincides with more than one point, then the problem has **alternate** solutions. If

the isoprofit or isocost line coincides with a line, then the problem will have **innumerable number** of solutions.

8. The different situation was found when the objective function could be made arbitrarily large. Of course, no corner was optimal in that case.

QUESTIONS

1. An aviation fuel manufacturer sells two types of fuel A and B. Type A fuel is 25 % grade 1 gasoline, 25 % of grade 2 gasoline and 50 % of grade 3 gasoline. Type B fuel is 50 % of grade 2 gasoline and 50 % of grade 3 gasoline. Available for production are 500 liters per hour grade 1 and 200 liters per hour of grade 2 and grade 3 each. Costs are 60 paise per liter for grade 1, 120 paise for grade 2 and100 paise for grade 3. Type A can be sold at Rs. 7.50 per liter and B can be sold at Rs. 9.00 per liter. How much of each fuel should be made and sold to maximise the profit.
2. A company manufactures two products X_1 and X_2 on three machines A, B, and C. X_1 require 1 hour on machine A and 1 hour on machine B and yields a revenue of Rs.3/-. Product X_2 requires 2 hours on machine A and 1 hour on machine B and 1 hour on machine C and yields revenue of Rs. 5/-. In the coming planning period the available time of three machines A, B, and C are 2000 hours, 1500 hours and 600 hours respectively. Find the optimal product mix.
3. Maximize $Z = 1\,x + 1\,y$ S.T.

   $1\,x + 2\,y \le 2000$

   $1\,x + 1\,y \le 1500$

   $0\,x + 1\,y \le 600$ and both x and y are ≥ 0.
4. Maximise $Z = 8000a + 7000b$ S.T.

   $3\,a + 1\,b \le 66$

   $1\,a + 1\,b \le 45$

   $1\,a + 0\,b \le 20$

   $0\,a + 1\,b \le 40$ and both a and b are ≥ 0.
5. Minimise $Z = 1.5\,x + 2.5\,y$ S.T.

   $1\,x + 3\,y \ge 3$

   $1\,x + 6\,y \ge 2$ and both x and $y \ge 0$
6. Maximise $Z = 3\,a + 2\,b$ S.T.

   $1a - 1\,b \le 1$

   $1\,a + 1b \ge 3$ and both x and y are ≥ 0
7. Maximise $Z = -3\,x + 2\,y$ S.T.

   $1\,x + 0\,y \le 3$

   $1\,x - 1\,y \le 0$ and both x and y are ≥ 0

8. Maximize $Z = -1\,a + 2b$ S.T.
$-1\,a + 1\,b \le 1$
$-1\,a + 2\,b \le 4$ and both a and b are ≥ 0.
9. Maximise $Z = 3\,x - 2\,y$ S.T.
$1x + 1\,y \le 1$
$2\,x + 2\,y \ge 4$ and both x and y are ≥ 0
10. Maximize $Z = 1x + 1\,y$ S.T.
$1\,x - 1y \ge 0$
$-3x + 1\,y \ge 3$ and both x and $y \ge 0$

***CHAPTER* – 3**

Linear Programming Models:

(Solution by Simplex Method)
Resource Allocation Model – Maximisation Case

INTRODUCTION

As discussed earlier, there are many methods to solve the Linear Programming Problem, such as Graphical Method, Trial and Error method, Vector method and Simplex Method. Though we use graphical method for solution when we have two problem variables, the other method can be used when there are more than two decision variables in the problem. Among all the methods, **SIMPLEX METHOD** is most powerful method. It deals with iterative process, which consists of first designing a **Basic Feasible Solution** or a **Programme** and proceed towards the **OPTIMAL SOLUTION** and testing each feasible solution for **Optimality** to know whether the solution on hand is optimal or not. If not an optimal solution, redesign the programme, and test for optimality until the test confirms **OPTIMALITY.** Hence we can say that the Simplex Method depends on two concepts known as **Feasibility** and **optimality**.

The simplex method is based on the property that the optimal solution to a linear programming problem, if it exists, can always be found in one of the basic feasible solution. The simplex method is quite simple and mechanical in nature. The iterative steps of the simplex method are repeated until a finite optimal solution, if exists, is found. If no optimal solution, the method indicates that no finite solution exists.

COMPARISION BETWEEN GRAPHICAL AND SIMPLEX METHODS

1. The graphical method is used when we have two decision variables in the problem. Whereas in Simplex method, the problem may have any number of decision variables.
2. In graphical method, the inequalities are assumed to be equations, so as to enable to draw straight lines. But in Simplex method, the inequalities are converted into equations by:
   (i) Adding a **SLACK VARIABLE** in maximisation problem and subtracting a **SURPLUS VARIABLE** in case of minimisation problem.
3. In graphical solution the **Isoprofit** line moves away from the origin to towards the far off point in maximisation problem and in minimisation problem, the **Isocost** line moves from far off distance towards origin to reach the nearest point to origin.
4. In graphical method, the areas outside the feasible area (area covered by all the lines of constraints in the problem) indicates idle capacity of resource where as in Simplex method, the presence of slack variable indicates the idle capacity of the resources.

5. In graphical solution, if the isoprofit line coincides with more than one point of the feasible polygon, then the problem has second alternate solution. In case of Simplex method the net-evaluation row has zero for non-basis variable the problem has alternate solution. (If two alternative optimum solutions can be obtained, the infinite number of optimum, solutions can be obtained).

However, as discussed in the forth coming discussion, the beauty of the simplex method lies in the fact that the relative exchange profitabilities of all the non-basis variables (vectors) can be determined simultaneously and easily; the replacement process is such that the new basis does not violate the feasibility of the solution.

MAXIMISATION CASE

Problem 3.1: A factory manufactures two products *A* and *B* on three machines *X*, *Y*, and *Z*. Product A requires 10 hours of machine *X* and 5 hours of machine *Y* a one our of machine *Z*. The requirement of product *B* is 6 hours, 10 hours and 2 hours of machine *X*, *Y* and *Z* respectively. The profit contribution of products *A* and *B* are Rs. 23/– per unit and Rs. 32 /– per unit respectively. In the coming planning period the available capacity of machines *X*, *Y* and *Z* are 2500 hours, 2000 hours and 500 hours respectively. Find the optimal product mix for maximizing the profit.

Solution:

The given data is:

Machines	*Products*		*Capacity in hours*
	A *Hrs.*	*B* *Hrs.*	
X	10	6	2500
Y	5	10	2000
Z	1	2	500
Profit/unit Rs.	23	32	—

Let the company manufactures *a* units of *A* and *b* units of *B*. Then the inequalities of the constraints (machine capacities) are:

Maximise $Z = 23\,a + 32\,b$ S.T. $\longrightarrow$ OBJECTIVE FUNCTION

$10\,a + 6\,b \leq 2500$

$5\,a + 10\,b \leq 2000$ $\longrightarrow$ STRUCTURAL CONSTRAINTS.

$1\,a + 2\,b \leq 500$

And both a and b are ≥ 0. $\longrightarrow$ NON-NEGATIVITY CONSTRAINT.

Now the above inequalities are to be converted into equations.

Take machine *X*: One unit of product *A* requires 10 hours of machine *X* and one unit of product *B* require 6 units. But company is manufacturing a units of *A* and *b* units of *B,* hence both put together must be **less than or equal to** 2,500 hours. Suppose a = 10 and *b* = 10 then the total consumption is $10 \times 10 + 6 \times 10 = 160$ hours. That is out of 2,500 hours, 160 hours are consumed, and 2,340 hours are still remaining idle. So if we want to convert it into an equation then 100 + 60 + 2,340 = 2,500. As we do not know the exact values of decision variables a and b how much to add to convert the inequality into an equation. For this we represent the idle capacity by means of a **SLACK VARIABLE** represented by **S**. Slack variable for first inequality is S_1, that of second one is S_2 and that of 'n'th inequality is S_n.

Regarding the objective function, if we sell one unit of *A* it will fetch the company Rs. 23/– per unit and that of *B* is Rs. 32/– per unit. If company does not manufacture *A* or *B*, all resources remain idle. Hence the profit will be Zero rupees. This clearly shows that the profit contribution of each hour of idle resource is zero. In Linear Programming language, we can say that the company has capacity of manufacturing 2,500 units of S_1, *i.e.*, S_1 is an **imaginary product**, which require one hour of machine *X* alone. Similarly, S_2 is an **imaginary product** requires one hour of machine *Y* alone and S_3 is an imaginary product, which requires one hour of machine Z alone. In simplex language S_1, S_2 and S_3 are idle resources. The profit earned by keeping all the machines idle is Rs.0/–. Hence the profit contributions of S_1, S_2 and S_3 are Rs.0/– per unit. By using this concept, the inequalities are converted into equations as shown below:

Maximise $Z = 23\,a + 32\,b + 0S_1 + 0S_2 + 0S_3$ S.T.

$10\,a + 6\,b + 1S_1 = 2500$

$5\,a + 10\,b + 1S_2 = 2000$

$1\,a + 2\,b + 1S_3 = 500$ and a, b, S_1, S_2 and S_3 all ≥ 0.

In Simplex version, all variables must be available in all equations. Hence the Simplex format of the model is:

Maximise $Z = 23\,a + 32\,b + 0S_1 + 0S_2 + 0S_3$ S.T.

$10\,a + 6\,b + 1S_1 + 0S_2 + 0S_3 = 2500$

$5\,a + 6\,b + 0S_1 + 1S_2 + 0S_3 = 2000$

$1\,a + 2\,b + 0S_1 + 0S_2 + 1S_3 = 500$ and a, b, S_1, S_2 and S_3 all ≥ 0.

The above data is entered in a table known as **simplex table (or tableau)**. There are many versions of table but in this book only one type is used.

In Graphical method, while finding the profit by Isoprofit line, we use to draw Isoprofit line at origin and use to move that line to reach the far off point from the origin. This is because starting from zero rupees profit; we want to move towards the maximum profit. Here also, first we start with zero rupees profit, *i.e.*, considering the slack variables as the basis variables (problem variables) in the initial programme and then improve the programme step by step until we get the optimal profit. Let us start the first programme or initial programme by rewriting the entries as shown in the above simplex table.

This column shows Basic or Problem variables.

This column shows objective co-efficients corresponding to basic variables in the programme

This column shows the values of the basic variables, the value of such non basic variable is = 0

This row shows C_j above each variable, the respective objective coefficient.

Variable row lists all the variable in the problem.

The numbers under non-basic variables represent substitution Ratios.

Every simplex tableau contains an identity Matrix under the basic variables.

Programme Variable or Basic variable	Profit per unit in Rs.	Quantity or Capacity	23	32	0	0	0
			a	*b*	S_1	S_2	S_3
S_1	0	2500	10	6	1	0	0
S_2	0	2000	5	10	0	1	1
S_3	0	500	1	2	0	0	1
Z_j			0	0	0	0	0
Net Evaluation $C_j - Z_j$			23	32	0	0	0

The number in Z_j row under each column variable gives the total gross amount of **outgoing profit** when we consider the exchange between one unit of the column variable and the basic Variable.

The numbers in the net-evaluation row, under each column represent the **opportunity cost** of not having one unit of the respective column variables in the solution. In other words, the number represent the potential improvement in the objective function that will result by introducing into the programme one unit of the respective column variable.

Table: 1. Initial Programme

Solution: $a = 0$, $b = 0$, $S_1 = 2500$, $S_2 = 2000$ and $S_3 = 500$ and $Z =$ Rs. 0.00.

Programme (Basic variables)	Profit per unit In Rs. C_b	Quantity in Units.	C_j 23 a	32 b	0 S_1	0 S_2	S_3	Replacement Ratio.
S_1	0	2500	10	6	1	0	0	2500/6 = 416.7
S_2	0	2000	5	**10**	0	1	0	2000/10 = 200
S_3	0	500	1	2	0	0	1	500/2 = 250
Z_j			0	0	0	0	0	
$C_j - Z_j$ = Opportunity cost in Rs. Net evaluation row.			23	32	0	0	0	

The interpretation of the elements in the first table

1. In the first column, programme column, are the problem variables or basis variables that are included in the solution or the company is producing at the initial stage. These are S_1, S_2 and S_3, which are known as **basic variables.**

2. The second column, labeled as Profit per unit in Rupees shows the profit co-efficient of the basic variables *i.e.*, C_b. In this column we enter the profit co-efficient of the variables in the program. In table 1, we have S_1, S_2 and S_3 as the basic variables having Rs.0.00 as the profit and the same is entered in the programme.

3. In the quantity column, that is 3rd column, the values of the basic variables in the programme or solution *i.e.*, quantities of the units currently being produced are entered. In this table, S_1, S_2 and S_3 are being produced and the units being produced (available idle time) is entered *i.e.*, 2500, 2000 and 500 respectively for S_1, S_2 and S_3. The variables that are not present in this column are known as **non-basic variables**. The values of non-basis variables are zero; this is shown at the top of the table (solution row).

4. In any programme, the profit contribution, resulting from manufacturing the quantities of basic variables in the quantity column is **the sum of product of quantity column element and the profit column element.**

In the present table the total profit is $Z = 2500 \times 0 + 2000 \times 0 + 500 \times 0 =$ Rs. 0.00.

5. The elements under column of non-basic variables, *i.e.*, a and b (or the main body of the matrix) are interpreted to mean **physical ratio of distribution** if the programme consists of only slack variables as the basic variables. Physical ratio of distribution means, at this stage, if company manufactures one unit of 'a' then 10 units of S_1, 5 units of S_2 and 1 unit of S_3 **will be reduced or will go out or to be scarified**. By sacrificing the basic variables, the company will lose the profit to an extent the sum of product of quantity column element and the profit column element. At the same time it earns a profit to an extent of **product of profit co-efficient of incoming variable and the number in the quantity column against the just entered (in coming) variable.**

6. Coming to the entries in the identity matrix, the elements under the variables, S_1, S_2 and S_3 are unit vectors, hence we apply the principle of **physical ratio of distribution**, one unit of S_1 replaces one unit of S_1 and so on. Ultimately the profit is zero only. In fact while doing successive modifications in the programme towards getting optimal; solution, finally the unit matrix transfers to the main body. **This method is very much similar with G.J. method (Gauss Jordan) method in matrices, where we solve simultaneous equations by writing in the form of matrix. The only difference is that in G.J method, the values of variables may be negative, positive or zero. But in Simplex method as there is non-negativity constraint, the negative values for variables are not accepted.**

7. C_j at the top of the columns of all the variables represent the coefficients of the respective variables *I* the objective function.

8. The number in the Z_j row under each variable gives the total gross amount of outgoing profit when we consider the exchange between one unit of column, variable and the basic variables.

9. The number in the **net evaluation row, $C_j - Z_j$** row gives the **net effect** of exchange between **one unit** of each variable and **basic variables**. This they are zeros under columns of S_1, S_2 and S_3. **A point of interest to note here is the net evaluation element of any basis variable (or problem variable) is ZERO only. Suppose variable '*a*' becomes basis**

variable, the entry in net evaluation row under '*a*' is zero and so on. Generally the entry in net evaluation row is known as OPPORTUNITY COST. Opportunity cost means for not including a particular profitable variable in the programme, the manufacturer has to lose the amount equivalent to the profit contribution of the variable. In the present problem the net evaluation under the variable '*a'* is Rs. 23 per unit and that of '*b*' is Rs, 32 per unit. Hence the if the company does not manufacture '*a'* at this stage it has to face a penalty of Rs. 23/– for every unit of *'a'* for not manufacturing and the same of product variable '*b'* is Rs. 32/–. Hence the opportunity cost of product '*b'* is higher than that of '*a'*, hence '*b'* will be the incoming variable. **In general, select the variable, which is having higher opportunity cost as the incoming variable (or select the variable, which is having highest positive number in the net evaluation row**.

In this problem, variable '*b'* is having higher opportunity cost; hence it is the incoming variable. This should be marked by an arrow (↑) at the bottom of the column and enclose the elements of the column in a rectangle this column is known as **KEY COLUMN**. The elements of the key column show the **substitution ratios**, *i.e*., how many units of slack variable goes out when the variable enters the programme.

Divide the capacity column elements by key column numbers to get REPLACEMENT RATIO COLUMN ELEMENTS, which show that how much of variable '*b*' can be manufactured in each department, without violating the given constraints. Select the lowest replacement ratio and mark a tick (√) at the end of the row, which indicates **OUT GOING VARIABLE**. Enclose the elements of this column in a rectangle, which indicates **KEY ROW**, indicating out going variable. We have to select the lowest element because this is the **limiting ratio**, so that, that much of quantity of product can be manufactured on all machines or in all departments as the case may be. In the problem 200 units is the limiting ratio, which falls against S_2, *i.e*., S_2 is the outgoing variable. This means that the entire capacity of machine *Y* is utilized. By manufacturing 200 units of '*b*', 6 × 200 = 1200 hours of machine *X* is consumed and 2 × 200 = 400 hours of machine *Z* is consumed. Still 2500 – 1200 = 1300 hours of machine *X* and 500 – 400 = 100 units of machine *Z* remains idle. This is what exactly we see in graphical solution when two lines of the machines are superimposed.

The element at the intersection of key column and key row is known as **KEY NUMBER**. This is known as key number because with this number we have to get the next table.

For getting the **revised programme**, we have to transfer the rows of table 1 to table 2. To do this the following procedure is used.

Step 1: To Write the incoming variable '*b'* in place of out going variable S_2. Enter the profit of '*b'* in profit column. Do not alter S_1 and S_3. While doing so DO NOT ALTER THE POSITION OF THE ROWS.

Step 2: DIVIDING THE ELEMENTS OF OLD COLUMN BY KEY COLUMN ELEMENTS obtains capacity column elements.

Step 3: Transfer of key row: DIVIDE ALL ELEMENTS OF KEY ROW BY RESPECTIVE KEY COLUMN NUMBER.

Step 4: Transfer of Non-Key rows: NEW ROW NUMBER = (old row number – corresponding key row number) × fixed ratio.

Fixed ratio = Key column number of the row/key number.

Step 5: Elements of Net evaluation row are obtained by:

Objective row element at the top of the row – Σ key column element × profit column element.

Step 6: Select the highest positive element in net evaluation row or highest opportunity cost and mark the column by an arrow to indicate key column (incoming variable).

Step 7: Find the replacement ratios by dividing the capacity column element in the row by key column element of the same row and write the ratios in replacement ratio column. Select the limiting (lowest) ratio and mark with a tick mark to indicate key row (out going variable). The element at the intersection of key column and key row is known as key number.

Continue these steps until we get:

(i) For maximisation problem all elements of net evaluation row must be either zeros or negative elements.

(ii) For Minimisation problem, the elements of net evaluation row must be either zeros or positive elements.

Table: 2.

Solution: S_1 = 1,300, S_2 = 0, S_3 = 100, a = 0, b = 200, Z = 32 × 200 = Rs. 6400.

Problem variable.	*Profit in Rs.*	*Capacity*	*C_j 23 a*	*32 b*	*0 S_1*	*0 S_2*	*0 S_3*	*Replacement Ratio (R.R)*
S_1	0	1,300	**7**	0	1	-0.6	0	1300/7 = 185.7
b	32	200	0.5	1	0	0.10	0	400
S_3	0	100	0	0	0	-0.5	1	--
Z_j			16	32	0	3.2	0	
C_j- Z_j	= net evaluation		7	0	0	- 3.2	0	

1. Transfer of Key row: 2000/10 , 5 /10, 10 /10, 0 /10, 1 / 10, 0 / 10
2. Transfer of Non key rows:

Rule: (Old row Number – corresponding key row number) – key column number / key number = new row no.

1st row. 2500 – 2000 × 6/10 = 1300 2nd row: 500 – 2000 × 2/10 = 100

10 – 10 × 6/10 = 0

6 – 10 × 6/10 = 0 1 – 5 × 2/10 = 0

1 – 0 × 6/10 = 1 2 – 10 × 2/10 = 0

0 – 1 × 6/10 = – 0.6 0 – 0 × 2/10 = 0

0 – 0 × 6/10 = 0 0 – 1 × 2/10 = – 0.2

1 – 0 × 2 / 10 = 1

Replacement ratios: 1300/7 = 185.7, 200/0.5 = 400, 100/0 = Infinity.

Net evaluation row elements =

Column under 'a' = 23 – (7 × 0 + 0.5 × 32 + 0 × 0) = 23 – 16 = 7

'b' = 32 – (0 × 0 + 1 × 32 + 0 × 0) = 32 – 32 = 0

S_1 = 0 – (1 × 0 + 0 × 32 + 0 × 0) = 0

S_2 = 0 – (– 0.6 × 0 + 0.1 × 32 + –0.2 × 0) = – 3.2

S_3 = 0 – (0 × 0 + 0 × 32 + 1 × 0) = 0

In the above table, the net evaluation under S_2 is – **3.2.** This resource is completely utilized to manufacture product B. The profit earned by manufacturing B is Rs. 6400/–. As per the law of economics, the **worth of resources** used must be equal to the profit earned. Hence the element **3.2** (ignore negative sign) is known as **economic worth or artificial accounting price (technically it can be taken as MACHINE HOUR RATE) of the resources or shadow price of the resource. (In fact all the elements of reevaluation row under slack variables are shadow prices of respective resources). This concept is used to check whether the problem is done correctly or not. To do this MULTIPLY THE ELEMENTS IN NET EVALUATION ROW UNDER SLACK VARIABLES WITH THE ORIGINAL CAPACITY CONSTRAINTS GIVEN IN THE PROBLEM AND FIND THE SUM OF THE SAME. THIS SUM MUST BE EQUAL TO THE PROFIT EARNED BY MANUFACTRUING THEPRODUCT.**

Σ **Shadow prices of resources used must be equal to the profit earned.**

Table: 3.

Problem variable	*Profit in Rs.*	*Capacity*	*C_j 23 a*	*32 b*	*0 S_1*	*0 S_2*	*0 S_3*	*Replacement ratio*
a	23	185.7	1	0	0.143	– 0.086	0	
b	32	107.14	0	1	– 0.07	0.143	0	
S_3	0	100	0	0	0	– 0.02	1	
Z_j			23	32	1	2.6	0	
$C_j – Z_j$	Net evaluation.		0	0	–1.0	– 2.6	0	

Transfer of key row: 1300/7 = 185.7, 7/7 = 1, 0/7 = 0, 1/7 = 0.143, –3 /5 = – 0.086 0/7 = 0

Row No. 2

200 – 1300 × 1/14 = 107.14

0.5 – 7 × 1/14 = 0

1 – 0 × 1/14 = 1

0 – 1 × 1/14 = – 0.07

0.1 – (– 0.6) × 1/14 = 0.143

0 – 0 × 1/14 = 0

Row No.3

As the fixed ratio will be zero for this row the row elements will not change.

Net evaluation row elements:

For 'a' = 23 – 1 × 23 + 0 × 32 + 0 × 0 = 0

For 'b' = 32 – 0 × 23 + 1 × 32 +0 × 0 = 0

For S_1 = 0 – 0.143 × 23 + (– 0.07 × 32) + 0 × 0 = –1

For S_2 = 0 – (–0.086 × 23) + 0.143 × 32 + (– 0.02 × 0) = – 2.6

For $S_3 = 0 - 0 \times 23 + 0 \times 32 + 1 \times 0 = 0$

Profit $Z = 185.7 \times 23 + 107.14 \times 32 =$ Rs. 7,700

Shadow price = 1 × 2500 + 2.6 × 2000 = Rs. 2500 + 5200 = Rs. 7700/–

As all the elements of net evaluation row are either negative elements or zeros, the solution is optimal.

Also the profit earned is equal to the shadow price.

The answer is the company has to manufacture:

185.7 units of *A* and 107.14 units of *B* and the optimal return is *Z* = Rs. 7,700/–

MINIMISATION CASE

Above we have discussed how to solve maximisation problem and the mechanism or simplex method and interpretation of various elements of rows and columns. Now let us see how to solve a minimization problem and see the mechanism of the simplex method in solving and then let us deal with some typical examples so as to make the reader confident to be confident enough to solve problem individually.

Comparison between maximisaton case and minimisation case

S.No.	*Maximisation case*	*Minimisation case*
	Similarities:	
1.	It has an objective function.	This too has an objective function.
2.	It has structural constraints.	This too has structural constraints.
3.	The relationship between variables and constraintsis linear.	Here too the relationship between and variables constraints is linear.
4.	It has non-negativity constraint.	This too has non-negativity constraints.
5.	The coefficients of variables may be positive or negative or zero.	The coefficient of variables may be positive, Negative or zero.
6.	For selecting out going variable (key row) lowest replacement ratio is selected.	For selecting out going variable (key row) lowest replacement ratio is selected.
	Differences:	
1.	The objective function is of maximisation type.	The objective function is of minimisation type.
2.	The inequalities are of ≤ type.	The inequalities are of ≥ type.
3.	To convert inequalities into equations, **slack variables** are added.	To convert inequalities into equations, **surplus Variables are subtracted and artificial surplus variables are added.**
4.	While selecting incoming variable, highest positive Opportunity cost is selected from net evaluation Row.	While slecting, incoming variable, lowest element in the net evaluation row is selected (highest number with negative sign).
5.	When the elements of net evaluation row are either Negative or zeros, the solution is optimal	When the element of net evaluation row are either positive or zeros the solution is optimal.

It is most advantageous to introduce minimisation problem by dealing with a well-known problem, known as **diet problem.**

Problem 3.2: In this problem, a patient visits the doctor to get treatment for ill health. The doctor examines the patient and advises him to consume **at leas**t 40 units of vitamin A and 50 units of vitamin B daily for a specified time period. He also advises the patient that to get vitamin A and vitamin B he has to drink tonic X and tonic Y that have both vitamin A and vitamin B in a proportion. One unit of tonic X consists 2 units of vitamin A and 3 units of vitamin B and one unit of tonic Y consists of 4 units of vitamin A and 2 units of vitamin B. These tonics are available in medical shops at a cost of Rs.3.00 and Rs.2.50 per unit of X and Y respectively. Now the problem of patient is how much of X and how much of Y is to be purchased from the shop to **minimise** the total cost and at the same time he can get **required amounts** of vitamins A and B.

First we shall enter all the data in the form of a table.

Vitamin	Tonic		Requirement
	X	Y	
A	2	4	40
B	3	2	50
Cost in Rs.	3	2.50	

Let the patient purchase 'x' units of X and 'y' units of Y then the inequalities are **(Note: the condition given in the problem is AT LEAST hence the inequalities are of ≥ type)**

Inequalities:

For vitamin A: Minimize $Z = 3x + 2.5y$ S.T

$2x + 4y \geq 40$

$3x + 2y \geq 50$

And both x and y are ≥ 0.

In the above inequalities, say $2x + 4y \geq 40$, if we give values to x and y such that the sum is greater than or equal to 40, for example, $x = 10$ and $y = 10$ then $2x + 4y = 60$ which is > 40. To make it equal to 40 we have to subtract 20, so that 20 + 40 – 20 = 40. When we know the values, we can do this. But as we do not know the values of x and y we have to subtract a **SURPLUS VARIABLE**, generally represented by 'p', 'q', 'r'……. etc. If we do this then the inequality $2x + 4y \geq 40$ will be $2x + 4y - 1p = 40$.

Now if we allocate value **zero** to x and y then $0 + 0 - 1p = 40$ or $p = -40$. Which is against to the rules of l.p.p. as every l.p.problem the values of variables must be ≥ 0. Hence in minimization problem, we introduce one more **Surplus variable**, known as **ARTIFICIAL SURPLUS VARIABLE** generally represented by A_1, A_2, A_3 … etc. Now by introducing artificial surplus variable, we can write $2x + 4y = 40$ as $\mathbf{2x + 4y - 1p + 1A_1 = 40}$.

If values of x, y, and p are equal to zero, then $\mathbf{1A_1 = 40.}$ The artificial surplus variable has the value 40, a positive integer. Hence we start our initial programme with the artificial variables, A_1, A_2, A_3 etc. and go on replacing them by x, y, z etc. that is decision variables.

Coming to the cost coefficients of surplus and artificial surplus variables, for example, p **is very similar to vitamin *A* and one unit of *p* consists of only one unit of vitamin *A***. It will come as **give away** product when we purchase vitamin A. That is the cost coefficient of 'p' is zero (it is very much similar to slack variable in maximization problem). But the artificial surplus variable has to be purchased by paying a very high price for it. In character it is very much similar to surplus variable '*p*' because one unit of A_1 consists of one unit of vitamin A. The cost coefficient of A_1 is represented by a very high value represented by M (which means one unit of A_1 cost Millions or Rupees). As we are introducing **CAPITAL 'M', THIS METHOD IS KNOWN AS BIG 'M' METHOD**.

By using the above concept, let us write the equations of the inequalities of the problem.

Minimise $Z = 3x + 2.5y + 0p + 0q + M A_1 + M A_2$ S.T. → Objective Function.

$2x + 4y - 1p + 1A_1 = 40$
$3x + 2y - 1q + 1A_2 = 50$ → Structural Constraints. →
And x, y, p, q, A_1, A_2 all ≥ 0 → Non negativity Constraint.

Simplex format of the above is:

Minimise $Z = 3x + 2.5y + 0p + 0q + MA_1 + MA_2$ S.T.
$2x + 4y - 1p + 0q + 1A_1 + 0A_2 = 40$
$3x + 2y + 0p - 1q + 0A_1 + 1A_2 = 50$
And x, y, p, q, A_1, A_2 all = 0.
Let us enter the data in the Initial table of Simplex method.

Table: 1.

$x = 0, y = 0, p = 0, q = 0$ $A_1 = 40$, Z = Rs. 40M + 50M = 90M

Programme variable	*Cost per unit in Rs.*	*Cost → C_j requirement*	3 x	2.5 y	0 p	0 q	M A_1	M A_2	*Replacement ratio*
A_1	M	40	2	**4**	– 1	0	1	0	40/4 = 10
A_2	M	50	3	2	0	– 1	0	1	50/2 = 25
Z_j			5 M	6 M	– M	– M	M	M	
Net evaluation	$C_j - Z_j$		3– 5 M	2.5 – 6 M	M	M	0	0	

Note: As the variables A_1 and A_2 are basis variables, their Net evaluation is zero.

Now take 6M and 5M, 6 M is greater and if we subtract 2.5 from that it is negligible. Hence –6m will be the lowest element. The physical interpretation is if patient purchases *Y* now, his cost will be reduced by an amount 6M. In other words, if the patient does not purchase the *Y* at this point, his penalty is 6M, *i.e.*, **the opportunity cost is 6M.** As the **non-basis variable *Y* has highest opportunity cost** (highest element with negative sign), *Y* is the **incoming variable**. Hence, the column under *Y* is **key column**. To find the out going variable, divide requirement column element by **key column** element and find the replacement ratio. Select the lowest ratio, *i.e.*, here it is **10**, falls in first row, hence **A_1 is the out going variable.**

To transfer key row, divide all the elements of key row by key number (= 4).

40/4 = 10, 2/4 = 0.5, –1/4 = – 0.25, 0/25 = 0, 1/25 = 0.25, 0/4 = 0.

To transfer non-key row elements:

New row element = old row element – corresponding Key row element × (Key column number/key number).

50 – 40 × 2/4 = 30
3 – 2 × 0.5 = 2
2 – 4 × 0.5 = 0
0 – (–1) × 0.5 = 0.5 .
–1 – 0 × 0.5 = – 1
0 – 1 × 0.5 = – 0.5
1 – 0 × 0.5 = 1

Note:

(i) **The elements under A_1 and A_2 *i.e.*, artificial variable column are negative versions of elements under artificial variable column.**

(ii) **The net evaluation row elements of basis variables are always zero. While writing the second table do not change the positions of the rows).**

Let us now enter the new elements of changed rows in the second simplex table.

Table: 2.

$x = 0$, $y = 10$, $p = 0$, $q = 0$, $A_1 = 0$, $A_2 = 30$ and Z = Rs. 10 × 2.5 = Rs. 25.00

Programme variable	*Cost per unit in Rs.*	*Cost→C_j requirement*	*3* x	*2.5* y	*0* p	*0* q	*M* A_1	*M* A_2	*Replacement ratio*
y	*2.5*	10	0.5	1	– 0.25	0	0.25	0	10/0.5 = 20
A_2	*M*	30	**2**	0	0.5	– 1	–0.5	1	30/2 = 15
Z_j			1.25 + 2 M	2.5	0.5 M – 0.625	– M	0.625 – 0.5 M	0	
$C_j – Z_j$			1.75 – 2 M	0	0.625 – .5 M	M	1.5 M – 0.625	0	

Changing the key row: 30/2 = 15, 2/2 = 1, 0/2 = 0, 0.5/2 = 0.25, –1/2 = –0.5, –0.5/2 = –0.25, 1/2 = 0.5.

Changing the non key row:
10 – 30 × 0.5/2 = 2.5
0.5 – 2 × 0.25 = 0
1 – 0 × 0.25 = 1
–0.25 – 0.5 × 0.25 = –0.375
0 – (–1) × 0.25 = 0.25
0.25 – (–0.5) × 0.25 = 0.375
0 – 1 × 0.25 = – 0.25

Entering the above in the simplex table 3.

Table: 3.

$x = 15$, $y = 2.5$, $p = 0$, $q = 0$, $A_1 = 0$, $A_2 = 0$ and Z = Rs. 15 × 3 + Rs. 2.5 × 2.5 = 45 + 6.25 = Rs. 51.25

Programme Variable	*Cost per unit in Rs.*	*Cost* $\rightarrow C_j$ *Requirement*	*3* x	*2.5* y	*0* p	*0* q	*M* A_1	*M* A_2	*Replacement ratio*
y	*2.5*	2.5	0	1	– 0.375	0.25	0.375	– 0.25	—
A_x	3	15	1	0	0.25	–0.5	–0.25	0.5	—
Z_j			3	2.5	– 0.188	0.875	0.188	0.875	
$C_j – Z_j$			0	0	0.188	0.875	M – 0.188	M– 0.875	

Optimal Cost = Z^* = 3 × 15 + 2.5 × 2.5 = 45 + 6.25 = Rs. 51.25

Imputed value = 0.1875 × 40 + 0.875 × 50 = 7.5 + 43.75 = Rs. 51.25.

As all the elements of net evaluation row are either zeros or positive elements the solution is optimal.

The patient has to purchase 15 units of *X* and 2.5 units of *Y* to meet the requirement and the cost is Rs. 51.25/–

While solving maximisation problem, we have seen that the elements in net evaluation row, *i.e.*, ($C_j – Z_j$) row associated with slack variables represent the marginal worth or shadow price of the resources. In minimisation problem, the elements associated with surplus variables in the optimal table, represent the marginal worth or imputed value of one unit of the required item. **In minimisation problem, the imputed value of surplus variables in optimal solution must be equal to the optimal cost.**

***Point to Note*:**

1. **In the mechanics of simplex method of minimization problem, once an artificial surplus variable leaves the basis, its exit is final because of its high cost coefficient (M), which will never permit the variable to reenter the basis. In order to save time or to reduce calculations, we can cross out the column containing the artificial surplus variable, which reduces the number of columns.**
2. **A better and easier method is to allocate a value for M in big M method; this value must be higher than the cost coefficients of the decision variables. Say for example the cost coefficients of the decision variable in the above problem are for X it is Rs.3/– and for Y it is Rs. 2.5. We can allocate a cost coefficient to M as Rs.10, which is greater than Rs.3/– and Rs. 2.5. Depending the value of decision variables, this value may be fixed at a higher level (Preferably the value must be multiples of 10 so that the calculation part will be easier.**

By applying the above note, let us see how easy to work the same problem:

Table: 1.

$x = 0, y = 0, p = 0, q = 0 = A_1 = 40, A_2 = 50$ and $Z = 10 \times 40 + 10 \times 50 =$ Rs.900/–

Problem variable	*Cost*	$C_j \longrightarrow$ *requirement*	*3* x	*2.5* y	*0* p	*0* q	*10* A_1	*10* A_2	*Replacement ratio*
A_1	10	40	2	**4**	– 1	0	1	0	10
A_2	10	50	3	2	0	– 1	0	1	25
NER			– 47	– 57.5	10	10	0	0	

Table: 2.

$x = 0, y = 25, p = 0, q = 0, A_1 = 0, A_2 = 30$ and $Z = 25 \times 10 + 30 \times 10 = 250 + 300 =$ Rs. 550/–

Problem variable	*Cost per*	$C_j \longrightarrow$ *requirement*	*3* x	*2.5* y	*0* p	*0* q		*10* A_2	*Replacement ratio*
y	2.5	10	0.5	1	– 0.5	0		0	20
A_2	10	30	**2**	0	0.5	– 1		1	15
			– 18.75	0	12.5	10		0	

Table: 3.

$x = 15, y = 2.5, p = 0, q = 0, A_1 = 0, A_2 = 0$ and $Z = 15 \times 3 + 2.5 \times 2.5 =$ Rs. 51.75

Problem variable	*Cost per*	$C_j \longrightarrow$ *requirement*	*3* x	*2.5* y	*0* p	*0* q			*Replacement ratio*
y	2.5	2.5	0	1	– 0.375	0.25			—
x	3	15	1	0	– 0.25	– 0.5			—
NER			0	0	0.1875	0.875			

Optimal Cost = $15 \times 3 + 2.5 \times 2.5$ = Rs. 51.25 /–

Imputed value = $0.1875 \times 40 + 0.875 \times 50$ = Rs. 51.25/–

CERTAIN IMPORTANT POINTS TO BE REMEMBERED WHILE SOLVING LINEAR PROGRAMMING PROBLEMS BY SIMPLEX METHOD:

1. **In the given inequalities, there should not be any negative element on right hand side ($b_i \geq 0$). If any b_i is negative, multiply the inequality by –1 and change the inequality sign.**
2. **Sometimes, the objective function may be maximisation type and the inequalities may be $\geq$ type. In such cases, multiply the objective function by –1 and convert it into minimisation type and vice versa.**
3. **While selecting, the incoming variable, *i.e.*, key column, in maximisation case, we have to select the highest positive opportunities cost and in minimisation case, select the highest element with negative sign (smallest element). While doing so, sometimes you may find the highest positive element in maximisation case or lowest element in minimisation case falls under the slack variable in maximisation case or under surplus variable in minimisation case. Do not worry. As per rule, select that element and take the column containing that element as key column.**
4. **Some times the columns of non-basis variables (decision variables) may have their net evaluation elements same. That is the net evaluation elements are equal. This is known as a TIE in Linear Programming Problem. To break the time, first select any one column of your choice as the key column. In the next table, everything will go right.**
5. **While selecting the out going variable *i.e.*, key row, we have to select limiting ratio (lowest ratio) in net evaluation row. In case any element of key column is negative, the replacement ratio will be negative. In case it is negative, do not consider it for operation. Neglect that and consider other rows to select out going variable.**
6. **Sometimes all the replacement ratios for all the rows or some of the rows may be equal and that element may be limiting ratio. This situation in Linear Programming Problem is known as DEGENERACY. We say that the problem is degenerating. When the problem degenerate, the following precautions are taken to get rid of degeneracy.**
   (a) **Take any one ratio of your choice to select key row or out going variable. If you do this, there is a possibility that the problem may cycle. Cycling means, after doing many iterations, you will get the first table once again. But it may not be the case all times.**
   (b) **Select the variable, whose subscript is small. Say S_1 is smaller than S_2 and S_2 is smaller than S_3 or X_1 is smaller than X_2 an so on or x is smaller than y or '*a*' is smaller than '*b*' and so on.**
   (c) **If we do above two courses of action, we may encounter with one problem. That one of the remaining variable in the next table (the one corresponding to the tied variable that was not considered) will be reduced to a magnitude of zero. This causes trouble in selecting key column in the next table.**

(d) **Identify the tied variable or rows. For each of the columns in the identity (starting with the extreme left hand column of the identity and proceeding one at a time to the right), compute a ratio by dividing the entry in each tied row by the key column number in that row.**

Compare these ratios, column by column, proceeding to the right. The first time the ratios are unequal, the tie is broken. Of the tied rows, the one in which the smaller algebraic ratio falls is the key row.

(e) **If the ratios in the identity do not break the tie, form similar ratios in the columns of the main body and select the key row as described in (*d*) above. The application of the above we shall see when we deal with degeneracy problems.**

7. **While solving the linear programming problems, we may come across a situation that the opportunity cost of more than one non- basic variables are zero, then we can say that the problem has got ALTERNATE SOLUTIONS.**
8. **If in a simplex table only one unfavourable $C_j - Z_j$ identifying the only incoming variable and if all the elements of that column are either negative elements or zeros, showing that no change in the basis can be made and no current basic variable can be reduced to zero. Actually, as the incoming arable is introduced, we continue to increase, without bounds, those basic variables whose ratios of substitutions are negative. This is the indication of UNBOUND SOLUTION.**
9. **In a problem where, the set of constraints is inconsistent *i.e.*, mutually exclusive, is the case of NO FEASIBLE SOLUTION. In simplex algorithm, this case will occur if the solution is optimal (*i.e.*, the test of optimality is satisfied) but some artificial variable remains in the optimal solution with a non zero value.**

WORKED OUT PROBLEMS

Example 3.3. A company manufactures two products *X* and *Y* whose profit contributions are Rs. 10 and Rs. 20 respectively. Product *X* requires 5 hours on machine I, 3 hours on machine II and 2 hours on machine III. The requirement of product *Y* is 3 hours on machine I, 6 hours on machine II and 5 hours on machine III. The available capacities for the planning period for machine I, II and III are 30, 36 and 20 hours respectively. Find the optimal product mix.

Solution: The given data:

Machine	*Products (Time required in hours)*		*Availability in hours*
	X	*Y*	
I	5	3	30
II	3	6	36
III	2	5	20
Profit per unit in Rs.	10	20	

Inequalities:

Maximize $Z = 10x + 20y$ s.t.

$5x + 3y \leq 30$

$3x + 6y \leq 36$

$2x + 5y \leq 20$ and

Both x and y are ≥ 0.

Simplex format:

Maximize $Z = 10x + 20y + 0S_1 + 0S_2 + 0S_3$ s.t.

$5x + 3y + 1S_1 + 0S_2 + 0S_3 = 30$

$3x + 6y + 0S_1 + 1S_2 + 0S_3 = 36$

$2x + 5y + 0S_1 + 0S_2 + 1S_3 = 20$ and

x, y, S_1, S_2 and S_3 all ≥ 0.

Table: I. $x = 0$, $y = 0$ $S_1 = 30$, $S_2 = 36$ and $S_3 = 20$, $Z =$ Rs. 0

Problem variable	*Profit in Rs.*	*Capacity*	$C_j=10$ x	20 y	0 S_1	0 S_2	0 S_3	*Replacement ratio*
S_1	0	30	5	3	1	0	0	30/3 = 10
S_2	0	36	3	6	0	1	0	36/6 = 6
S_3	0	20	2	**5**	0	0	1	20/5 = 4
Opportunity cost.			10	20	0	0	0	

Step 1: For the first table the net evaluation row elements are same as profit row element. For successive rows, the key column element is given by Profit – (Sum of key column element x respective object row element). For maximization problem, select the highest element among the key column element and mark an arrow as shown to indicate incoming variable. For minimization problems select the lowest element or highest element with negative sign and write an arrow to indicate the incoming variable. Enclose key column elements in a rectangle. Here they are shown in red colour. It is known as key row because it gives the clue about incoming variable.

Step 2: Divide the capacity column numbers with respective key column number to get the replacement ratio. Select the lowest ratio as the indicator of out going variable. The lowest ratio is also known as limiting ratio. In the above table the limiting ratio elements are 10, 6, 4. We select 4 as the indicator of outgoing variable. It is because, if we select any other number the third machine cannot compete more than 4 units of product. Though the machine has got capacity to manufacture 10 units and second machine has got capacity of 6 units, only 4 units can be manufactured completely. Rest of the capacity of machine 1 and 2 will become idle resource. Enclose the elements of key row in a rectangle. It is known as key row because it gives the clue about out going variable. Mark this row with a tick mark. (Here the elements are marked in thick.)

Step 3: The element at the intersection of key row and key column is known as Key number, in the table it is marked in bold thick number. It is known as the key number, because, the next table is written with the help of this key element.

Table: II. $x = 0$, $y = 4$, $S_1 = 18$, $S_2 = 12$, $S_3 = 0$. Z = Rs. 4 × 20 = Rs. 80.

Problem variable	*Profit in Rs.*	*Capacity*	*C_j = 10* *x*	*20* *y*	*0* *S_1*	*0* *S_2*	*0* *S_3*	*Replacement ratio*
S_1	0	18	**3.8**	0	1	0	– 0.6	18/3.8 = 4.8
S_2	0	12	0.6	0	0	1	– 1.2	12/0.6 = 2.0
y	20	4	0.4	1	0	0	0.2	4/0.4 = 10
Opportunity cost.			2	0	0	0	– 4	

Step 4: To improve the progeramme or to get the new table the following procedure is adopted:

(i) Transfer of key row of old tableau: Divide all the elements of key row of old tableau by key number, which gives the key row elements of the new tableau.

(ii) To transfer non key rows: New row number = old row number – (corresponding key row number × fixed ratio.)

Here, fixed ratio = (key column number of the row/key number).

While transferring remembers you should not alter the positions of the rows. Only incoming variable replaces the outgoing slack variable or any other outgoing basis variable as the case may be.

The net evaluation row element of the variable entered into the programme will be zero. When all the variables are transferred, the identity matrix will come in the position of main matrix.

To check whether, the problem is done in a correct manner, check that whether profit earned at present stage is equal to shadow price at that stage. Multiplying the net evaluation row element under non-basis can get shadow price variable (identity matrix) by original capacities of resources given in the problem.

Above explained procedure of transferring key row and non-key rows is worked out below:
Transfer of Key row: 20 / 5 = 4, 2 / 5 = 0.4, 5 / 5 = 1, 0 /5 = 0, 0 / 5 = 0, and 1 / 5 = 0.2.

Transfer of non-key rows:

30 – 20 × 3/5 = 18	36 – 20 × 6 / 5 = 12
5 – 2 × 3/5 = 3.8	3 – 2 × 1.2 = 0.6
3 – 5 = 3/5 = 0	6 – 5 × 1,2 = 0
1 – 0 × 3/5 = 1	0 – 0 × 1.2 = 0
0 – 0 × 3/5 = 0	1 – 0 × 1.2 = 1
0 – 1 × 3/5 = – 0.6	0 – 1 × 1.2 = – 1.2

Step 5: Once the elements of net evaluation row are either negatives or zeros (for maximization problem), the solution is said to be optimal. For minimization problem, the net evaluation row elements are either positive elements or zeros.

As all the elements of net evaluation row are either zeros or negative elements, the solution is optimal. The company will produce 4.8 units of *X* and 3.6 units of *Y* and earn a profit of Rs. 120/–.

Shadow price is 2.6 × 30 + 2 × 20 = Rs. 128/–. The difference of Rs. 8/– is due to decimal values, which are rounded off to nearest whole number.

Table: III. $x = 4.8$, $y = 3.6$, $S_2 = 9$, $S_1 = 0$, $S_3 = 0$ and $Z = 4.8 \times 10 + 3.6 \times 20 =$ Rs. 120/–

Problem variable	*Profit in Rs.*	*Capacity*	*$C_j = 10$ x*	*20 y*	*0 S_1*	*0 S_2*	*0 S_3*	*Replacement ratio*
x	10	4.8	1	0	0.26	0	– 0.16	
S_2	0	9	0	0	– 0.16	1	– 1.1	
y	20	3.6	0	1	0	0	0.18	
Opportunity cost.			0	0	– 2.6	0	–2.0	

Problem. 3.4: A company manufactures three products namely *X*, *Y* and *Z*. Each of the product require processing on three machines, Turning, Milling and Grinding. Product *X* requires 10 hours of turning, 5 hours of milling and 1 hour of grinding. Product *Y* requires 5 hours of turning, 10 hours of milling and 1 hour of grinding, and Product Z requires 2 hours of turning, 4 hours of milling and 2 hours of grinding. In the coming planning period, 2700 hours of turning, 2200 hours of milling and 500 hours of grinding are available. The profit contribution of *X*, *Y* and Z are Rs. 10, Rs.15 and Rs. 20 per unit respectively. Find the optimal product mix to maximize the profit.

Solution: The given data can be written in a table.

Machine	*Product Time required in hours per unit*			*Available hours*
	X	*Y*	*X*	
Turning.	10	5	2	2,700
Milling	5	10	4	2,200
Grinding.	1	1	2	500
Profit contribution in Rs. per unit.	10	15	20	

Let the company manufacture *x* units of *X*, *y* units of *Y* and *z* units of *Z*

Inequalities:

Maximise $Z = 10x + 15y + 20z$ S.T.

$10x + 5y + 2z \leq 2{,}700$

$5x + 10y + 4z \leq 2{,}200$

$1x + 1y + 2z \leq 500$ and

All x, y and z are ≥ 0

Equations:

Maximise $Z = 10x + 15y + 20x$ S.T

$10x + 5y + 2z + 1S_1 = 2700$

$5x + 10y + 4z + 1S_2 = 2200$

$1x + 1y + 2z + 1S_3 = 500$ and

x, y and z all ≥ 0

Simplex format:

Maximise $Z = 10x + 15y + 20z + 0S_1 + 0S_2 + 0S_3$ S.t.

$10x + 5y + 2z + 1S_1 + 0S_2 + 0S_3 = 2700$

$5x + 10_y + 4z + 0S_1 + 1S_2 + 0S_3 = 2200$
$1x + 1y + 2z + 0S_1 + 0S_2 + 1S_3 = 500$
And all x, y, z, S_1, S_2, S_3 are ≥ 0

Table I. $x = 0$, $y = 0$, $z = 0$, $S_1 = 2700$, $S_2 = 2200$, $S_3 = 500$. Profit Z = Rs. 0

Programme	*Profit*	*Capacity*	C_j=10 *x*	15 *y*	20 *z*	0 S_1	0 S_2	0 S_3	*Replacement ratio*	*Check column.*
S_1	0	2700	10	5	2	1	0	0	2700/2 = 13500	2718
S_2	0	2200	5	10	4	0	1	0	2200/4 = 550	2220
S_3	0	500	1	1	**2**	0	0	1	500/2 = 250	505
Net evaluation			10	15	20	0	0	0		

{Note: The check column is used to check the correctness of arithmetic calculations. The check column elements are obtained by adding the elements of the corresponding row starting from capacity column to the last column (avoid the elements of replacement ration column). As far as treatment of check column is concerned it is treated on par with elements in other columns. In the first table add the elements of the row as said above and write the elements of check column. In the second table onwards, the elements are got by usual calculations. Once you get elements, add the elements of respective row starting from capacity column to the last column of identity, then that sum must be equal to the check column element.}

Table: II. $x = 0$, $y = 0$, $z = 250$ units, $S_1 = 2200$, $S_2 = 1200$, $S_3 = 0$ and Z = Rs. 20 × 250 = Rs. 5,000.

Programme	*Profit*	*Capacity*	C_j=10 *x*	15 *y*	20 *z*	0 S_1	0 S_2	0 S_3	*Check column.*	*Replacement ratio*
S_1	0	2210	9	4	0	1	0	– 1	2213	552.5
S_2	0	1200	3	**8**	0	0	1	– 2	1210	150
Z	20	250	0.5	0.5	1	0	0	0.5	500	500
Net Evaluation.			0	5	0	0	0	– 10		

Profit at this stage = Rs. 20 × 250 = Rs. 5,000 and Shadow price = 10 × 500 = Rs. 5000.

Table: III. $x = 0$, $y = 150$, $z = 174.4$, $S_1 = 1600$, $S_2 = 0$, $S_3 = 0$ and Z = Rs. 5738/–

Programme	*Profit*	*Capacity*	C_j=10 x	15 y	20 z	0 S_1	0 S_2	0 S_3	*Check column*	*Replacement Ratio*
S_1	0	1600	7.5	0	0	1	–0.5	0	1608	
Y	15	150	0.375	1	0	0	0.125	–0.25	151.25	
Z	20	174.4	0.311	0	1	0	–0.063	0.626	423.7	
Net Evn.			–1.85	0	0	0	–0.615	– 8.77		

As all the elements of Net evaluation row are either zeros or negative elements, the solution is optimal. The firm has to produce 150 units of Y and 174.4 units of Z. The optimal profit = 15 × 150 + 20 × 174.4 = Rs.5738 /–

To check the shadow price = 0.615 × 2200 + – 8.77 × 500 = 1353 + 4385 = Rs. 5738 /–.

Problem 3.5: A company deals with three products *A*, *B* and *C*. They are to be processed in three departments *X*, *Y* and *Z*. Products *A* require 2 hours of department *X*, 3 hours of department *Y* and product *B* requires 3 hours,2 hours and 4 hours of department *X*, *Y* and *Z* respectively. Product *C* requires 2 hours in department *Y* and 5 hours in department *Z* respectively. The profit contribution of *A*, *B* and *C* are Rs. 3/–, Rs.5/– and Rs. 4/– respectively. Find the optimal product mix for maximising the profit. In the coming planning period, 8 hours of department *X*, 15 hours of department *Y* and 10 hours of department *Z* are available for production.

The Data *Departments*	*Product* *Hours required per unit*			*Available capacity in hours*
	A	*B*	*C*	
X	2	3	0	8
Y	3	2	4	15
Z	0	2	5	10
Profit per unit in Rs..	3	5	4	

Inequalities:

Maximise $Z = 3a + 5b + 4c$ s.t.

$2a + 3b + 0c \leq 8$

$3a + 2b + 4c \leq 15$

$0a + 2b + 5c \leq 10$ and

$a, b,$ and c all ≥ 0

Equations.

Maximise $Z = 3a + 5b + 4c + 0S_1 + 0S_2 + 0S_3$ s.t.

$2a + 3b + 0c + 1S_1 + 0S_2 + 0S_3 = 8$

$3a + 2b + 4c + 0S_1 + 1S_2 + 0S_3 = 15$

$0a + 2b + 5c + 0S_1 + 0S_2 + 1S_3 = 10$ and

a, b, c, S_1, S_2, S_3 all ≥ 0.

Table: I. $a = 0$, $b = 0$, $c = 0$, $S_1 = 8$, $S_2 = 15$ and $S_3 = 10$ and Z = Rs.0

Programme	*Profit*	*Capacity*	*C_j= 3* *a*	*5* *b*	*4* *c*	*0* *S_1*	*0* *S_2*	*0* *S_3*	*Check column.*	*Replacement Ratio*
S_1	0	8	2	**3**	0	1	0	0	14	2.6
S_2	0	15	3	2	4	0	1	0	25	7.5
S_3	0	10	0	2	5	0	0	1	18	5
Net.Ev.			3	5	4	0	0	0		

Table: II. $a = 0$, $b = 2.6$, $c = 0$, $S_1 = 0$, $S_2 = 9.72$, $S_3 = 4.72$ and Z = Rs.5 × 2.6 = Rs. 13/–

Programme	*Profit*	*Capacity*	*C_j= 3* *a*	*5* *b*	*4* *c*	*0* *S_1*	*0* *S_2*	*0* *S_3*	*Check column.*	*Replacement Ratio*
B	5	2.6	0.66	1	0	0.33	0	0	4.6	—
S_2	0	9.72	1.68	0	4	0.66	1	0	15.76	2.43
S_3	0	4.72	– 1.32	0	**5**	– 0.66	0	1	8.76	0.944
Net.Evn			3	0	4	– 1.65	0	0		

Note: as the key column element is equal to 0 for column under '*c*' replacement ratio is not calculated, as it is equals to zero.

Profit at this stage is Rs. 2 × 2.6 = Rs. 13/ –. Shadow price = 1.65 × 8 = Rs. 13 / –.

Table: III. $a = 0$, $b = 2.6$, $c = 0.944$, $S_2 = 5.94$, $S_1 = 0$, $S_3 = 0$, Profit Z = Rs. 5 × 2.6 + 4 × 0.944 = Rs. 16.776 /–

Programme	*Profit*	*Capacity*	*C_j= 3* *a*	*5* *b*	*4* *c*	*0* *S_1*	*0* *S_2*	*0* *S_3*	*Check column.*	*Replacement ratio*
B	5	2.6	0.66	1	0	0.33	0	0	4.6	
S_2	0	5.94	2.74	0	0	1.19	1	– 0.8	8.72	
C	4	0.944	– 0.264	0	1	– 0.132	0	0.2	1.792	
Net Evn.			– 8.64	0	0	– 1.122	0	– 0.8	—	

As the elements of net evaluation row are either zeros or negative elements, the solution is optimal. Now the optimal profit = Z = Rs. 5 × 2.6 + Rs. 4 × 0.944 = Rs. 16.776. The company manufactures units of *B* and one unit of *C*.

The shadow price = 1.122 × 8 + 0.8 × 10 = Rs. 17.976. The small difference is due to decimal calculations. Both of them are approximately equal hence the solution is correct.

Problem 3.6: A company manufactures two types of products *X* and *Y* on facilities, *A*, *B*, *C*, *D*, *E*, and *F* having production capacities as under.

Facilities.	Production capacity to produce
A	100 of *X* OR 150 of *Y*
B	80 of *X* OR 80 of *Y*
C	100 of *X* OR 200 of *Y*
D	120 of *X* OR 90 of *Y*
E	60 of *X* only (Testing facility for product *X*)
F	60 of *Y* only.(Testing facility for product *Y*)

If the profit contribution of product *X* is Rs.40/– per unit and that of *Y* is Rs. 30 per unit, find the optimal product mix for maximising the profit.

Solution:

Let the company manufactures *x* units of *X* and *y* units of *Y*.

Each unit of product '*x*' uses1/100 capacity of A therefore capacity A used by product '*x*' is $(A / 100)\, x$.

Similarly capacity of *A* used by '*y*' is $(A / 150)\, y$. Therefore,

$(A/100)\, x + (A/150)\, y \leq A$ *i.e.*, $150\, x + 100\, y \leq 15000$ OR $3\, x + 2\, y \leq 300$ is the equation for *A*. Similarly equations for other facilities can be written. Objective function is to Maximise $Z = 40x + 30y$

Inequalities:	Simplex version:
Maximise $Z = 40x + 30y$ s.t.	Maximise $Z = 40x + 30y + 0S_1 + 0S_2 + 0S_3 + 0S_4 + 0S_5 + 0S_6$ s.t.
$3x + 2y \leq 300$	$3x + 2y + 1S_1 + 0S_2 + 0S_3 + 0S_4 + 0S_5 + 0S_6 = 300$
$1x + 1y \leq 80$	$1x + 1y + 0S_1 + 1S_2 + 0S_3 + 0S_4 + 0S_5 + 0S_6 = 80$
$2x + 1y \leq 200$	$2x + 1y + 0S_1 + 0S_2 + 1S_3 + 0S_4 + 0S_5 + 0S_6 = 200$
$3x + 4y \leq 360$	$3x + 4y + 0S_1 + 0S_2 + 0\, S_3 + 1S_4 + 0S_5 + 0S_6 = 360$
$1x + 0y \leq 60$	$1x + 0y + 0S_1 + 0S_2 + OS_3 + 0S_4 + 1S_5 + 0S_6 = 60$
$0x + 1y \leq 60$ and	$0x + 1y + 0S_1 + 0S_2 + 0S_3 + 0S_4 + 0S_5 + 1S_6 = 60$ and
Both *x* and *y* are ≥ 0.	All x, y, S_i $(i = 1,2,3,4,5,6)$ are ≥ 0

Table: I. $x = 0$, $y = 0$, $S_1 = 300$, $S_2 = 80$, $S_3 = 200$, $S_4 = 360$, $S_5 = 60$, $S_6 = 60$ and Profit Z = Rs.0

Prog.	*Profit*	*Capacity* C_j =	*40* *x*	*30* *y*	*0* S_1	*0* S_2	*0* S_3	*0* S_4	*0* S_5	*0* S_6	*Check Col.*	*Replacement ratio.*
S_1	0	300	3	2	1	0	0	0	0	0	306	102
S_2	0	80	1	1	0	1	0	0	0	0	83	80
S_3	0	200	2	1	0	0	1	0	0	0	203	100
S_4	0	360	3	4	0	0	0	1	0	0	367	120
S_5	0	60	**1**	0	0	0	0	0	1	0	61	60
S_6	0	60	0	1	0	0	0	0	0	1	61	Infinity.
	Net	Evaluation	40	30	0	0	0	0	0	0	—	

Table: III. $x = 60$, $y = 20$, $S_1 = 80$, $S_2 = 0$, $S_3 = 60$, $S_4 = 100$, $S_5 = 0$, $S_6 = 40$, Profit: Rs. 40 × 60 + 30 × 20 = Rs. 3000 /–

Prog.	*Profit*	*Capacity* C_j =	40 x	30 y	0 S_1	0 S_2	0 S_3	0 S_4	0 S_5	0 S_6	*Check col.*	*Replacement ratio.*
S_1	0	80	0	0	1	–2	0	0	– 1	0	78	
y	30	20	0	1	0	1	0	0	– 1	0	21	
S_3	0	60	0	0	0	–1	1	0	– 1	0	60	
S_4	0	100	0	0	0	–4	1	1	1	0	99	
X	40	60	1	0	0	0	0	0	1	0	59	
S_6	0	40	0	0	0	–1	0	0	1	1	39	
	Net	Evaluation	0	0	0	–30	0	0	– 10	0	—	

As all the elements of net evaluation row are either negative elements or zeros the solution is optimal. The company will produce 60 units of X and 20 units of Y and the optimal Profit = Z Rs. 40 × 60 + Rs. 30 × 20 = Rs. 3000/–

Shadow price = 30 × 80 + 10 × 6 = 2400 + 600 = Rs. 3000 /–. Shadow price and profit are equal.

Problem 3.7: A company produces three products A, B and C by using two raw materials X and Y. 4000 units of X and 6000 units of Z are available for production. The requirement of raw materials by each product is given below:

Raw material	*Requirement per unit of product*		
	A	*B*	*C*
X	2	3	5
Y	4	2	7

The labour time for each unit of product A is twice that of product B and three times that of product C. The entire labour force of the company can produce the equivalent of 2500 units of product A. A market survey indicates the minimum demand of the three products are 500, 500 and 375 respectively for A, B and C. However, their ratio of number of units produced must be equal to 3: 2: 5. Assume that the profit per units of product A, B and C are Rupees 60/–, 40/– and 100 respectively. Formulate the L.P.P. for maximizing the profit.

Solution:

Let the company manufactures a units of A, b units of B and c units of C. The constraints for raw materials are

$$2a + 3b + 5c \leq 4000 \qquad \ldots(1)$$

$$4a + 2b + 7c \leq 6000 \qquad \ldots(2)$$

Now let 't' be the labour time required for one unit of product A, then the time required for per unit of product B is $t/2$ and that for product C is $t/3$. As 2500 units of A are produced, the total time available is 2500 t. Hence the constraints for time are:

$$ta + t/2\ b + t/3\ c \leq 2500\text{t}$$

i.e., $a + 1/2\ t\ b + 1/3\ t\ c \ \leq\ 2500$ ...(3)

Now the market demand constraints are: $a \geq 500$, $b \geq 500$ and $c \geq 375$ ...(4)

As the ratio of production must be 3 : 2 : 5,

$A = 3k$. $b = 2k$ and $c = 5k$ which gives the equations:

$1/3\ a = 1/2/\ b$ and $1/2\ b = 1/5\ c$ …(5)

The objective function is Maximise $Z = 60\ a + 40\ b + 100\ c$

Hence the Linear programme in the form inequalities for the above problem is:

Maximise $Z = 60\ a + 40b + 100c$ s.t.

$2a + 3b + 5c \ \leq\ 4000$

$4a + 2b + 7c \ \leq\ 6000$

$1a + 1/2\ b + 1/3\ c \leq 2500$

$1/3\ a = 1/3\ b$

$½\ b = 1/5\ c$

$a \geq 500$, $b \geq 500$ and $c \geq 375$

As the last constraint shows that the values of all the variables are ≥ 0, the same constraint will become non-negativity constraint.

Problem 3.8: A product consists of two components *A* and *B*. These components require two different raw materials *X* and *Y*. 100 units of *X* and 200 units of *Y* are available for production. The materials are processed in three departments. The requirement of production time in hours and materials in units are given in the table below.

Departments	Raw material input per run in unit		Output of components per run (units)	
	X	*Y*	*A*	*B*
1	7	5	6	4
2	4	8	5	8
3	2	7	7	3

Formulate a progrmme to determine the number of production runs for each department, which will maximise the total number of components *A* and *B* for the product.

Solution: Formulation of L.P.P

Let *a*, *b* and c is the production runs for departments 1, 2 and 3 respectively. Therefore the total production is $6a + 5b + 7c$ of components of *A* and $4a + 8b + 3c$ of components B.

The raw material restrictions are :

$$7a + 4b + 2c \ \leq\ 100$$

$$5a + 8b + 7c \ \leq\ 200$$

Now the final product requires 4 units of *A* and 3 units of *B* for assembly. Hence the total production of final product will be the smaller of the quantities: $1/4\ (6a + 5b + 7c)$ and $1/3\ (4a + 8b + 3c)$.

Our objective is to maximize the production of final product. Hence the objective function would be:

Maximise Z = Minimum of $\{¼\ (6a + 5b + 7c),\ 1/3\ (4a + 8b + 3c)\}$

Let Minimum $\{1/4\ (6a + 5b + 7c),\ 1/3\ (4a + 8b + 3c)\} = v$ *i.e*.

¼ $(6a + 5b + 7c) \geq v$ and $1/3\ (4a + 8b + 3c) = v$

Then the required l.p.p. is : Find a, b, c and v which maximise $Z = v$, subject to the constraints

$7a + 8b + 2c \leq 100$

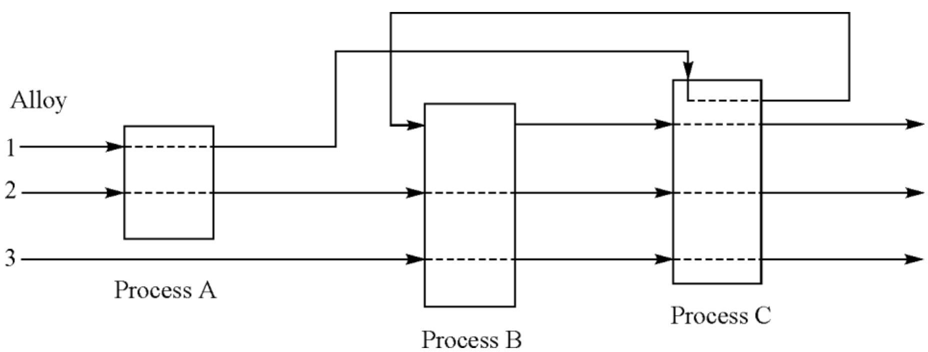
$5a + 8b + 7c \leq 200$

$6a + 5b + 7c - 4v \geq 0$

$4a + 8b + 3c - 3v \geq 0$, and

a, b, c and v all ≥ 0.

Problem 3.9: A firm manufactures three types of coils each made of a different alloy. The flow process chart is given in the figure below. The problem is to determine the amount of each alloy to produce, within the limitations of sales and machine capacities, so as to maximise the profits.



The further data given is:

Table: I.

Machine	*Number of machines*	*8–hour shift per week*	*Down time %*
A	3	18	5
B	2	26	10
C	1	22	0

Table: II.

Alloy	*Operation*	*Machine rate*	*Sales potential*	*Profit per ton in Rs.*
1	A C (1) B C (2)	30 hours/30 ton 40 feet per minute 25 feet per minute 30 feet per minute	1500 tons per month.	80
2	A B C	25 hours per 10 tons 25 feet per minute 30 feet per minute	800 tons per month.	400
3	B C	15 feet per minute 20 feet per minute	1000 tons per	250

Coils for each alloy are 400 feet long and weigh 5 tons. Set up objective function and restrictions to set up matrix.

Solution: Let the company produce '*a*' units of alloy 1, *b* units of alloy 2 and *c* units of alloy 3. Then the objective functions is Maximise $Z = 80a + 400b + 250c$ subject to the limitations imposed by the available machine capacity and sales potential.

Constraint of Machine Capacity per month:

Table: III.

Process	*Number of machines*	*shifts per week*	*% of useful time*	*Capacity in hours per month*
A	3	18	95	3 × 18 × 8 × 4 ½ × 0.95 = 1778.3
B	2	26	90	2 × 26 × 8 × 4 ½ × 0.90 = 1662.4
C	1	22	100	1 × 22 × 8 × 4 ½ × 1 = 726.7

To convert machine rates into tons per hour:

Table: IV.

Alloy	*Process*	*Machine rates*
1	*A*	30 hour per 10 tons = 0.333 tons per hour.
	C (1)	40 feet per minute = (40 × 60 × 5)/400 = 30 tons per hour.
	B	25 feet per minute = (25 × 60 × 5)/400 = 18.75 tons per hour.
	C (2)	30 feet per minute = (30 × 60 × 5)/400 = 22.5 tons per hour.
2	*A*	25 hours per 10 tons. = 0.4 tons per hour.
	B	25 feet per minute = (25 × 60 × 5)/400 = 18.75 tons per hour.
	C	30 feet per minute = (30 × 60 × 5)/400 = 22.5 tons per hour.
3	*B*	15 feet per minute = (15 × 60 × 5)/400 = 11.25 tons per hour.
	C	20 feet per minute = (20 × 60 × 5)/400 = 15 tons per hour.

Now let us calculate the times required for *a*, *b* and *c* tons of alloys and use these machine times for a formal statement of capacity constraints.

For process *A* : $(a/0.333) + (b/0.4) \leq 1778.3$

For process *B* : $(a/18.75) + (b/18.75) + (c/11.25) \leq 1662.4$

For process *C* : $(a/30) + (a/22.5) + (b/22.5) + (c/15) \leq 762.7$ OR

$(7a/90) + (b/22.5) + 1c \leq 762.7$

Limitations imposed by sales potential:

$a \leq 1500$, $b \leq 800$ and $c \leq 1000$.

Hence the l.p.p is:

Maximise $Z = 80a + 400\,b + 250c$ s.t.

$(a/0.333) + (b/0.4) \leq 1778.3$

$(a\,/\,18.75) + (b/18.75) + (c\,/\,11.25) \leq 1662.4$

$(7a/90) + (b/22.5) + 1c \leq 762.7$
$a \leq 1500$, $b \leq 800$ and $c \leq 1000$. And
a,b and c all ≥ 0

MINIMISATION PROBLEMS

Problem 3.10: A small city of 15,000 people requires an average of 3 lakhs of gallons of water daily. The city is supplied with water purified at a central water works, where water is purified by filtration, chlorination and addition of two chemicals softening chemical *X* and health chemical *Y*. Water works plans to purchase two popular brands of products, product *A* and product *B*, which contain these two elements. One unit of product A gives 8 Kg of *X* and 3 Kg of *Y*. One unit of product B gives 4 Kg of *X* and 9 Kg of *Y*. To maintain the water at a minimum level of softness and meet a minimum in health protection, it is decided that 150 Kg and 100 Kg of two chemicals that make up each product must be added daily. At a cost of Rs. 8/– and Rs. 10/– per unit respectively for A and B, what is the optimum quantity of each product that should be used to meet consumer standard?

Before discussing solution, let us have an idea of what is known as **Big M–n Method**, which is generally used to solve minimization problems.

While solving the linear programming problems by graphical method, we have seen an isoprofit line is drawn and at the origin and then it is moved away from the origin to find the optima point. Similarly an isocost line is drawn away from the origin in minimization problem and moved towards the origin to find the optimal point.

But in simplex method of solving the minimization problem, a highest cost is allocated to artificial surplus variable to remove it form the matrix. This high cost is Big – *M*. *M* stands for millions of rupees. If we use big *M* some times we feel it difficult while solving the problem. Hence, we can substitute a big numerical number to *M*, which is bigger than all the cost coefficients given in the problem. This may help us in numerical calculations.

Solution: Let the water works purchase *x* units of *X* and *y* units of *Y*, then:

Inequalities:	Simplex Format:
Minimise $Z = 8x + 10y$ s.t	Minimise $Z = 8x + 10y + 0p + 0q + M A_1 + MA_2$ s.t.
$3x + 9y \geq 100$	$3x + 9y - 1p + 0q + 1A_1 + 0A_2 = 100$
$8x + 4y \geq 150$ and	$8x + 4y + 0p - 1q + 0A_1 + 1A_2 = 150$ and
Both x and $y \geq 0$	x, y, p, q, A_1, A_2 all ≥ 0

Table: I. $x = 0$, $y = 0$, $p = 0$, $a = 0$, $A_1 = 100$, $A_2 = 150$ and $Z = 100M + 150M =$ Rs. $250\ M$.

Programe	*Cost in Rs.*	*C_j = requirement*	*8* *x*	*10* *y*	*0* *p*	*0* *q*	*M* *A_1*	*M* *A_2*	*Replacement ratio*
A_1	M	100	3	**9**	–1	0	1	0	100/9 = 11.11
A_2	M	150	8	4	0	–1	0	1	150 / 4 = 37.5
Net	Evaluation		8 – 11M	10 – 13M	M	M	0	0	—

Table II: $x = 0$, $y = 11.11$, $p = 0$, $q = 0$, $A_1 = 1.32$, $A_2 = 0$, Z = Rs. 11.11 x 10 + 1.32M = 111.1 +1.32 M

Program	*Cost in Rs.*	*C_j = requirement*	*8* x	*10* y	*0* p	*0* q	*M* A_1	*M* A_2	*Replacement ratio*
y	10	11.1	0.33	1	–0.11	0	0.11	0	33.6
A_2	M	106	**6.88**	0	0.44	–1	–0.44	1	15.4
Net	Evaluation		4.3–6.88M	0	–1.1+0.44M	M	–1.1+5.4M	0	

Table III. $x = 0.5$, $y = 15.4$, $p = 0$, $q = 0$, $A_1 = 0$, $A_2 = 0$, Z = Rs. 10 x 0.50 + 8 x 15.4 = Rs. 128.20

Program	*Cost in Rs.*	*C_j = requirement*	*8* x	*10* y	*0* p	*0* q	*M* A_1	*M* A_2	*Replacement ratio*
y	10	0.5	0	1	–0.154	0.1	0.154	–0.1	—
x	8	15.4	1	0	0.06	–0.14	–0.06	0.14	—
Net	Evaluation	—	0	0	1.062	0.12	M–1.06	M– 0.12	

Water works purchases 0.5 Kg of *Y* and 15.4 Kg of *X* at a cost of Rs. 128.20. The shadow price will be Rs. 107/–. The difference is due to decimal numbers. (Note: We can avoid the artificial variables as and when they go out to reduce the calculations. We can use a numerical value for *M*, which is higher than the cost of variables given in the problem so that we can save time.).

Problem 3.11: 10 grams of Alloy A contains 2 grams of copper, 1 gram of zinc and 1 gram of lead. 10 grams of Alloy B contains 1 gram of copper, 1 gram of zinc and 1 gram of lead. It is required to produce a mixture of these alloys, which contains at least 10 grams of copper, 8 grams of zinc, and 12 grams of lead. Alloy B costs 1.5 times as much per Kg as alloy A. Find the amounts of alloys *A* and *B*, which must be mixed in order to satisfy these conditions in the cheapest way.

Solution: The given data is: (Assume the cost of Alloy *A* as Re.1/– then the cost of Alloy *B* will be Rs. 1.50 per Kg.

Metals	*Alloys (In grams per 10 grams)*		*Requirement in Grams*
	A	*B*	
Copper	2	1	10
Zinc	1	1	8
Lead	1	1	12
Cost in Rs. per Kg.	1	1.5	

Let the company purchase *x* units of Alloy *A* and *y* units of Alloy *B*. (Assume a value of 10 for *M*)

Inequalities:	Simplex Format:
Minimise Z = $1x + 1.5\ y$ s.t.	Minimise $Z = 1x + 1.5y + 0p + 0q + 0r + 10A_1 + 10\ A_2 + 10A_3$ s.t.
$2x + 1y \geq 10$	$2x + 1y - 1p + 0q + 0r + 1A_1 + 0A_2 + 0A_3 = 10$
$1x + 1y \geq 8$	$1x + 1y + 0p - 1q + 0r + 0A_1 + 1A_2 + 0A_3 = 8$
$1x + 1y \geq 12$ and	$1x + 1y + 0p + 0q - 1r + 0A_1 + 0A_2 + 1A_3 = 12$ and
x, y both ≥ 0	$x, y, p, q, r, A_1, A_2, A_3$ all ≥ 0

Table: I. $x = 0$, $y = 0$, $p = 0$, $q = 0$, $r = 0$, $A_1 = 10$, A_2 8 and $A_3 = 12$ and Profit Z = Rs. 10 × 10 + 10 × 8 + 10 × 12 = Rs. 300

Program	*Cost in Rs.*	*C_j = Require–ment*	*1 x*	*1.5 y*	*0 p*	*0 q*	*0 r*	*10 A_1*	*10 A_2*	*10 A_3*	*Replace–ment ratio*
A_1	10	10	**2**	1	–1	0	0	1	0	0	5
A_2	10	8	1	1	0	–1	0	0	1	0	8
A_3	10	12	1	1	0	0	–1	0	0	1	12
Net	Evaluation		–39	–28.5	10	10	10	0	0	0	

Table: II. $x = 5$, $y = 0$, $p = 0$, $q = 0$, $r = 0$, $A_1 = 0$, $A_2 = 3$, $A_3 = 7$ and Z = Rs. 1 × 5 + 10 x 3 + 7 × 10 = Rs. 105/–

Program	*Cost in Rs.*	*C_j = Require–ment*	*1 x*	*1.5 y*	*0 p*	*0 q*	*0 r*	*10 A_1*	*10 A_2*	*10 A_3*	*Replace–ment ratio*
x	1	5	1	0.5	– 0.5	0	0	0.5	0	0	– 10 (neglect)
A_2	10	3	0	0.5	**0.5**	– 1	0	– 0.5	1	0	6
A_3	10	7	0	0.5	0.5	0	– 1	– 0.5	0	1	14
Net	Evaluation		0	– 9	–9.5	10	10	19.5	0	0	

Table: III. $x = 8$, $y = 0$, $p = 6$, $q = 0$, $r = 0$, $A_1 = 0$, $A_2 = 0$, $A_3 = 4$, Z = Rs. 8 + 40 = Rs. 48 /–

Program	*Cost in Rs.*	*C_j = Require–ment*	*1 x*	*1.5 y*	*0 p*	*0 q*	*0 r*	*10 A_1*	*10 A_2*	*10 A_3*	*Replace–ment ratio*
x	1	8	1	1	0	– 1	0	0	1	0	– 8 (neglect)
p	0	6	0	1	1	– 2	0	– 1	2	0	– 3 (neglect)
A_3	10	4	0	0	0	**1**	– 1	0	–1	1	4
Net	Evaluation		0	0.5	0	– 9	10	10	19	0	

Table: IV. $x = 12$, $y = 0$, $p = 14$, $q = 4$, $r = 0$, Z = Rs. 1×12 = Rs. 12/ –

Program	*Cost in Rs.*	C_j = *Require– ment*	*1* x	*1.5* y	*0* p	*0* q	*0* r	*10* A_1	*10* A_2	*10* A_3	*Replace– ment ratio*
x	1	12	1	1	0	0	–1	0	0	1	
p	0	14	0	1	1	0	–2	–1	1	2	
q	0	4	0	0	0	1	–1	0	–1	1	
Net	Evaluation		0	0.5	0	0	1	0	0	9	

As all the net evaluation elements are either zeros or positive element, the solution is optimal. The company can purchase 12 units of X at a cost of Rs. 12/–

Problem 3.12: Minimise $Z = 4a + 2b$ s.t.

$3a + 1b \geq 27$

$–1a – 1b \leq –21$

$1a + 2b \geq 30$ and both a and b are ≥ 0.

Inequalities:	Equations:
Minimise $Z = 4a + 2b \geq 27$	Minimise $Z = 4a + 2b + 0p + 0q + 0r + MA_1 + MA_2 + MA_3$ s.t.
$3a + 1b \geq 27$	$3a + 1b – 1p + 0q + 0r + 1A_1 + 0A_2 + 0A_3 = 27$
$1a + 1b \geq 21$	$1a + 1b + 0p – 1q + 0r + 0A_1 + 1A_2 + 0A_3 = 21$
$1a + 2b \geq 30$	$1a + 2b + 0p$ +– $+ 0q – 1r + 0A_1 + 0A_2 + 1A_3 = 30$
And a, b both ≥ 0	a, b, p, q, r, A_1, A_2 and A_3 all ≥ 0

(**Note: converting the objective function conveniently we can solve the minimization or maximization problems.** For example, if the objective function given is minimization type, we can convert it into maximization type by multiplying the objective function by –1. For example, in the problem 3.12, the objective function may be written as Maximise $Z = –4a – 2b$ s.t. But the inequalities are in the form of $\geq$ type. In such cases when artificial surplus variable (A_1) is introduced then the cost co–efficient of the artificial surplus variable will be $–M$ instead of $+M$. Rest of the procedure of solving the problem is same. Similarly, any maximization problem can be converted into minimization problem by multiplying the objective function by –1. If the inequalities are in $\geq$ form, subtracting the surplus variable and adding the artificial surplus variable is done to inequalities to convert them into equations. In case the inequalities are of $\leq$ type, slack variable is added to convert them into equations. Let us see this in next example).

Solution: Let M be represented by a numerical value Rs.10/– that is higher than the cost coefficients given in the problem. (*i.e.* 4 and 2).

Table: I. $a = 0$, $b = 0$, $p = 0$, $q = 0$, $r = 0$, $A_1 = 27$, $A_2 = 21$ and $A_3 = 30$ and the cost Z = Rs. 780/–

Program	*Cost in Rs.*	*C_j Require–ment*	*4 a*	*2 b*	*0 p*	*0 q*	*0 r*	*10 A_1*	*10 A_2*	*10 A_3*	*Replace–ment ratio*
A_1	10	27	**3**	1	– 1	0	0	1	0	0	9
A_2	10	21	1	1	0	– 1	0	0	1	0	21
A_3	10	30	1	2	0	0	– 1	0	0	1	30
	Net	Evaluation	– 46	– 38	10	10	10	0	0	0	—

Table: II. $a = 9$, $b = 0$, $p = 0$, $q = 0$, $r = 0$, $A_1 = 0$, $A_2 = 12$, $A_3 = 21$ and Z = Rs. 366/–

Program	*Cost in Rs.*	*C_j Require–ment*	*4 a*	*2 b*	*0 p*	*0 q*	*0 r*	*10 A_1*	*10 A_2*	*10 A_3*	*Replace–ment ratio*
a	4	9	1	0.33	– 0.33	0	0	0.33	0	0	27.27
A_2	10	12	0	0.67	0.33	– 1	0	– 0.33	1	0	17.91
A_3	10	21	0	**1.67**	0	0	– 1	0	0	1	12.51
	Net	Evaluation	0	– 22.72	– 1.98	10	10	8.22	0	0	

Table: III. $a = 4.84$, $b = 12.51$, $p = 0$, $q = 0$, $r = 0$, $A_1 = 0$, $A_2 = 3.6$, $A_3 = 0$, Z = Rs. 80.50.

Program	*Cost in Rs.*	*C_j Require–ment*	*4 a*	*2 b*	*0 p*	*0 q*	*0 r*	*10 A_1*	*10 A_2*	*10 A_3*	*Replace–ment ratio*
a	4	4.84	1	0	– 0.33	0	– 0.198	0.33	0	0.198	Neglect
A_2	10	3.6	0	0	0.33	– 1	– 0.4	– 0.33	**1**	0.4	10.9
b	2	12.57	0	1	0	0	– 0.6	0	0	0.6	Infinity
	Net	Evaluation	0	0	– 1.98	10	3.592	8.02	0	4.008	—

Table: IV. $a = 3$, $b = 18.05$, $p = 9$, $q = 0$, $r = 0$, $A_1 = 0$, $A_2 = 0$, $A_3 = 0$, Cost Z = Rs. 48.10

Program	*Cost in Rs.*	*C_j requirement*	*4 x*	*2 y*	*0 p*	*0 q*	*0 r*	*10 A_1*	*10 A_2*	*10 A_3*	*Replacement ratio*
a	4	3	1	0	– 0.485	0.5	0	0.485	– 0.5	0	
p	0	9	0	0	0.425	– 2.5	1	– 0.425	0.25	– 1	
b	2	18.05	0	1	0.445	– 1.5	0	– 0.445	1.5	0	
	Net	Evaluation	0	0	1.05	1	0	8.95	9	0	

As the elements of net evaluation row are either zeros or positive elements, the solution is optimal.

$A = 3$ and $B = 12.57$ and the optima cost Z = Rs. 48.10. The shadow price = $1.05 \times 27 + 1 \times 21$ = Rs. 49.35.

The difference is due to decimal calculations.

Problem 3. 13: Solve the Minimization L.P.P. given below:

Min. $Z = 1x - 3y + 2z$ S.t.

$3x - 1y - + 3z \leq 7$

$-2x + 4y + 0z \leq 12$

$-4x + 3y + 8z \leq 10$ and x, y, and z all ≥ 0.

Solution: As the objective function is of minimization type and the constraints are of $\leq$ type, we can rewrite the problem in simplex format as:

Maximize $Z = -1x + 3y - 2z + 0S_1 + 0S_2 + 0S_3$ S.t.

$3x - 1y + 3z + 1S_1 + 0S_2 + 0\,S_3 = 7$

$-2x + 4y + 0z + 0S_1 + 1S_2 + 0S_3 = 12$

$-4x + 3y + 8z + 0S_1 + 0S_2 + 1S_3 = 10$ and x, y, z, S_1, S_2 and S_3 all ≥ 0.

Table: I. $x = 0$, $y = 0$, $z = 0$, $S_1 = 7$, $S_2 = 0$, $S_3 = 0$ and Z = Rs.0./–

Problem variable	*Profit Rs.*	*Capacity,= Z units*	*– 1 x*	*3 y*	*– 2 z*	*0 S_1*	*0 S_2*	*0 S_3*	*Replacement ratio*
S_1	0	7	3	– 1	3	1	0	0	—
S_2	0	12	– 2	**4**	0	0	1	0	12/4 = 3
S_3	0	10	– 4	3	8	0	0	1	10/3 = 3.3.
	Net	Evaluation.	– 1	3	– 2	0	0	0	

Table: II. $x = 0$, $y = 3$, $z = 0$, $S_1 = 10$, $S_2 = 0$, $S_3 = 1$ and $Z = 3 \times 3$ = Rs.9.00.

Problem variable	*Profit Rs.*	*Capacity, Z = units*	*– 1* x	*3* y	*– 2* z	*0* S_1	*0* S_2	*0* S_3	*Replacement ratio*
S_1	0	10	**3.5**	0	3	1	0.25	0	2.86
y	3	3	– 0.5	1	0	0	0.25	0	—
S_3	0	1	– 2.5	0	8	0	– 0.75	1	—
	Net	Evaluation.	0.5	0	– 2	0	– 0.75	0	

Table III. $x = 2.86$, $y = 4.43$, $z = 0$, $S_1 = 0$, $S_2 = 0$ and $S_3 = 8.14$, Z = Rs. 10.43.

Problem variable	*Profit Rs.*	*Capacity, Z = units*	*– 1* x	*3* y	*– 2* z	*0* S_1	*0* S_2	*0* S_3	*Replacement ratio*
x	– 1	2.86	1	0	0.86	0.29	0.07	0	—
y	3	4.43	0	1	0.43	0.143	0.285	0	—
S_3	0	8.14	0	0	10.14	0.174	– 0.57	1	—
	Net	Evaluation.	0	0	– 2.43	– 0.14	– 0.68	0	

Answer: $X = 2.86$, $Y = 4.43$, $Z = 0$ and Profit Z = Rs. 10.43.

MIXED PROBLEMS

As a mathematical interest, we may deal with some problems which have the characteristics of both maximization and minimization problems. These problems may not exist in real world, but they are significantly important as far as mathematical interest. These problems are generally known as **Mixed** problems. Let us work out some problems of this nature. (by using **big – M method**).

Problem 3.14: Solve the following L.P.P.:

Minimize $Z = 4a + 2b$ S.t.

$3a + 1b \geq 27$

$-1a - 1b \leq -21$

$1a + 2b \geq 30$ and both a and b are ≥ 0

The right hand side of any inequality or equation should not be negative. Hence we have to multiply the second inequality by –1. Then the given problem becomes:

Minimize $Z = 4a + 2b$ s.t.	OR Maximize $Z = -4a - 2b$ s.t
$3a + 1b \geq 27$	$3a + 1b \geq 27$
$1a + 1b \geq 21$	$1a + 1b \geq 21$
$1a + 2b \geq 30$ and both a and $b \geq 0$	$1a + 2b \geq 30$ and both a and $b \geq 0$.

The simplex version of the problem is:

Minimize $Z = 4a + 2b + 0p + 0q + 0r + MA_1 + MA_2 + MA_3$ s.t.

$3a + 1b - 1p + 0q + 0r + 1A_1 + 0A_2 + 0A_3 = 27$

$1a + 1b + 0p - 1q + 0r + 0A_1 + 1A_2 + 0A_3 = 21$

$1a + 2b + 0p + 0q - 1r + 0A_1 + 0A_2 + 1A_3 = 30$ and $a, b, p, q, r, A_1, A_2, A_3$ all ≥ 0

The simplex format of Maximization version is: **In maximization version we use negative sign for big –M.**

Maximize $Z = -4a - 2b + 0p + 0q + 0r - MA_1 - MA_2 - MA_3$ s.t.

$3a + 1b - 1p + 0q + 0r + 1A_1 + 0A_2 + 0A_3 = 27$

$1a + 1b + 0p - 1q + 0r + 0A_1 + 1A_2 + 0A_3 = 21$

$1a + 2b + 0p + 0q - 1r + 0A_1 + 0A_2 + 1A_3 = 30$ and $a, b, p, q, r, A_1, A_2, A_3$ all ≥ 0

Let us solve the maximization version.

Table: I. $a = 0$, $b = 0$ $p = 0$ $q = 0$ $r = 0$ $A_1 = 27$ $A_2 = 21$ $A_3 = 30$ and $Z =$ Rs. 78 M.

Problem variable	*Profit Rs.*	*Capacity C = Units*	*– 4 a*	*– 2 b*	*0 p*	*0 q*	*0 r*	*–M A_1*	*–M A_2*	*–M A_3*	*Replace–ment ratio*
A_1	$-M$	27	**3**	1	– 1	0	0	1	0	0	27/3 = 9
A_2	$-M$	21	1	1	0	– 1	0	0	1	0	21/1 = 21
A_3	$-M$	30	1	2	0	0	– 1	0	0	1	30/1 = 30
		Net evaluation	– 4 + 5M	– 2 + 4M	– M	– M	– M	0	0	0	

Table: II. $a = 9$, $b = 0$, $p = 0$, $q = 0$, $r = 0$, $A_1 = 0$, $A_2 = 12$, $A_3 = 21$, $Z = -33M - 36$

Problem variable	*Profit Rs.*	*Capacity C = Units*	*– 4 a*	*– 2 b*	*0 p*	*0 q*	*0 r*	*– M A_1*	*– M A_2*	*– M A_3*	*Replace–ment ratio*
a	– 4	9	1	1/3	– 1/3	0	0		0	0	
A_2	$-M$	12	0	2/3	1/3	– 1	0		1	0	
A_3	$-M$	21	0	**5/3**	1/3	0	– 1		0	1	
		Net evaluation	0	–2 + 7/3M	4/3 + 2/3M	– M	– M		0	0	

Note: Artificial variable removed is not entered.

Table: III. $a = 24/5$, $b = 63/5$, $p = 0$, $q = 0$, $r = 0$, $A_1 = 0$, $A_2 = 18/5$, $A_3 = 0$

Problem variable	*Profit Rs.*	*Capacity C = Units*	*– 4 a*	*– 2 b*	*0 p*	*0 q*	*0 r*	*– M A_1*	*– M A_2*	*– M A_3*	*Replace–ment ratio*
a	– 4	24/5	1	0	– 2/5	0	1/5		0		
A_2	– M	18/5	0	0	1/5	–1	**2/5**		1		
b	– 2	63/5	0	1	1/5	0	– 3/5		0		
		Net evaluation	0	0	– 6/5 + 1/5M	– M	– 2/5 + 2/5M		0		

Table: IV. $a = 3$, $b = 18$, $p = 0$, $q = 0$, $r = 9$ and Z = Rs. 48.00

Problem variable	*Profit Rs.*	*Capacity C = Units*	*– 4 a*	*– 2 b*	*0 p*	*0 q*	*0 r*	*–M A_1*	*–M A_2*	*–M A_3*	*Replace–ment ratio*
a	– 4	3	1	0	– 1/2	1/2	0				
r	0	9	0	0	1/2	– 5/2	1				
b	– 2	18	0	1	1/2	– 3/2	0				
		Net evaluation	0	0	– 1	– 1	0				

***A* = 3, *B* = 18 and *Z* = Rs. 48/–**. That is for minimization version; the total minimum cost is Rs. 48/–

Problem 3.15: Solve the following L.P.P.

Maximize $Z = 1a + 2b + 3c - 1d$ S.t.

$1a + 2b + 3c = 15$

$2a + 1b + 5c = 20$

$1a + 2b + 1c + 1f = 10$ and a, b, c, f all are ≥ 0.

In this problem given constraints are equations rather than inequalities. Also by careful examination, we can see that in the third equation variable '*f*' exists and it also exists in objective function. Hence we consider it as a surplus variable and we add two more slack variables '*d*' and '*e*' is added to first and second equations. But the cost coefficient in objective function for variables *d* and *e* will be $-M$. When we use big *M* in maximization problem, we have to use $-M$ in objective function. While solving the problem, all the rules related to solving maximization problem will apply. Hence now the simplex format of the problem is as follows:

Maximize $Z = 1a + 2b + 3c - 1d - Me - Mf$ s.t.

$1a + 2b + 3c + 1d + 0e + 0f = 15$

$2a + 1b + 5c + 0d + 1e + 0f = 20$

$1a + 2b + 1c + 0d + 0e + 1f = 10$ and a, b, c, d, e, f all = 0.

Table I. $a = 0$, $b = 0$, $c = 0$, $d = 15$, $e = 20$, $f = 10$ and Z = Rs. 0.

Problem variable	*Profit Rs.*	C_j *Capacity, units*	*1* *a*	*2* *b*	*3* *c*	*– 1* *d*	*– M* *e*	*– M* *f*	*Replacement ratio*
d	– 1	15	1	2	3	1	0	0	5
e	– M	20	2	1	**5**	0	1	0	4
f	– M	10	1	2	1	0	0	1	10
		Net evaluation.	2 + 3M	4 + 3M	4 + 8M	0	0	0	

Table II. $A = 0$, $b = 0$, $c = 4$, $d = 0$, $e = 0$, $f = 0$., Z = Rs. 3 × 4 = Rs. 12.

Problem Variable	*Profit Rs.*	C_j *Capacity, units*	*1* *a*	*2* *b*	*3* *c*	*– 1* *d*	*– M* *e*	*– M* *f*	*Replacement Ratio*
d	– 1	6	3/5	9/5	0	1	0	0	30/9
c	3	4	2/5	1/5	1	0	0	0	20
f	– M	3	– 1/5	**7/5**	0	0	1	1	15/7
		Net evaluation.	2/5 + M /5	16/5 + 7/5M	0	M	– M	0	

TableL: III. $a = 0$, $b = 15/7$, $c = 25/7$, $d = 15/7$, $e = 0$, $f = 0$, Z = Rs. 107 – 15/7 = Rs.92.

Problem variable	*Profit Rs.*	*Profit capacity*	*1* *a*	*2* *b*	*3* *c*	*– 1* *d*	*– M* *e*	*– M* *f*
d	– 1	15/7	6/7	0	0	**1**	0	0
c	3	25/7	3/7	0	1	0	– 5/7	– 5/7
b	2	15/7	– 1/7	1	0	0	5/7	5/7
		Net evaluation	6/7	0	0	0	– M + 5/7	– M + 5/7

Table: IV. a =5/2, b = 5/2, c = 5/2, d = 0, e = 0, f = 0, Z = Rs. 15/–

Problem Variable	*Profit Rs.*	*Profit capacity*	*1* *a*	*2* *b*	*3* *c*	*– 1* *d*	*– M* *e*	*– M* *f*
a	1	5/2	1	0	0	7/6	0	0
c	3	5/2	0	0	1	– 1/2	– 5/7	– 5/7
b	2	5/2	0	1	0	1/6	5/7	5/7
			0	0	0	– 4/12	– M + 5/7	– M = 5/7

As all the elements of net evaluation row are either zeros or negative elements, the solution is optimal.

Z = Rs. 15/–. And $a = b = c = 5/2$.

Problem 3.16: Solve the given l.p.p:

Maximize $4x + 3y$ s.t.

$1x + 1y \leq 50$

$1x + 2y \geq 80$

$3x + 2y \geq 140$

And both x and $y \geq 0$

Simplex format is:

Maximize $Z = 4x + 3y + 0S + 0p + 0q - MA_1 - MA_2$ s.t.

$1x + 1y + 1S + 0p + 0q + 0A_1 + 0A_2 = 50$

$1x + 2y + 0S - 1p + 0q + 1A_1 + 0A_2 = 80$

$3x + 2y + 0S + 0p - 1q + 0A_1 + 1A_2 = 140$

and x, y, S, p, q, A_1 and A_2 all ≥ 0

Table: I. $X = 0$, $y, = 0$, $S = 50$, $p = 0$ $q = 0$, $A_1 = 80$, $A_2 = 140$ and Z = Rs. 220 M.

Problem variable	*Profit Rs.*	*Profit: capacity units*	*4* *x*	*3* *y*	*0* *S*	*0* *p*	*0* *q*	*–M* A_1	*–M* A_2	*Replace– ment ratio*
S	0	50	1	1	1	0	0	0	0	50
A_1	– M	80	1	**2**	0	– 1	0	1	0	40
A_2	– M	140	3	2	0	0	– 1	0	1	70
		Net evaluation.	4 + 4M	3 + 4M	0	M	M	0	0	

Now in the net evaluation row the element under variable 'x' is $4 + 4M$ is greater than the element $3 + 3M$. But if we take 'x' as the incoming variable we cannot send artificial variable out first. Hence we take 'y' as the incoming variable, so that A_1 go out first.

Table: II. $X = 0$, $y = 40$, $S = 10$, $A_1 = 0$, $A_2 = 60$, $p = 0$, $q = 0$ and $Z =$ Rs. $120 - 60\ M$.

Problem variable	*Profit Rs.*	*Profit: capacity units*	*4* x	*3* y	*0* S	*0* p	*0* q	$-M$ A_1	$-M$ A_2	*Replace–ment ratio*
S	0	10	**0.5**	0	1	0.5	0	– 0.5	0	20
y	3	40	0.5	1	0	– 0.5	0	0.5	0	80
A_2	– M	60	2	0	0	2	– 1	– 2	1	30
		Net evaluation.	2.5 + 8M	0	0	1.5 – 2M	– M	– 1.5 – 3M	0	

Table: III. $x = 20$, $y = 30$, $p = 0$, $q = 0$, $A_1 = 0$, $A_2 = 20$, $Z =$ Rs. 170/–

Problem variable	*Profit Rs.*	*Profit: capacity units*	*4* x	*3* y	*0* S	*0* p	*0* q	$-M$ A_1	$-M$ A_2	*Replace–ment ratio*
x	4	20	1	0	2	1	0	– 1	0	
y	3	30	0	1	– 1	– 1	0	1	0	
A_2	– M	20	0	0	– 4	0	– 1	0	1	
		Net evaluation.	0	0	– 5 –4M	– 1	– M	– M = 1	0	

In the last table though basis variables have the opportunity cost as 0, still artificial variable A_2 exists in the problem, hence the original problem has no feasible solution.

$X = 20$, $Y = 30$.

Problem 3.17: Solve the given l.p.p.

Maximize $Z = 1a + 1.5b + 5c + 2d$ s.t.

$3a + 2b + 1c + 4d \leq 6$

$2a + 1b + 5c + 1d \leq 4$

$2a + 6b - 4c + 8d = 0$

$1a + 3b - 2c + 4d = 0$

And a, b, c, d all ≥ 0

Simplex version of the problem is:

Maximize $Z = 1a + 1.5b + 5c + 2d + 0S_1 + 0S_2 - MA_1 - MA_2$ s.t.

$3a + 2b + 1c + 4d + 1S_1 + 0S_2 + 0A_1 + 0A_2 = 6$

$2a + 1b + 5c + 1d + 0S_1 + 1S_2 + 0A_1 + 0A_2 = 4$

$2a + 6b - 4c + 8d + 0S_1 + 0S_2 + 1A_1 + 0A_2 = 0$

$1a + 3b - 2c + 4d + 0S_1 + 0S_2 + 0A_1 + 1A_2 = 0$

and $a, b, c, d, S_1, S_2, A_1, A_2$ all ≥ 0

Table I. $A = 0$, $b = 0$, $c = 0$, $d = 0$, $S_1 = 6$, $S_2 = 0$, $A_1 = 0$, $A_2 = 0$ and Z = Rs. 0.00

Problem variable	*Profit Rs.*	C_j *Capacity units*	*1* *a*	*1.5* *b*	*5* *c*	*2* *d*	*0* S_1	*0* S_2	$-M$ A_1	$-M$ A_2	*Replace–ment ratio*
S_1	0	6	3	2	1	4	1	0	0	0	6/4 = 1.5
S_2	0	4	2	1	5	1	0	1	0	0	4/1 = 4
A_1	– M	0	2	6	– 4	8	0	0	1	0	0
A_2	– M	0	1	3	– 2	**4**	0	0	0	1	0
		Net evaluation	1 + 3M	1.5 + 9M	5 – 6M	2 + 12M	0	0	0	0	

Note: Both A_1 and A_2 have the same replacement ratio. That is to say there is a tie in outgoing variables. This type of situation in Linear Programming Problem is known as DEGENERACY. To Solve degeneracy refer to the rules stated earlier.

Now let us remove as and when a surplus variable goes out, which will ease our calculation and also save time.

Table: II. $a = 0$, $b = 0$, $c = 0$, $d = 0$, $S_1 = 6$, $S_2 = 4$, $A_1 = 0$, $A_2 = 0$, Z = Rs. 0/–

Problem variable	*Profit Rs.*	C_j *Capacity units*	*1* *a*	*1.5* *b*	*5* *c*	*2* *d*	*0* S_1	*0* S_2	$-M$ A_1	$-M$ A_2	*Replace–ment ratio*
S_1	0	6	2	– 1	3	0	1	0	0		
S_2	0	4	7/4	1/4	11/2	0	0	1	0		
A_1	– M	0	0	0	0	0	0	0	1		
d	2	0	1/4	3/4	– 1/2	1	0	0	0		
		Net evaluation	1/2	0	6	0	0	0	0		

In the given problem the inequalities number 3 and 4 *i.e.*, $2a + 6b - 4c + 8d = 0$ and

$1a + 3b - 2c + 4d = 0$ appears to be similar. If you carefully examine, we see that

$2a + 6b - 4c + 8d = 0$ is $2 \times (1a + 3b - 2c + 4d = 0)$. Hence one of them may be considered as redundant and cancelled. *i.e.*, third constraint is double the fourth constraint hence we can say it is not independent. Hence we can eliminate the third row and the column under A_1 from the tableau. For identifying the redundancy, one need not wait for final optimal table.

Table: III. (Second reduced second table) $S_1 = 6$, $S_2 = 4$, $d = 0$, $a = 0$, $b = 0$, $c = 0$, Z = Rs. 0

Problem variable	*Profit Rs.*	C_j *Capacity units*	*1* *a*	*1.5* *b*	*5* *c*	*2* *d*	*0* S_1	*0* S_2	$-M$ A_1	$-M$ A_2	*Replace–ment ratio*
S_1	0	6	2	– 1	3	0	1	0			2
S_2	0	4	7/4	1/4	**11/2**	0	0	1			8/11
d	2	0	1/4	3/4	– 1/2	1	0	0			—
		Net evaluation.	1/2	0	6	0	0	0			

Table: IV. $S_1 = 42/11$, $C = 5$, $d = 4/11$ and Z = Rs. 4.36.

Problem variable	*Profit Rs.*	C_j *Capacity units*	*1* *a*	*1.5* *b*	*5* *c*	*2* *d*	*0* S_1	*0* S_2	$-M$ A_1	$-M$ A_2	*Replace–ment ratio*
S_1	0	42/11	23/22	– 25/22	0	0	1	– 6/11			
c	5	8/11	7/22	1/22	1	0	0	2/11			
d	2	4/11	9/22	17/22	0	1	0	1/11			
		Net evaluation	– 31/22	– 3/11	0	0	0	– 12/11			

As all the elements of net evaluation row are either zeros or negative elements the solution at this stage is optimal. Hence $c = 8/11$, $d = 4/11$ and Z = Rs. 48/11 = Rs. 4.36.

ARTIFICIAL VARIABLE METHOD OR TWO PHASE METHOD

In linear programming problems sometimes we see that the constraints may have $\geq$, $\leq$ or = signs. In such problems, basis matrix is not obtained as an identity matrix in the first simplex table; therefore, we introduce a new type of variable called, the artificial variable. These variables are fictitious and cannot have any physical meaning. The introduction of artificial variable is merely to get starting basic feasible solution, so that simplex procedure may be used as usual until the optimal solution is obtained. Artificial variable can be eliminated from the simplex table as and when they become zero *i.e*, non–basic. This process of eliminating artificial variable is performed in **PHASE I of the solution. PHASE II** is then used for getting optimal solution. Here the solution of the linear programming problem is completed in two phases, this method is known as **TWO PHASE SIMPLEX METHOD**. Hence, the two–phase method deals with removal of artificial variable in the fist phase and work for optimal solution in the second phase. If at the end of the first stage, there still remains artificial variable in the basic at a positive value, it means there is no feasible solution for the problem given. In that case, it is not necessary to work on phase II. If a feasible solution exists for the given problem, the value of objective function at the end of phase I will be zero and artificial variable will be non-basic. In phase II original objective coefficients are introduced in the final tableau of phase I and the objective function is optimized.

Problem 3.18: By using two phase method find whether the following problem has a feasible solution or not?

Maximize $Z = 4a + 5b$ s.t.
$2a + 4b \leq 8$
$1a + 3b \geq 9$ and both a and b are ≥ 0.

Simplex version is: Max. $Z = 4a + 5b + 0S_1 + 0S_2 - MA$ s.t.
$2a + 4b + 1S_1 + 0S_2 + 0A = 8$
$1a + 3b + 0S_1 - 1S_2 + 1A = 9$ and
a, b, S_1, S_2, A all are ≥ 0

Phase I

Maximize $Z = 0a + 0b + 0S_1 + 0S_2 - 1A$ s.t.
$2a + 4b + 1S_1 + 0S_2 + 0A = 8$
$1a + 3b + 0S_1 - 1S_2 + 1A = 9$ and a, b, S_1, S_2 and A all ≥ 0.

Table: I. $a = 0$, $b = 0$, $S_1 = 8$, $S_2 = 0$, $A = 9$ and $Z = -$ Rs 9

Problem variable	*Profit Rs.*	C_j *Capacity, units*	*0* a	*0* b	*0* S_1	*0* S_2	*– 1* A	*Replacement ratio.*
S_1	0	8	2	**4**	1	0	0	8/4 = 2
A	– 1	9	1	3	0	– 1	1	9/3 = 3
		Net evaluation	1	3	0	– 1	0	

Table: II. $a = 0$, $b = 2$, $S_1 = 0$, $S_2 = 0$, $A = 3$ and $Z =$ Rs. – 3/–

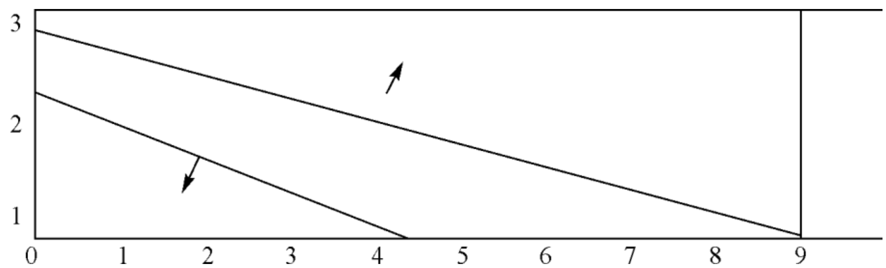
Problem variable	*Profit Rs.*	C_j *Capacity, units*	*0* a	*0* b	*0* S_1	*0* S_2	*– 1* A	*Replacement ratio.*
b	0	2	**0.5**	1	0.25	0	0	4
A	– 1	3	0.5	0	– 0.75	– 1	1	8
		Net evaluation	0.5	0	– 0.75	– 1	0	

Table: III. $a = 4$, $b = 0$, $S_1 = 0$, $S_2 = 0$, $A = 1$ and $z =$ Rs. –1/–

Problem variable	*Profit Rs.*	C_j *Capacity, units*	*0* a	*0* b	*0* S_1	*0* S_2	*– 1* A	*Replacement ratio.*
a	0	4	1	2	0.5	0	0	
A	– 1	1	0	– 1	– 1	– 1	1	
		Net evaluation	0	– 1	– 1	– 1	0	

As the artificial variable still remains as the basic variable and has a positive value, the given problem has no feasible solution.

If we examine the same by graphical means, we can see that the problem has no feasible region.



Problem 3.18:

Maximize $Z = 4x + 3y$ s.t.

$2x + 3y \leq 6$

$3x + 1y \geq 3$

Both x and y all ≥ 0

Simplex version:

Maximize $Z = 4x + 3y + 0S_1 + 0S_2 - MA$ s.t.

$2x + 3y + 1S_1 + 0S_2 + 0A = 6$

$3x + 1y + 0S_1 - 1S_2 + 1A = 3$

x, y, S_1, S_2, and A all $= 0$.

Phase I

$0x + 0y + 0S_1 + 0S_2 - 1A$ s.t.

$2x + 3y + 0S_1 + 0S_2 + 0A = 6$

$3x + 1y + 0S_1 - 1S_2 + 1A_3$, x, y, S_1, S_2, and A all ≥ 0

Table: I. $x = 0$, $y = 0$, $S_1 = 6$, $S_2 = 0$, $A = 3$ and $Z = 3 \times -1 = -$ Rs. 3/–

Problem variable	*Profit Rs.*	*C_j Capacity, units*	*0 x*	*0 y*	*0 S_1*	*0 S_2*	*– 1 A*	*Replacement ratio.*
S_1	0	6	2	3	1	0	0	3
A	– 1	3	**3**	1	0	– 1	1	1
		Net evaluation	3	1	0	– 1	0	

Table: II. $x = 1$, $y = 0$, $S_1 =$, $S_2 = 0$, $A = 0$. $Z =$ Rs. 0.

Problem variable	*Profit Rs.*	*C_j Capacity, units*	*0 x*	*0 y*	*0 S_1*	*0 S_2*	*– 1 A*	*Replacement ratio.*
S_1	0	0	0	7/3	1	2/3	– 2/3	
x	0	1	1	1/3	0	– 1/3	1/3	
		Net evaluation	0	0	0	0	1	

As there is no artificial variable in the programme, we can get the optimal solution for the given problem.

Hence Phase II is:

Phase II

Table: III. $S_1 = 4$, $x = 1$, $y = 0$, $S_2 = 0$, $A = 0$ and Z = Rs. 1 × 4 = Rs.4 /–

Problem variable	*Profit Rs.*	C_j *Capacity, units*	*0* x	*0* y	*0* S_1	*0* S_2	*– 1* A	*Replacement ratio.*
S_1	0	4	0	7/3	1	**2/3**	– 2/3	6
x	4	1	1	1/3	0	– 1/3	1/3	– 3 (neglect)
		Net evaluation	0	1/3	0	4/3	4/3 – M	

Table: IV. $x = 3$, $y = 0$, $S_1 = 6$, $S_2 = 0$, $A = 0$ and Z = Rs. 3x4 = Rs. 12/–

Problem variable	*Profit Rs.*	C_j *Capacity, units*	*0* x	*0* y	*0* S_1	*0* S_2	*– 1* A	*Replacement ratio.*
S_2	0	6	0	7/2	3/3	1	–1	
x	4	3	1	3/2	½	0	0	
		Net evaluation	0	–1	–2	0	– M	

Optimal solution is $x = 4$ and Z = Rs. 12/–. Graphically also student can work to find optimal solution.

Problem 3.19:

Solve the following L.P.P. by two–phase method:

Maximize $Z = 2a - 1b + 1c$ s.t.

$1a + 1b - 3c \leq 8$

$4a - 1b + 1c \geq 2$

$2a + 3b - 1c \geq 4$

And a, b, c all ≥ 0.

Simplex version:

Maximize $Z = 2a - 1b + 1c + 0S_1 + 0S_2 + 0S_3 - 1A_1 - 1A_2$ s.t

$1a + 1b - 3c + 1S_1 + 0S_2 + 0S_3 + 0A_1 + 0A_2 = 8$

$4a - 1b + 1c + 0S_1 - 1S_2 + 0S_3 + 1A_1 + 0A_2 = 2$

$2a + 3b - 1c + 0S_1 + 0S_2 - 1S_3 + 0A_1 + 1A_2 = 4$

a, b, c, S_1, S_2, S_2 A_1 and A_2 all are ≥ 0.

Phase I

The objective function is Maximize $Z = 0a + 0b + 0c + 0S_1 + 0S_2 + 0S_3 + (-1)\,A_1 + (-1)\,A_2$

The structural constraints will remain same as shown in simplex version.

Table: I. $a = 0$, $b = 0$, $c = 0$, $S_1 = 8$, $S_2 = 0$, $S_3 = 0$, $A_1 = 2$, $A_2 = 4$, $Z = –$ Rs. 6/–

Problem variable	*Profit Rs.*	$C_j \rightarrow$ *Capacity units*	*0* *a*	*0* *b*	*0* *c*	*0* S_1	*0* S_2	*0* S_3	*– 1* A_1	*– 1* A_2	*Replace–ment ratio*
S_1	0	8	1	1	– 3	1	0	0	0	0	8
A_1	– 1	2	**4**	– 1	1	0	– 1	0	1	0	1/2
A_2	– 1	4	2	3	– 1	0	0	– 1	0	1	2
		Net evaluation	6	2	0	0	– 1	– 1	0	0	

Table: II. $a = 1/2$, $b = 0$, $c = 0$, $S_1 = 15/2$, $S_2 = 0$, $S_3 = 0$, $A_1 = 0$, $A_2 = 3$, $Z = –$ Rs.3/–

Problem variable	*Profit Rs.*	$C_j \rightarrow$ *Capacity units*	*0* *a*	*0* *b*	*0* *c*	*0* S_1	*0* S_2	*0* S_3	*– 1* A_1	*– 1* A_2	*Replace–ment ratio*
S_1	0	15/2	0	3/4	– 11/4	1	1/4	0		0	
a	0	1/2	1	– 1/4	1/4	0	– 1/4	0		0	
A_2	– 1	3	0	**7/2**	– 5/2	0	1/2	– 1		1	
		Net evaluation	0	7/2	– 3/2	0	1/2	– 1		0	

Table: III. $a = 5/7$, $b = 6/7$, $c = 0$, $S_1 = 45/7$, $S_2 = 0$, $S_3 = 0$, $A_1 = 0$, $A_2 = 0$, $Z =$ Rs.0

Problem variable	*Profit Rs.*	$C_j \rightarrow$ *Capacity units*	*0* *a*	*0* *b*	*0* *c*	*0* S_1	*0* S_2	*0* S_3	*– 1* A_1	*– 1* A_2	*Replace–ment ratio*
S_1	0	45/7	0	0	– 19/7	1	1/14	5/14			
a	0	5/7	1	0	1/2	0	– 3/14	– 1/14			
b	0	6/7	0	1	– 3/7	0	1/7	– 2/7			
		Net evaluation	0	0	0	0	0	0			

As there are no artificial variables, we can go for second phase.

Phase II.

Table: I. $a = 5/7$, $b = 6/7$, $c = 0$, $S_1 = 45/7$, $S_2 = 0$, $S_3 = 0$, $A_1 = 0$, $A_2 = 0$, Z = Rs. 4/7.

Problem variable	*Profit Rs.*	$C_j \rightarrow$ *Capacity units*	*2* *a*	*– 1* *b*	*1* *c*	*0* S_1	*0* S_2	*0* S_3	*– 1* A_1	*– 1* A_2	*Replace–ment ratio*
S_1	0	45/7	0	0	– 19/7	1	1/14	5/14			
a	2	5/7	1	0	1/7	0	– 3/14	– 1/14			
b	– 1	6/7	0	1	– 3/7	0	**1/7**	3/7			
		Net evaluation	0	0	2/7	0	4/7	– 1/7			

Table: II.

Problem variable	*Profit Rs.*	$C_j \rightarrow$ *Capacity units*	*2* *a*	*– 1* *b*	*1* *c*	*0* S_1	*0* S_2	*0* S_3	*– 1* A_1	*– 1* A_2	*Replace–ment ratio*
S_1	0	6	0	– 7/2	– 5/2	1	0	1/2			
a	2	2	1	3/2	– 1/2	0	0	– 7/2			
S_2	0	6	0	7	– 3	1	1	–2			
		Net evaluation	0	– 4	2	0	0	7			

Highest positive element under S_3 in net evaluation row shows that the problem has unbound solution.

Problem 3.20: This can be written as:

Minimize $Z = 15/2a – 3b + 0c$ s.t. Maximize $Z = –15/2\, a + 3b – 0c$ s.t

$3a – 1b –1c \geq 3$ $\quad$ $3a – 1b – 1c \geq 3$

$1a – 1b +1c \geq 2$ $\quad$ $1a – 1b + 1c \geq 2$ and a, b, c all ≥ 0

a, b, c all ≥ 0

Simplex version is:

Maximize $Z = 15/2\, a – 3b – 0c + 0S_1 + 0S_2 – 1A_1 – 1A_2$ s.t.

$3a – 1b – 1c – 1S_1 + 0S_2 + 1A_1 + 0A_2 = 3$

$1a – 1b + 1c + 0S_1 – 1S_2 + 0A_1 + 1A_2 = 2$ and a, b, c, S_1, S_2, A_1 and A_2 all ≥ 0

In Phase I we give profit coefficients of variables as zero.

Maximize: $Z = 0a + 0b + 0c +0S_1 + 0S_2 –1A_1 – 1\, A_2$ s.t.

The constraints remain same.

Phase I

Table: I. $a = 0$, $b = 0$, $c = 0$, $S_1 = 0$, $S_2 = 0$, $A_1 = 3$, $A_2 = 2$ and $Z = -$ Rs.5/–

Problem variable	*Profit Rs.*	C_j *Capacity*	*0* *a*	*0* *b*	*0* *c*	*0* S_1	*0* S_2	*–1* A_1	*–1* A_2	*Replace–ment ratio*
A_1	– 1	3	**3**	1	1	1	0	1	0	3/3 = 1
A_2	– 1	2	1	– 1	1	0	– 1	0	1	2/1 = 2
		Net evaluation	4	0	2	1	1	0	0	

Table: II. $a = 1$, $b = 0$, $c = 0$, $S_1 = 0$, $S_2 = 0$, $A_1 = 0$, $A_2 = 1$, and $Z = -$ Rs.1/–

Problem variable	*Profit Rs.*	C_j *Capacity*	*0* *a*	*0* *b*	*0* *c*	*0* S_1	*0* S_2	*–1* A_1	*–1* A_2	*Replace–ment ratio*
a	0	1	1	– 1/3	– 1/3	– 1/3	0	1/3	0	—
A_2	– 1	1	0	– 2/3	**4/3**	1/3	– 1	– 4/3	1	3/4
		N.E.	0	– 2/3	4/3	1/3	– 1	– 4/3	0	

Table: III. $a = 5/4$, $b = 0$, $c = ¾$, $S_1 = 0$, $S_2 = 0$, $A_1 = 0$, $A_2 = 0$ and $Z =$ Rs.0/–

Problem variable	*Profit Rs.*	C_j *Capacity*	*0* *a*	*0* *b*	*0* *c*	*0* S_1	*0* S_2	*–1* A_1	*–1* A_2	*Replace–ment ratio*
a	0	5/4	1	– 1/2	0	– 1/4	– 1/4	1/4	1/4	
c	0	3/4	0	– 1/2	1	1/4	– 3/4	– 1/4	3/4	
		Net evaluation	0	0	0	0	0	– 1	– 1	

Phase II

Table: I. $a = 5/4$, $b = 0$, $c = ¾$, $S_1 = 0$, $S_2 = 0$, $A_1 = 0$, $S_2 = 0$, $Z =$ Rs. 75/8.

Problem variable	*Profit Rs.*	C_j *Capacity*	*– 15/2* *a*	*3* *b*	*0* *c*	*0* S_1	*0* S_2	*– 1* A_1	*– 1* A_2	*Replace–ment ratio*
a	– 15/2	5/4	1	– 1/2	0	– 1/4	– 1/4			
c	0	3/4	0	– 1/2	1	1/4	– 3/4			
		Net evaluation	0	– 3/4	0	– 15/8	– 15/8			

As all the net evaluation row elements are negative or zeros, the solution is optimal.
$a = 5/4$, $c = 3/5$, and minimum Z = Rs. 75/8.

The Disadvantages of Big M method over Two–phase method:

1. **Big M method can be used to find the existence of feasible solution. But it is difficult and many a time one gets confused during computation because of manipulation of constant M. In two–phase method big M is eliminated and calculations will become easy.**
2. **The existence of big M avoids the use of digital computer for calculations.**

DEGENERACY IN LINEAR PROGRAMMING PROBLEMS

The degeneracy in linear programming problems and the methods of solving degeneracy, if it exists, are discussed earlier in the chapter. To recollect the same a brief discussion is given below:

While improving the basic feasible solution to achieve optimal solution, we have to find the key column and key row. While doing so, we may come across two situations. One is **Tie** and the other is **Degeneracy**.

The tie occurs when two or more net evaluation row elements of variables are equal. In maximization problem, we select the highest positive element to indicate incoming variable and in minimization we select lowest element to indicate incoming variable (or highest numerical value with negative sign). When two or more net evaluation row elements are same, to break the tie, we select any one of them to indicate incoming variable and in the next iteration the problem of tie will be solved.

To select the out going variable, we have to select the lowest ratio or limiting ratio in the replacement ratio column. Here also, some times during the phases of solution, the ratios may be equal. This situation in linear programming problem is known as degeneracy. To solve degeneracy, the following methods are used:

1. Select any one row as you please. If you are lucky, you may get optimal solution, otherwise the problem cycles.

OR

2. Identify the rows, which are having same ratios. Say for example, S_1 and S_3 rows having equal ratio. In such case select the row, which contains the variable with smaller subscript. That is select row containing S_1 as the key row. Suppose the rows of variable x and z are having same ratio, then select the row-containing x as the key row.
3. (*a*) Divide the elements of unit matrix by corresponding elements of key column. Verify the ratios column-wise in unit matrix starting from left to right. Once the ratios are unequal, the degeneracy is solved. Select the minimum ratio and the row containing that element is the key row. (This should be done to the rows that are in tie).

(*b*) If the degeneracy is not solved by 3 (*a*), then divide the elements of the main matrix by the corresponding element in the key column, and verify the ratios. Once the ratios are unequal, select the lowest ratios. (This should be done only to rows that are in tie).

Problem 3.21: A company manufactures two product *A* and *B*. These are machined on machines *X* and *Y*. *A* takes one hour on machine *X* and one hour on Machine *Y*. Similarly product *B* takes 4 hours on Machine *X* and 2 hours on Machine *Y*. Machine *X* and *Y* have 8 hours and 4 hours as idle capacity.

The planning manager wants to avail the idle time to manufacture *A* and *B*. The profit contribution of *A* is Rs. 3/– per unit and that of *B* is Rs.9/– per unit. Find the optimal product mix.

Solution:

Maximize $Z = 3a + 9b$ s.t.

$1a + 4b \leq 8$

$1a + 2b \leq 4$ both a and b are ≥ 0

Simplex format is:

Maximize $Z = 3a + 9b + 0S_1 + 0S_2$ s.t.

$1a + 4b + 1S_1 + 0S_2 = 8$

$1a + 2b + 0S_1 + 1S_2 = 4$ and

a, b, S_1, S_2 all ≥ 0.

Table: I. $A = 0$, $b = 0$, $S_1 = 8$, $S_2 = 4$ and Z = Rs.0/–

Problem variable	*Profit Rs.*	*C_j →* *Capacity*	*3* *a*	*9* *b*	*0* *S_1*	*0* *S_2*	*Replacement ratio*
S_1	0	8	1	4	1	0	8/4 = 2
S_2	0	4	1	**2**	0	1	4/2 = 2
		Net evaluation	3	9	0	0	

Now to select the out going variable, we have to take limiting ratio in the replacement ratio column. But both the ratios are same *i.e.* = 2. Hence there exists a tie as an indication of degeneracy in the problem. To solve degeneracy follow the steps mentioned below:

(*i*) Divide the elements of identity column by column from left to right by the corresponding key column element.

Once the ratios are unequal select the lowest ratio and the row containing that ratio is the key row.

In this problem, for the first column of the identity (*i.e*. the S_1 column) the ratios are: 1/4, and 0/2. The lowest ratio comes in row of S_2. Hence S_2 is the outgoing variable. In case ratios are equal go to the second column and try.

Table: II. $a = 0$, $b = 2$, $S_1 = 0$, $S_2 = 0$, Z = Rs. 18/–

Problem variable	*Profit Rs.*	*C_j →* *Capacity*	*3* *a*	*9* *b*	*0* *S_1*	*0* *S_2*	*Replacement ratio*
S_1	0	0	1	– 2	– 1	0	
b	9	2	0	1/2	1/2	1	
		Net evaluation	0	– 9/2	– 3/2	0	

Optimal solution is $b = 2$ and Profit is 2×9 = Rs. 18/–

Problem 3.22:

Maximize $Z = 2x + 1y$ s.t

$4x + 3y \leq 12$

$4x + 1y \leq 8$

$4x - 1y \leq 8$

Both x and y are ≥ 0

Simplex version is:

Maximize $Z = 2x + 1y + 0S_1 + 0S_2 + 0S_3$ s.t.

$4x + 3y + 1S_1 + 0S_2 + 0S_3 = 12$

$4x + 1y + 0S_1 + 1S_2 + 0S_3 = 8$

$4x - 1y + 0S_1 + 0S_2 + 1S_3 = 8$

x, y, S_1, S_2, S_3 all ≥ 0

Table: I. $x = 0$, $y = 0$, $S_1 = 12$, $S_2 = 8$, $S_3 = 8$ and Z = Rs. 0/–

Problem variable	*Profit Rs.*	*$C_j \longrightarrow$ Capacity units*	*2 x*	*1 y*	*0 S_1*	*0 S_2*	*0 S_3*	*Replacement ratio*
S_1	0	12	4	3	1	0	0	12/4 = 3
S_2	0	8	4	1	0	1	0	8/4 = 2
S_3	0	8	**4**	– 1	0	0	1	8/4 = 2
		Net evaluation	2	1	0	0	0	

As the lowest ratio (= 2) is not unique, degeneracy occurs. Hence divide the elements of the identity column by column from left to right and verify the ratios.

In this problem, the elements of the first column of identity under S_1 (for 2nd and 3rd row) are 0,0. If we divide them by respective key column element, the ratios are 0/4 and 0/4 which are equal, hence we cannot break the tie.

Now try with the second column *i.e.*, column under S_2. The elements of 2nd and 3rd row are 1 and 0. If we divide them by respective elements of key column, we get 1/4 and 0/4. The ratios are unequal and the lowest being zero for the third row. Hence S_3 is the outgoing variable.

Table: II. $x = 2$, $y = 0$, $S_1 = 4$, $S_2 = 0$, $S_3 = 0$, and Z = Rs. 4/–

Problem variable	*Profit Rs.*	*$C_j \longrightarrow$ Capacity units*	*2 x*	*1 y*	*0 S_1*	*0 S_2*	*0 S_3*	*Replacement ratio*
S_1	0	4	0	4	1	0	– 1	1/4
S_2	0	0	0	**2**	0	1	– 1	0/2
X	2	2	1	– 1/4	0	0	1/4	—
		Net evaluation	0	3/2	0	0	– 1/2	

Table: III. $x = 2$, $y = 0$, $S_1 = 4$, $S_2 = 0$, $S_3 = 0$, $Z =$ Rs. 4/–

Problem variable	*Profit Rs.*	$C_j \longrightarrow$ *Capacity units*	*2* *x*	*1* *y*	*0* S_1	*0* S_2	*0* S_3	*Replacement ratio*
S_1	0	4	0	0	**1**	– 2	1	4/1
y	1	0	0	1	0	1/2	– 1/2	—
x	2	2	1	0	0	1/8	1/8	2 × 8/1 = 16
		Net evaluation	0	0	0	– 3/4	1/4	

Table: IV. $x = 3/2$, $y = 2$, $S_1 = 0$, $S_2 = 0$, $S_3 = 4$ and $Z =$ Rs. 3 + 2 = Rs. 5/–

Problem variable	*Profit Rs.*	$C_j \longrightarrow$ *Capacity units*	*2* *x*	*1* *y*	*0* S_1	*0* S_2	*0* S_3	*Replacement ratio*
S_3	0	4	0	0	1	– 2	1	
y	1	2	0	1	1/2	– 1/2	0	
x	2	3/2	1	0	– 1/8	3/8	0	
		Net evaluation	0	0	– 1/4	– 1/4	0	

As the elements of net evaluation row are either zeros or negative elements, the solution is optimal.

$x = 3/2$ and $y = 2$, and $Z =$ Rs. $2 \times 3/2 + 2 \times 1 =$ Rs. 5/–

Shadow price = ¼ × 12 + ¼ × 8 = Rs. 5/–.

Special Cases

Some times we come across difficulties while solving a linear programming problem, such as alternate solutions and unbound solutions. Let us solve some problems of special nature.

Problem. 3.23:

Solve the given l.p.p. by big – M method.

Maximize $Z = 6a + 4b$ s.t.

$2a + 3b \le 30$

$3a + 2b \le 24$

$1a + 1b \ge 3$ and x, y both ≥ 0.

Simplex version is:

Maximize $Z = 6a + 4b + 0S_1 + 0S_2 + 0S_3 - MA$

$2a + 3b + 1S_1 + 0S_2 + 0S_3 + 0A = 30$

$3a + 2b + 0S_1 + 1S_2 + 0S_3 + 0A = 24$

$1a + 1b + 0S_1 + 0S_2 - 1S_3 + 1A = 3$

a, b, S_1, S_2, S_3, A all ≥ 0.

Table: I. $a = 0$, $b = 0$, $S_1 = 30$, $S_2 = 24$, $S_3 = 0$, $A = 3$ and $Z =$ Rs. $3M$

Problem variable	*Profit Rs.*	$C_j \rightarrow$ *Capacity units*	*6* *a*	*4* *b*	*0* S_1	*0* S_2	*0* S_3	$-M$ *A*	*Replacement ratio.*
S_1	0	30	2	3	1	0	0	0	30/2 = 15
S_2	0	24	3	2	0	1	0	0	24/3 = 8
A	$-M$	3	**1**	1	0	0	– 1	1	3/1 = 3
		Net evaluation	$6 + M$	$4 + M$	0	0	$-M$	0	

Table II. $a = 3$, $b = 0$, $S_1 = 24$, $S_2 = 15$, $S_3 = 0$, $A = 0$, $Z =$ Rs. 18/– (out going column eliminated)

Problem variable	*Profit Rs.*	$C_j \rightarrow$ *Capacity units*	*6* *a*	*4* *b*	*0* S_1	*0* S_2	*0* S_3	$-M$ *A*	*Replacement ratio*
S_1	0	24	0	1	1	0	2		24/2 = 12
S_2	0	15	0	– 1	0	1	**3**		15/3 = 5
A	6	3	1	1	0	0	– 1		—
		Net evaluation	0	– 2	0	0	6		

Table: III. $a = 8$, $b = 0$, $S_1 = 14$, $S_2 = 5$, $S_3 = 0$, $A = 0$, $Z =$ Rs. 48

Problem variable	*Profit Rs.*	$C_j \rightarrow$ *Capacity units*	*6* *a*	*4* *b*	*0* S_1	*0* S_2	*0* S_3	$-M$ *A*	*Replacement ratio*
S_3	0	14	0	**5/3**	1	– 2/3	0		42/5
S_2	0	5	0	– 1/3	0	1/3	0		—
a	6	8	1	2/3	0	1/3	0		24/2 = 12
		Net evaluation	0	0	0	– 3	0		

In the above table, as all the net evaluation elements are either zeros or negative elements the optimal solution is obtained. Hence the answer is: $a = 8$ and the profit $Z =$ Rs. 48/–. But the net evaluation element in the column under '*b*' is zero. This indicates that the problem has alternate solution. If we modify the solution we can get the values of basis variables, but the profit Z will be the same. This type of situation is very helpful to the production manager and marketing manager to arrange the production schedules and satisfy the market demands of different segments of the market. As the profits of all alternate solutions are equal, the manager can select the solution, which is more needed by him. Now let us workout the alternate solution.

Table: IV. $a = 12/5$, $b = 42/5$, $S_1 = 0$, $S_2 = 0$, $S_3 = 39/5$, $A = 0$, $Z = $ Rs. 48/–

Problem variable	*Profit Rs.*	C_j *Capacity units*	*6* *a*	*4* *b*	*0* S_1	*0* S_2	*0* S_3	*– M* *A*	*Replacement ratio*
b	4	42/5	0	1	3/5	– 2/5	0		
S_3	0	39/5	0	0	1/5	1/5	1		
a	6	12/5	1	0	– 2/5	3/5	0		
		Net evaluation	0	0	0	– 2	0		

All the elements of net evaluation rows are either zeros or negative elements so the solution is Optimal. The answer is : $a = 12/5$, $b = 42/5$ and $Z = 12/5 \times 6 + 42/5 \times 4 =$ Rs. 48/– . Shadow price = element in column under S_2 multiplied by element on the R.H.S of second inequality *i.e.*, $24 \times 2 =$ Rs. 48/–. Once we get one alternate solution then any number of solutions can be written by using the following rule.

(a) One alternate value of the basis variable is: First value $\times d$ + second value $\times (1–d)$. In the given example, the first value of '*a*' is 8 and second value of '*a*' is 12/5. Hence next value is $8d + (1 – d) \times 12/5$. Like this we can get any number of values. Here '*d*' is any positive fraction number: for example 2/5, 3/5 etc. It is better to take $d = 1/2$, so that next value is 1/2 × first value + 1/2 × second value. Similarly the values of other variables can be obtained.

Unbound Solutions

In linear programming problem, we come across certain problem, where feasible region is unbounded *i.e.*, the value of objective function can be increased indefinitely. Then we say that the solution is **UNBOUND**. But it is not necessary, however, that an unbounded feasible region should yield an unbounded value of the objective function. Let us see some examples.

Problem 3.24: Maximize $Z = 107a + 1b + 2c$ s.t

$14a + 1b – 6c + 3d = 7$

$16a + ½ b + 6c \leq 5$

$3a – 1b – 6c \leq 0$ and a, b, c and d all ≥ 0.

We find that there is variable '*d'* in the first constraint, with coefficient as 3. Second and third constraints do not have variable '*d'*. Hence we can divide the first constraint by 3 we can write as: $14/3\, a + 1/3\, b – 6/3\, c + 3/3\, d = 7/3$. If we write like this, we can consider the variable '*d*' as slack variable. Hence the given l.p.p. becomes as:

Maximize $Z = 107a + 1b + 2c$ s.t

$14/3a + 1/3b – 2c + 1d = 7/3$

$16a + ½ b – 6c \leq 5$

$3a – 1b – 1c \leq 0$

and a, b, c, and d all ≥ 0

Simplex version is

Maximize $Z = 107a + 1b + 2c + 0d + 0S_1 + 0S_2$ s.t.

$14/3\ a + 1/3b - 2c + 1d + 0S_1 + 0S_2 = 7/3$

$16a + ½\ b - 6c + 0d + 1S_1 + 0S_2 = 5$

$3a - 1b - 1c + 0d + 0S_1 + 1S_2 = 0$

and a, b, c, d, S_1, S_2 all ≥ 0.

TableL: I. $a = 0$, $b = 0$, $c = 0$, $d = 7/3$, $S_1 = 5$, $S_2 = 0$ and Z = Rs. 0.

Problem variable	*Cost Rs.*	C_j *Capacity units*	*107 a*	*1 b*	*2 c*	*0 d*	*0* S_1	*0* S_2	*Replacement ratio*
d	0	7/3	14/3	1/3	– 2	1	0	0	7/14
S_1	0	5	16	1/2	6	0	1	0	5/16
S_2	0	0	3	– 1	– 1	0	0	1	0/3
		Net evaluation	107	1	2	0	0	0	

Table: II. $a = 0$, $b = 0$, $c = 0$, $d = 7/3$, $S_1 = 5$, $S_2 = 0$ and Z = Rs.0/–

Problem variable	*Cost Rs.*	C_j *Capacity units*	*107 a*	*1 b*	*2 c*	*0 d*	*0* S_1	*0* S_2	*Replacement ratio*
d	0	7/3	0	17/9	– 4/9	1	0	– 14/9	—
S_1	0	5	0	35/6	– 2/3	0	1	– 16/3	—
a	107	0	1	– 1/3	– 1/3	0	0	1/3	—
		Net evaluation	0	110/3	113/3	0	0	107/3	

As the elements of incoming variable column are negative we cannot workout replacement ratios and hence the problem cannot be solved. **This is an indication of existence of unbound solution to the given problem.**

Problem 3.25:

Maximize $Z = 6x - 2y$ s.t.

$2x - 1y \leq 2$

$1x + 0y \leq 4$ and both x and y are ≥ 0.

Simplex version is:

Maximize $Z = 6x - 2y + 0S_1 + 0S_2$ s.t.

$2x - 1y + 1S_1 + 0S_2 = 2$

$1x + 0y + 0S_1 + 1S_2 = 4$

And x, y, S_1 and S_2 all ≥ 0.

Table: I. $x = 0$, $y = 0$, $S_1 = 2$, $S_2 = 0$ and $Z =$ Rs. 0

Problem variable	*Profit Rs.*	*$C_j \longrightarrow$ Capacity units*	*6 x*	*– 2 y*	*0 S_1*	*0 S_2*	*Replacement ratio*
S_1	0	2	**2**	1	1	0	2/2 = 1
S_2	0	4	1	0	0	1	4/1 = 4
		Net evaluation	6	– 2	0	0	

Table: II. $x = 1$, $y = 0$, $S_1 = 0$, $S_2 = 3$, $Z =$ Rs. 6/–

Problem variable	*Profit Rs.*	*$C_j \longrightarrow$ Capacity units*	*6 x*	*– 2 y*	*0 S_1*	*0 S_2*	*Replacement ratio*
x	6	1	1	– 1/2	1/2	0	—
S_2	0	3	0	**1/2**	– 1/2	1	7/2
		Net evaluation	0	1	3	0	

Table: III. $x = 4$, $y = 6$, $S_1 = 0$, $S_2 = 0$, $Z = 24 – 12 =$ Rs. 12/–

Problem variable	*Profit Rs.*	*$C_j \longrightarrow$ Capacity units*	*6 x*	*– 2 y*	*0 S_1*	*0 S_2*	*Replacement ratio*
x	6	4	1	0	0	1	
y	– 2	6	0	1	– 1	2	
		Net evaluation	0	0	– 2	– 2	

$X = 4$, $Y = 6$ and $Z =$ Rs. 12/–

(Note: Students can verify the answer by graphical method).

(Note that in the first table the elements in column under '*y*' are –1 and 0. This indicates that the feasible region is unbound. But the problem has solution. This clearly states that the problem may have unbound region but still it will have solution).

Problem 3.26:

Maximize $Z = 3a + 2b$ S.t.

$2a + 1b \leq 2$

$3a + 4b \geq 12$

Both a and b are > 0

Simplex version is:

Maximize $Z = 3a + 2b + 0S_1 + 0S_2 – MA$ s.t

$3a + 1b + 1S_1 + 0S_2 + 0A = 2$

$3a + 4b + 0S_1 – 1S_2 + 1A = 12$

a, b, S_1, S_2 and A all ≥ 0.

Table: I. $a = 0$, $b = 0$, $S_1 = 2$, $S_2 = 0$, $A = 12$ and $Z =$ Rs. $- 12\ M$

Problem variable	*Profit Rs.*	C_j *Capacity units*	*3* *a*	*2* *b*	*0* S_1	*0* S_2	$-M$ *A*	*Replacement ratio*
S_1	0	2	3	**1**	1	0	0	2/1 = 2
A	$-M$	12	3	4	0	– 1	1	12/4 = 3
		Net evaluation	3 + 3M	2 + 4M	0	– M	0	

Table: II. $b = 2$, $a = 0$, $S_1 = 0$, $S_2 = 0$, $A = 4$, $Z =$ Rs. $4 - 4M$

Problem variable	*Profit Rs.*	C_j *Capacity units*	*3* *a*	*2* *b*	*0* S_1	*0* S_2	$-M$ *A*	*Replacement ratio*
b	2	2	2	1	1	0	0	
A	$-M$	4	– 5	0	– 4	– 1	0	
		Net evaluation	– 1 –5M	0	– 2 – 4M	– M	0	

In the above solution, though the elements of net evaluation row are either negative or zeros, the presence of artificial variable '*A*' as problem variable with value 4 shows that the problem has no feasible solution because the positive value of A violates the second constraint of the problem.

(Students can verify the solution by graphical method).

Problem with Unrestricted Variables

The non–negativity constraint of a linear programming problem restricts that the values of all variables in the problem say for example *x*, *y* and *z* or *a*, *b*, *c* and d etc must be = 0. Some times we may come across a situation that the values of the variables are unrestricted, *i.e*., may assume any value (0 or > 1 or < 1) *i.e.*, to say that the ≥ sign is not required. In such cases to maintain non-negativity restriction for all variables, each variable is replaced by two non-negative variables, say for example: *x*, *y* and *z* are replaced by x' and x'', y' and y'', z' and z'' respectively. If $x' > x''$, **then *x* is positive, while $x' < x''$ then *x* is negative.**

Problem 3.27: Maximize $Z = 2x + 3y$ s.t.

$-1x + 2y \leq 4$

$1x + 1y \leq 6$

$1x + 3y \leq 9$ and x and y are unrestricted.

As it is given that both *x* and *y* are unrestricted in sign they are replaced by non–negative variables, x' and x'', y' and y'' respectively. This is subjected to $x = x' - x''$ and $y = y' - y''$. By introducing slack variables and non–negative variables the simplex format is:

Maximize $Z = 2(x' - x'') + 3(y' - y'') + 0S_1 + 0S_2 + 0S_3$ s.t.

$-1(x'-x'') + 2(y'-y'') + 1S_1 + 0S_2 + 0S_3 = 4$
$1(x'-x'') + 1(y'-y'') + 0S_1 + 1S_2 + 0S = 6$
$1(x'-x'') + 3(y'-y'') + 0S_1 + 0S_2 + 1S_3 = 9$
x', x'', y', y'' and S_1, S_2 and S_3 all ≥ 0.

Table: I. $x' = x = y' = y'' = 0$, $S_1 = 4$, $S_2 = 6$, S_3 9 and Z = Rs. 0/–

Problem variable	*Profit Rs.*	*C_j Capacity units*	*2 x'*	*– 2 x''*	*3 y'*	*– 3 y''*	*0 S_1*	*0 S_2*	*0 S_3*	*Replace–ment ratio*
S_1	0	4	– 1	1	**2**	– 2	1	0	0	2
S_2	0	6	1	– 1	1	– 1	0	1	0	6
S_3	0	9	1	– 1	3	– 3	0	0	1	3
		Net evaluation	2	– 2	3	– 3	0	0	0	

Table: II. $x' = 0$, $x'' = 0$, $y' = 3$, $y'' = 0$, $S_1 = 0$, $S_2 = 4$, $S_3 = 3$, Z = Rs. 6/–

Problem variable	*Profit Rs.*	*C_j Capacity units*	*2 x'*	*– 2 x''*	*3 y'*	*– 3 y''*	*0 S_1*	*0 S_2*	*0 S_3*	*Replace–ment ratio*
y'	3	2	– 1/2	½	1	– 1	1/2	0	0	– 4
S_2	0	4	3/2	– 3/2	0	0	– 1/2	1	0	8/3
S_3	0	3	**5/2**	– 5/2	0	0	– 3/2	0	1	6/5
		Net evaluation	7/2	– 7/2	0	0	– 3/2	0	0	

Table:III. $x' = 6/5$, $x'' = 0$, $y' = 13/3$, $y'' = 0$, $S_1 = 0$, $S_2 = 11/5$, $S_3 = 0$, Z = Rs. 10.20

Problem variable	*Profit Rs.*	*C_j Capacity units*	*2 x'*	*– 2 x''*	*3 y'*	*– 3 y''*	*0 S_1*	*0 S_2*	*0 S_3*	*Replace–ment ratio*
y'	3	13/5	0	0	1	– 1	1/5	0	1/5	13
S_2	0	11/5	0	0	0	0	**2/5**	1	– 3/5	11
x'	2	6/5	1	– 1	0	0	–3/5	0	2/5	—
		Net evaluation	0	0	0	0	3/5	0	– 7/5	

Table: IV. $x' = 9/2$, $x'' = 0$, $y' = 3/2$, $y'' = 0$, $S_1 = 0$, $S_2 = 11/2$, $S_2 = 0$, $S_2 = 0$, $Z =$ Rs. 13.50

Problem variable	*Profit Rs.*	C_j *Capacity units*	*2* x'	*– 2* x''	*3* y'	*– 3* y''	*0* S_1	*0* S_2	*0* S_3	*Replace–ment ratio*
y'	3	3/2	0	0	1	– 1	0	– 1/2	1/2	
S_1	0	11/2	0	0	0	0	1	5/2	– 3/2	
x'	2	9/2	1	– 1	0	0	0	3/2	– 1/2	
		Net evaluation	0	0	0	0	0	– 3/2	– 1/2	

Shadow price = 3/2 × 6 + ½ × 9 = 9 + 4.5 = Rs. 13.5
$x = x' – x'' = 9/2 – 0 = 9/2$
$y = y' – y'' = 3/2 – 0 = 3/2$ and $Z = 2 \times 9/2 + 3 \times 3/2 =$ Rs. 13.50.

Problem 3.28: Minimize $Z = 1a + 1b + 1c$ s.t
$1a – 3b + 4c = 5$
$1a – 2b \leq 3$
$2a – 1c \geq 4$ and
a and b are ≥ 0 and c is unrestricted.

(Note: Whenever the range of a variable is not given in the problem, it should be understood that such variable is unrestricted in sign).

Solution: Replace variable 'c' by $c' – c''$ and introducing slack variable for inequality 2 and surplus variable and artificial variable for inequalities 1 and 3 respectively, we can write simplex version of the problem.

Simplex version:

Maximize $Z = – 1a – 1b – 1c' + 1c'' + 0A_1 + 0S_2 – MA_1 – MA_2$ s.t.
$1a – 3b + 4c' – 4c'' + 1A_1 + 0S_1 + 0S_2 + 0A_2 = 5$
$1a – 2b + 0c' – 0c'' + + 0A_1 + 1S_1 + 0S_2 + 0A_2 = 3$
$0a + 2b – 1c' + 1c'' + 0A_1 + 0S_1 – 1S_2 + 1A_2 = 4$
And $a, b, c', c'', S_1, S_2, S_3$ and A_1, A_2 all ≥ 0.

Table: I. $A = 0$, $b = 0$, $c' = 0$, $c'' =$ o, $S_1 = 3$, $S_2 = 0$, $A_1 = 5$, $A_2 = 4$ and $Z =$ Rs. – 9M

Problem variable	*Profit Rs.*	C_j *Capacity Units*	*– 1* a	*– 1* b	*– 1* c'	*1* c''	*0* S_1	*0* S_2	*– M* A_1	*– M* A_2	*Replace–ment ratio*
A_1	$– M$	5	1	– 3	4	– 4	0	0	1	0	5/4
S_1	0	3	1	– 2	0	0	1	0	0	0	3/0
A_2	$– M$	4	0	2	– 1	1	0	– 1	0	1	negative
		Net evaluation	$– 1 + M$	$– 1 – M$	$– 1 + 3M$	$1 – 3M$	0	$– M$	0	0	

Table: II. $A = 0$, $b = 0$, $c' = 5/4$, $c'' = 0$, $S_1 = 3$, $S_2 = 0$, $A_1 = 0$, $A_2 = 21/4$, Z = Rs. $-5/4 - 21/4$ M

Problem variable	*Profit Rs.*	C_j *Capacity units*	-1 a	-1 b	-1 c'	1 c''	0 S_1	0 S_2	$-M$ A_1	$-M$ A_2	*Replace–ment ratio*
c'	– 1	5/4	1/4	– 3/4	1	– 1	0	0		0	Negative
S_1	0	3	1	– 2	0	0	1	0		0	Negative
A_2	– M	21/4	1/4	**5/4**	0	0	0	– 1		1	21/5
		Net evaluation	– 3/4 + M/4	– 7/4 + 5/4 M	0	0	0	– M		0	

Table: III. $a = 0$, $b = 21/5$, $c' = 22/5$, $c'' = 0$, $S_1 = 57/5$, $S_2 = 0$, $A_1 = 0$, $A_2 = 0$, Z = Rs. 43/5

Problem variable	*Profit Rs.*	C_j *Capacity units*	-1 a	-1 b	-1 c'	1 c''	0 S_1	0 S_2	$-M$ A_1	$-M$ A_2	*Replace–ment ratio*
c'	– 1	22/5	2/5	0	1	– 1	0	– 3/5			
S_1	0	57/5	7/5	0	0	0	1	– 8/5			
b	– 1	21/5	1/5	1	0	0	0	– 4/5			
		Net evaluation	– 2/5	0	0	0	0	– 7/5			

As the elements of net evaluation row are either negative or zeros, the solution is optimal. The answer is $a = 0$, $b = 21/5$, $c = 22/5 - 0 = 22/5$, and $Z = 21/5 + 22/5$ = Rs. 43/5.

(In this example, the columns of artificial variable is eliminated whenever it goes out of the programme)

Example 3.29: Solve the given l.p.p

Maximize $Z = 0a + 8b$ s.t.

$a - b \geq 0$

$2a + 3b \leq -6$ and both a and b are unrestricted.

As a and b are unrestricted, they may be + ve or –ve or zero. As non–negativity constraint is a condition for simplex method, this can be solved by writing $a = a' - a''$ and $b = b' - b''$ so that a', a'', b' and b'' all ≥ 0.

Now the revised problem is: Simplex version is:

Maximize $Z = 0a' + 0a'' + 8b' - 8b''$ s.t. Maximize $Z = 0a' + 0a'' + 8b' - 8b'' + 0S_1 + 0S_2 - \text{M}A_1 - MA_2$ s.t.

$(1a' - 1a'') - 1\,(b' - b'') \geq 0$ $\qquad a' - a'' - b' + b'' - 1S_1 + 0S_2 + 1A_1 + 0A_2 = 0$

$-2\,(a' - a'') - 3\,(b' - b'') > 6$ $\qquad -2a' + 2a'' - 3b' + 3b'' + 0S_1 - 1S_2 + 0A_1 + 1A_2 = 6$

And a', a'', b', b'' all ≥ 0 $\qquad a'$, a'', b', b'', S_1, S_2, A_1 and A_2 all ≥ 0.

Table: I. $a' = 0$, $a'' = 0$, $b' = 0$, $b'' = 0$, $S_1 = 0$, $S_2 = 0$, $A_1 = 0$ and $A_2 = 6$ and Z = Rs.–6M

Problem variable	*Profit Rs.*	C_j *Capacity units*	0 a'	0 a''	8 b'	– 8 b''	0 S_1	0 S_2	– M A_1	– M A_2	*Replace–ment ratio*
A_1	– M	0	1	– 1	– 1	**1**	1	0	0	1	0
A_2	– M	6	– 2	2	– 3	3	0	– 1	0	1	6/3 = 2
		Net evaluation	– M	M	8 – 4M	4M – 8	– M	– M	0	1	

Table: II. $a' = 0$, $a'' = 0$, $b' = 0$, $b'' = 0$, $S_1 = 0$, $S_2 = 0$, $A_1 = 0$, $A_2 = 6$ and Z = Rs. –6M

Problem variable	*Profit Rs.*	C_j *Capacity units*	0 a'	0 a''	8 b'	– 8 b''	0 S_1	0 S_2	– M A_1	– M A_2	*Replace–ment ratio*
b''	–8	0	1	–1	–1	1	–1	0		0	Negative
A_2	–M	6	–5	**5**	0	0	3	–1		1	6/5
		Net evaluation	8 – 5M	–8 + 5M	0	0	–8 + 3M	– M		0	

Table: III. $a'' = 6/5$, $b'' = 6/5$, $a' = 0$, $b' = 0$, $S_1 = 0$, $S_2 = 0$, $A_1 = 0$, $A_2 = 0$, Z = Rs. –8/5

Problem variable	*Profit Rs.*	C_j *Capacity units*	0 a'	0 a''	8 b'	– 8 b''	0 S_1	0 S_2	– M A_1	– M A_2	*Replace–ment ratio*
b''	–8	6/5	0	0	–1	1	–2/5	1/5			
a''	0	6/5	–1	1	0	0	3/5	–1/5			
		Net Evaluation.	0	0	0	0	–16/5	–8/5			

Answer: $a'' = 6/5$, $b'' = 6/5$ and Z = Rs.– 48/5. The shadow price = 8/5 × 6 = Rs. 48/5. = Rs. 9.60

Problem 3.30: Solve the given l.p.p.

Maximize $Z = 4x + 5y - 3z$ s.t.

$$1x + 1y + 1z = 10 \quad \text{...(1)}$$

$$1x - 1y \geq 1 \quad \text{...(2)}$$

$$2x + 3y + 1z \leq 40 \qquad ...(3)$$

And x, y, z all ≥ 0.

Solution: (Note: In case there is equality in the constraints, then one variable can be eliminated from all inequalities with $\leq$ or $\geq$ sign by subtracting the equality from the inequality. Then the variable, which is eliminated from equality, is treated as slack variable for further calculations).

Now, subtracting (1) from (3) to eliminate z from (3) and retaining z to work as slack variable for the equality (1), the given problem becomes:

Given Problem:	Simplex version:
Maximize $Z = 4x + 5y - 3z$ s.t.	Maximize: $Z = 4x + 5y - 3z + 0S_1 - MA + 0S_2$ s.t.
$1x + 1y + 1z = 10$	$1x + 1y + 1z + 0S_1 + 0A + 0S_2 = 10$
$1x - 1y + 0z \geq 1$	$1x - 1y + 0z - 1S_1 + 1A + 0S_2 = 1$
$1x + 2y + 0z \leq 30$	$1x + 2y + 0z + 0S_1 + 0A + 1S_2 = 30$
$x, y,$ and z all ≥ 0.	$x, y, z, S_1, S_2\, A$ all ≥ 0.

Table: I. $X = 0$, $y = 0$, $z = 10$, $S_1 = 0$, S_2 30, $A = 1$ and $Z =$ Rs. $-30 - M$

Problem variable	*Profit Rs.*	C_j *Capacity units*	*4* x	*5* y	*– 3* z	*0* S_1	*– M* A	*0* S_2	*Replacement ratio*
z	–3	10	1	1	1	0	0	0	10/1 = 10
A	–M	1	**1**	–1	0	–1	1	0	1/1 = 1
S_2	0	30	1	2	0	0	0	1	30/1 = 30
		Net evaluation	7 + M	8 – M	0	– M	0	0	

Table: II. $x = 1$, $y = 0$, $z = 9$, $S_1 = 0$, $S_2 = 29$, $A = 0$, $Z =$ Rs. – 23/–.

Problem variable	*Cost Rs.*	C_j *Capacity units*	*4* x	*5* y	*– 3* z	*0* S_1	*– M* A	*0* S_2	*Replacement ratio*
z	–3	9	0	**2**	1	1		0	9/2 = 4.5
x	4	1	1	–1	0	–1		0	Negative
S_2	0	29	0	3	0	1		1	29/3 = 9.33
		Net evaluation	0	9	0	4		0	

Table: III. $x = 11/2$, $y = 9/2$, $z = 0$, $S_1 = 0$, $S_2 = 31/2$, $A = 0$, Z = Rs. 45/2 + 44/2 = 89/2 = Rs. 44.50.

Problem variable	*Cost Rs.*	C_j *Capacity units*	*4* x	*5* y	*– 3* z	*0* S_1	*– M* A	*0* S_2	*Replacement ratio*
y	5	9/2	0	1	1/2	1/2		0	
x	4	11/2	1	0	1/2	– 1/2		0	
S_2	0	31/2	0	0	– 3/2	– 1/2		1	
		Net evaluation	0	0	– 15/2	– 1/2		0	

As all the net evaluation row elements are either negative or zeros, the solution is optimal.

$x = 11/2$, $y = 9/2$ and Z = Rs. 44.50.

Problem 3.31: Maximize $Z = 4x + 5y – 3z + 50$ s.t

$1x + 1y + 1z = 10$,

$1x – 1y \geq 1$

$2x + 3y + 1z \leq 40$

x, y, z all ≥ 0.

Note: If any constant is included in the objective function, it should be deleted in the beginning and finally adjusted in optimum value of *Z* and if there is equality in the constraints, then one variable can be eliminated from the inequalities with ≥ or ≤ sign. In this example constant 50 presents in the objective function. Also one equality is present in the constraints. As done earlier, variable 'z' is eliminated from third constraint and is considered as slack variable in first equality.

The problem 3.30 and 3.31 are one and the same except a constant present in the objective function of problem 3.31.

The solution for the problem 3.30 is $x = 11/2$, $y = 9/2$ and $Z = 89/2$.

While solving the problem 3.31, neglect 50 from the objective function, and after getting the final solution, add 50 to that solution to get the answer. That is the solution for the problem 3.31 is Z = Rs. 89/2 + 50 = Rs. 189/2.

DUALITY IN LINEAR PROGRAMMING PROBLEMS

Most important finding in the development of Linear Programming Problems is the existence of **duality** in linear programming problems. Linear programming problems exist in pairs. That is in linear programming problem, every maximization problem is associated with a minimization problem. Conversely, associated with every minimization problem is a maximization problem. Once we have a problem with its objective function as maximization, we can write by using duality relationship of linear programming problems, its minimization version. The original linear programming problem is known as ***primal problem***, and the derived problem is known as dual problem.

The concept of the **dual problem** is important for several reasons. Most important are (*i*) the variables of dual problem can convey important information to managers in terms of formulating their future plans and (*ii*) in some cases the dual problem can be instrumental in arriving at the optimal solution to the original problem in many fewer iterations, which reduces the labour of computation.

Whenever, we solve the primal problem, may be maximization or minimization, we get the solution for the dual automatically. That is, the solution of the dual can be read from the final table of the primal and vice versa. Let us try to understand the concept of dual problem by means of an example. Let us consider the diet problem, which we have discussed while discussing the minimization case of the linear programming problem.

Example: The doctor advises a patient visited him that the patient is weak in his health due to shortage of two vitamins, *i.e*., vitamin *X* and vitamin *Y*. He advises him to take at least 40 units of vitamin *X* and 50 units of Vitamin *Y* everyday. He also advises that these vitamins are available in two tonics *A* and *B*. Each unit of tonic *A* consists of 2 units of vitamin *X* and 3 units of vitamin *Y*. Each unit of tonic *B* consists of 4 units of vitamin *X* and 2 units of vitamin *Y*. Tonic *A* and *B* are available in the medical shop at a cost of Rs. 3 per unit of *A* and Rs. 2,50 per unit of *B*. The patient has to fulfill the need of vitamin by consuming *A* and *B* at a minimum cost.

The problem of **patient is the primal problem**. His problem is to minimize the cost. The tonics are available in the medical shop. The medical shop man wants to maximize the sales of vitamins A and B; hence he wants to maximize his returns by fixing the competitive prices to vitamins. The problem of medical shop person is the **dual problem. Note that the primal problem is minimization problem and the dual problem is the maximization problem.**

If we solve and get the solution of the primal problem, we can read the answer of dual problem from the primal solution.

Primal problem:

Minimize $Z = 3a + 2.5b$. s.t.

$2a + 4b \geq 40$

$3a + 2b \geq 50$

both a and b are ≥ 0.

Dual Problem:

Maximize $Z = 40x + 50y$ s.t.

$2x + 3y \leq 3$

$4x + 2y \leq 2.50$

both x and y are ≥ 0.

Solution to Primal: (Minimization problem *i.e.*, patient's problem)

Problem variable	*Cost Rs.*	C_j *Requirement*	*3* a	*2.50* b	*0* p	*0* q	*M* A_1	*M* A_2
b	2.50	5/2	0	1	–3/8	1/4	3/8	–1/4
a	3	15	1	0	1/4	–1/2	–1/4	1/2
		Net evaluations	0	0	3/16	7/8	M – 3/16	M – 7/8

Answer: a = 15 units, b = 2.5 units and total minimum cost is Rs. 51.25

Solution to Dual: (Maximization problem *i.e* medical shop man's problem)

Problem variable	*Profit Rs.*	C_j *Capacity units*	*40* x	*50* y	*0* S_1	*0* S_2
y	50	7/8	0	1	1/2	–1/4
x	40	3/16	1	0	–1/4	3/8
		Net evaluation	0	0	–15	–5/8

Answer: $x = 3/16$, $y = 7/8$, and maximum profit is Rs. 51.25

The patient has to minimize the cost by purchasing vitamin *X* and *Y* and the shopkeeper has to increase his returns by fixing competitive prices for vitamin *X* and *Y*. Minimum cost for patient is Rs. 51.25 and the maximum returns for the shopkeeper is Rs. 51.25. The competitive price for tonics is Rs.3 and Rs.2.50. Here we can understand the concept of **shadow price or economic worth of resources** clearly. If we multiply the original elements on the right hand side of the constraints with the net evaluation elements under slack or surplus variables we get the values equal to the minimum cost of minimization problem or maximum profit of the maximization problem. The concept of shadow price is similar to the economist's concept of the **wort**h of a marginal resource. In other words, we can see for a manufacturing unit it is **machine hour rate**. It is also known as **imputed value of the resources.** One cannot earn more than the economic worth of the resources he has on his hand. **The fact that the value of the objective function in the optimal program equals to the imputed value of the available resources has been called the FUNDAMENTAL THEOREM OF LINEAR PROGRAMMING.**

By changing the rows of the primal problem (dual problem) into columns we get the dual problem (primal problem) and vice versa.

To understand the rationale of dual problem and primal problem, let us consider another example.

Problem 3.32: A company manufactures two products *X* and *Y* on three machines Turning, Milling and finishing machines. Each unit of *X* takes, 10 hours of turning machine capacity, 5 hours of milling machine capacity and 1 hour of finishing machine capacity. One unit of *Y* takes 6 hours of turning machine capacity, 10 hours of milling machine capacity and 2 hours of finishing machine capacity. The company has 2500 hours of turning machine capacity, 2000 hours of milling machine capacity and 500 hours of finishing machine capacity in the coming planning period. The profit contribution of product *X* and *Y* are Rs. 23 per unit and Rs. 32 per unit respectively. Formulate the linear programming problems and write the dual.

Solution:

Department	*Product*		*Available* Capacity
	X	*Y*	
Turning	10	6	2500
Milling	5	10	2000
Finishing	1	2	500
Profit per unit in Rs.	23	32	

Let us take the maximization problem stated to be primal problem. Associated with this maximization problem is the minimization problem that is the dual of the given primal problem. Let us try to formulate the dual by logical argument.

The primal problem is the **seller's maximization problem**, as the seller wants to maximize his profit. Now the technology, *i.e.*, the machinery required are with the seller and they are his available resources. Hence he has to prepare the plans to produce the products to derive certain profit and he wants to know what will be his profit he can get by using the available resources. Hence the buyer's problem is:

Maximize $23x + 32y$ s.t

$10x + 6y \leq 2500$

$5x + 10y \leq 2000$

$1x + 2y \leq 500$ and both x and $y \geq 0$ (This we shall consider as **primal** problem).

Associated with seller's maximization problem is a **buyer's minimization problem**. Let us assume seller will pretend to be the buyer. The rationale is the buyer, it is assumed, will consider the purchase of the resources in full knowledge of the technical specifications as given in the problem. If the buyer wishes to get an idea of his total outlay, he will have to determine how much must he pay to buy all the resources. Assume that he designates variables by a, b, c, to represent per unit price or value that he will assign to turning, milling and finishing capacities, respectively, while making his purchase plans. The total outlay, which the buyer wishes to minimize, will be determined by the function $2500a + 2000b + 500c$, which will be the objective function of the buyer. The linear function of the objective function mentioned, must be minimized in view of the knowledge that the current technology yields a profit of Rs. 23 by spending 10 machine hours of turning department, 5 machine hours of milling department and 1 man-hour of finishing department. Similarly we can interpret other constraints also. Now the buyer's minimization problem will be:

Minimize. $Z = 2500a + 2000b + 500c$ s.t.

$10a + 5b + 1c \geq 23$

$6a + 10b + 2c \geq 32$ and all a, b, c, are ≥ 0 (This minimization version is the **dual of seller's primal problem given above**), where a, b, and c are dual variables and x, y, z are primal variables.

The values assigned to the dual variables in the optimal tableau of the dual problem, represent **artificial accounting prices, or implicit prices or shadow prices, or marginal worth or machine hour rate of various resources**. Because of this, we can read the values of dual variables from the net evaluation row of final tableau of primal problem. The values will be under slack variable column in net evaluation row.

The units of the constraint to which the dual variable corresponds determine the dimension of any dual variable.

In this problem the dimension of variable 'a' as well as 'b' is rupees per machine hour and that of variable 'c' is rupees per man-hour.

Another important observation is by definition, the entire profit in the maximization must be traced to the given resources, the buyer's total outlay, at the equilibrium point, must equal to the total profit. That is, optimal of the objective function of the primal equals to the optimal value of the objective function of the dual. This observation will enable the problem solver to check whether his answer is correct or not. The total profit (cost) of maximization problem (minimization problem) must be equal to the shadow price (or economic worth) of resources.

The given **primal** problem will have **symmetrical dual**. The symmetrical dual means all given structural constraints are inequalities. All variables are restricted to nonnegative values.

Now let us write primal and dual side by side to have a clear idea about both.

Primal problem:	**Dual Problem:**
Maximize $Z = 23x + 32y$ s.t.	**Minimize $Z = 2500a + 2000b + 5000c$ s.t.**
$10x + 6y \leq 2500$	**$10a + 5b + 1c \geq 23$**
$5x + 10y \leq 2000$	**$6a + 10b + 2c \geq 32$**
$1x + 2y \leq 500$	
Both x and y are ≥ 0.	**All a, b, c, are ≥ 0.**

Now let us discuss some of the important points that are to be remembered while dealing with primal and dual problem. Hence the characteristics are:

Note:

1. **If in the primal, the objective function is to be maximized, then in the dual it is to be minimized.**

   Conversely, if in the primal the objective function is to be minimized, then in the dual it is to be maximized.
2. **The objective function coefficients of the prima appear as right-hand side numbers in the dual and vice versa.**
3. **The right hand side elements of the primal appear as objective function coefficients in the dual and vice versa.**
4. **The input - output coefficient matrix of the dual is the transpose of the input – output coefficient matrix of the primal and vice versa.**
5. **If the inequalities in the primal are of the "less than or equal to" type then in the dual they are of the "greater than or equal to" type. Conversely, if the inequalities in the primal are of the "greater than or equal to" type; then in the dual they are of the "less than or equal to" type.**
6. **The necessary and sufficient condition for any linear programming problem and its dual to have optimum solution is that both have feasible solution. Moreover if one of them has a finite optimum solution, the other also has a finite optimum solution. The solution of the other (dual or primal) can be read from the net evaluation row (elements under slack/surplus variable column in net evaluation row). Then the values of dual variables are called *shadow prices*.**
7. **If the primal (either) problem has an unbound solution, then the dual has no solution.**
8. **If the i th dual constraints are multiplied by –1, then i th primal variable computed from net evaluation row of the dual problem must be multiplied by –1.**
9. **If the dual has no feasible solution, then the primal also admits no feasible solution.**
10. **If k th constraint of the primal is equality, then the k th dual variable is unrestricted in sign.**
11. **If p th variable of the primal is unrestricted in sign, then the p th constraint of the dual is a strict equality.**

Summary:

	Primal	Dual
(a)	Maximize.	Minimize
(b)	Objective Function.	Right hand side.
(c)	Right hand side.	Objective function.
(d)	i th row of input-output coefficients.	i th column of input output coefficients.
(e)	j th column of input-output coefficients.	j the row of input-output coefficients.
(f)	i th relation of inequality ($\leq$).	i th variable non-negative.
(g)	i th relation is an equality (=).	i the variable is unrestricted in sign.
(h)	j th variable non-negative.	j relation an inequality ($\geq$).
(i)	j th variable unrestricted in sign.	j th relation an equality.

Note:

1. **Primal of a Prima is Primal**
2. **Dual of a Dual is Primal.**
3. **Primal of a Dual is Primal.**
4. **Dual of a Primal is Dual.**
5. **Dual of a Dual of a Dual is Primal.**

Procedure for converting a primal into a dual and vice versa

Case 1: When the given problem is maximization one:

The objective function of primal is of maximization type and the structural constraints are of $\leq$ type. Now if the basis variables are x, y and z, give different name for variables of dual. Let them be a, b, and c etc. Now write the structural constraints of dual reading column wise. The coefficients of variables in objective function will now become the left hand side constants of structural constraints. And the left hand side constants of primal will now become the coefficients of variables of objective function of dual. Consider the example given below:

Primal	**Dual**
Maximize: $Z = 2x + 3y$ s.t.	Minimize: $Z = 10a + 12b$ s.t.
$1x + 3y \leq 10$	$1a + 2b \geq 2$
$2x + 4y \leq 12$ and	$3a + 4b \geq 3$ and
Both x and y are ≥ 0	both a and b are ≥ 0

Case 2: When the problem is of Minimization type:

The procedure is very much similar to that explained in case 1. Consider the example below:

Primal	**Dual**
Minimize $Z = 10x + 12y$ s.t	Maximize $Z = 2a + 3b$ s.t.
$1x + 2y \geq 2$	$1a + 3b \leq 10$
$3x + 4y \geq 3$ and	$2a + 4b \leq 12$ and
Both x and y are ≥ 0	Both a and b are ≥ 0.

Case 3: When the problem has got both $\geq$ and $\leq$ constraints, then depending upon the objective function convert all the constraints to either $\geq$ or $\leq$ type. That is, if the objective function is of minimization type, then see that all constraints are of $\geq$ type and if the objective function is of maximization type, then see that all the constraints are of $\leq$ type. To convert $\geq$ to $\leq$ or $\leq$ to $\geq$, simply multiply the constraint by –1. Once you convert the constraints, then write the dual as explained in case 1 and 2. Consider the example:

Primal:	**The primal can be written as:**
Maximize $Z = 2a + 3b$ s.t.	Maximize $Z = 2a + 3b$ s.t.
$1a + 4b \leq 10$	$1a + 4b \leq 10$
$2a + 3b \geq 12$ and	$-2a - 3b \leq -12$ and both a and b are ≥ 0.
Both a and b are ≥ 0	

Now dual is:

Minimize $Z = 10x - 12y$ s.t.

$1x - 2y \geq 2$

$4x - 3y \geq 3$ and both x and y are ≥ 0.

Similarly when the objective function of primal is of minimization type, then same procedure is adopted. See that all the constraints are of $\geq$ type.

Primal:

Minimise $Z = 4a + 5b$ s.t.

$3a + 2b \leq 10$

$2a + 4b \geq 12$ and

Both a and b are ≥ 0.

Primal can be written as:

Minimise $Z = 4a + 5b$ s.t.

$-3a - 2b \geq -10$

$2a + 4b \geq 12$ and both a and b are ≥ 0.

The dual is:

Maximize $Z = -10x + 12y$ s.t.

$-3x + 2y \leq 4$

$-2x + 4y \leq 5$ and

Both x and y are ≥ 0.

Case 4: When one of the constraint is an equation, then we have to write two versions of the equation, that is remove equality sign and write $\geq$ and $\leq$ sign for each one of them respectively and then write the dual as usual. Consider the example given below:

Primal:

Maximize $Z = 2x + 3y$ s.t.

$1x + 2y \geq 10$

$2x + 2y = 20$ and

Both x and y are ≥ 0.

This can be written as:

Maximise $Z = 2x + 3y$ s.t.

$-1x - 2y \leq -10$

$-2x - 2y \leq -20$

$2x + 2y \leq 20$ and both x and y are ≥ 0.

Primal can be written as:

Maximise $Z = 2x + 3y$ s.t.

$1x + 2y \geq 10$

$2x + 2y \geq 20$

$2x + 2y \leq 20$ and both x and y are ≥ 0

The dual is:

Minimise $Z = -10a - 20b + 20c$ s.t.

$-1a - 2b + 2c \geq 2$

$-2a - 2b + 2c \geq 3$ and a, b, c all ≥ 0.

Problem 3.32:

Write dual of the given l.p.p.

Minimize $Z = 3x + 1y$ s.t.

$2x + 3y \geq 2$

$1x + 1y \geq 1$ and

Both x and y are ≥ 0.

Dual of the given problem is:

Maximize $Z = 2a + 1b$ s.t.

$2a + 1b \leq 3$

$3a + 1b \leq 1$ and

Both a and b are ≥ 0

Simplex version **(Primal)**

Minimize $Z = 3x + 1y + 0p + 0q + MA_1 + MA_2$ s.t.

$2x + 3y - 1p + 0q + 1A_1 + 0A_2 = 2$

Simplex version: **(Dual)**

Maximize $Z = 2a + 1b + 0S_1 + 0S_2$ s.t.

$2a + 1b + 1S_1 + 0S_2 = 3$

$1x + 1y + 0p – 1q + 0A_1 + 1A_2 = 1$ and
x, y, p, q, A_1 and A_2 all ≥ 0

$3a + 1b + 0S_1 + 1S_2 = 1$ and
a, b, S_1, S_2 all ≥ 0

Problem 3.33: Write the dual of the primal problem given and solve the both and interpret the results.

Primal Problem:
Maximize $Z = 5a + 20b$ s.t.
$5a + 2b \leq 20$
$1a + 2b \leq 8$
$1a + 6b \leq 12$ and
Both a and $b \geq 0$

Simplex version:
Maximize $Z = 5a + 20b + 0S_1 + 0S_2 + 0S_3$ s.t.
$5a + 2b + 1S_1 + 0S_2 + 0S_3 = 20$
$1a + 2b + 0S_1 + 1S_2 + 0S_3 = 8$
$1a + 6b + 0S_1 + 0S_2 + 1S_3 = 12$
and $a, b. S_1, S_2$, and S_3 all ≥ 0

First let us solve the Primal problem by using simplex method and then write the dual and solve the same.

Table: I. $a = 0, b = 0, S_1 = 20, S_2 = 8, S_3 = 12$ and Z = Rs. 0

Problem variable	*Profit Rs.*	C_j *Capacity units*	5 a	20 b	0 S_1	0 S_2	0 S_3	*Replacement ratio*
S_1	0	20	5	2	1	0	0	10
S_2	0	8	1	2	0	1	0	4
S_3	0	12	1	**6**	0	0	1	2
		Net evaluation	5	20	0	0	0	

Table: II. $a = 0, b = 2, S_1$ 16, $S_2 = 4, S_3 = 0$, and Z = Rs. 40/–

Problem variable	*Profit Rs.*	C_j *Capacity units*	5 a	20 b	0 S_1	0 S_2	0 S_3	*Replacement ratio*
S_1	0	16	14/4	0	1	0	– 1/3	24/7 = 21 3/7
S_2	0	4	**2/3**	0	0	1	– 1/3	12/2 = 6
B	20	2	1/6	1	0	0	– 1/6	12
		Net evaluation	5/3	0	0	0	– 10/3	

Table: III. $a = 24/7, b = 10/7, S_1 = 0, S_2 = 12/7, S_3 = 0$, and Z = Rs. 45.75.

Problem variable	*Profit Rs.*	C_j *Capacity units*	5 a	20 b	0 S_1	0 S_2	0 S_3	*Replacement ratio*
a	5	24/7	1	0	3/14	0	– 1/14	
S_2	0	12/7	0	0	– 1/7	1	– 2/7	
b	20	10/7	0	1	– 1/28	0	5/28	
		Net evaluation	0	0	– 5/14	0	– 45/14	

As all the values in net evaluation row are either zeros or negative elements, the solution is optimal.

Hence, $a = 24/7$, $b = 10/7$ and optimal profit is Rs. 45.75.

Now let us solve the dual of the above.

Dual of the given problem:

Minimize $Z = 20x + 8y + 12z$ s.t.

i.e., Maximize $Z = -20x - 8y - 12z$ s.t.

$5x + 1y + 1z \geq 5$

$2x + 2y + 6z \geq 20$

and x, y, and z all ≥ 0

Simplex version:

Maximize $Z = -20x - 8y - 12z + 0S_1 + 0S_2 - MA_1 - M A$ s.t.

$5x + 1y + 1z - 1S_1 + 0S_2 + 1A_1 + 0A_2 = 5$

$2x + 2y + 6z + 0S_1 - 1S_2 + 0A_1 + 1A_2 = 20$

and $x, y, z, S_1, S_2, A_1, A_2$ all ≥ 0.

Two Phase version is: Maximize: $Z = 0x + 0y + 0z + 0S_1 + 0S_2 - 1A_1 - 1A_2$ s.t.

$5x + 1y + 1z - 1S_1 + 0S_2 + 1A_1 + 0A_2 = 5$

$2x + 2y + 6z + 0S_1 - 1S_2 + 0A_1 + 1A_2 = 20$

And $x, y, z, S_1, S_2, A_1, A_2$ all ≥ 0.

Table: I. $x = 0$, $y = 0$, $z = 0$, $S_1 = 0$, $S_2 = 0$, $A_1 = 5$, $A_2 = 20$, $Z = -$ Rs 25/–

Problem variable	*Profit Rs.*	C_j *Capacity units*	*0* x	*0* y	*0* z	*0* S_1	*0* S_2	*– 1* A_1	*– 1* A_2	*Replace–ment ratio*
A_1	–1	5	**5**	1	1	–1	0	1	0	1
A_2	–1	20	2	2	6	0	–1	0	1	10
		Net evaluation	7	3	7	–1	–1	0	0	

Table: II. $a = 1$, $y = 0$, $z = 0$, $S_1 = 0$, $S_2 = 0$, $A_1 = 0$, $A = 20$, $Z = -$ Rs.18/–

Problem variable	*Profit Rs.*	C_j *Capacity units*	*0* x	*0* y	*0* z	*0* S_1	*0* S_2	*– 1* A_1	*– 1* A_2	*Replace–ment ratio*
x	0	1	1	1/5	1/5	–1/5	0	1/5	0	5
A_2	–1	18	0	8/5	**28/5**	2/5	–1	–2/5	1	45/14
		Net evaluation	0	8/5	28/5	2/5	–1	–2/5	0	

Table: III. $x = 5/14$, $y = 0$, $z = 45/14$, $S_1 = 0$, $S_2 = 0$, $A_1 = 0$, $A_2 = 0$, Z = Rs. 0

Problem variable	*Profit Rs.*	C_j *Capacity units*	*0* x	*0* y	*0* z	*0* S_1	*0* S_2	*– 1* A_1	*– 1* A_2	*Replace–ment ratio*
x	0	5/14	1	1/7	0	–3/14	1/28	3/14	–1/28	
z	0	45/14	0	2/7	1	1/14	–5/28	–1/14	5/28	
		Net evaluation	0	0	0	0	0	–1	–1	

Table: IV. $x = 5/14$, $y = 0$, $z = 45/14$, $S_1 = 0$, $S_2 = 0$, $A_1 = 0$, $A_2 = 0$ and Z = – Rs. 45.75.

Problem variable	*Profit Rs.*	C_j *Capacity units*	*0* x	*0* y	*0* z	*0* S_1	*0* S_2	*– 1* A_1	*– 1* A_2	*Replace–ment ratio*
x	–20	5/14	1	1/7	0	–3/14	1/28	3/14	–1/28	
z	–12	45/14	0	2/7	1	1/14	–5/28	–1/14	5/28	
		Net evaluation	0	–12/7	0	–24/7	–10/7	24/7 – M	10/7 – M	

$x = 5/14$, $z = 45/14$ and Z = Rs. 45.75.

Now let us compare both the final (optimal solution) table of primal and dual.

Optimal solution table of Primal

Table: III. $a = 24/7$, $b = 10/7$, $S_1 = 0$, $S_2 = 12/7$, $S_3 = 0$, and Z = Rs. 45.75.

Problem variable	*Profit Rs.*	C_j *Capacity units*	*5* a	*20* b	*0* S_1	*0* S_2	*0* S_3	*Replacement ratio*
a	5	**24/7**	1	0	3/14	0	–1/14	
S_2	0	**12/7**	0	0	–1/7	1	–2/7	
b	20	**10/7**	0	1	–1/28	0	5/28	
		Net evaluation	0	0	**–5/14**	0	**–45/14**	

Optimal Solution table of Dual:

Table: IV. $x = 5/14$, $y = 0$, $z = 45/14$, $S_1 = 0$, $S_2 = 0$, $A_1 = 0$, $A_2 = 0$ and $Z = –$ Rs. 45.75.

Problem variable	*Profit Rs.*	C_j *Capacity units*	*0* x	*0* y	*0* z	*0* S_1	*0* S_2	*– 1* A_1	*– 1* A_2	*Replace–ment ratio*
x	–20	5/14	1	1/7	0	–3/14	1/28	3/14	–1/28	
z	–12	45/14	0	2/7	1	1/14	–5/28	–1/14	5/28	
		Net evaluation	0	**–12/7**	0	**–24/7**	**–10/7**	**24/7** – M	**10/7** – M	

Construction of Dual problem:

Problem 3.34: Construct the dual of the given l.p.p.

Maximize $Z = 5w + 2x + 6y + 4z$ s.t

$1w + 1x + 1y + 1z \le 140$

$2w + 5x + 6y + 1z \ge 200$

$1w + 3x + 1y + 2z \le 150$

And w, x, y, z all are ≥ 0.

Solution: Dual: As the constraint 2 is of $\ge$ type and other two are $\le$ type and the objective function is maximization, we have to write the constraint 2 also as $\le$ type. Hence multiply the constraint by –1 and write the dual problem.

Hence the given problem is:

Maximize $Z = 5w + 2x + 6y + 4z$ s.t.

$1w + 1x + 1y + 1z \le 140$

$–2w – 5x – 6y – 1z \le – 200$

$1w + 3x + 1y + 2z \le 150$

And w, x, y, z all are ≥ 0.

Dual is: Minimize $Z = 140\,a – 200\,b + 150c$ s.t.

$1a – 2b + 1c \ge 5$

$1a – 5b + 3c \ge 2$

$1a – 6b + 1z \ge 6$

$1a – 1b + 2z \ge 4$

And a, b, c all ≥ 0.

Problem 3.35: (Primal): Maximize $Z = 4a + 3b$ s.t.

$2a + 9b \le 100$

$3a + 6b \le 120$

$1a + 1b = 50$ and both a and b are ≥ 0.

When there is equality in the constraints, we have to write two versions of the same, *i.e.*, both $\le$ and $\ge$ version and then we have to write the dual. As the objective function is maximization type, we must take care that all the inequalities are of ≤ type. In case the objective function is minimization type, then all the inequalities must be of ≥ type.

Dual of the given problem:

The given problem is written as:

Maximize $4a + 3b$ s.t.

$2a + 9b \leq 100$

$3a + 6b \leq 120$

$1a + 1b \leq 50$

$1a + 1b \geq 50$ and both a and b are ≥ 0

This can be written as:

Maximize $Z = 4a + 3b$ s.t.

$2a + 9b \leq 100$

$3a + 6b \leq 120$

$1a + 1b \leq 50$

$-1a - 1b \leq -50$ and both a and b are ≥ 0.

Dual of the primal:

Minimize $100w + 120x + 50y - 50z$ s.t

$2w + 3x + 1y - 1z \geq 4$

$9w + 6x + 1y - 1z \geq 3$ and w, x, y, and z all are > 0.

As the number of rows is reduced, the time for calculation will be reduced. Some times while solving the l.p.p it will be better to write the dual and solve so that one can save time. And after getting the optimal solution, we can read the answer of the primal from the net evaluation row of the dual.

Problem 3.36: Write the dual of the given primal problem:

Minimize: $1a + 2b + 3c$ s.t.

$2a + 3b - 1c \geq 20$

$1a + 2b + 3c \leq 15$

$0a + 1b + 2c = 10$ and

a, b, c all ≥ 0

As the objective function is minimization type all the inequalities must be of ≥ type. Hence convert the second constraint by multiplying by –1 into ≥ type. Rewrite the third constraint into ≤ and ≥ type and multiply the ≤ type inequality by –1 and convert it into ≥ type. And then wire the dual.

Given problem can be written as:

Minimize $1a + 2b + 3c$ s.t

$2a + 3b - 1c \geq 20$

$-1a - 2b - 3c \geq -15$

$0a + 1b + 2c \geq 10$

$-0a - 1b - 2c \geq -10$ and

$a, b,$ and c all ≥ 0.

Dual is:

Maximize $Z = 20w - 15x + 10y - 10z$ s.t

$2w - 1x + 0y - 0z \leq 1$

$3w - 2x + 1y - 1z \leq 2$

$-1w - 3x + 2y - 2z \leq 3$ and

w, x, y and z all ≥ 0.

Points to Remember

1. **While writing dual see that all constraints agree with the objective function. That is, if the objective function is maximization, then all the inequalities must be ≤ type. In case the objective function is Minimization, then the inequalities must be of ≥ type.**
2. **If the objective function is maximization and any one or more constraints are of ≥ type then multiply that constraint by –1 to convert it into ≤ type. Similarly, if**

the objective function is minimization type and one or more constraints are of ≤ type, then multiply them by –1 to convert them into ≥ type.

3. In case any one of the constraint is an equation, then write the two versions of the same *i.e.*, ≤ and ≥ versions, then depending on the objective function, convert the inequalities to agree with objective function by multiplying by –1.

SENSITIVITY ANALYSIS

While solving a linear programming problem for optimal solution, we assume that:

(*a*). Technology is fixed, (*b*). Fixed prices, (*c*). Fixed levels of resources or requirements, (*d*). The coefficients of variables in structural constraints (*i.e*. time required by a product on a particular resource) are fixed, and (*e*) profit contribution of the product will not vary during the planning period. These assumptions, implying certainty, complete knowledge, and static conditions, permit us to design an optimal programme. The condition in the real world however, might be different from those that are assumed by the model. It is, therefore, desirable to determine ***how sensitive*** the optimal solution is to different types of changes in the problem data and parameters. The changes, which have effect on the optimal solution are: (*a*) Change in objective function coefficients (a_{ij}), (*b*) Resource or requirement levels (b_i), (*c*) Possible addition or deletion of products or methods of production. The process of checking the sensitivity of the optimal solution for changes in resources and other components of the problem, is given various names such as: ***Sensitivity Analysis, Parametric Programming and Post optimality analysis or what if analysis.***

Post optimality test is an important analysis for a manager in their planning process, when they come across certain uncertainties, say for example, shortage of resources due to absenteeism, breakdown of machinery, power cut off etc. They may have to ask question 'what if', a double–edged sword. They are designed to project the consequences of possible changes in the future, as well as the impact of the possible errors of estimation of the past. The need for sensitivity analysis arises due to (*i*) To know the effect of and hence be prepared for, possible future changes in various parameters and components of the problem, (*ii*) To know the degree of error in estimating certain parameters that could be absorbed by the current optimal solution. Or to put in other way, sensitivity **analysis answers questions regarding what errors of estimation could have been committed, or what possible future changes can occur, without disturbing the optimality of the current optimal solution.**

The outcome of sensitivity analysis fixes **ranges** *i.e*., upper limits and lower limits of parameters like C_j, a_{ij}, b_i etc. within which the current optimal programme will remain optimal. Hence, we can say that the sensitivity analysis is a major guide to managerial planning and control. Also sensitivity analysis arises the need for **reworking of the entire problem from the very beginning each time a change is investigated or incorporated.** The present optimal solution can be used to study the changes with minimum computational effort. By adding or deleting a new column (product) or adding or deleting a new row (new process) we can analyze the changes with respect to C_j, a_{ij}, and b_i.

To summarize the Sensitivity analysis include:

1. Coefficients (C_j) of the objective function, which include:
   (a) Coefficients of basic variables (C_j).
   (b) Confidents of non–basic variables.
2. Changes in the right hand side of the constraints (b_i).

3. Changes in a_{ij}, the components of the matrix, which include:
   (a) Coefficients of the basic variables, a_{ij}.
   (b) Coefficients of non-basic variables.
4. Addition of new variables to the problem.
5. Addition of new or secondary constraint.

The above changes may results in one of the following three cases:

Case I. The optimal solution remains unchanged, that is the basic variables and their values remain essentially unchanged.

Case II. The basic variables remain the same but their values are changed.

Case III. The basic solution changes completely.

Change in the Objective Coefficient

(a) Non-basic Variables

Consider a change in the objective coefficient of the non-basic variable in the optimal solution. Any change in the objective coefficient of the non-basic variable will affect only its index row coefficient and not others.

Problem 3.37: Maximize $Z = 2a + 2b + 5c + 4d$ s.t

$1a + 3b + 4c + 3d \leq 10$

$4a + 2b + 6c + 8d \leq 25$ and a, b, c, and d all are ≥ 0.

Table: I. $a = 0$, $b = 0$, $c = 0$, $d = 0$, $S_1 = 10$, $S_2 = 25$, $Z =$ Rs.0 /–

Problem variable	*Profit Rs.*	C_j *Capacity units*	2 *a*	2 *b*	5 *c*	4 *d*	0 S_1	0 S_2	*Replacement ratio*
S_1	0	10	1	3	**4**	3	1	0	10/4
S_2	0	25	4	2	6	8	0	1	25/6
		Net evaluation	2	2	5	4	0	0	

Table: II. $a = 0$, $b = 0$, $c = 5/2$, $d = 0$, $S_1 = 0$, $S_2 = 10$, $Z =$ Rs. 12.50

Problem variable	*Profit Rs.*	*Capacity units*	*a*	*b*	*c*	*d*	S_1	S_2	*Replacement ratio*
c	5	5/2	1/4	3/4	1	3/4	1/4	0	10
S_2	0	10	**5/2**	– 5/2	0	7/5	3/2	1	4
		Net evaluation	3/4	– 1.75	0	1/4	– 5/4	0	

Table: III. $a = 4$, $b = 0$, $c = 1.5$, $d = 0$, $S_1 = 0$, $S_2 = 0$, Z = Rs. 15.50 /–

Problem variable	*Profit Rs.*	C_j *Capacity units*	*2* *a*	*2* *b*	*5* *c*	*4* *d*	*0* S_1	*0* S_2
c	5	3/2	0	1	1	2/5	2/5	– 1/10
a	2	4	1	– 1	0	7/5	– 3/5	2/5
		Net evaluation	0	– 1	0	– 4/5	– 4/5	– 3/10

Here '*a*' and '*c*' are basic variables and '*b*' and '*d*' are non–basic variables. Consider a small change x_1 in the objective coefficient of the variable '*b*', then its index row (net evaluation row) element becomes:

$(2 + x_1) - (-2 + 5) = x_1 - 1$. If variable 'b' wants to be an incoming variable $x_1 - 1$ must be positive. Then the value of x_1 should be > 1. Hence when the value of the increment is > 1 then the present optimal solution changes.

Similarly for D, if x_2 is the increment, then $(4 + x_2) - (14/5 + 10/5) = (4 + x_2) - 24/5$, hence if '*d*' wants to become incoming variable then the value of $x_2 > 4/5$. To generalize, one can easily conclude that for non-basic variables when its objective coefficient just exceeds its index row coefficient in the optimal solution, the present solution ceases to be optimal.

(b) Basic variables:

Now let us consider a change in the objective coefficient of the basic variable in the optimal solution. Here, it affects the net evaluation row coefficients of all the variables. Hence, as soon as the net evaluation row coefficients of basic variables become negative, it leaves the solution, and that of non-basic variable becomes positive, it becomes an incoming variable. In either case, the present optimal solution changes.

Consider the above example. Let us say that there is a small reduction 'x_1' in the objective coefficient of variable '*a*' *i.e.*, $(2 - x_1)$ then the net evaluation row coefficients of variables are:

Variable	Corresponding change in net evaluation row element.
a	$(2 - x_1) - 1\,(2 - x_1) = 0$
b	$2 - \{-1\,(2 - x_1) + 5) = -(x_1 + 1)$
c	0
d	$-(4/5 + 7/5\, x_1)$
S_1	$-4/5 + 3/5\, x_1$
S_2	$-3/10 + 2/5\, x_1$

Results:

(a) For any value of x_1 variable '*b*' cannot enter the solution.

(b) As soon as x_1 is > 4/7, variable '*d*' enters the solution.

(c) For any value of x_1, S_1 will not enter into solution.

(d) As soon as $x_1 > 3/4$, S_2 claims eligibility to enter into solution.

A reduction in objective coefficient of variable '*a*' by more than 4/7, the present optimal solution change. *i.e.*, the value is $2 - 4/7 = 10/7$.

When the objective coefficient of variable 'a' increases by a value x_2, the changes are:

Variable	Corresponding change.
a	$(2 + x_2) - (2 + x_2) = 0$
b	$2 - (-1) \{(2+x_2) + 5)\} = -1 + x_2$
c	0
d	$-4/5 - 7/5\ x_2$
S_1	$-4/5 + 3/5\ x_2$
S_2	$-3/10 - 2/5\ x_2$

If x_2 is ≥ 1 the variable 'b' claims the entry into solution and the optimal solution changes.

For any value of x_2 variables 'd' and S_1 are not affected.

If x_2 is $> 4/3$, S_1 claims the entry into solution, and the optimal changes.

Hence, as soon as objective coefficient of variable 'a' increases by more than 1 the present optimal solution changes. Hence the maximum permissible value of objective coefficient of 'a' is 2 + 1 = 3 for the present optimal solution to remain. That is the range for objective coefficient of variable 'a' is 10/7 to 3.

Change in the right – hand side of the constraint

The right hand side of the constraint denotes present level of availability of resources (or requirement in minimization problems). When this is increased or decreased, it will have effect on the objective function and it may also change the basic variable in the optimal solution.

Example 3.38: A company manufactures three products: X, Y and Z by using three resources. Each unit of product X takes three man hours and 10 hours of machine capacity and 1 cubic meter of storage place. Similarly, one unit of product Y takes 5 man-hours and 2 machine hours on 1cubic meter of storage place and that of each unit of products Z is 5 man-hours, 6 machine hours and 1 cubic meter of storage place. The profit contribution of products X, Y and Z are Rs. 4/–, Rs.5/– and Rs. 6/– respectively. Formulate the linear programming problem and conduct sensitivity analysis when

Maximize $Z = 4x + 5y + 6z$ s.t.

$3x + 5y + 5z \leq 900$

$10x + 2y + 6z \leq 1400$

$1x + 1y + 1z \leq 250$ and all x, y, and z are ≥ 0

The final table of the solution is:

$x = 50$, $y = 0$, $z = 150$, $S_1 = 0$, $S_2 = 0$, $S_3 = 50$ and $Z =$ Rs. 1100/–

Problem variable	*Profit Rs.*	*C_j Capacity units*	*4 x*	*2 y*	*6 z*	*0 S_1*	*0 S_2*	*0 S_3*
z	6	150	0	11/8	1	5/16	– 3/32	0
x	4	50	1	– 5/8	0	– 3/16	5/32	0
S_3	0	50	0	1/4	0	– 1/8	– 1/16	1
		Net evaluation	0	– 23/4	0	– 9/8	–1/16	0

The solution is $x = 50$, $y = 0$, $z = 150$, $S_1 = 0$, $S_2 = 0$, $S_3 = 50$ and profit Z = Rs. 1100/–

Here man- hours are completely utilized hence $S_1 = 0$, Machine hours are completely utilized, hence $S_3 = 0$ but the storage capacity is not completely utilized hence still we are having a balance of 50 cubic meters of storage place *i.e.*, $S_3 = 50$.

Value of dual variable under S_1 in net evaluation row is 9/8. This is the shadow price or per unit price of the resource. The resource is man- hours. Hence it means to say that as we go on increasing one hour of man-hour resource, the objective function will go on increasing by Rs. 9/8 per hour. Similarly the shadow price of machine hour is Rs. 1/16 and that of storage space is Rs.0. Similar reasoning can be given. That is every unit increase in machine hour resource will increase the objective function by Rs. 1/16 and that of storage space is Rs.0/–

Now let us ask ourselves what the management wants to do:

Question No.1. If the management considers to increase man-hours by 100 hours *i.e.*, from 900 hours to 1000 hours and machine hours by 200 hours *i.e.*, 1400 hours to 1600 hours will the optimal solution remain unchanged?

Now let us consider the elements in the identity matrix and discuss the answer to the above question.

Problem variable	S_1	S_2	S_3	*Capacity*	$B^{-1} \times b$	=	
z	5/16	– 3/32	0	1000	5000/16 – 4800/32 + 0	=	325/2
x	– 3/16	5/32	0	1600	– 3000/16 + 8000/32 + 0	=	125/2
S_3	– 1/8	– 1/16	1	250	– 1000/8 – 1600/16 +250	=	25

Now $x = 125/2$, and $z = 325/2$ and $S_3 = 25$. As x and z have positive values the current optimal solution will hold well. Note that the units of x and z have been increased from 50 and 150, to 125/2 and 325/2. These extra units need the third resource, the storage space. Hence storage space has been reduced from 50 to 25.

Shadow price indicates that resource of man-hours can be increased to increase objective function. A solution to question No.1 above, showed that with increase of man- hours by 100 (*i.e.*, from 900 to 1000 hours), the basic variables remain the same (*i.e.*, x and z and S_3) with different values at the optimal stage.

Question No. 2: Up to what values the resource No.1, *i.e.*, man-hours can be augmented without affecting the basic variables? And up to what value the resource man-hours can be without affecting the basic variables?

Let α be the increment in man- hour resource, then:

Problem variable	S_1	S_2	S_3	*Capacity*	$B^{-1} \times b$	=	
z	5/16	–3/32	0	$900 + \alpha$	$5/16(900 + \alpha) – 3/32 \times 1400 + 0 \times 250$	=	$150 + 5/16\alpha$
x	–3/16	5/32	0	1400	$– 3/16 (900 + \alpha) + 5/32 \times 1400 + 0 \times 250$	=	$50 – 3/16\alpha$
S_3	–1/8	–1/16	1	250	$– 1/8 (900 + \alpha) – 1/16 \times 1400 + 1 \times 250$	=	$50 – 1/8 \alpha$

As $z = 150 + 5/16\alpha$, z remains positive for any value of $\alpha \geq 0$.

As $x = 50 - 3/16\alpha$, x remains positive for any value of $\alpha \leq 800/3$.

As $S_3 = 50 - \alpha/8$, S_3 remains positive for any value of $\alpha \leq 400$.

Therefore, the present basis remains feasible for the increment of resource 1, *i.e.,* man-hours by 800/3 only. For further increase, basis will change. Similarly if the resource 1 being contemplated, the solution would be:

Problem variable	S_1	S_2	S_3	*Capacity*	$B^{-1} \times b$	=	
z	5/16	– 3/32	0	$900 - \alpha$	$5/16\,(900 - \alpha) - 3/32 \times 1400 + 0 \times 250$	=	$150 - 5/16\alpha$
x	– 3/16	5/32	0	1400	$-3/16\,(900 - \alpha) + 5/32 \times 1400 + 0 \times 250$	=	$50 + 3/16\alpha$
S_3	– 1/8	– 1/16	1	250	$-1/8\,(900 - \alpha) - 1/16 \times 1400 + 1 \times 250$	=	$50 + 1/8\alpha$

Thus z will be negative, when $5/16\alpha > 150$ *i.e.,* $\alpha > 480$. Hence when resource No.1 *i.e.* Man-hour is reduced to the level below 420 = (900 – 480) the present basis will change.

Hence the range for resource No. 1, *i.e.,* man-hours to retain the present basic variables is:

(900 + 800/3) to (900 – 480) = 3500 /3 to 420.

Dual Simplex Method

We remember that while getting optimal solution for a linear programming problem, by using simplex method, we start with initial basic feasible solution (with slack variables in the programme for maximization problem and artificial variables in the programme for minimization problem) and through stages of iteration along simplex algorithm we improve the solution till we get optimal solution. An optimal solution is one, in terms of algorithm for maximization, in which, the net evaluation row are either negative elements are zeros, *i.e.,* the dual variables are feasible. Now with the knowledge of Primal and Dual relationship, we know that the net evaluation row elements are the values of dual variables and hence it suggests dual feasible solution. There are situations, where the primal solution may be infeasible, but corresponding variables indicate that dual is feasible. Thus, solution of primal is infeasible but optimum. In a dual simplex method initial solution is infeasible but optimum, and through iteration it reaches feasibility at which stages it also reaches true optimum.

Problem 3.39:	**The problem can be written as:**
Minimize $Z = 2a + 1b$ s.t.	Maximize $Z = -2a - 1b$ s.t.
$3a + 1b \geq 3$	$-3a - 1b \leq -3$
$4a + 3b \geq 6$	$-4a - 3b \leq -6$
$1a + 2b \leq 3$ and both a and b are ≥ 0.	$1a + 2b \leq 3$ and both a and b are ≥ 0.

The linear programming version is:

Maximize $Z = -2a - 1b + 0S_1 + 0S_2 + 0S_3$ s.t

$-3a - 1b + 1S_1 + 0S_2 + 0S_3 = -3$

$-4a - 3b + 0S_1 + 1S_2 + 0S_3 = -6$

$1a + 2b + 0S_1 + 0S_2 + 1S_3 = 3$

And a, b, S_1, S_2, and S_3 all ≥ 0.

Table: I. $a = 0$, $b = 0$, $S_1 = -3$, $S_2 = -6$, $S_3 = 3$ and Z = Rs.0/–

Problem variable	*Profit Rs.*	*C_j Capacity units*	*– 2 a*	*– 1 b*	*0 S_1*	*0 S_2*	*0 S_3*	*Replacement ratio*
S_1	0	– 3	– 3	– 1	1	0	0	3
S_2	0	– 6	– 4	**– 3**	0	1	0	1/2
S_3	0	3	1	2	0	0	1	3/2
		Net evaluation	–2	–1	0	0	0	

Quotient Row: –4/–2 –1/–3

Now observe here, both $S_1 = -3$ and $S_2 = -6$ are negative and $S_3 = 3$ positive. This shows that the basic variables of primal (S_1 and S_3) are infeasible. Moreover the net evaluation row elements are negative or zeros, the solution is optimal. Now let us change the solution towards feasibility without disturbing optimality. That is deciding the incoming variable and outgoing variable. **In regular simplex method, we first decide incoming variable. In dual simplex we first decide outgoing variable, *i.e.*, key row.**

(a) Criterion for out going variable:

The row, which has got the largest negative value (highest number with negative sign) in the capacity column, becomes the key row and the variable having a solution in that row becomes out going variable. If all the values in the capacity column are non-negative and if all net evaluation row elements are negative or zeros, the solution is optimal basic feasible solution. In the given example, the row containing S_2 is having highest number with negative sign; hence S_2 is the out going variable.

(b) Criterion for incoming variable:

Divide the net evaluation row elements by corresponding coefficients (if negative) of the key row. The column for which the coefficient is smallest becomes key column and the variable in that column becomes entering variable. (If all the matrix coefficients in the key row are positive, the problem has no feasible solution).

In the problem given, variable satisfies the condition, hence variable '*b*' is the incoming variable. The rest of the operation is similar to regular simplex method.

Table: II. $a = 0$, $b = 2$, $S_1 = -1$, $S_2 = 0$, $S_3 = -1$ and Z = Rs. –2

Problem variable	*Profit Rs.*	*C_j Capacity units*	*– 2 a*	*– 1 b*	*0 S_1*	*0 S_2*	*0 S_3*	*Replacement ratio*
S_1	0	– 1	**– 5/3**	0	1	– 1/3	0	3/5
b	– 1	2	4/3	1	0	– 1/3	0	6/4
S_3	0	– 1	– 5/3	0	0	2/3	1	3/5
		Net evaluation quotient:	– 2/3 – 1/5	0 –	0 –	– 1/3 – 1	 –	 –

The solution is infeasible. As net evaluation row elements are negative the solution remains optimal and as basic variables are negative, it is infeasible.

Now both rows of S_1 and S_3 are having –1 in capacity column, any one of them becomes key row. Let us select first row as key row. The quotient row shows that '*a*' as the incoming variable.

Table: III. $a = 3/5$, $b = 6/5$, $S_1 = 0$, $S_2 = 0$, $S_3 = 0$, $Z = –$Rs. 12/5

Problem variable	*Profit Rs.*	C_j *Capacity units*	– 2 *a*	– 1 *b*	0 S_1	0 S_2	0 S_3	*Replacement ratio*
a	– 2	3/5	1	0	– 3/5	1/5	0	
b	– 1	6/5	0	1	4/5	– 3/5	0	
S_3	0	0	0	0	– 1	1	1	
		Net evaluation	0	0	– 2/5	– 1/5	0	

As the net evaluation row elements are negatives or zeros, the solution is optimal and feasible. Answer is $a = 3/5$ and $b = 6/5$ and profit $Z = –$ Rs. 12/5. That is for minimization problem the minimum cost is Rs. 12/5.

To Summarize:

(i) The solution associated with a basis is optimal if all basic variables are ≥ 0.

(ii) The basic variable having the largest negative value (highest number with negative sign), is the outgoing variable and the row containing it is the key row.

(iii) If α is the matrix coefficient of the key row and is < 0, the variable for which (index row coefficient/α) is numerically smallest will indicate incoming variable.

(iv) If $\alpha > 0$, for all the variables in the key row the problem is infeasible.

Problem 3.40:

Minimize $Z = 20x + 16y$ s.t.

$1x + 1y \geq 12$

$2x + 1y \geq 17$

$2x + 0y \geq 5$

$0x + 1y \geq 6$ and

Both x and y are ≥ 0

The problem can be written as:

Maximize $Z = – 20x – 16y + 0S_1 + 0S_2 + 0S_3 + 0S_4$ s.t.

$–1x – 1y + 1S_1 + 0S_2 + 0S_3 + 0S_4 = –12$

$–2x –1y + 0S_1 + 1S_2 + 0S_3 + 0S_4 = –17$

$–2x +0y + 0S_1 + 0S_2 + 1S_3 + 0S_4 = –5$

$0x – 1y + 0S_1 + 0S_2 + 0S_3 + 1S_4 = – 6$

and x, y, S_1, S_2, S_3 and S_4 all ≥ 0.

Solution:

Table: I. $x = 0$, $y = 0$, $S_1 = -12$, $S_2 = -17$, $S_3 = -6$, $S_4 -6$, $Z =$ Rs.0/–

Problem variable	*Profit Rs.*	C_j *Capacity units*	*– 20* x	*– 16* y	*0* S_1	*0* S_2	*0* S_3	*0* S_4	*Replacement ratio*
S_1	0	– 12	– 1	– 1	1	0	0	0	12
S_2	0	– 17	**– 2**	– 1	0	1	0	0	8.5
S_3	0	– 5	– 2	0	0	0	1	0	2.5
S_4	0	– 6	0	– 1	0	0	0	1	0
		Net evaluation	– 20	– 16	0	0	0	0	
		Quotient	10	16					

As all the elements of net evaluation row are negative, the solution is optimal but as slack variables have –ve values, the solution is infeasible. The row with highest number with negative sign becomes outgoing variable. Here S_2 is out going variable *i.e.*, it becomes key row. By dividing net evaluation row elements by corresponding key row elements, quotient row elements are obtained, which show that x is the incoming variable (lowest number).

Table: II. $x = 17/2$, $y = 0$, $S_1 = -7/2$, $S_2 = 0$, $S_3 = 12$, $S_4 = -6$ and $Z = -$ Rs. 170/–

Problem variable	*Profit Rs.*	C_j *Capacity units*	*– 20* x	*– 16* y	*0* S_1	*0* S_2	*0* S_3	*0* S_4	*Replacement ratio*
S_1	0	– 7/2	0	– 1/2	1	–1/2	0	0	7
x	– 20	17/2	1	1/2	0	–1/2	0	0	Negative
S_3	0	12	0	1	0	–1	1	0	12
S_4	0	– 6	0	**– 1**	0	0	0	1	6
		Net evaluation	0	– 6	0	– 10	0	0	
		Quotient	–	6	–	–	–	–	

Solution is optimal and infeasible as net evaluation row elements are negative and slack variables are negative.

Table: III. $x = 11/2$, $y = 6$, $S_1 = -1/2$, $S_2 = 0$, $S_3 = 6$, $S_4 = 0$, $Z = -$ Rs.206/–

Problem variable	*Profit Rs.*	C_j *Capacity units*	*– 20* *x*	*– 16* *y*	*0* S_1	*0* S_2	*0* S_3	*0* S_4	*Replacement ratio*
S_1	0	– 1/2	0	0	**1**	– 1/2	0	– 1/2	Negative
x	– 20	11/2	1	0	0	– 1/2	0	1/2	
S_3	0	6	0	0	0	– 1	1	1	
y	– 16	6	0	1	0	0	0	– 1	
		Net evaluation	0	0	0	– 10	0	– 6	
		Quotient				20		12	

Table: IV. $X = 5$, $y = 7$, and $Z = -$ Rs. 212/–

Problem variable	*Profit Rs.*	C_j *Capacity units*	*– 20* *x*	*– 16* *y*	*0* S_1	*0* S_2	*0* S_3	*0* S_4	*Replacement ratio*
S_4	0	1	0	0	– 2	1	0	1	
x	– 20	5	1	0	1	– 1	0	0	
S_3	0	5	0	0	2	– 2	1	0	
y	– 16	7	0	1	– 2	1	0	0	
		Net evaluation	0	0	– 12	– 4	0	0	
		Quotient							

As all the elements of net evaluation row are either negative or zeros the solution is optimal and as all the slack variables and basic variables have positive values the solution is feasible.

$x = 5$, $y = 7$, $S_1 = 0$, $S_2 = 0$. $S_3 = 5$, $S_4 = 1$ and $Z = -$ Rs. 212/– *i.e* minimum optimal is Rs.212/

Problem 3.41:

Minimize $Z = 10a + 6b + 2c$ s.t.

$-1a + 1b + 1c \geq 1$

$3a + 1b - 1c \geq 2$

And all $a, b, c, \geq 0$.

The problem can be written as:

Maximize $Z = -10a - 6b - 2c + 0S_1 + 0S_2$

$1a - 1b - 1c + 1S_1 + 0S_2 = -1$

$-3a - 1b + 1c + 0S_1 + 1S_2 = -2$

And a, b, c, S_1, S_2 all ≥ 0

Solution:

Table: I. $a = 0$, $b = 0$, $X = 0$, $S_1 = -1$, $S_2 = -2$, $Z =$ Rs. 0/–

Problem variable	*Profit Rs.*	C_j *Capacity units*	*– 10* *a*	*– 6* *b*	*– 2* *c*	*0* S_1	*0* S_2	*Replacement ratio*
S_1	0	–1	1	–1	–1	1	0	–1
S_2	0	–2	–3	–1	1	0	1	2/3
		Net evaluation	–10	–6	–2	0	0	
		Quotient	10/3	6				

Net evaluation row elements are negative hence solution is optimal but slack variables are negative, hence the solution is infeasible. Variable '*a*' has lowest quotient hence incoming variable and S_2 has got highest element with negative sign, it is out going variable and the row having S_2 is the key row.

Table: II. $a = 2/3$, $b = 0$, $z = 0$, $S_1 = -5/3$, $S_2 = 0$, $Z = -$ Rs. 20/3.

Problem variable	*Profit Rs.*	C_j *Capacity units*	*– 10* *a*	*– 6* *b*	*– 2* *c*	*0* S_1	*0* S_2	*Replacement ratio*
S_1	0	–5/3	0	–4/3	–2/3	1	1/3	Negative
a	–10	2/3	1	1/3	–1/3	0	–1/3	
		Net evaluation	0	–8/3	–16/3	0	–10/3	
		Quotient		2	8			

Variable '*b*' has lowest positive quotient, it is incoming variable, S_1 has highest number with negative sign, it is the out going variable. The solution is infeasible optimal.

Table: III. $a = 1/4$, $b = 5/4$, $c = 0$, $S_1 = 0$, $S_2 = 0$, $Z = -$ Rs. 10/–

Problem Variable	*Profit Rs.*	C_j *Capacity units*	*– 10* *a*	*– 6* *b*	*– 2* *c*	*0* S_1	*0* S_2	*Replacement ratio*
b	– 6	5/4	0	1	1/2	–3/4	–1/4	
a	– 10	1/4	1	0	–1/2	1/4	–1/4	
		Net evaluation	0	0	–4	–2	–4	

The solution is optimal and feasible. $a = ¼$, $b = 5/4$, $c = 0$ and $Z = -$ Rs. 10/–. Hence the minimum cost is Rs. 10/–

Addition of a New Constraint

Whenever a linear programming problem is formulated, the constraints, which are considered important and significant, are considered and the problem is solved to find the optimal solution. After obtaining the optimal solution for the problem, the optimal solution is checked to see whether it satisfies the remaining constraints of secondary importance. This approach reduces the size of the problem to be handled in the first instance and reduces the calculation part. We may come across a situation that at a later stage, newly identified significant constraints have to be introduced into the problem. The situation in either case amounts addition of one or several constraints. With the values of optimal basic variables it is first checked whether they satisfy the new constraint (s). If so, the solution remains optimal. If not, the constraints are introduced in the optimal tableau, and by elimination of coefficients of basic variables in the new constraints are reduced to zero. With that, if solution is feasible, and non–optimal, regular simplex method is used for optimization. On the other hand, if solution is infeasible but optimal, dual simplex method is used for optimization. New constraint, which is not satisfied by the previous optimal solution, is called **tighter constraint**, because it changes the solution.

Problem 3.42: (Repetition of problem 3.39)

Minimize $Z = 2a + 1b$ s.t.

$3a + 1b \geq 3$

$4a + 3b \geq 6$

$1a + 2b \leq 3$ and both a and b are ≥ 0.

Optima solution obtained by dual simplex method is: $a = 3/5$ and $b = 6/5$.

Let us suppose that new constraint added is $5a + 5b \geq 10$. The simplex version of this inequality is:

$5a + 5b - S_4 = 10$.

Let us substitute the values of 'a' and 'b' in the above.

$5 \times 3/5 + 5 \times 6/5 - S_4 = 10$. This gives a value of –1 to S_4, which violates the non-negativity constraint and hence the solution is not optimal. Hence by introducing the new constraint in the final table, we have to get a new optimal solution.

Table: I.

Problem variable	*Profit Rs.*	*C_j Capacity units*	*– 2 a*	*– 1 b*	*0 S_1*	*0 S_2*	*0 S_3*	*0 S_4*	*Replacement ratio*
a	–2	3/5	1	0	–3/5	1/5	0	0	Row 1
b	–1	6/5	0	1	4/5	–3/5	0	0	Row 2
S_3	0	0	0	0	–1	1	1	0	Row 3
S_4	0	–10	–5	–5	0	0	0	1	Row 4
		Net evaluation	0	0	–2/5	–1/5	0	10	

Multiplying the row 1 by 5 and adding it to row 4 and multiplying row 2 by 5 and adding it to row 4, we get

Table: II.

Problem vriable	*Profit Rs.*	C_j *Capacity units*	*– 2* *a*	*– 1* *b*	*0* S_1	*0* S_2	*0* S_3	*0* S_4	*Replacement ratio*
a	–2	3/5	1	0	–3/5	1/5	0	0	
b	–1	6/5	0	1	4/5	–3/5	0	0	
S_3	0	0	0	0	–1	1	1	0	
S_4	0	–1	0	0	1	**–2**	0	1	
		Net evaluation	0	0	– 2/5	– 1/5	0	0	
		Quotient				1/10			

a = 3/5, b = 6/5, S_3 = 0, S_4 = –1, Hence the solution is infeasible. As the net evaluation row elements are negative or zeros the solution is optimal. Hence by using dual simplex method, we get S_3 as incoming variable and S_4 as the out going variable.

Table: III.

Problem variable	*Profit Rs.*	C_j *Capacity units*	*– 2* *a*	*– 1* *b*	*0* S_1	*0* S_2	*0* S_3	*0* S_4	*Replacement ratio*
a	–2	1/2	1	0	–1/2	0	0	1/10	
b	–1	3/2	0	1	1/2	0	0	–3/10	
S_3	0	–1/2	0	0	**–1/2**	0	1	1/2	Negative
S_2	0	1/2	0	0	–1/2	1	0	–1/2	
		Net evaluation	0	0	–1/2	0	0	1/10	
		Quotient			1				

Solution is infeasible and optimal. S_1 is the incoming variable and S_3 is the outgoing variable.

Table: IV.

Problem variable	*Profit Rs.*	C_j *Capacity units*	*– 2* *a*	*– 1* *b*	*0* S_1	*0* S_2	*0* S_3	*0* S_4	*Replacement ratio*
a	–2	1	1	0	0	–1	0	–2/5	
b	–1	1	0	1	0	1	1	1/5	
S_1	0	1	0	0	1	0	–2	–1	
S_2	0	1	0	0	0	1	–1	–1	
		Net evaluation	0	0	0	–1	0	–3/5	

As the net evaluation row elements are either negatives or zeros the solution is optimal. As all variables have positive values the solution is feasible.

$a = 1$, $b = 1$, and Z min is Rs. 3/–. Here the basic variables remain same but the optimal value of cost is changed.

Problem 3.42: (Extension of problem 3.40)

Add a new constraint $4x + 3y \geq 40$ to problem 3.40 and examine whether basic variables change and if so what are the new values of basic variables?

The optimal solution obtained is $x = 5$, $y = 7$ and Z = Rs. 212/–

Substituting the values in the new constraint, $4 \times 5 + 3 \times 7 = 41$ which is ≥ 40. Hence the condition given in the new constraint is satisfied. Therefore it is not **tighter constraint. This will not have any effect on the present basis.** Let us verify the same.

The simplex version of the new constraint is: $-4x - 3y + 1S_5 = 40$. The optimal table is:

Table: I.

Problem variable	*Profit Rs.*	C_j *Capacity units*	*– 20* x	*– 16* y	*0* S_1	*0* S_2	*0* S_3	*0* S_4	*0* S_5	*Replace–ment ratio*
S_4	0	1	0	0	–2	1	0	1	0	Row 1
x	–20	5	1	0	1	–1	0	0	0	Row 2
S_3	0	5	0	0	2	–2	1	0	0	Row 3
y	–16	7	0	1	–2	1	0	0	0	Row 4
S_5	0	–40	–4	–3	0	0	0	0	1	Row 5
		Net evaluation	0	0	–12	–4	0	0	0	
		Quotient								

Now to convert matrix coefficients of basic variables in row 5 to zero, multiply row 4 by 3 and adding it to row 5 and multiplying row 2 by 4 and adding it to row 5, we get:

Table: II.

Problem variable	*Profit Rs.*	C_j *Capacity units*	*– 20* x	*– 16* y	*0* S_1	*0* S_2	*0* S_3	*0* S_4	*0* S_5	*Replace–ment ratio*
S_4	0	1	0	0	–2	1	0	1	0	
x	–20	5	1	0	1	–1	0	0	0	
S_3	0	5	0	0	2	–2	1	0	0	
y	–16	7	0	1	–2	1	0	0	0	
S_5	0	1	0	0	–2	–1	0	0	1	
		Net evaluation	0	0	–12	–4	0	0	0	

$x = 5$, $y = 7$ and minimum Z = Rs. 212/– As the net evaluation row is negative or zeros, and all problem variables have positive value, the solution is feasible and optimal. Values of basic variables have not changed.

Problem 3.43: (Repetition of Problem 3.41).

Add the following two new constraints to problem No. 3.41 and find the optimal solution.

Minimize $Z = 10a + 6b + 2c$ s.t

$-1a + 1b + 1c \geq 1$

$3a + 1b - 1c \geq 2$

New constraints are:

$4a + 2b + 3c \leq 5$

$8a - 1b + 1c \geq 5$

The simplex format of new constraints is:

$4a + 2b + 3c + 1S_3 = 5$

$8a - 1b + 1c - 1S_4 = 4$

Earlier values of $a = 1/4$ and $b = 5/4$ and $z = 0$

$4 \times ¼ + 2 \times 5/4 + 3 \times 0 + 1S_3 = 5$, gives $S_3 = 3/2$, which is feasible.

$8 \times ¼ - 5/4 \times 1 + 0 - S_4 = 4$ gives $S_4 = -13/4$ is not feasible. Hence the earlier basic solution is infeasible.

We have seen that the first additional constraint has not influenced the solution and only second additional constraint will influence the solution, by introducing both the constraints in optimal table we get:

Table: I.

Problem variable	*Profit Rs.*	C_j *Capacity units*	*– 16* *a*	*– 6* *b*	*– 2* *c*	*0* S_1	*0* S_2	*0* S_3	*0* S_4	*Replace–ment ratio*
b	–6	5/4	0	1	½	–3/4	–1/4	0	0	Row 1
a	–16	1/4	1	0	–1/2	1/4	–1/4	0	0	Row 2
S_3	0	5	4	2	3	0	0	1	0	Row 3
S_4	0	–4	–8	1	–1	0	0	0	1	Row 4
		Net evaluation								

Multiply row 1 by 2 and subtract it from row 3.

Subtract row 1 from row 4.

Multiply row 2 by 4 and subtract from row 3.

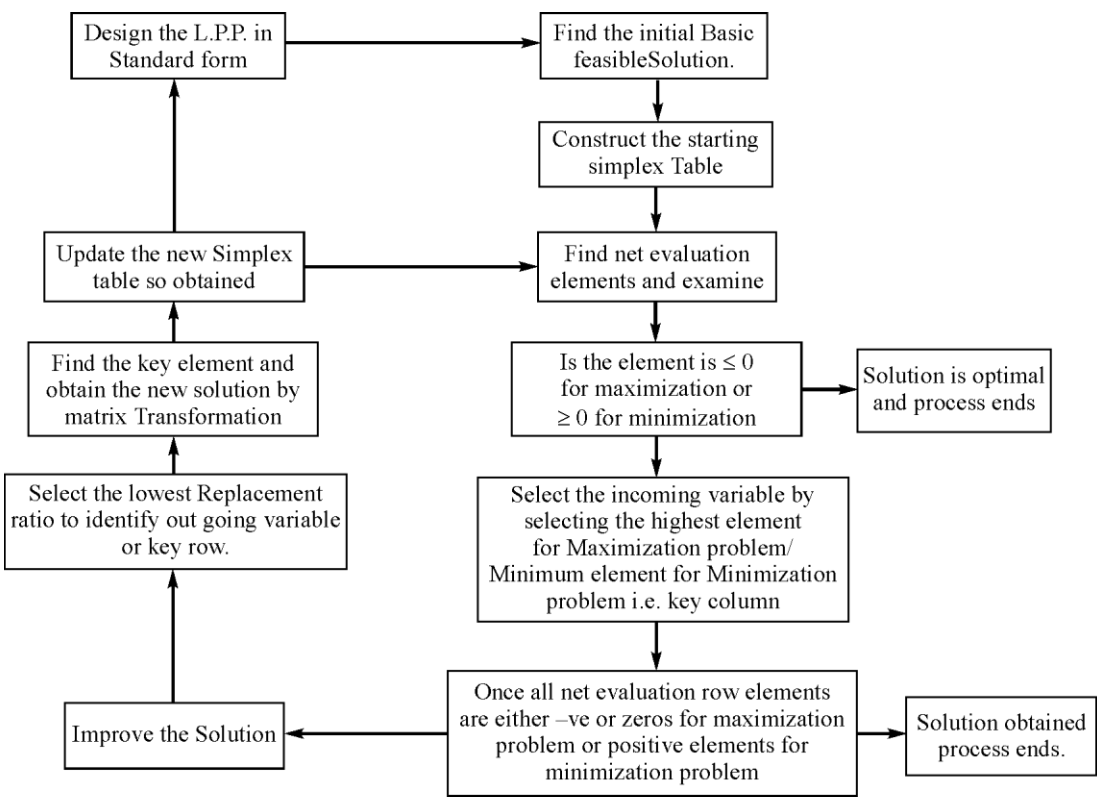
Multiply row 2 by 8 and add it to row 4

Table II.

Problem variable	*Profit Rs.*	C_j *Capacity units*	*– 10* *a*	*– 6* *b*	*– 2* *c*	*0* S_1	*0* S_2	*0* S_3	*0* S_4	*Replace– ment ratio*
b	–6	5/4	0	1	1/2	–3/4	–1/4	0	0	
a	–10	1/4	1	0	–1/2	1/4	–1/4	0	0	
S_3	0	3/2	0	0	4	1/2	3/2	1	0	
S_4	0	–13/4	0	0	**–11/2**	11/4	–7/4	0	1	
		Net evaluation	0	0	–4	–2	–4	0	0	
		Quotient			8/11		16/7			

As S_4 = – 13/4, the basic solution is not feasible. As net evaluation row elements are either negative or zeros the solution is optimal. Students can see that by using dual simplex method, variable '*c*' will enter the solution and S_4 will leave the solution.

FLOW CHART FOR SIMPLEX METHOD



QUESTIONS

1. (*a*) "Operations Research is a bunch of Mathematical Techniques" Comment.
   (b) Explain the steps involved in the solution of an Operations Research problem.
   (c) Give a brief account of various types of Operations Research models and indicate their application to Production – inventory – distribution system.
2. A company makes four products, *v*, *x*, *y* and *z* which flow through four departments– drilling, milling and turning and assembly. The variable time per unit of various products are given below in hours.

Products	*Drilling*	*Milling*	*Turning*	*Assembly*
v	3	0	3	4
x	7	2	4	6
y	4	4	0	5
z	0	6	5	3

The unit contributions of the four products and hours of availability in the four departments are as under:

Product	*Contribution in Rs.*		*Department*	*Hours available.*
v	9		Drilling	70
x	18		Milling	80
y	14		Turning	90
z	11		Assembly	100

   (a) Formulate a Linear Programme for Maximizing the Contribution.
   (b) Give first two iterations of the solution by Simplex method.
3. Metal fabricators limited manufactures 3 plates in different sizes, *A*, *B*, and C through Casting, Grinding and Polishing processes for which processing time in minutes per unit are given below:

Processing time/unit in minutes.

Sizes	*Casting*	*Grinding*	*Polishing*
A	5	5	10
B	8	7	12
C	10	12	16
Available Hours per Day.	16	2 4	32

If the contribution margins are Rs.1/–, 2/– and 3/– for *A*, *B* and *C* respectively, find contribution maximizing product mix.

4. A manufacturer can produce three different products *A*, *B*, and *C* during a given time period. Each of these products requires four different manufacturing operations: Grinding, Turning, Assembly and Testing. The manufacturing requirements in hours per unit of the product are given below for *A*, *B*, and *C*:

	A	B	C
Grinding	1	2	1
Turning	3	1	4
Assembly	6	3	4
Testing	5	4	6

The available capacities of these operations in hours for the given time period are as follows: Grinding 30 hours, Turning: 60 hours, Assembly: 200 hours and Testing: 200 hours.

The contribution of overheads and profit is Rs.4/– for each unit of *A*, Rs.6/– for each unit of *B* and Rs.5/– for each unit of *C*. The firm can sell all that it produces. Determine the optimum amount of *A*, *B*, and *C* to produce during the given time period for maximizing the returns.

5. (*a*) Briefly trace the major developments in Operations Research since World War II.
   (b) Enumerate and explain the steps involved in building up various types of mathematical models for decision making in business and industry.
   (c) State and explain the important assumptions in formulating a Linear Programming Model.

6. An oil refinery wishes its product to have at least minimum amount of 3 components: 10% of *A*, 20% of *B* and 12% of *C*. It has available three different grades of crude oil: *x*, *y*, and *z*.
   Grade *x* contains: 15% of *A*, 10% of *B* and 9% of C and costs Rs. 200/– per barrel.
   Grade *y* contains: 18% of *A*, 25% of *B* and 3% of C and costs Rs. 250/– per barrel.
   Grade *z* contains 10% of *A*, 15% of *B* and 30% of C and costs Rs. 180/– per barrel.
   Formulate the linear programming for least cost mix and obtain the initial feasible solution.

7. (*a*) Define Operations Research.
   (b) List the basic steps involved in an Operations Research study.
   (c) List the areas in which Operations Research Techniques can be employed.
   (d) Give two examples to show how Work Study and Operations Research are complementary to each other.

8. You wish to export three products *A*, *B*, and *C*. The amount available is Rs. 4,00,000/–. Product *A* costs Rs. 8000/– per unit and occupies after packing 30 cubic meters. Product *B* costs Rs. 13,000/– per unit and occupies after packing 60 cubic meters and product C costs Rs. 15,000/–per unit and occupies 60 cubic meters after packing. The profit per unit of *A* is Rs. 1000/–, of B is Rs. 1500/– and of C is Rs. 2000/–.

   The shipping company can accept a maximum of 30 packages and has storage space of 1500 cubic meters. How many of each product should be bought and shipped to maximize profit? The export potential for each product is unlimited. Show that this problem has two basic optimum solutions and find them. Which of the two solutions do you prefer? Give reasons.

9. A fashion company manufactures four models of shirts. Each shirt is first cut on cutting process in the trimming shop and next sent to the finishing shop where it is stitched, button holed and packed. The number of man–hours of labour required in each shop per hundred shirts is as follows:

Shop	*Shirt A*	*Shirt B*	*Shirt C*	*Shirt D*
Trimming shop	1	1	3	40
Finishing shop	4	9	7	10

Because of limitations in capacity of the plant, no more than 400 man–hours of capacity is expected in Trimming shop and 6000 man – hours in the Finishing shop in the next six months. The contribution from sales for each shirt is as given below: Shirt. *A*: Rs. 12 /– per shirt, Shirt *B*: Rs.20 per shirt, Shirt *C*: Rs. 18/– per shirt and Shirt *D*: Rs. 40/– per shirt. Assuming that there is no shortage of raw material and market, determine the optimal product mix.

10. A company is interested in manufacturing of two products *A* and *B*. *A* single unit of Product *A* requires 2.4 minutes of punch press time and 5 minutes of assembly time. The profit for product *A* is Rs. 6/– per unit. A single unit of product *B* requires 3 minutes of punch press time and 2.5 minutes of welding time. The profit per unit of product *B* is Rs. 7/–. The capacity of punch press department available for these products is 1,200 minutes per week. The welding department has idle capacity of 600 minutes per week; the assembly department can supply 1500 minutes of capacity per week. Determine the quantity of product *A* and the quantity of product *B* to be produced to that the total profit will be maximized.

11. A manufacturing firm has discontinued production of a certain unprofitable product line. This created considerable excess production capacity. Management is considering devote this excess capacity to one or more of three products *X*. *Y* and *Z*. The available capacity on the machines, which might limit output, is given below:

Machine type	*Available time in machine hours per week.*
Milling machine	200
Lathe	100
Grinder	50

The number of machine hours required for each unit of the respective product is as follows.

Machine type.	*Productivity (in machine hours per unit)*		
	Product X	*Product Y*	*Product Z*
Milling machine.	8	2	3
Lathe	4	3	0
Grinder	2	0	1

The sales department indicates that the sales potential for products X and Y exceeds the maximum production rate and that of sales potential for product Z is 20 units per week. The unit profit would be Rs. 20/–, Rs.6/– and Rs.8/– respectively for products X, Y and Z. Formulate a linear programming model and determine how much of each product the firm should produce in order to maximize profit.

12. A jobbing firm has two workshops and the centralized planning department is faced with the problem of allocating the two sets of machines, in the workshops, to meet the sales demand. The sales department has committed to supply 80 units of product and 100 units of product Q and can sell any amount of product R. Product Q requires special selling force and hence sales department does not want to increase the sale of this product beyond the commitment. Cost and selling price details as well as the machine availability details are given in the following tables:

Product	*Sellig price Rs. per unit*	*Raw material Cost in Rs. per unit*	*Labour cost In Rs. per unit*	*Labour costs In Rs. per unit.*
			Work shop I	*Work shop II*
P	25	5	12	14
Q	32	8	17	19
R	35	10	23	24

Machine	*Hours per unit*						*Available hours.*	
	Workshop I			*Work shop II*			*Workshop I*	*Workshop II.*
	P	*Q*	*R*	*P*	*Q*	*R*		
I	2	1	3	2	1.5	3	250	300
II	1	2	3	1.5	3	3.5	150	175

(a) What is the contribution in Rs. per unit for each of the products when made in workshop II and I?

(b) Formulate a linear programming model.

(c) Write the first simplex tableau (Need not solve for optimality).

13. A manufacturer manufactures three products P, Q and R, using three resources A, B and C. The following table gives the amount of resources required per unit of each product, the availability of resource during a production period and the profit contribution per unit of each product.

(a) The object is to find what product–mix gives maximum profit. Formulate the mathematical model of the problem and write down the initial table.

(b) In the final solution it is found that resources A and C are completely consumed, a certain amount of B is left unutilized and that no R is produced. Find how much of X and Y are to be produced and that the amount of B left unutilized and the total profits.

(c) Write the dual of the problem and give the answers of dual from primal solution.

Products. → *Resources* ↓	*P*	*Q*	*R*	*Availability in units*
A	3	2	7	1000
B	3	5	6	2500
C	2	4	2	1600
Profit per Unit in Rs.	8	9	10	

14. The mathematical model of a linear programming problem, after introducing the slack variables is:

Maximize $Z = 50a + 60b + 120c + 0S_1 + 0S_2$ s.t.

$2a + 4b + 6c + S_1 = 160$

$3a + 2b + 4c + S_2 = 120$ and a, b, c, S_1 and S_2 all ≥ 0.

In solving the problem by using simplex method the last but one table obtained is given below:

50	*60*	*120*	*0*	*0*	*Capacity*
a	*b*	*c*	S_1	S_2	
1/3	2/3	1	1/6	0	80/3
5/3	–2/3	0	–2/3	1	40/3

(a) Complete the above table.

(b) Complete the solution y one more iteration and obtain the values of *A*, *B*, and *C* and the optimal profit *Z*.

(c) Write the dual of the above problem.

(d) Give the solution of the dual problem by using the entries obtained in the final table of the primal problem.

(e) Formulate the statement of problem from the data available.

15. The India Fertilizer company manufactures 2 brands of fertilizers, Sulpha–*X* and Super–Nitro. The Sulpher, Nitrate and Potash contents (in percentages) of these brands are 10–5–10 and 5–10–10 respectively. The rest of the content is an inert matter, which is available in abundance. The company has made available, during a given period, 1050 tons of Sulpher, 1500 tons of Nitrates, and 2000 tons of Potash respectively. The company can make a profit of Rs. 200/– per tone on Sulpha – X and Rs. 300/– per ton of Super– Nitro. If the object is to maximise the total profit how much of each brand should be procured during the given period?

(a) Formulate the above problem as a linear programming problem and carry out the first iteration.

(b) Write the dual of the above problem.

16. Solve:

Minimize $S = 1a - 3b + 2c$ S.t

$3a - 1b + 3c \leq 7$

$-2a + 4b + 0c \leq 12$

$-4a + 3b + 8c \leq 10$ and $a, b, c,$ all ≥ 0.

17. Solve:

Maximize $Z = 3x + 2y + 5z$ s.t.

$1x + 2y + 1z \leq 430$

$3x + 0y + 2z \leq 460$

$1x + 4y + 0z \leq 420$ and x, y, z all ≥ 0.

18. A manufacturer of three products tries to follow a polity producing those, which contribute most to fixed costs and profit. However, there is also a policy of recognizing certain minimum sales requirements currently, these are: Product X = 20 units per week, Product Y 30 units per week, and Product Z 60 units per week. There are three producing departments. The time consumed by products in hour per unit in each department and the total time available for each week in each department are as follows:

Time required per unit in hours.

Departments	*X*	*Y*	*Z*	*Total Hours Available*
1	0.25	0.20	0.15	420
2	0.30	0.40	0.50	1048
3	0.25	0.30	0.25	529

The contribution per unit of product X, Y, and Z is Rs. 10.50, RS. 9.00 and Rs. 8.00 respectively. The company has scheduled 20 units X, 30 units of Y and 60 units of Z for production in the following week, you are required to state:

(a) Whether the present schedule is an optimum from a profit point of view and if it is not, what it should be?

(b) The recommendations that should be made to the firm about their production facilities (from the answer of (a) above).

19. Minimize $Z = 1a - 2b - 3c$ s.t.

$-2a + 1b + 3c = 2$

$2a + 3b + 4c = 1$ and all a, b, and c are ≥ 0.

(*b*) Write the dual of the above and give the answer of dual from the answer of the primal.

20. Minimize $Z = 2x + 9y + 1z$ s.t
$1x + 4y + 2z \geq 5$
$3x + 1y + 2z \geq 4$ and x, y, z all are ≥ 0, Solve for optimal solution.

21. Minimize $Z = 3a + 2b + 1c$ s.t.
$2a + 5b + 1c = 12$
$3a + 4b + 0c = 11$ and a is unrestricted and b and c are ≥ 0, solve for optimal values of a, b and c.

22. Max $Z = 22x + 30y + 25z$ s.t
$2x + 2y + 0z \leq 100$
$2x + 1y + 1z \leq 100$
$1x + 2y + 2z \leq 100$ and x, y, z all ≥ 0 Find the optimal solution.

23. Obtain the dual of the following linear programming problem.
Maximize $Z = 2x + 5y + 6z$ s.t.
$5x + 6y - 1z \leq 3$
$-1x + 1y + 3z \geq 4$
$7x - 2y - 1x \leq 10$
$1x - 2y + 5z \geq 3$
$4x + 7y - 2z = 2$ and x, y, z all ≥ 0

24. Use dual simplex method for solving the given problem.
Maximize $Z = 2a - 2b - 4c$ s.t
$2a + 3b + 5c \geq 2$
$3a + 1y + 7z \leq 3$
$1a + 4b + 6c \leq 5$ and a, b, c all ≥ 0

25. Find the optimum solution to the problem given:
Maximize $Z = 15x + 45y$ s.t.
$1x + 16y \leq 240$
$5x + 2y \leq 162$
$0x + 1y \leq 50$ and both x and $y \geq 0$
If Z_{max} and c_2 is kept constant at 45, find how much c_1 can be changed without affecting the optimal solution.

26. Maximize $Z = 3a + 5b + 4c$ s.t.
$2a + 3b + 0c \leq 8$
$0a + 2b + 5c \leq 10$
$3a + 2b + 4c < 15$ and a, b, c all ≥ 0
Find the optimal solution and find the range over which resource No.2 (*i.e.*, b_2) can be changed maintaining the feasibility of the solution.

27. Define and explain the significance of Slack variable, Surplus variable, Artificial variable in linear programming resource allocation model.

28. Explain how a linear programming problem can be solved by graphical method and give limitations of graphical method.
29. Explain the procedure followed in simplex method of solving linear programming problem.
30. Explain the terms:
    (a) Shadow price,
    (b) Opportunity cost,
    (c) Key column,
    (d) Key row
    (e) Key number and
    (*f*) Limiting ratio.

***CHAPTER* – 4**

Linear Programming: II
Transportation Model

INTRODUCTION

In operations Research Linear programming is one of the model in mathematical programming, which is very broad and vast. Mathematical programming includes many more optimization models known as Non - linear Programming, Stochastic programming, Integer Programming and Dynamic Programming - each one of them is an efficient optimization technique to solve the problem with a specific structure, which depends on the assumptions made in formulating the model. We can remember that the general linear programming model is based on the assumptions:

(a) Certainty

The resources available and the requirement of resources by competing candidates, the profit coefficients of each variable are assumed to remain unchanged and they are certain in nature.

(b) Linearity

The objective function and structural constraints are assumed to be linear.

(c) Divisibility

All variables are assumed to be continuous; hence they can assume integer or fractional values.

(d) Single stage

The model is static and constrained to one decision only. And planning period is assumed to be fixed.

(e) Non-negativity

A non-negativity constraint exists in the problem, so that the values of all variables are to be ≥ 0, *i.e*. the lower limit is zero and the upper limit may be any positive number.

(f) Fixed technology

Production requirements are assumed to be fixed during the planning period.

(g) Constant profit or cost per unit

Regardless of the production schedules profit or cost remain constant.

Now let us examine the applicability of linear programming model for **transportation** and **assignment models**.

TRANSPORTATION MODEL

The transportation model deals with a special class of linear programming problem in which the objective is to **transport a homogeneous commodity from various origins or factories to different destinations or markets at a total minimum cost.**

To understand the problem more clearly, let us take an example and discuss the rationale of transportation problem. Three factories *A*, *B* and *C* manufactures sugar and are located in different regions. Factory *A* manufactures, b_1 tons of sugar per year and *B* manufactures b_2 tons of sugar per year and *C* manufactures b_3 tons of sugar. The sugar is required by four markets *W*, *X*, *Y* and *Z*. The requirement of the four markets is as follows: Demand for sugar in Markets *W*, *X*, Yand *Z* is d_1, d_2, d_3 and d_4 tons respectively. The transportation cost of one ton of sugar from each factory to market is given in the matrix below. The objective is to transport sugar from factories to the markets at a minimum total transportation cost.

	Markets	*Transportation cost per ton in Rs.*				*Availability in tons*
		W	*X*	*Y*	*Z*	
Factories	*A*	c_{11}	c_{12}	c_{13}	c_{14}	b_1
	B	c_{21}	c_{22}	c_{23}	c_{24}	b_2
	C	c_{31}	c_{32}	c_{33}	c_{34}	b_3
Demand in Tons.		d_1	d_2	d_3	d_4	$\Sigma b_j / \Sigma d_j$

For the data given above, the mathematical model will be:

Minimize $Z = c_{11} x_{11} + c_{12} x_{12} + c_{13} x_{13} + c_{14} x_{14} + c_{21} x_{21} + c_{22} x_{22} + c_{23} x_{23} + c_{24} x_{24} + c_{31} x_{31} + c_{32} x_{32} + c_{33} x_{33} + c_{34} x_{34}$ subject to a condition: ⟶ **OBJECTIVE FUNCTION.**

$a_{11} x_{11} + a_{12} x_{12} + a_{13} x_{13} + a_{14} x_{14} \leq b_1$ (because the sum must be less than or equal to the available capacity)

$a_{21} x_{21} + a_{22} x_{22} + a_{23} x_{23} + a_{24} x_{24} \leq b_2$

$a_{31} x_{31} + a_{32} x_{32} + a_{33} x_{33} + a_{34} x_{34} \leq b_3$ ⟶ **MIXED STRUCTURAL CONSTRAINTS.**

$a_{11} x_{11} + a_{21} x_{21} + a_{31} x_{31} \geq d_1$

$a_{12} x_{12} + a_{22} x_{22} + a_{32} x_{32} \geq d_2$ (because the sum must be greater than or equal to the demand of the market. We cannot send less than what is required)

$a_{13} x_{13} + a_{23} x_{23} + a_{33} x_{33} \geq d_3$

$a_{14} x_{14} + a_{24} x_{24} + a_{34} x_{34} \geq d_4$ and

All x_{ij} and x_{ji} are ≥ 0 where i = 1,2,3 and j = 1,2,3,4. (This is because we cannot supply negative elements). ⟶ **NON-NEGATIVITY CONSTRAINT.**

The above problem has got the following properties:

1. It has an objective function.
2. It has structural constraints.
3. It has a non-negativity constraint.
4. The relationship between the variables and the constraints are linear.

We know very well that these are the properties of **a linear programming problem**. Hence the transportation model is also a **linear programming problem**. But a **special type of linear programming problem.**

Once we say that the problem has got the characteristics of linear programming model, and then we can solve it by simplex method. Hence we can solve the transportation problem by using the simplex method. As we see in the above given transportation model, the structural constraints are of mixed type. That is some of them are of ≤ type and some of them are of ≥ type. When we start solving the transportation problem by simplex method, it takes more time and laborious. Hence we use **transportation algorithm or transportation method** to solve the problem. Before we discuss the transportation algorithm, let us see how a general model for transportation problem appears. The general problem will have '*m*' rows and '*n*' columns *i.e.*, $m \times n$ matrix.

$$\text{Minimize } Z = \sum_{j=1}^{n}\sum_{i=1}^{m} c_{ij}\, x_j \quad \text{s.t. where } i = 1 \text{ to } m \text{ and } j = 1 \text{ to } n.$$

$$\sum_{i=1}^{m} a_{ij}\, x_{ij} \le b_i \quad \text{where } i = 1 \text{ to } m \text{ and } j = 1 \text{ to } n$$

$$\sum_{j=1}^{n} a_{ij}\, x_{ji} \ge d_j \quad \text{where } i = 1 \text{ to } m \text{ and } j = 1 \text{ to } n$$

COMPARISON BETWEEN TRANSPORTATION MODEL AND GENERAL LINEAR PROGRAMMING MODEL

Similarities

1. Both have objective function.
2. Both have linear objective function.
3. Both have non - negativity constraints.
4. Both can be solved by simplex method. In transportation model it is laborious.
5. A general linear programming problem can be reduced to a transportation problem if (*a*) the a_{ij}'s (coefficients of the structural variables in the constraints) are restricted to the values 0 and/or 1 and (*b*) There exists homogeneity of units among the constraints.

Differences

1. Transportation model is basically a minimization model; where as general linear programming model may be of maximization type or minimization type.

2. The resources, for which, the structural constraints are built up is homogeneous in transportation model; where as in general linear programming model they are different. That is one of the constraint may relate to machine hours and next one may relate to man-hours etc. In transportation problem, all the constraints are related to one particular resource or commodity, which is manufactured by the factories and demanded by the market points.
3. The transportation problem is solved by transportation algorithm; where as the general linear programming problem is solved by simplex method.
4. The values of structural coefficients (*i.e.* x_{ij}) are not restricted to any value in general linear programming model, where as it is restricted to values either 0 or 1 in transportation problem. Say for example:

   Let one of the constraints in general linear programming model is: $2x - 3y + 10z \leq 20$. Here the coefficients of structural variables x, y and z may negative numbers or positive numbers of zeros. Where as in transportation model, say for example $x_{11} + x_{12} + x_{13} + x_{14} = b_i = 20$. Suppose the value of variables x_{11}, and x_{14} are 10 each, then $10 + 0.\ x_{12} + 0.\ x_{13} + 10 = 20$. Hence the coefficients of x_{11} and x_{14} are 1 and that of x_{12} and x_{13} are zero.

APPROACH TO SOLUTION TO A TRANSPORTATION PROBLEM BY USING TRANSPORTATION ALGORITHM

The steps used in getting a solution to a transportation problem is given below:

Initial Basic Feasible Solution

Step 1. Balancing the given problem. Balancing means check whether sum of availability constraints must be equals to sum of requirement constraints. That is $\Sigma b_i = \Sigma d_j$. Once they are equal, go to step two. If not by opening a *Dummy row* or *Dummy column* balance the problem. The *cost coefficients of dummy cells are zero*. If Σb_i is greater than Σd_j, then open a dummy column, whose requirement constraint is equals to $\Sigma b_i - \Sigma d_j$ and the cost coefficient of the cells are zeros. In case if Σd_j is greater than Σb_i, then open a dummy row, whose availability constraint will be equals to $\Sigma d_j - \Sigma b_i$ and the cost coefficient of the cells are zeros. Once the balancing is over, then go to second step. Remember while solving general linear programming problem to convert an inequality into an equation, we add (for maximization problem) a slack variable. In transportation problem, the dummy row or dummy column, exactly similar to a slack variable.

Step II. A .Basic feasible solution can be obtained by three methods, they are

(a) North - west corner method.

(b) Least - cost cell method. (Or Inspection method Or Matrix minimum - row minimum - column minimum method)

(c) Vogel's Approximation Method, generally known as VAM.

After getting the basic feasible solution (*b.f.s.*) give **optimality test** to check whether the solution is optimal or not.

There are two methods of giving optimality test:

(a) Stepping Stone Method.

(b) Modified Distribution Method, generally known as **MODI** method.

Properties of a Basic feasible Solution

1. The allocation made must satisfy the rim requirements, *i.e.*, it must satisfy availability constraints and requirement constraints.
2. It should satisfy non negativity constraint.
3. Total number of allocations must be equal to $(m + n - 1)$, where 'm' is the number of rows and 'n' is the number of columns. Consider a value of $m = 4$ and $n = 3$, *i.e.* 4×3 matrix. This will have four constraints of $\leq$ type and three constraints of $\geq$ type. Totally it will have 4 + 3 (*i.e* $m + n$) inequalities. If we consider them as equations, for solution purpose, we will have 7 equations. In case, if we use simplex method to solve the problem, only six rather than seen structural constraints need to be specified. In view of the fact that the sum of the origin capacities (availability constraint) equals to the destination requirements (requirement constraint) *i.e.*, $\Sigma b_i = \Sigma d_j$, any solution satisfying six of the seven constraints will automatically satisfy the last constraint. In general, therefore, if there are 'm' rows and 'n' columns, in a given transportation problem, we can state the problem completely with $m + n - 1$ equations. This means that one of the rows of the simplex tableau represents a redundant constraint and, hence, can be deleted. This also means that a basic feasible solution of a transportation problem has only $m + n - 1$ positive components. If $\Sigma b_i = \Sigma d_j$, it is always possible to get a basic feasible solution by North-west corner method, Least Cost cell method or by VAM.

Basic Feasible Solution by North - West corner Method

Let us take a numerical example and discuss the process of getting basic feasible solution by various methods.

Example 4.1. Four factories, *A*, *B*, *C* and *D* produce sugar and the capacity of each factory is given below: Factory *A* produces 10 tons of sugar and *B* produces 8 tons of sugar, *C* produces 5 tons of sugar and that of *D* is 6 tons of sugar. The sugar has demand in three markets *X*, *Y* and *Z*. The demand of market *X* is 7 tons, that of market *Y* is 12 tons and the demand of market *Z* is 4 tons. The following matrix gives the transportation cost of 1 ton of sugar from each factory to the destinations. Find the Optimal Solution for least cost transportation cost.

Factories.	*Cost in Rs. per ton (× 100) Markets.*			*Availability in tons.*
	X	*Y*	*Z*	
A	4	3	2	10
B	5	6	1	8
C	6	4	3	5
D	3	5	4	6
Requirement in tons.	7	12	4	$\Sigma b = 29$, $\Sigma d = 23$

Here Σb is greater than Σd hence we have to open a dummy column whose requirement constraint is 6, so that total of availability will be equal to the total demand. Now let get the basic feasible solution by three different methods and see the advantages and disadvantages of these methods. After this let us give optimality test for the obtained basic feasible solutions.

a) North- west corner method

(i) Balance the problem. That is see whether $\Sigma b_i = \Sigma d_j$. If not open a dummy column or dummy row as the case may be and balance the problem.

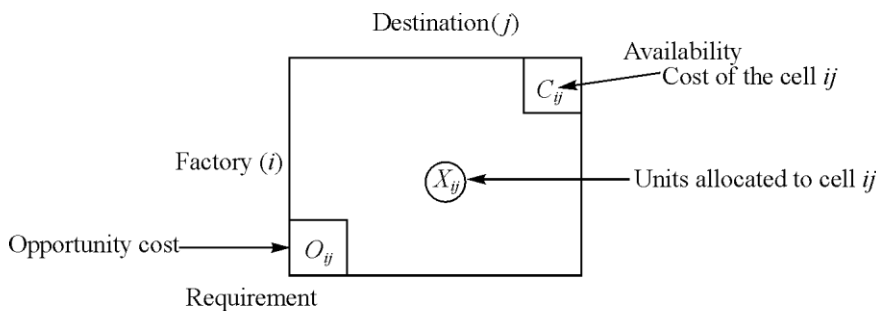
(ii) Start from the left hand side top corner or cell and make allocations depending on the availability and requirement constraint. If the availability constraint is less than the requirement constraint, then for that cell make allocation in units which is equal to the availability constraint. In general, verify which is the smallest among the availability and requirement and allocate the smallest one to the cell under question. Then proceed allocating either sidewise or downward to satisfy the rim requirement. Continue this until all the allocations are over.

(iii) Once all the allocations are over, *i.e.,* both rim requirement (column and row *i.e.*, availability and requirement constraints) are satisfied, write allocations and calculate the cost of transportation.

Solution by North-west corner method:

	X	Y	Z	Dummy	Availability
A	$\boxed{4}$ 7	$\boxed{3}$ 3	$\boxed{2}$	$\boxed{0}$	10
B	$\boxed{5}$	$\boxed{6}$ 8	$\boxed{1}$	$\boxed{0}$	8
C	$\boxed{6}$	$\boxed{4}$ 1	$\boxed{3}$ 4	$\boxed{0}$	5
D	$\boxed{3}$	$\boxed{5}$	$\boxed{4}$ 1	$\boxed{0}$ 5	6
Requirement.	7	12	5	5	29

For cell *AX* the availability constraint is 10 and the requirement constraint is 7. Hence 7 is smaller than 10, allocate 7 to cell *AX*. Next 10 – 7 = 3, this is allocated to cell *AY* to satisfy availability requirement. Proceed in the same way to complete the allocations. Then count the allocations, if it is equals to $m + n - 1$, then the solution is ***basic feasible solution***. The solution, we got have **7** allocations which is = 4 + 4 – 1 = 7. Hence the solution is basic feasible solution.



Now allocations are:

From	To	Units in tons	Cost in Rs.
A	*X*	7	7 × 4 = 28
A	*Y*	3	3 × 3 = 09
B	*Y*	8	8 × 6 = 48
C	*Y*	1	1 × 4 = 04
C	*Z*	4	4 × 3 = 12
D	*Z*	1	1 × 4 = 04
D	DUMMY	5	5 × 0 = 00
	Total in Rs.		105

Solution by Least cost cell (or inspection) Method: (Matrix Minimum method)

(*i*) Identify the lowest cost cell in the given matrix. In this particular example it is = 0. Four cells of dummy column are having zero. When more than one cell has the same cost, then both the cells are competing for allocation. This situation in transportation problem is known as **tie**. To break the tie, select any one cell of your choice for allocation. Make allocations to this cell either to satisfy availability constraint or requirement constraint. Once one of these is satisfied, then mark crosses (×) in all the cells in the row or column which ever has completely allocated. Next search for lowest cost cell. In the given problem it is cell BZ which is having cost of Re.1/- Make allocations for this cell in similar manner and mark crosses to the cells in row or column which has allocated completely. Proceed this way until all allocations are made. Then write allocations and find the cost of transportation. As the total number of allocations are **7** which is equals to 4 + 4 – 1 = 7, the solution is basic feasible solution.

	X	Y	Z	Dummy	Availability
A	× $\boxed{4}$	8 $\boxed{3}$	2 $\boxed{2}$	× $\boxed{0}$	10(8) (0) iv
B	× $\boxed{5}$	× $\boxed{6}$	3 $\boxed{1}$	5 $\boxed{0}$	8(3) (0) ii
C	1 $\boxed{6}$	4 $\boxed{4}$	× $\boxed{3}$	× $\boxed{0}$	5(4) (0) viii
D	6 $\boxed{3}$	× $\boxed{5}$	× $\boxed{4}$	× $\boxed{0}$	6 (0) v
Requirement.	7 (1) (0) vi	12 (4) (0) vii	5 (2) (0) iii	5 (0) i	29

(**Note:** The numbers under and side of rim requirements shows the sequence of allocation and the units remaining after allocation)

Allocations are:

From	*To*	*Units in tons*	*Cost in Rs.*
A	Y	8	8 × 3 = 24
A	*Z*	2	2 × 2 = 04
B	*Z*	3	3 × 1 = 03
B	DUMMY	5	5 × 0 = 00
C	*X*	1	1 × 6 = 06
C	*Y*	4	4 × 4 = 16
D	*X*	6	6 × 3 = 18
		Total in Rs.	**71**

Solution by Vogel's Approximation Method: (Opportunity cost method)

(i) In this method, we use concept of **opportunity cost**. Opportunity cost is the penalty for not taking correct decision. ***To find the row opportunity cost in the given matrix deduct the smallest element in the row from the next highest element. Similarly to calculate the column opportunity cost, deduct smallest element in the column from the next highest element. Write row opportunity costs of each row just by the side of availability constraint and similarly write the column opportunity cost of each column just below the requirement constraints. These are known as penalty column and penalty row.***

The rationale in deducting the smallest element form the next highest element is:

Let us say the smallest element is 3 and the next highest element is 6. If we transport one unit through the cell having cost Rs.3/-, the cost of transportation per unit will be Rs. 3/-. Instead we transport through the cell having cost of Rs.6/-, then the cost of transportation will be Rs.6/- per unit. That is for not taking correct decision; we are spending Rs.3/- more (Rs.6 – Rs.3 = Rs.3/-). ***This is the penalty for not taking correct decision and hence the opportunity cost. This is the lowest opportunity cost in that particular row or column as we are deducting the smallest element form the next highest element.***

Note: If the smallest element is three and the row or column having one more three, then we have to take next highest element as three and not any other element. Then the opportunity cost will be zero. In general, if the row has two elements of the same magnitude as the smallest element then the opportunity cost of that row or column is zero.

(ii) Write row opportunity costs and column opportunity costs as described above.

(iii) Identify the highest opportunity cost among all the opportunity costs and write a tick (√) mark at that element.

(iv) If there are two or more of the opportunity costs which of same magnitude, then select any one of them, to break the tie. While doing so, see that both availability constraint and requirement constraint are simultaneously satisfied. If this happens, we may not get basic feasible solution *i.e* solution with $m + n - 1$ allocations. As far as possible see that both are not satisfied simultaneously. In case if inevitable, proceed with allocations. We may not get a solution with, $m + n - 1$ allocations. For this we can allocate a small element epsilon ($\in$) to any one of the empty cells. This situation in transportation problem is known as degeneracy. (This will be discussed once again when we discuss about optimal solution).

In transportation matrix, all the cells, which have allocation, are known as **loaded cells** and those, which have no allocation, are known as **empty cells**.

(Note: All the allocations shown in matrix 1 to 6 are tabulated in the matrix given below:**)**

	X	Y	Z	Dummy	Availability
A	3 (cost 4)	7 (cost 3)	(cost 2)	(cost 0)	10
B	3 (cost 5)	(cost 6)	5 (cost 1)	(cost 0)	8
C	(cost 6)	5 (cost 4)	(cost 3)	(cost 0)	5
D	1 (cost 3)	(cost 5)	(cost 4)	5 (cost 0)	6
Requirement.	7	12	5	5	29

(1)

	X	Y	Z	DMY		
A	4	3	2	0	10	(2)
B	5	6	1	0	8	(1)
C	6	4	3	0	5	**(3)**
D	3	5	4	0 **5**	6	**(3)**
	7	12	5	5	29	
	(1)	(1)	(1)	(0)		

(4)

	X	Y		
A	4	3	10	(1)
B	5	6	3	(1)
C	6	4 **5**	5	**(2)**
	6	12	18	
	(1)	(1)		

(2)

	X	Y	Z	
A	4	3	2	10
B	5	6	1 **5**	8
C	6	4	3	5
D	3	5	4	1
	5	12	5	24
	(1)	(1)	(1)	

(5)

	X	Y		
A	4	**7** 3	10	(1)
B	5	6	3	(1)
	6	7	13	
	(1)	**(3)**		

(3)

	X	Y		
A	4	3	10	(1)
B	5	6	3	(1)
C	6	4	5	**(2)**
D	**1** 3	5	1	**(2)**
	7	12	19	
	(1)	(1)		

(6)

		X	
(1)	A	4 **3**	3
(4)			
(1)	B	**3** 5	3
(1)		6	6

Consider matrix (1), showing cost of transportation and availability and requirement constraints. In the first row of the matrix, the lowest cost element is 0, for the cell A-Dummy and next highest element is 2, for the cell AZ. The difference is 2 – 0 = 2. The meaning of this is, if we transport the load through the cell A-Dummy, whose cost element is 0, the cost of transportation will be = Rs.0/- for each unit transported. Instead, if we transport the load through the cell, AZ whose cost element is Rs. 2/- the transportation cost is = Rs.2/- for each unit we transport. This means to say if we take decision to send the goods through the cell AZ, whose cost element is Rs.2/- then the management is going to loose Rs. 2/- for every unit it transport through *AZ*. Suppose, if the management decide to send load through the cell *AX*, Whose cost element is Rs.4/-, then the penalty or the opportunity cost is *Rs*.4/-. We write the minimum opportunity cost of the row outside the matrix. Here it is shown in brackets. Similarly, we find the column opportunity costs for each column and write at the bottom of each corresponding row (in brackets). After writing all the opportunity costs, then we select the highest among them. In the given matrix it is Rs.3/- for the rows *D* and *C*. **This situation is known as tie.** When tie exists, select any of the rows of your choice. At present, let us select the row *D*. Now in that row select the lowest cost cell for allocation. This is because; our objective is to minimize the transportation cost. For the problem, it is *D*-dummy, whose cost is zero. For this cell examine what is available and what is required? Availability is 6 tons and requirement is 5 tons. Hence allocate 5 tons to this cell and cancel the dummy row from the problem. Now the matrix is reduced to 3 × 4. Continue the above procedure and for every allocation the matrix goes on reducing, finally we get all allocations are over. Once the allocations are over, count them, if there are $m + n – 1$ allocations, then the solution is basic feasible solution. Otherwise, the **degeneracy** occurs in the problem. To solve degeneracy, we have to **add epsilon** ($\in$), a small element to one of the empty cells. This we shall discuss, when we come to discuss optimal solution. Now for the problem the allocations are:

From	*To*	*Load*	*Cost in Rs.*
A	*X*	3	3 × 4 = 12
A	*Y*	7	7 × 3 = 21
B	*X*	3	3 × 5 = 15
B	*Z*	5	5 × 1 = 05
C	*Y*	5	5 × 4 = 20
D	*X*	1	1 × 3 = 03
D	DUMMY	5	5 × 0 = 00
		Total Rs.	**76**

Now let us compare the three methods of getting basic feasible solution:

North – west corner method.	*Inspection or least cost cell method*	*Vogel's Approximation Method.*
1. The allocation is made from the left hand side top corner irrespective of the cost of the cell.	The allocations are made depending on the cost of the cell. Lowest cost is first selected and then next highest etc.	The allocations are made depending on the opportunity cost of the cell.
2. As no consideration is given to the cost of the cell, naturally the total transportation cost will be higher than the other methods.	As the cost of the cell is considered while making allocations, the total cost of transportation will be comparatively less.	As the allocations are made depending on the opportunity cost of the cell, the basic feasible solution obtained will be very nearer to optimal solution.
3. It takes less time. This method is suitable to get basic feasible solution quickly.	The basic feasible solution, we get will be very nearer to optimal solution. It takes more time than northwest coroner method.	It takes more time for getting basic Feasible solution. But the solution we get will be very nearer to Optimal solution.
4. When basic feasible solution alone is asked, it is better to go for northwest corner method.	When optimal solution is asked, better to go for inspection method for basic feasible solution and MODI for optimal solution.	VAM and MODI is the best option to get optimal solution.

In the problem given, the total cost of transportation for Northwest corner method is Rs. 101/-. The total cost of transportation for Inspection method is Rs. 71/- and that of VAM is Rs. 76/-. The total cost got by inspection method appears to be less. That of Northwest coroner method is highest. The cost got by VAM is in between.

Now let us discuss the method of getting optimal solution or methods of giving optimality test for basic feasible solution.

Optimality Test: (Approach to Optimal Solution)

Once, we get the basic feasible solution for a transportation problem, the next duty is to test whether the solution got is an optimal one or not? This can be done by two methods. (*a*) By Stepping Stone Method, and (*b*) By Modified Distribution Method, or MODI method.

(a) Stepping stone method of optimality test

To give an optimality test to the solution obtained, we have to find the opportunity cost of empty cells. As the transportation problem involves decision making under certainty, we know that an optimal solution must not incur any positive opportunity cost. Thus, we have to determine whether any positive opportunity cost is associated with a given progarmme, *i.e.,* for empty cells. **Once the opportunity cost of all empty cells are negative, the solution is said to be optimal.** In case any one cell has got positive opportunity cost, then the solution is to be modified. The Stepping stone method is used for finding the opportunity costs of empty cells. Every empty cell is to be evaluated for its opportunity cost. To do this the methodology is:

1. Put a small '+' mark in the empty cell.

2. Starting from that cell draw a loop moving horizontally and vertically from loaded cell to loaded cell. Remember, there should not be any diagonal movement. We have to take turn only at loaded cells and move to vertically downward or upward or horizontally to reach another loaded cell. In between, if we have a loaded cell, where we cannot take a turn, ignore that and proceed to next loaded cell in that row or column.
3. After completing the loop, mark minus (–) and plus (+) signs alternatively.
4. Identify the lowest load in the cells marked with negative sign.
5. This number is to be added to the cells where plus sign is marked and subtract from the load of the cell where negative sign is marked.
6. Do not alter the loaded cells, which are not in the loop.
7. The process of adding and subtracting at each turn or corner is necessary to see that rim requirements are satisfied.
8. Construct a table of empty cells and work out the cost change for a shift of load from loaded cell to loaded cell.
9. If the cost change is positive, it means that if we include the evaluated cell in the programme, the cost will increase. If the cost change is negative, the total cost will decrease, by including the evaluated cell in the programme.
10. The negative of cost change is the opportunity cost. Hence, in the optimal solution of transportation problem empty cells should not have positive opportunity cost.
11. Once all the empty cells have negative opportunity cost, the solution is said to be optimal.

One of the drawbacks of stepping stone method is that we have to write a loop for every empty cell. Hence it is tedious and time consuming. Hence, for optimality test we use MODI method rather than the stepping stone method.

Let us take the basic feasible solution we got by Vogel's Approximation method and give optimality test to it by stepping stone method.

Basic Feasible Solution obtained by VAM:

	X	Y	Z	Dummy	Availability
A	4; 3; –	3; 7; +	2; –2	0; +1	10
B	5; 3	6; –2	1; 5	0; +2	8
C	6; –1	4; 5; –	3; –1	0; +; 0	5
D	3; 1; +; 3	5; –3; 5	4; –5; 4	0; 5; –; 0	6
Requirement.	7	12	5	5	29

Table showing the cost change and opportunity costs of empty cells:

Table I.

S.No.	*Empty Cell*	*Evalution Loop formation*	*Cost change in Rs.*	*Opportunity cost -(Cost change)*
1.	AZ	+AZ – AX + BX – BZ	+2 – 4 + 5 – 1 = + 2	–2
2	A Dummy	+ A DUMMY – AX + BX – B DUMMY	+0 – 4 + 3 – 0 = – 1	+1
3	BY	+ BY – AY + AX – BX	+6 – 3 + 4 – 5 = +2	-2
4	B DUMMY	+ B DUMMY – BX + DX – D DUMMY	+0 – 5 +3 – 0 = –2	+2
5	CX	+CX – CY + AX – AY	6 – 4 + 3 – 4 = +1	– 1
6	CZ	+CZ – BZ + BX –AX + AY – CY	+2 –1 +5 – 4 +5 – 4 =+1	– 1
7	C DUMMY	+ C DUMMY – D DUMMY + DX – AX + AY – CY	+ 0 – 0 +3 – 4 +3 – 4 = –2	+2
8	DY	+DY – DX + AX – AY	+5 – 3 +4 – 3 = +3	– 3
9	DZ	+DZ – DX +BX – BZ	+4 – 3 + 5 – 1 = +5	– 5

In the table 1 cells A DUMMY, B DUMMY, C DUMMY are the cells which are having positive opportunity cost. Between these two cells B DUMMY and C DUMMY are the cells, which are having higher opportunity cost i.e Rs. 2/ - each. Let us select any one of them to include in the improvement of the present programme. Let us select C DUMMY.

	X	Y	Z	Dummy	Availability
A	4 –8	3 (10)	2 –4	0 +1	10
B	5 (3) –	6 0	1 (5)	0 +2 +	8
C	6 –3	4 (2)	3 –3	0 (3)	5
D	3 (4) +	5 –1	4 –5	0 (2)	6
Requirement.	7	12	5	5	29

Table II.

S.No.	*Empty Cell*	*Evalution Loop formation*	*Cost change in Rs.*	*Opportunity Cost*
1	AX	+AX –DX + D DUMMY – C DUMMY + CY – AY	+ 4 – 3 + 0 – 0 + 4 – 3 = +2	–2
2	AX	AZ – AY + CY – C DUMMY + D DUMMY – DX+ BX – BZ	+ 2 – 3 + 4 – 0 + 0 – 3 + 3 – 0 = + 4	–4
3	ADUMMY	+ A DUMMY – AY + DX – D DUMMY	+ 0 – 4 + 3 – 0 = – 1	+1
4	BY	+BY – BX + DX – D DUMMY + C DUMMY – CY	+ 6 –5 + 3 –0 + 0 – 4 = 0	0
5	B DUMMY	+ B DUMMY – BX + DX – D DUMMY	+ 0 – 5 + 3 – 0 = –2	+2
6	CX	+ CX – DX + D DUMMY – C DUMMY	+ 6 –3 + 0 – 0 = +3	–3
7	CZ	+ CZ – C DUMMY + D DUMMY – DX + BX – BZ	+ 2 – 0 + 0 – 3 + 5 – 1 = +3	–3
8	DY	DY – CY + C DUMMY – D DUMMY	+ 5 – 4 + 0 – 0 = 1	–1
9	DZ	+ DZ – DX + BX – BZ	+ 4 – 3 +5 – 1 = + 5	–5

Cells A DUMMY and B DUMMY are having positive opportunity costs. The cell B DUMMY is having higher opportunity cost. Hence let us include this cell in the next programme to improve the solution.

Table III.

S.No.	*Empty Cell*	*Evaluation Loop formation*	*Cost change in Rs.*	*Opportunity Cost*
1	AX	+AX – AY + CY – C DUMMY + B DUMMY – BX	+4 – 3 + 4 – 0 + 0 – 5 = 0	0
2	AZ	+ AZ – BZ + B DUMMY – C DUMMY + CX – AX	+2 –1 + 0 – 0 + 4 – 3 =+ 2	–2
3	A DUMMY	+ A DUMMY – C DUMMY + CY – AY	+0 – 0 + 4 – 3 = +1	–1
4	BY	+ BY – B DUMMY + C DUMMY – CY	+6 – 0 + 0 – 4 = + 2	–2
5	CX	+ CX – BX + B DUMMY – C DUMMY	+6 – 5 + 0 – 0 = +1	–1
6	CZ	+ CZ – BZ + B DUMMY – C DUMMY	+2 – 1 + 0 – 0 = +1	–1
7	DY	+DY – CY + C DUMMY – B DUMMY + BX – DX	+5 – 4 + 0 – 0 + 5 – 3 = +3	–3
8	DZ	+ DZ – BZ + BX – DX	+4 – 1 + 5 – 3 = +5	–5
9	D DUMMY	+ D DUMMY – DX + BX – B DUMMY	+ 0 – 3 + 5 – 0 = +2	–2

All the empty cells have negative opportunity cost hence the solution is optimal. The **allocations are:**

S.No	*Loaded cell*	*Load*	*Cost in Rs.*
1	AY	10	10 × 3 = 30
2	BX	01	01 × 5 = 05
3	BZ	05	05 × 1 = 05
4	B DUMMY	02	02 × 0 = 00
5	CY	02	02 × 4 = 08
6	C DUMMY	03	03 × 0 = 00
7	DX	06	06 × 3 = 18
	Total in Rs.		66

Total minimum transportation cost is Rs. 66/-

Optimal allocation.

	X	Y	Z	Dummy	Availability
A	4 0	3 10	2 0	0 –1	10
B	5 1	6 –2	1 5	0 2	8
C	6 –1	4 2	3 –1	0 3	5
D	3 6	5 –2	4 –5	0 –2	6
Requirement.	7	12	5	5	29

(b) Modified Distribution Method of Optimality test

In stepping stone method, we have seen that to get the opportunity cost of empty cells, for every cell we have to write a loop and evaluate the cell, which is a laborious process. In MODI (Modified DIstribution method, we can get the opportunity costs of empty cells without writing the loop. After getting the opportunity cost of all the cells, we have to select the cell with highest positive opportunity cost for including it in the modified solution.

Steps in MODI method:

1. Select row element (u_i) and Column element (v_j) for each row and column, such that $u_i + v_j$ = the actual cost of loaded cell. In MODI method we can evaluate empty cells simultaneously and get the opportunity cost of the cell by using the formula $(u_i + v_j) – C_{ij}$, where C_{ij} is the actual cost of the cell.
2. In resource allocation problem (maximization or minimization method), we have seen that once any variable becomes basis variable, *i.e.*, the variable enters the programme; its opportunity cost or net evaluation will be zero. Here, in transportation problem also, once any cell is loaded, its opportunity cost will be zero. Now the opportunity cost is given by $(u_i + v_j) – C_{ij}$, which is, equals to zero for a loaded cell.
   i.e. $(u_i + v_j) – C_{ij} = 0$ which means, $(u_i + v_j) = C_{ij}$. Here $(u_i + v_j)$ is known as **implied cost of the cell. For any loaded cell the implied cost is equals to actual cost of the cell as its opportunity cost is zero. For any empty cell, (implied cost – actual cost) will give opportunity cost.**
3. How to select u_i and v_j? The answer is:
   (a) Write arbitrarily any one of them against a row or against a column. The written u_i or vj may be any whole number *i.e* u_i or v_j may be ≤ or ≥ to zero. By using the formula $(u_i + v_j) = C_{ij}$ for a loaded cell, we can write the other row or column element. For example, if the actual cost of the cell $C_{ij} = 5$ and arbitrarily we have selected $u_i = 0$, then v_j is given by $u_i + v_j = 0 + v_j = 5$. Hence $v_j = –5$. Like this, we can go from loaded cell to loaded cell and complete entering of all u_i *s* and v_j *s*.
   (b) Once we get all u_i *s* and v_j *s*, we can evaluate empty cells by using the formula $(u_i + v_j)$ – Actual cost of the cell = opportunity cost of the cell, and write the opportunity cost of each empty cell at left hand bottom corner.
   (c) **Once the opportunity costs of all empty cells are negative, the solution is said to be optimal. In case any cell is having the positive opportunity cost, the programme is to be modified.**
   Remember the formula that IMPLIED COST OF A CELL = $u_i + v_j$
   Opportunity cost of loaded cell is zero *i.e* $(u_i + v_j)$ = Actual cost of the cell.
   Opportunity cost of an empty cell = implied cost – actual cost of the cell = $(u_i + v_j) – C_{ij}$
   (d) **In case of degeneracy, *i.e.* in a basic feasible solution, if the number of loaded cells are not equals to $m + n – 1$, then we have to add a small element *epsilon* (∈), to any empty cell to make the number of loaded cells equals to $m + n – 1$. While adding '∈' we must be careful enough to see that this ∈ should not form a closed loop when we draw horizontal and vertical lines from loaded cell to loaded cell. In case the cell to which we have added ∈ forms a closed loop, then if we cannot write all u_i s and v_j *s*.**
   ∈ is such a small element such that $a + ∈ = a$ or $a – ∈ = a$ and $∈ – ∈ = 0$.

Implied cost	Actual cost	Action
$u_i + v_j >$	C_{ij}	A better programme can be designed by including this cell in the solution.
$u_i + v_j =$	C_{ij}	Indifferent; however, an alternative programme with same total cost can be written by including this cell in the programme.
$u_i + v_j <$	C_{ij}	Do not include this cell in the programme.

Now let us take the basic feasible solution obtained by VAM method and apply MODI method of optimality test.

Basic feasible solution got by VAM method.

	X	Y	Z	Dummy	Availability	Row No. u_i
A	4 (3) −	3 (7) +	2	0	10	4
B	5 (3)	6	1 (5)	0	8	5
C	6	4 (5)	3	0 +	5	5
D	3 (1) +	5	4	0 (5) −	6	3
Requirement.	7	12	5	5	29	
Column No v_j	0	−1	−4	−3		

The cell C DUMMY is having a positive opportunity cost. Hence we have to include this cell in the programme. The solution has $m + n - 1$ allocations.

	X	Y	Z	DUMMY	Availability	Row No. u_i
A	4 −2	3 (10)	2 0	0 −1	10	0
B	5 (3) −	6 0	1 (5)	0 +2 +	8	3
C	6 −3	4 (2)	3 −2	0 (3)	5	1
D	3 (4) +	5 −1	4 −5	0 (2) −	6	1
Requirement .	7	12	5	5	29	
Column Number: v_j	2	3	−2	−1		

The cell B DUMMY is having a positive opportunity cost. Thïs is to be included in the modified programme.

	X	Y	Z	DUMMY	Availability	Row No. u_i
A	$\boxed{4}$ 0	$\boxed{3}$ 10	$\boxed{2}$ –2	$\boxed{0}$ –1	10	–1
B	$\boxed{5}$ 1	$\boxed{6}$ –2	$\boxed{1}$ 5	$\boxed{0}$ 2	8	0
C	$\boxed{6}$ –1	$\boxed{4}$ 2	$\boxed{3}$ –2	$\boxed{0}$ 3	5	0
D	$\boxed{3}$ 6	$\boxed{5}$ –1	$\boxed{4}$ –3	$\boxed{0}$ 0	6	0
Requirement .	7	12	5	5	29	
Column Number: v_j	5	4	1	0		

As the opportunity cost of all empty cells are negative, the solution is optimal. The solution has $m + n - 1$ allocations.

The allocations are:

S.No	*Loaded Cell*	*Load*	*Cost in Rs.*
1	AY	10	10 × 3 = 30
2.	BX	01	01 × 5 = 05
3.	BZ	05	05 × 1 = 05
4.	B DUMMY	02	02 × 0 = 00
5.	CY	02	02 × 4 = 08
6.	C DUMMY	03	03 × 0 = 00
7.	CX	06	06 × 3 = 18
	Total Cost in Rs.		66

Readers can verify the optimal solution got by Stepping stone method and the MODI method they are same. And they can also verify the opportunity costs of empty cells they are also same. This is the advantage of using MODI method to give optimality test. Hence the combination of VAM and MODI can be conveniently used to solve the transportation problem when optimal solution is asked.

Alternate Solutions

By principle, we know that the opportunity cost of a loaded cell or a problem variable is always equals to zero. In case any empty cell of the optimal solution of a transportation problem got zero as the opportunity cost, it should be understood that it is equivalent to a loaded cell. Hence by including that cell, we can derive another solution, which will have same total opportunity cost, but different allocations. Once one alternate solution exists, we can write any number of alternate solutions. The methodology is:

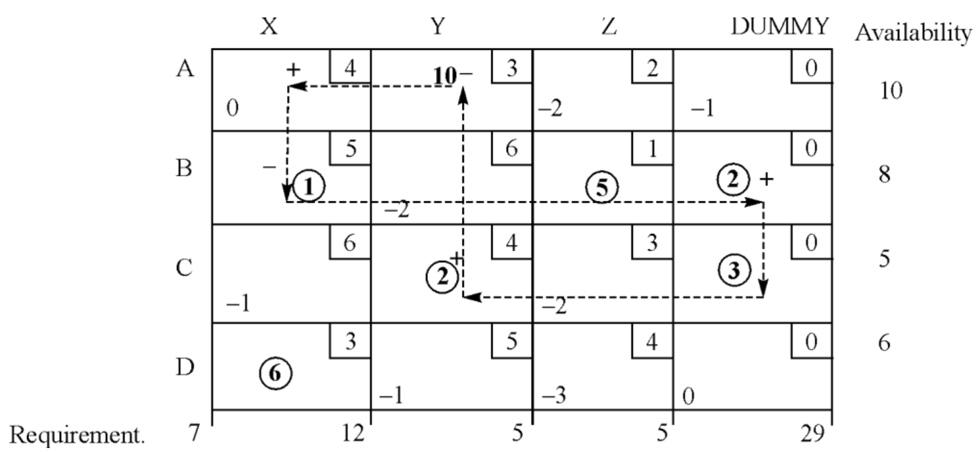
1. Let the Optimal solution is matrix A with one or more empty cells having zero as the opportunity cost.

2. By including the cell having zero as the opportunity cost, derive one more optimal solution, let it be the matrix B.
3. The new matrix *C* is obtained by the formula: ***C* = *d A* + (1– *d*) *B*,** where '*d*' is a positive fraction less than 1.

It is better to take always *d* = 1/2, so that ***C* = 1/2 *A* + 1/2 *B*.**

Now we shall take the optimal solution of the problem above and write the alternate optimal solutions.

Matrix A (First optimal Solution).



The cell AX, having zero opportunity cost is included in revised solution. The loop is:
+ AX – BX + B DUMMY – C DUMMY + CY – AY = + 4 – 5 + 0 – 0 + 4 – 3 = 0
Allocation:

S.No	*Loaded Cell*	*Load*	*Cost in Rs.*
1.	AX	01	01 × 4 = 04
2.	AY	09	09 × 3 = 18
3.	BZ	05	05 × 1 = 05
4.	B Dummy	03	03 × 0 = 00
5.	CY	03	03 × 4 = 12
6.	C Dummy	02	02 × 0 = 00
7.	DX	06	06 × 3 = 18
	Total cost in Rs.		66

Matrix B (First alternative solution):

	X	Y	Z	Dummy	Availability
A	$\boxed{4}$ 1 0	$\boxed{3}$ 9	$\boxed{2}$ –2	$\boxed{0}$ –1	10
B	$\boxed{5}$	$\boxed{6}$ –2	$\boxed{1}$ **5**	$\boxed{0}$ 3	8
C	$\boxed{6}$ –1	$\boxed{4}$ **3**	$\boxed{3}$ –2	$\boxed{0}$ 2	5
D	$\boxed{3}$ 6	$\boxed{5}$ –1	$\boxed{4}$ –3	$\boxed{0}$ 0	6
Requirement	7	12	5	5	29

Matrix C (Second alternate solution)

	X	Y	Z	Dummy	Availability
A	$\boxed{4}$ **0.5**	$\boxed{3}$ **9.5**	$\boxed{2}$	$\boxed{0}$	10
B	$\boxed{5}$ 0.5	$\boxed{6}$	$\boxed{1}$ 5	$\boxed{0}$ 2.5	8
C	$\boxed{6}$	$\boxed{4}$ 2.5	$\boxed{3}$	$\boxed{0}$ 2.5	5
D	$\boxed{3}$ 6	$\boxed{5}$	$\boxed{4}$	$\boxed{0}$	6
Requirement	7	12	5	5	29

The total cost is $0.5 \times 4 + 9.5 \times 3 + 0.5 \times 5 + 5 \times 1 + 2.5 \times 0 + 2.5 \times 0 + 2.5 \times 0 + 6 \times 3 =$ Rs. 66/-

Once we get one alternate solution we can go on writing any number of alternate solutions until we get the first optimal solution.

MAXIMIZATION CASE OF TRANSPORTATION PROBLEM

Basically, the transportation problem is a minimization problem, as the objective function is to minimize the total cost of transportation. Hence, when we would like to maximize the objective function. There are two methods.

maximization

(i) The given matrix is to be multiplied by –1, so that the problem becomes maximization problem. Or ii) Subtract all the elements in the matrix from the highest element in the matrix. Then the problem becomes maximization problem. Then onwards follow all the steps of maximization problem to get the solution. Let us consider the same problem solved above.

Problem 4.2. Four factories, *A*, *B*, *C* and *D* produce sugar and the capacity of each factory is given below: Factory *A* produces 10 tons of sugar and *B* produces 8 tons of sugar, *C* produces 5 tons of sugar and that of *D* is 6 tons of sugar. The sugar has demand in three markets *X*, *Y* and *Z*. The demand of market *X* is 7 tons, that of market *Y* is 12 tons and the demand of market *Z* is 4 tons. The following matrix gives the returns the factory can get, by selling the sugar in each market. Formulate a transportation problem and solve for maximizing the returns.

	Profit in Rs. per ton (× 100)			*Availability in tons.*
		Markets.		
	X	*Y*	*Z*	
Factories.				
A	4	3	2	10
B	5	6	1	8
C	6	4	3	5
D	3	5	4	6
Requirement in tons.	7	12	4	$\Sigma\, b = 29, \Sigma\, d = 23$

Here $\Sigma\, b$ is greater than $\Sigma\, d$ hence we have to open a dummy column whose requirement constraint is 6, so that total of availability will be equal to the total demand. Now let get the basic feasible solution by VAM and then give optimality test by MODI method. The balanced matrix of the transportation problem is:

Profit per ton in Rs.

	X	Y	Z	Dummy	Availability	Row No. u_i
A	–4; 2	–3; 3	–2; 0	–0; 5	10	0
B	–5; 2	–6; 8	–1; 4	–0; 3	8	3
C	–6; 5	–4; 1	–3; 2	–0; 2	5	2
D	–3; 3	–5; 1	–4; 5	–0; 2	6	2
Requirement	7	12	5	5	29	
column. no. v_j	4	3	2	0		

By multiplying the matrix by –1, we can convert it into a maximisation problem. Now in VAM we have to find the row opportunity cost and column opportunity costs. In minimisation problem, we use to subtract the smallest element in the row from next highest element in that row for finding row opportunity cost. Similarly, we use to subtract smallest element in the column by next highest element in that column to get column opportunity cost. Here as we have multiplied the matrix by –1 the highest element will become lowest element. Hence subtract the lowest element from the next highest element as usual. Otherwise, instead of multiplying by –1 simply find the difference between highest element and the next lowest element and take it as opportunity cost of that row or column. For example in the given problem in the row A, the highest element is 4 and the next lowest element is 3 and hence the opportunity cost is 4 – 3 = 1. (Or smallest element is –4 and the next highest element is –3 and the opportunity cost is –3 – (– 4) = –3 + 4 = 1). Similarly, we can write all opportunity costs. Once we find the opportunity costs, rest of the procedure is same. That is, we have to select highest opportunity cost and select the highest profit element in that row or column for allocation. Obtain the basic feasible solution. As usual the basic feasible solution must have $m + n - 1$ allocations. If the allocations are not equal to $m + n - 1$, the problem degenerate. In that case, add $\in$ to an empty cell, which do not form loop with other loaded cells. Once we have basic feasible solution, the optimality test by MODI method, is followed. Here, once the opportunity costs of all the cells are positive, (as we have converted the maximistion problem into minimisation problem) the solution is said to be optimal.

In the given problem as the opportunity costs of all empty cells are positive, the solution is optimal. And the optimal return to the company is Rs. 125/-.

Allocations:

S.No	Loaded Cell	*Load*	*Cost in Rs.*
1.	*AX*	02	02 × 4 = 08
2.	*AY*	03	03 × 3 = 09
3.	*A Dmy*	05	05 × 0 = 00
4.	*BY*	08	08 × 6 = 48
5.	*CX*	05	05 × 6 = 30
6.	*DY*	01	01 × 5 = 05
7.	DZ	05	05 × 4 = 20
	Total returns in Rs.		125

(1)

	X	Y	Z	DMY	
A	4	3	2	0	10(1)
B	5	6	1	0	8 (1) (2)
C	**5** 6	4	2	0	5 (2)
D	3	5	4	0	6 (1)
	7	12	5	5	29
	1	1	2	0	

(2)

	X	Y	Z	DMY		
A	4	3	2	0	10	1
B	5	6	1	0	8	1
D	3	5	4 **5**	0	6	1
	7	12	5	5	24	
	1	1	2	0		

(3)

	X	Y	DMY		
A	4	3	0	10	1
B	5	6	0	8	1
D	3	5 **1**	0	1	2
	2	12	5	19	
	1	1	0		

(4)

	X	Y	DMY		
A	4	3	0	10	1
B	5	6 **8**	0	8	1
	2	11	5	18	
	1	3	0		

(5)

	X	Y	DMY	
A	4 **2**	3 **3**	0 **5**	10
	2	3	5	10

DEGENERACY IN TRANSPORTATION PROBLEM

Earlier, it is mentioned that the basic feasible solution of a transportation problem must have $(m + n - 1)$ basis variables or allocations. This means to say that the number of occupied cells or loaded cells in a given transportation problem is 1 less than the sum of number of rows and columns in the transportation matrix. Whenever the number of occupied cells is less than $(m + n - 1)$, the transportation problem is said to be degenerate.

Degeneracy in transportation problem can develop in two ways. First, the problem becomes degenerate when the initial programme is designed by northwest corner or inspection or VAM, *i.e*. at the stage of initial allocation only.

To solve degeneracy at this stage, we can allocate extremely small amount of goods (very close to zero) to one or more of the empty cells depending on the shortage, so that the total occupied cells becomes $m + n - 1$. The cell to which small element (load) is allocated is considered to be an occupied cell. In transportation problems, Greek letter '$\in$' represents the small amount. One must be careful enough to see that the smallest element epsilon is added to such an empty cell, which will enable us to write row number 'u_i' and column number 'v_j' without any difficulty while giving optimality test to the basic feasible solution by MODI method. That is care must be taken to see that the epsilon is added to such a cell, which will not make a **closed loop**, when we move horizontally and vertically from loaded cell to loaded cell.

(Note: Epsilon is so small so that if it is added or subtracted from any number, it does not change the numerical value of the number for which it added or from which it is subtracted.).

Secondly, the transportation problem may become degenerate during the solution stages. This happens when the inclusion of a most favorable empty cell *i.e*. cell having highest opportunity cost results in simultaneous vacating of two or more of the currently occupied cells. Here also, to solve degeneracy, add epsilon to one or more of the empty cells to make the number of occupied cells equals to $(m + n - 1)$.

To understand the procedure let us solve one or two problems.

Problem. 4.3. Solve the transportation problem given below

(Cost in Rs. per unit)
Destinations.

Origins	A	B	C	Available capacity	Row number u_i
X	2	1	2	20	
Y	3	4	1	40	
Requirement	20	15	25	60	
Column element v_j					

Solution by Northwest corner method:

Initial allocation show that the solution is not having $(m+n-1)$ allocations. Hence degeneracy occurs.

(Cost in Rs. per unit)
Destinations.

Origins	A	B	C	Available capacity	Row number u_i
X	$\boxed{2}$ 20	$\boxed{1}$	$\boxed{2}$	20	
Y	$\boxed{3}$	$\boxed{4}$ 15	$\boxed{1}$ 25	40	
Requirement	20	15	25	60	

(Cost in Rs. per unit)
Destinations.

Origins	A	B	C	Available capacity	Row number u_i
X	$\boxed{2}$ 20	$\boxed{1}$ ε	$\boxed{2}$	20	
Y	$\boxed{3}$	$\boxed{4}$ 15	$\boxed{1}$ 25	40	
Requirement	20	15	25	60	
Column element v_j					

The smallest load $\in$ is added to cell *XB* which does not make loop with other loaded cells.

(Cost in Rs. per unit)
Destinations.

Origins	A	B	C	Available capacity	Row number u_i
X	$\boxed{2}$ 20 –	$\boxed{1}$ ε +	$\boxed{2}$	20	
Y	$\boxed{3}$ +	$\boxed{4}$ – 15	$\boxed{1}$ 25	40	
Requirement	20	15	25	60	
Column element v_j					

Shifting of load by drawing loops to cell YA.

(Cost in Rs. per unit)
Destinations.

Origins	A	B	C	Available capacity	Row number u_i
X	5 [2]	15 [1]	[2]	20	
Y	15 [3]	[4]	25 [1]	40	
Requirement	20	15	25	60	
Column element v_j					

The basic feasible solution is having four loaded cells. As the number of columns is 3 and number of rows is 2 the total number of allocations must be 2 + 3 – 1 = 4. The solution got has four allocations. Hence the basic feasible solution. Now let us give optimality test by MODI method.

(Cost in Rs. per unit)
Destinations.

Origins	A	B	C	Available capacity	Row number u_i
X	5 [2]	15 [1]	[2] –2	20	0
Y	15 [3]	[4] –2	25 [1]	40	1
Requirement	20	15	25	60	
Column element v_j	2	1	0		

Row numbers u_i s and column numbers v_j s are written in the matrix and opportunity cost of empty cells are evaluated. As the opportunity cost of all empty cells are negative, the solution is optimal. The allocations and the total cost of transportation is:

S.No	*Loaded Cell*	*Load*	*Cost in Rs.*
1.	*XA*	05	05 × 2 = 50
2.	*XB*	15	15 × 1 = 15
3.	*YA*	15	15 × 3 = 45
4.	*YC*	25	25 × 1 = 25
	Total cost in Rs.		135

Problem. 4.4. Solve the transportation problem given below:

Cost in Rs. per unit

To → / From ↓	D_1	D_2	D_3	D_4	D_5	Availability	u_i
O_1	4 (30)	3 (10)	1 2	2	6	40	0
O_2	5	2 (20)	3 (10)	4	5	30	–1
O_3	3	5	6 (5)	3 (15)	2	20	2
O_4	2	4	4	5 (5)	3 (5)	10	4
Requirement	30	30	15	20	5	100	
v_j	4	3	4	1	–1		

Let us make initial assignment by using Northwest corner method. To modify the solution we include the cell O_1D_3 in the programme, as it is having highest opportunity cost.

Improved solution:

Cost in Rs. per unit

To → / From ↓	D_1	D_2	D_3	D_4	D_5	Availability	u_i
O_1	4 (20)	3 (ε)	1 (10) 2	2	6	40	0
O_2	5	2 (30)	3	4	5	30	0
O_3	3	5	6 (5)	3 (15)	2	20	5
O_4	2 **9**	4	4	5 (5)	3 (5)	10	7
Requirement	30	30	15	20	5	100	
v_j	4	2	1	–2	–4		

Total number of allocations are less than $m + n - 1$. Hence we have to add one epsilon to an empty cell. Remember, in transportation problem, which has minimization of cost as its objective function a, we have to add epsilon to recently vacated cell, which is having lowest shipping cost. We have a tie between two cells, *i.e.* O_1D_2 and O_2D_3. Let us select O_1D_2 to add epsilon. To improve the solution, let us take empty cell O_4D_1 in the programme.

Improved Programme: The solution is not having $m + n - 1$ allocations. We have to add epsilon; in the programme epsilon is added to cell 0_4D_4

Revised Programme.

Cost in Rs. per unit

To → From ↓	D_1	D_2	D_3	D_4	D_5	Availability	u_i
O_1	$\boxed{4}$ + 25	$\boxed{3}$ ε	$\boxed{1}$ 15	$\boxed{2}$ + 5	$\boxed{6}$	40	0
O_2	$\boxed{5}$	$\boxed{2}$ 30	$\boxed{3}$	$\boxed{4}$	$\boxed{5}$	30	−1
O_3	$\boxed{3}$	$\boxed{5}$	$\boxed{6}$	$\boxed{3}$ 20	$\boxed{2}$	20	−4
O_4	$\boxed{2}$ 5 −	$\boxed{4}$	$\boxed{4}$	$\boxed{5}$ ∈ +	$\boxed{3}$ 5	10	−2
Requirement	30	30	15	20	5	100	
v_j	4	3	1	7	5		

The epsilon is shifted to an empty cell. The improved solution is having 8 allocations. Hence a feasible solution.

As the cell O_1D_4 having positive opportunity cost, let us include and revise the programme. Revised programme. Cell 0_3D_5 having positive opportunity cost is included in revised programme.

Cost in Rs. per unit

To → From ↓	D_1	D_2	D_3	D_4	D_5	Availability	u_i
O_1	$\boxed{4}$ 25 −	$\boxed{3}$ ε	$\boxed{1}$ 15	$\boxed{2}$ + ε	$\boxed{6}$	40	0
O_2	$\boxed{5}$	$\boxed{2}$ 30	$\boxed{3}$	$\boxed{4}$	$\boxed{5}$	30	−1
O_3	$\boxed{3}$	$\boxed{5}$	$\boxed{6}$	$\boxed{3}$ 20 −	$\boxed{2}$ + 4	20	1
O_4	$\boxed{2}$ + 5	$\boxed{4}$	$\boxed{4}$	$\boxed{5}$	$\boxed{3}$ 5 −	10	−2
Requirement	30	30	15	20	5	100	
v_j	4	3	1	2	5		

Revised programme: Cell O_3D_1 having positive opportunity cost is included in the revised programme.

Cost in Rs. per unit

To → From ↓	D_1	D_2	D_3	D_4	D_5	Availability	u_i
O_1	4 − 20	3 ε	1 15	2 + 5	6	40	0
O_2	5	2 30	3	4	5	30	+–1
O_3	3 + 2	5	6	3 5 −	2 2	20	1
O_4	2 10	4	4	5	3	10	–2
Requirement	30	30	15	20	5	100	
v_j	4	2	1	2	1		

Revised Programme.

Cost in Rs. per unit

To → From ↓	D_1	D_2	D_3	D_4	D_5	Availability	u_i
O_1	4 5	3 ε	1 15	2 20	6 –3	40	0
O_2	5 –2	2 30	3 –3	4 –3	5 –3	30	–1
O_3	3 15	5 –3	6 –6	3 –2	2 5	20	–1
O_4	2 10	4 –4	4 –6	5 –6	3 –3	10	–3
Requirement	30	30	15	20	5	100	
v_j	4	3	1	2	3		

As the opportunity costs of all empty cells are negative, the solution is optimal. The allocations and the total cost of transportation is:

S.No	*Loaded Cell*	*Load*	*Cost in Rs.*
1.	O_1D_1	5	5 × 4 = 20
2.	O_1D_2	ε	----
3.	O_1D_3	15	15 × 1 = 15
4.	O_1D_4	20	20 × 2 = 40
5.	O_2D_2	30	30 × 2 = 60
6.	O_3D_1	15	15 × 3 = 45
7.	O_3D_5	5	5 × 2 = 10
8.	O_4D_1	10	10 × 2 = 20
	Total Cost in Rs.		210/-

The same problem, if we solve by VAM, the very first allocation will be feasible and optimality test shows that the solution is optimal.

Roc: Row opportunity cost, COC= Column opportunity cost, Avail: Availability, Req: Requirement.

	D_1	D_2	D_3	D_4	D_5	*Avail*	*ROC*
O_1	4	3	1 **15**	2	6	40	1
O_2	5	2	3	4	5	30	1
O_3	3	5	6	3	2	20	1
O_4	2	4	4	5	3	10	1
REQ	30	30	15	20	5	100	
COC	1	1	2	1	1		

	D_1	D_2	D_4	D_5	*Avail*	*ROC*
O_1	4	3	2	6	25	1
O_2	5	2 **30**	4	5	30	2
O_3	3	5	3		20	1
O_4	2	4	5	3	10	1
REQ	30	30	20	5	85	
COC	1	1	1	1		

	D_1	D_4		D_5	*Avail*	*ROC*
O_1	4	2	**20**	6	25	2
O_3	3	3		2	20	1
O_4	2	5		3	10	1
REQ	30	20		5	55	
ROC	1	1		1		

	D_1	D_5	*Avail*	*ROC*
O_1	4 **5**	6	5	2 ←
O_3	3	2	20	1
O_4	2	3	10	1
REQ	30	5	35	
COC	1	1		

	D_1	D_5	*Avail*	*ROC*
O_3	3	2	20	1
~~O_4~~	~~2~~ **10**	~~3~~	~~10~~	~~1~~ ←
REQ	25	5	30	
COC	1	1		

	D_1	D_5	*AVAIL*
O_3	3 **15**	2 **5**	20
REQ	15	5	20

Allocation by VAM:

Cost in Rs. per unit

To → / From ↓	D_1	D_2	D_3	D_4	D_5	Availability	u_i
O_1	[4] 5	[3] ε	[1] 15	[2] 20	[6] –1	40	0
O_2	[5] –2	[2] 30	[3] –3	[4] –3	[5] –1	30	–1
O_3	[3] 15	[5] –3	[6] –6	[3] –2	[2] 5	20	–1
O_4	[2] 10 9	[4] –3	[4] –5	[5] –5	[3] 0	10	–2
Requirement	30	30	15	20	5	100	
v_j	4	3	1	2	5		

Allocations are same as in the optimal solution got by northwest corner method. All opportunity costs of empty cells are negative. Hence the total transportation cost is Rs. 210/-

TIME MINIMISATION MODEL OR LEAST TIME MODEL OF TRANSPORTATION TIME.

It is well known fact that the transportation problem is cost minimization model, *i.e* we have to find the least cost transportation schedule for the given problem. Some times the cost will become secondary factor when the time required for transportation is considered. This type of situation we see in military operation. When the army want to send weapons or food packets or medicine to the war front, then the time is important than the money. They have to think of what is the least time required to transport the goods than the least cost of transportation. Here the given matrix gives the time elements, *i.e*. time required to reach from one origin to a destination than the cost of transportation of one unit from one origin to a destination. A usual, we can get the basic feasible solution by Northwest corner method or by least time method or by VAM. To optimize the basic feasible solution, we have to identify the highest time element in the allocated cells, and try to eliminate it from the schedule by drawing loops and encouraging to take the cell, which is having the time element less than the highest one. Let us take a problem and work out the solution. Many a time, when we use VAM for basic feasible solution, the chance of getting an optimal solution is more. Hence, the basic feasible solution is obtained by Northwest corner method.

Problem 4.5. The matrix given below shows the time required to shift a load from origins to destinations. Formulate a least time schedule. Time given in hours.

Roc: Row opportunity cost, Coc: Column opportunity cost, Avail: Availability, Req: Requirement.

Destinations (Time in hours)

Origins

	D_1	D_2	D_3	D_4	Avail
O_1	7	8	4	5	5
O_2	8	10	2	3	7
O_3	7	6	17	8	8
O_4	19	10	11	3	10
Req	10	5	10	5	

1. Initial assignment by Northwest corner method: The Maximum time of allocated cell is 17 hours. Any cell having time element greater than 17 hours is cancelled, so that it will not in the programme.

Destinations (Time in hours)

Origins

	D_1	D_2	D_3	D_4	Avail
O_1	7, 5	8	4	5	5
O_2	8, 5	10, 2 (−)	2 (+)	3	7
O_3	7	6, 3 (+)	17, 5 (−)	8	8
O_4	~~19~~	10	11, 5	3, 5	10
Req	10	5	10	5	

By drawing loops, let us try to avoid 17 hours cell and include a cell, which is having time element less than 17 hours. The basic feasible solution is having $m + n - 1$ allocations.

Destinations (Time in hours)

Origins	D1	D2	D3	D4	Avail	Roc
O1	7 (5)	8	4	5	5	
O2	8 (5)	10	2 (2)	3	7	
O3	7	6 (5)	17 (3)	8	8	
O4	19	10	11 (5)	3 (5)	10	
Req	10	5	10	5		
Coc.						

Here also the maximum time of transport is 17 hours.

Destinations (Time in hours)

Origins	D1	D2	D3	D4	Avail	Roc
O1	7 (5)	8	4	5	5	
O2	8 (2)	10	2 (5)	3	7	
O3	7 (3)	6 (5)	~~17~~	8	8	
O4	~~19~~	10	11 (5)	3 (5)	10	
Req	10	5	10	5		
Coc.						

In this allocation highest time element is 11 hours. Let us try to reduce the same.

Destinations (Time in hours)

Origins	D1	D2	D3	D4	Avail	Roc
O1	7 (5)	8	4	5	5	
O2	8	10	2 (7)	3	7	
O3	7 (5)	6 (3)	~~17~~	8	8	
O4	~~19~~	10 (2)	11 (3)	3 (5)	10	
Req	10	5	10	5		
Coc.						

In this allocation also the maximum time element is 11 hours. Let us try to avoid this cell.

Destinations (Time in hours)

Origins

	D_1	D_2	D_3	D_4	Avail
O_1	7 (2)	8	4 (3)	5	5
O_2	8	10	2 (7)	3	7
O_3	7 (8)	6	~~17~~	8	8
O_4	~~19~~	10 (5)	~~11~~	3 (5)	10
Req	10	5	10	5	

No more reduction of time is possible. Hence the solution is optimal and the time required for completing the transportation is 10 Hours. T_{max} = 10 hours.

PURCHASE AND SELL PROBLEM: (TRADER PROBLEM)

Problem. 4.7 M/S Epsilon traders purchase a certain type of product from three manufacturing units in different places and sell the same to five market segments. The cost of purchasing and the cost of transport from the traders place to market centers in Rs. per 100 units is given below:

Place of Manufacture.	*Availability In units x 10000.*	*Manufacturing cost in Rs. per unit*	*Market Segments. (Transportation cost in Rs.per 100 units).*				
			1	*2*	*3*	*4*	*5*
Bangalore (*B*)	10	40	40	30	20	25	35
Chennai (*C*)	15	50	30	50	70	25	40
Hyderabad (*H*)	5	30	50	30	60	55	40
	Requirement in units × 10000		6	6	8	8	4

The trader wants to decide which manufacturer should be asked to supply how many to which market segment so that the total cost of transportation and purchase is minimized.

Solution

Here availability is 300000 units and the total requirement is 320000 units. Hence a dummy row (*D*) is to be opened. The following matrix shows the cost of transportation and purchase per unit in Rs. from manufacturer to the market centers directly.

	1	2	3	4	5	*Availability*
B	4040	4030	4020	4025	4035	10
C	5030	5050	5070	5025	5040	15
H	3050	3030	3060	3055	3040	5
D	0	0	0	0	0	2
Requirement.	6	6	8	8	4	32

Let us multiply the matrix by 100 to avoid decimal numbers and get the basic feasible solution by VAM.

Table. Avail: Availability. Req: Requirement, Roc: Row opportunity cost, Coc: Column opportunity cost.

Tableau. I Cost of transportation and purchase Market segments.

	1	2	3	4	5	Avail.	Roc
B	4040	4030	4020	4025	4035	10	
C	5030	5050	5070	5025	5040	15	
H	3050	3030	3060	3055	3040	5	
D	0	0	0	0	0	2	
Req.	6	6	8	8	4	32	
Coc.							

Tableau. II Cost of transportation and purchase Market segments.

	1	2	3	4	5	Avail	Roc
B	4040	4030	4020	4025	4035	10	5
C	5030	5050	5070	5025	5040	15	5
H	3050	3030	3060	3055	3040	5	10
D	0	0	0 2	0	0	2	0
Req	6	6	8	8	4		
Coc	3050	3030	3060	3055	3040		

Tableau. II Cost of transportation and purchase Market segments.

	1	2	3	4	5	Avail.	Roc
B	4040	4030	4020	4025	4035	10	5
C	5030	5050	5070	5025	5040	15	5
H	3050	3030 5	3060	3055	3040	5	10
D	0	0	0 2	0	0	2	
Req.	6	6	6	8	4	30	
Coc.	990	1000	960	970	995		

Tableau. II Cost of transportation and purchase
Market segments.

	1	2	3	4	5	Avail.	Roc
B	4040	4030	4020 5	4025	4035	10	5
C	5030	5050	5070	5025	5040	15	5
H	3050	3030 5	3060	3055	3040	5	5
D	0	0	0 2	0	0	2	0
Req.	6	1	6	8	4	27	
Coc.	990	1020	1050	1000	1005		

Tableau. II Cost of transportation and purchase
Market segments.

	1	2	3	4	5	Avail.	Roc
B	4040	4030 1	4020 6	4025	4035	4	5
C	5030	5050	5070	5025	5040	15	5
H	3050	3030 5	3060	3055	3040		
D	0	0	0 2	0	0		
Req.	6	1		8	4	19	
Coc.	990	1020		1000	1005		

Tableau. II Cost of transportation and purchase
Market segments.

	1	2	3	4	5	Avail.	Roc
B	4040	4030 1	4020 6	4025	4035 **3**	3	5
C	5030	5050	5070	5025	5040	15	5
H	3050	3030 5	3060	3055	3040		5
D	0	0	0 **2**	0	0		0
Req.	6			8	4	18	
Coc.	990			1000	1005		

Tableau. II Cost of transportation and purchase
Market segments.

	1	2	3	4	5	Avail.
B	4040	4030 1	4020 6	4025	4035 3	10
C	5030 6	5050	5070	5025 8	5040 1	15
H	3050	3030 5	3060	3055	3040	5
D	0	0	0 **2**	0	0	2
Req.	6	6	8	8	4	32

Final Allocation by MODI method.

Tableau. II Cost of transportation and purchase
Market segments.

	1	2	3	4	5	Avail.
B	4040	4030 1	4020 8	4025	4035 1	10
C	5030 6	5050	5070	5025 8	5040 1	15
H	3050	3030 5	3060	3055	3040	5
D	0	0	0	0	0 2	2
Req.	6	6	8	8	4	32

Allocation:

From	*To*	*Load*	*Cost in Rs.*
Bangalore	2	10,000	4,03,000
Bangalore	3	80,000	32, 16,000
Bangalore	5	10,000	4, 03,000
Chennai	1	60,000	30, 18,000
Chennai	4	80,000	40, 20,000
Chennai	5	10,000	5, 04,000
Hyderabad	2	50,000	15, 15,000
Total cost in Rs.			1,30, 79,000

MAXIMISATION PROBLEM: (PRODUCTION AND TRANSPORTATION SCHEDULE FOR MAXIMIZATION)

This type of problems will arise when a company having many units manufacturing the same product and wants to satisfy the needs of various market centers. The production manager has to work out for transport of goods to various market centers to cater the needs. Depending on the production schedules and transportation costs, he can arrange for transport of goods from manufacturing units to the market centers, so that his costs will be kept at minimum. At the same time, this problem also helps him to prepare schedules to aim at maximizing his returns.

Problem.4.8. A company has three manufacturing units at *X*, *Y* and *Z* which are manufacturing certain product and the company supplies warehouses at *A*, *B*, *C*, *D*, and *E*. Monthly regular capacities for regular production are 300, 400 and 600 units respectively for *X*, *Y* and *Z* units. The cost of production per unit being Rs.40, Rs.30 and Rs. 40 respectively at units *X*, *Y* and *Z*. By working overtime it is possible to have additional production of 100, 150 and 200 units, with incremental cost of Rs.5, Rs.9 and Rs.8 respectively. If the cost of transportation per unit in rupees as given in table below, find the allocation for the total minimum production cum transportation cost. Under what circumstances one factory may have to work overtime while another may work at under capacity?

Transportation cost in Rs. To

From	*A*	*B*	*C*	*D*	*E*
X	12	14	18	13	16
Y	11	16	15	11	12
Z	16	17	19	16	14
REQ	400	400	200	200	300

(a) If the sales price per unit at all warehouses is Rs. 70/- what would be the allocation for maximum profit? Is it necessary to obtain a new solution or the solution obtained above holds valid?

(b) If the sales prices are Rs.70/-, Rs. 80/-, Rs. 72/-, Rs. 68/- and Rs. 65/- at *A*, *B*, *C*, *D* and *E* respectively what should be the allocation for maximum profit?

Solution: Total production including the overtime production is 1750 units and the total requirement by warehouses is 1500 units. Hence the problem is unbalanced. This can be balance by opening a Dummy Row (*DR*), with cost coefficients equal to zero and the requirement of units is 250. The cost coefficients of all other cells are got by adding production and transportation costs. The production cum transportation matrix is given below:

	A	*B*	*C*	*D*	*E*	*DC*	*Availability*
X	52	54	58	53	56	0	300
Y	41	46	45	41	42	0	400
Z	56	57	59	56	54	0	600
XOT	57	59	63	58	61	0	100
YOT	50	55	54	50	51	0	150
ZOT	64	65	67	64	62	0	200
Requirement:	400	400	200	200	300	250	1750

Initial Basic feasible solution by VAM:

	A	B	C	D	E	DC	Avail.	u_i
X	52 300	54 0	58 –2	53 0	56 –4	0 –7	300	52
Y	41 –1	46 –4	45 –1	41 100	42 300	0 –17	400	40
Z	56 –1	57 400	59 100	56 100	54 –1	0 –2	600	55
XOT	57 50	59 0	63 –2	58 0	61 –4	0 **50**	100	57
YOT	50 50	55 –3	54 100	50 –1	51 –1	0 –7	150	50
ZOT	64 –7	65 –6	67 –6	64 –6	62 –5	0 200	200	57
REQ. v_i	400 0	400 2	200 4	200 1	300 0	250 –57	1750	

As we have $m + n - 1$ (= 11) allocations, the solution is feasible and all the opportunity costs of empty cells are negative, the solution is optimal.

Allocations:

Cell	Load	Cost in Rs.
XA	300	300 × 52 = 15,600
YD	100	100 × 41 = 4,100
YE	300	300 × 40 = 12,000
ZB	400	400 × 54 = 21,000
ZC	100	100 × 59 = 5,900
ZD	100	100 × 56 = 5,600
XOT A	50	50 × 57 = 2,850
XOT DR	50	50 × 0 = 0
YOT A	50	50 × 50 = 5,500
YOT C	100	100 × 54 = 5,400
ZOT DR	50	50 × 0 = 0
Total Cost in Rs.		75, 550

Allocation by VAM:

(1)

	A	*B*	*C*	*D*	*E*	*DC*	*AVAIL*	*ROC*
X	52	54	58	53	56	0	300	52
Y	41	46	45	41	42	0	400	41
Z	56	57	59	56	54	0	600	54
XOT	57	59	63	58	61	0	100	50
YOT	50	55	54	50	51	0	150	50
ZOT	64	65	67	64	62	0 **(200)**	2 00	62 ←
REQ	400	400	2 00	2 00	300	2 50	1750	
COC	9	8	9	9	9	0		

As for one allocation a row and column are getting eliminated. Hence, the degeneracy occurs.

(2)

	A	*B*	*C*	*D*	*E*	*DC*	*AVAIL*	*ROC*
X	52	54	58	53	56	0	300	52
Y	41	46	45	41	42	0	400	41
Z	56	57	59	56	54	0	600	54
XOT	57	59	63	58	61	0 **(50)**	100	57 ←
YOT	50	55	54	50	51	0	150	50
REQ	400	400	2 00	2 00	300	2 50	15 50	
COC	9	8	9	9	9	0		

(3)

	A	*B*	*C*	*D*	*E*	*AVAIL*	*ROC*
X	52	54	58	53	56	300	1
Y	41	46	45	41	42 **(300)**	400	0
Z	56	57	59	56	54	600	2
XOT	57	59	63	58	61	50	2
YOT	50	55	54	50	51	150	0
REQ	400	400	2 00	2 00	300	1500	
COC	9	8	9	9	9 ↑		

Here also for one allocation, a row and a column are getting eliminated. Degeneracy will occur. In all we may have to allocate two ∈ s to two empty cells.

(4)

	A	*B*	*C*	*D*	*AVAIL*	*ROC*
X	52	54	58	53	300	1
Y	41	46	45	41 **(100)**	100	0
Z	56	57	59	56	600	0
XOT	57	59	63	58	50	1
YOT	50	55	54	50	150	0
REQ	400	400	2 00	2 00	1200	
COC	9	8	9	9		

(5)

	A	*B*	*C*	*D*	*Avail*	*Roc*
X	52	54	58	53	300	1
Z	56	57	59	56	600	0
XOT	57	59	63	58	50	1
YOT	50	55	54 **(150)**	50	150	0
Req	400	400	200	100	1100	
Coc	2	1	4	3		

(6)

	A	*B*	*C*	*D*	*Avail*	*Roc*
X	52 **(300)**	54	58	53	300	1
Z	56	57	59	56	600	0
XOT	57	59	63	58	50	1
Req	400	400	50	100	950	
Coc	4	3	1	3		

(6)

	A	*B*	*C*	*D*	*Avail*	*Roc*
Z	56	57	59 **(50)**	56	550	0
XOT	57	59	63	58	50	1
Req	100	400	50	100	600	
Coc	1	2	4	2		

(7)

	A	*B*	*D*	*Avail*	*Roc*
Z	56	57	56 **(100)**	550	0
XOT	57	59	58	50	1
Req	100	400	100	600	
Coc	1	2	2		

(8)

	A	*B*	*Avail*	*Roc*
Z	56	57 **(400)**	450	1
XOT	57	59	50	2
Req	100	400	500	
Coc	1	2		

(9)

	A	*Avail*
Z	56 **(50)**	50
XOT	57 **(50)**	50
	100	

In the table showing optimal solution, we can understand that the company *X* has to work 50% of its over time capacity, and company *Y* has to work 100% of its overtime capacity and company Z will not utilize its overtime capacity.

(a) Here the total profit or return that the trading company gets is equals to Sales revenue – total expenses, which include manufacturing cost and transportation cost. Hence,

Profit = (Total Sales Revenue) – (Manufacturing cost + transportation cost).

In the question given the sales price is same in all market segments, hence, the profit calculated is independent of sales price. Hence the programme, which minimizes the total cost will, maximizes the total profit. Hence the same solution will hold good. We need not work a separate schedule for maximization of profit.

(b) Here sales price in market segments will differ. Hence we have to calculate the total profit by the formula given above for all the markets and work for solution to maximise the profit.

The matrix showing the total profit earned by the company:

	A	B	C	D	E	DC	Avail.	u_j
X	$\boxed{18}$ 3	$\boxed{26}$ 300	$\boxed{14}$ 5	$\boxed{15}$ 3	$\boxed{9}$ 8	$\boxed{0}$ 6	900	6
Y	$\boxed{29}$ 300	$\boxed{34}$ 14	$\boxed{27}$ ε	$\boxed{27}$ 1	$\boxed{25}$ 0	$\boxed{0}$ 14	400	14
Z	$\boxed{14}$ 1 ↑	$\boxed{23}$ ∈	$\boxed{13}$ 200	$\boxed{12}$ 50	$\boxed{11}$ 300	$\boxed{0}$ 50	600	0
XOT	$\boxed{13}$ 3	$\boxed{21}$ 100	$\boxed{9}$ 5	$\boxed{10}$ 3	$\boxed{4}$ 8	$\boxed{0}$ 1	100	1
YOT	$\boxed{20}$ 1	$\boxed{25}$ 1	$\boxed{18}$ 1	$\boxed{18}$ 150	$\boxed{14}$ 3	$\boxed{0}$ 6	150	6
ZOT	$\boxed{6}$ 9	$\boxed{15}$ 5	$\boxed{5}$ 8	$\boxed{4}$ 8	$\boxed{3}$ 8	$\boxed{0}$ 200	200	0
Req.	400	400	200	200	300	250	1750	
Coc.	15	23	13	12	11	0		

As all the opportunity cost of empty cells are positive (maximization problem), the solution is optimal.

The allocations are:

Cell	*Load*	*Cost in Rs.*
XB	300	300 × 26 = 7, 800
YA	400	400 × 29 = 11, 600
ZC	200	200 × 13 = 2, 600
ZD	50	50 × 12 = 600
ZE	300	300 × 11 = 3, 300
Z DR	50	50 × 0 = 0
XOT B	100	100 × 21 = 2, 100
YOT D	150	150 × 18 = 2, 700
ZOT DR	200	200 × 0 = 0
Profit in Rs.		= 30, 700

	A	*B*	*C*	*D*	*E*	*DC*	*Avail*	*Coc*
X	18	26	14	15	9	0	300	8
Y	29 **400**	34	27	27	25	0	400	5 ←
Z	14	23	13	12	11	0	600	9
XOT	13	21	9	10	4	0	100	8
YOT	20	25	18	18	14	0	150	5
ZOT	6	15	5	4	3	0	200	9
Req	400	400	200	200	300	250	1750	
Coc	11	8	9	9	9	0		

As for one allocation a row and column are getting eliminated. Hence, the degeneracy occurs.

(2)

	B	*C*	*D*	*E*	*DC*	*Avail*	*Coc*
X	26 **300**	14	15	9	0	300	11 ←
Z	23	13	12	11	0	600	10
XOT	21	9	10	4	0	100	11
YOT	25	18	18	14	0	150	7
ZOT	15	5	4	3	0	200	10
Req	400	200	200	300	250	1350	
Coc	1	4	3	5	0		

(3)

	B	*C*	*D*	*E*	*DC*	*Avail*	*Coc*
Z	23	13	12	11	0	600	10
XOT	21 **100**	9	10	4	0	100	11 ←
YOT	25	18	18	14	0	150	7
ZOT	15	5	4	3	0	200	10
Req	100	200	200	300	250	1050	
Coc	2 ↑	5	6	3	0		

Here also for one allocation, a row and a column are getting eliminated. Degeneracy will occur. In all we may have to allocate two $\in$ s to two empty cells.

(4)

	C	*D*	*E*	*DC*	*Avail*	*Coc*
Z	13	12	11	0	600	1
YOT	18	18 **150**	14	0	150	0
ZOT	5	4	3	0	200	1
Req	200	200	300	250	950	
Coc	5	6	3	0		

(5)

	C	*D*	*E*	*DC*	*Avail*	*Coc*
Z	13 **200**	12	11	0	600	1
ZOT	5	4	3	0	200	1
Req	200	50	300	250	800	
Coc	8	8	8	0		

(6)

	D	*E*	*DC*	*Avail*	*Coc*
Z	12	11 **300**	0	400	1
ZOT	4	3	0	200	1
Req	50	300	250	600	
Coc	8	8	0		

(7)

	D	DC		
Z	12 **50**	0	100	12 ←
ZOT	4	0	200	4
	50	250	300	
	8	0		

(8)

	DC	
Z	**50** 0	50
ZOT	**200** 0	200
	250	250

Problem. 4.9. A company has booked the orders for its consignment for the months of April, May, June and July as given below:

April: 900 units, May: 800 units, June: 900 units and July: 600 units. The company can produce 750 units per month in regular shift, at a cost of Rs. 80/- per unit and can produce 300 units per month by overtime production at a cost of Rs. 100/- per unit. Decide how much the company has to produce in which shift for minimizing the cost of production. It is given that there is no holding cost of inventory.

Solution: Remember here the production of April is available to meet the orders of April and subsequent months. But the production of May cannot be available to meet the demand of April. Similarly, the production of June is not available to meet the demand of April, May, but it can meet the demand of June and subsequent months and so on. Hence very high cost of production is allocated to the cells (Infinity or any highest number greater than the costs given in the problem), which cannot meet the demands of previous months (*i.e.* back ordering is not allowed). Here total availability is 4200 units and the total demand is for 3200 units. Hence we have to open a dummy column (DC), with cost coefficients equal to zero. The balanced matrix is shown below. Let us find the initial basic feasible solution by Northwest corner method and apply optimality test by MODI method.

A: April, M: May, J: June, Jl: July, AT: April Over time, MT: May overtime, JT: June overtime, JLT: July over time. DC : Dummy column.

Tableau 1.

Month of Demand (Cost in Rs)

	A	M	J	Jl	DC	Avail.	u_j
A	80 750	80 0	80 0	80 0	0 0	750	0
AT	100 150	100 150	100 0	100 0	0 20	300	20
M	X	80 650	80 100	80 0	0 0	750	0
MT	X	100	100 300	100 0	0 20	300	20
J	X	X	80 500	80 250	0 0	750	0
JT	X	X	100 0	100 – 300	0 20 +	300	20
JL	X	X	X	80 50 +	0 – 700	750	0
JLT	X	X	X	100 –20	0 300	300	0
Req.	900	800	900	600	1000	4200	
v_j	80	80	80	80	0		

Here the cell JT DC is having highest opportunity cost. Hence let us include the cell in the revised programme. To find the opportunity costs of empty cells, the row number u_i and column number v_j are shown. The cells marked with (*X*) are avoided from the programme. We can also allocate very high cost for these cells, so that they will not enter into the programme.

Tableau II. Revised programme.

Month of Demand (Cost in Rs)

	A	M	J	JI	DC	Avail.	u_j
A	80 / 750	80 / 0	80 / 0	80 / 0	0 / 0	750	0
AT	100 / 150	100 / 150	100 / 0	100 / 0	0 / 0	300	20
M	X	80 / 650	80 / 100	80 / 0	0 / 0	750	0
MT	X	100	100 / 300	100 / 0	0 / 0	300	20
J	X	X	80 / 500	80 / 250	0 / 0	750	0
JT	X	X	100 / –20	100 / **–20**	0 / 300	300	20
JL	X	X	X	80 / 350	0 / 400	750	0
JLT	X	X	X	100 / –20	0 / 300	300	0
Req.	900	800	900	600	1000	4200	
v_j	80	80	80	80	0		

As the opportunity costs of all empty cells are either zeros or negative elements, the solution is optimal. As many empty cells are having zero as the opportunity cost, they can be included in the solution and get alternate solution.

Allocations:

Demand month.	*Production of the month*	*Load*	*Cost in Rs.*
April	April regular	750	750 × 80 = 60,000
April	April over time	150	150 × 100 = 15,000
May	April over time	150	150 × 100 = 15,000
May	May regular	650	650 × 80 = 52,000
June	May regular	100	100 × 80 = 8,000
June	May over time	300	300 × 100 = 30,000
June	June Regular	500	500 × 80 = 40,000
July	June regular	250	250 × 80 = 20,000
July	July regular	350	350 × 80 = 28,000
Dummy column	June over time	300	300 × 0
Dummy Column	July regular	300	300 × 0
Dummy column	July over time	300	300 ×0
	Total cost in Rs.:		2,68,000

Problem: 4.10. Let us slightly change the details given in the problem 4.9. It is given that production of a month could be stored and delivered in next month without extra costs. Let us now consider that there is a cost associated with inventory holding or inventory carrying cost. Let the inventory carrying cost is Rs. 20 per month decide the new allocation.

Solution: In the cost matrix, for regular production, the cost is Rs. 80/-, for overtime production, the cost is Rs. 100 and for the stock held the inventory carrying cost is Rs. 20/ per month. If the stock is held for two months the inventory carrying cost is Rs. 40/-. That is if the production of April is supplied in June the cost will be Rs. 80/- + Rs. 40/- =

Rs. 120/- and do on. The initial basic feasible solution is obtained by Northwest corner method.

Production cost in Rs.

	April	May	June	July	DC	Avail.	u_i	
April	80 750	100 0	120 0	140 –20	0 40	750	120	
AOT	100 150	100 150 –	140 0	160 –20	0 + 60	300	140	
May	X	80 650 +	100 100 –	120 –20	0 20	750	100	
MOT	X	100	120 50 +	140 –20	0 250 –	300	120	
June	X	X	80 750	100 –20	0 0	750	80	
JOT	X	X	100	120 –40	0 300	300	80	
July	X	X	X	80 300	0 150	750	80	
JLOT	X	X	X	100 –20	0 300	300	80	
Req.	900	800	900	600	1000	4200		
v_j	–40	–20	0	0	–80			

Cell AOT DC is having highest positive opportunity cost. Hence we have to include this in the revised programme.

Production cost in Rs.

	April	May	June	July	DC	Avail.	u_i	
April	80 750	100 0	120 –20	140 –80	0 –20	750	–20	
AOT	100 150	120 50	140 –20	160 –80	0 **150**	300	0	
May	X	80 750	100 **–20**	120 –20	0 –40	750	–40	
MOT	X	100 20	120 150	140 –60	0 **150**	300	0	
June	X	X	80 750	100 –60	0 –40	750	–40	
JOT	X	X	100 20	120 –40	0 300	300	0	
July	X	X	X	80 **600**	0 150	750	0	
JLOT	X	X	X	100 –20	0 300	300	0	
Req.	900	800	900	600	1000	4200		
v_j	100	120	120	80	0			

In the above matrix, two cells, MO M and JO J are having positive opportunity costs = 20. Hence, they may be included in the revised programme. If we include them in the programme, the final optimal solution will be as follows:

Production cost in Rs.

	April	May	June	July	DC	Avail.	u_i	
April	80 (750)	100 –20	120 –40	140 –80	0 –20	750	–20	
AOT	100 (150)	120 –20	140 –40	160 –80	0 (150)	300	0	
May	X	80 (750)	100 –20	120 –80	0 –20	750	–20	
MOT	X	100 (50)	120 –20	140 –80	0 (250)	300	0	
June	X	X	80 (750)	100 –40	0 –20	750	–20	
JOT	X	X	100 (150)	120 –40	0 (150)	300	0	
July	X	X	X	80 (600)	0 (150)	750	0	
JLOT	X	X	X	100 –20	0 **300**	300	0	
Req.	900	800	900	600	1000	4200		
v_j	100	100	100	80	0			

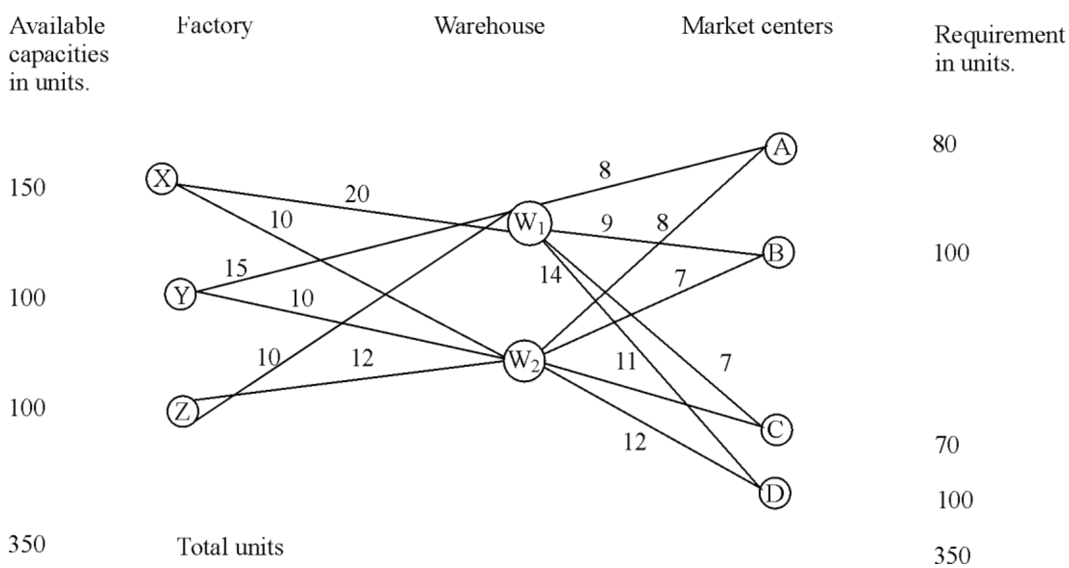
As all the opportunity costs of empty cells are negative, the solution is optimal. The optimal allocations are:

Month of demand	*Month of production*	*load*	*cost in Rs.*		*Rs.*
April	April regular	750	750 × 80	=	60,000
April	April over time	150	150 × 100	=	15,000
Dummy Col	April over time	150	150 × 0	=	0
May	May regular	750	750 × 80	=	60,000
May	May over time	50	50 × 100	=	5,000
Dummy Column	May over time	250	250 × 0	=	0
June	June regular	750	750 × 80	=	60,000
June	June over time	150	150 × 100	=	15,000
Dummy column	June over time	150	150 × 0	=	0
July	July regular	600	600 × 80	=	48,000
Dummy column	July regular	150	150 × 0	=	0
Dummy column	July overtime	300	300 × 0	=	0
	Total cost in Rs.				2,63,000

TRANSSHIPMENT PROBLEM

We may come across a certain situation, that a company (or companies) may be producing the product to their capacity, but the demand arises to these products during certain period in the year or the demand may reach the peak point in a certain period of the year. This is particularly true that products like Cool drinks, Textbooks, Notebooks and Crackers, etc. The normal demand for such products will exist, throughout the year, but the demand may reach peak points during certain months in the year. It may not possible for all the companies put together to satisfy the demand during peak months. It is not possible to produce beyond the capacity of the plant. Hence many companies have their regular production throughout the year, and after satisfying the existing demand, they stock the excess production in a warehouse and satisfy the peak demand during the peak period by releasing the stock from the warehouse. This is quite common in the business world. Only thing that we have to observe the inventory carrying charges of the goods for the months for which it is stocked is to be charged to the consumer. Take for example crackers; though their production cost is very much less, they are sold at very high prices, because of inventory carrying charges. When a company stocks its goods in warehouse and then sends the goods from warehouse to the market, the problem is known as **Transshipment problem**. Let us work one problem and see the methodology of solving the Transshipment Problem.

Problem. 4.11. A company has three factories *X*, *Y* and *Z* producing product *P* and two warehouses to stock the goods and the goods are to be sent to four market centers *A*, *B*, *C* and *D* when the demand arises. The figure given below shows the cost of transportation from factories to warehouses and from warehouses to the market centers, the capacities of the factories, and the demands of the market centers. Formulate a transportation matrix and solve the problem for minimizing the total transportation cost.



Solution:

To formulate a transportation problem for three factories and four market centers, we have to find out the cost coefficients of cells. For this, if we want the cost of the cell *XA*, the cost of transportation from *X* to warehouse W_1 + Cost transportation from W_1 to market center *A* are calculated and as our objective is to minimize the cost, the least of the above should be entered as the cost coefficient of cell *XA*. Similarly, we have to workout the costs and enter in the respective cells.

Cell *XA*: Route X-W_1-A and X- W_2- A minimum of these two (28 and 18) i.*e* 18
Cell *XB* Route X - W_1 - B and X - W_2 - B Minimum of the two is (29, 17) *i.e* 17
Cell *XC* Route X - W_1 - C and X- W_2 - C Minimum of the two is (27, 11) *i.e* 11
Cell *XD* Route X- W_1 - D and X- W_2 - D Minimum of the two is (34, 22) *i.e* 22

Similarly we can calculate for other cells and enter in the matrix. The required transportation problem is:

	A	*B*	*C*	*D*	*Available*
X	18	17	21	22	150
Y	18	17	21	22	100
Z	18	19	17	24	100
Required.	80	100	70	100	350

Basic Feasible Solution by VAM:

Market centers (Cost in Rs.)

	A	B	C	D	Available	u_i
X	18 (80)	17 + (70)	21	22	150	0
Y	18	17 − (30)	21	22 + (70)	100	0
Z	18	19	17 (70)	24 − (30)	100	2
REQUIRED	80	199	70	100	350	
v_j	18	17	15	22		

Market centers (Cost in Rs.)

	A	B	C	D	Available	u_i
X	18 (50)	17 (100)	21 –4	22 0	150	0
Y	18 0	17 (ε)	21 –4	22 (100)	100	0
Z	18 (30)	19 –2	17 (70)	24 –2	100	0
REQUIRED	80	100	70	100	350	
v_j	18	17	17	22		

As the opportunity costs of all empty cells are negative, the solution is optimal. The optimal allocation is:

Cell	*Route*	*Load*	*Cost in Rs.*		*Rs.*
XA	*X*-W_2-*A*	50	50 × 18	=	900
XB	*X* - W_2 - *B*	100	100 × 17	=	1700
YB	*Y*- W_2 - *B*	ε	---	=	----
YD	*Y* - W_2- *D*	100	100 × 22	=	2200
ZA	*Z*- W_1- *A*	30	30 × 18	=	540
ZC	*Z* - W_1 - *C*	70	70 × 17	=	1190
			Total Cost in Rs.		6530

(The answer shows that the capacity of W_2 is 250 units and capacity of W_1 is100 units).

(1)

	A	*B*		*C*	*D*	*Avail*	*Roc*
X	18	17		21	22	150	0
Y	18	17		21	22	100	0
Z	18	19		17	24	100	2
Req.	80	100		70	100	350	
Coc	18	17		15	22		

(2)

		A	*B*	*D*	*Avail*	*Roc*
X		18	17	22	150	1
Y	**80**	18	17	22	100	1
Z		18	19	24	30	1
Req.		80	100	100	280	
Coc		0	0	0		

(3)

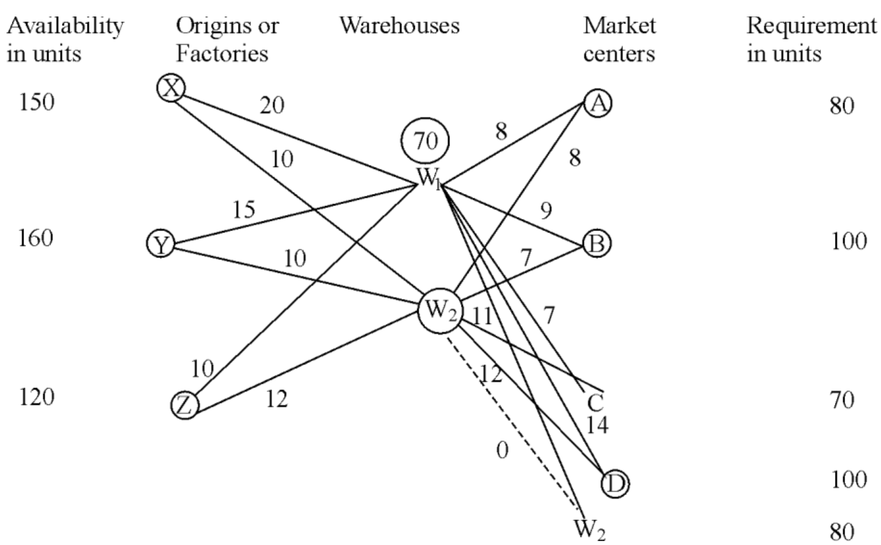
	B	*D*	*Avail*	*Roc*
X	17	22	70	5
	70			
Y	17	22	100	5
Z	19	24	30	5
Req.	100	100	200	
Coc	0	0		

(4)

	B	*D*	*Avail*	*Roc*
Y	17 3	22	100	5 ←
Z	19	24	30	5
Req.	30	100	200	
Coc	2	2		

	D	*Avail*	*Roc*
Y	22 **70**	70	←
Z	24 **30**	30	
Req.	100	100	
Coc			

Problem 4.12.



In the above some restrictions are imposed. The restrictions are:

Let warehouse W_1 be pure transshipment warehouse and W_2 is transshipment as well as distribution point.

(i) The capacity limitation on W_1 = 70 units.

(ii) The warehouse W_2 also deals with direct distribution of 80 units.

As per the given conditions, the following discussion will hold good.

Solution:

1. As a source and intermediate transshipment node, W_1 has the capacity limitations of 70 units. Hence, availability of W_1 and requirement of destination W_1 is 70 units.

2. W_2 has no capacity limitation. However, it deals partial direct distribution of 80 units. Therefore, as a source its availability should be the difference between the total availability from all factories *i.e* X, Y and Z less its own direct distribution. 430 – 80 = 350.
3. As an intermediate destination, it should have the capacity to route entire production *i.e*. 430 units.
4. Unit cost of transportation from X, Y, and Z to destinations A, B, C and D, through W_1 and W_2 can be had from figure given, this can be entered in the table- 1 showing the initial transportation matrix.
5. There is no direct transportation from X,Y, and Z to destinations A, B, C and D. To avoid this direct routes we can allocate very high cost of transportation costs for these cells or we can avoid these cells by crossing them, *i.e*. eliminating them from the programme.
6. W_1 as source giving to W_1 as warehouse or sink, and W_2 as a source giving to W_2 as warehouse or sink will have zero cost.

	W_1	W_2	A	B	C	D	Avail.	u_i
X	$\boxed{20}$ –12	$\boxed{10}$ 150	X	X	X	X	150	10
Y	$\boxed{15}$ –7	$\boxed{10}$ 160	X	X	X	X	160	10
Z	$\boxed{10}$ 70	$\boxed{12}$ 50	X	X	X	X	120	12
W_1	$\boxed{0}$ –2	X	$\boxed{8}$ ε	$\boxed{9}$ –2	$\boxed{7}$ 70	$\boxed{14}$ –2	70	0
W_2	X	$\boxed{0}$ 70	$\boxed{8}$ 80	$\boxed{7}$ 100	$\boxed{11}$ 0	$\boxed{12}$ 100	350	0
Req.	70	430	80	100	70	100	850	
v_j	–2	0	8	7	11	12		

As the total number of allocations are $m + n – 1$ after allocating $\in$ to cell W_1A, the solution is a basic feasible solution. By giving the optimality test by MODI method, we see that all the opportunity costs of empty cells are negative and hence the solution is optimal.

The allocation:

Cell	*Load*	*Cost in Rs.*		*Rs.*
XW_2	150	150 × 19	=	1500
YW_2	160	160 × 10	=	1600
ZW_1	70	70 × 10	=	700
ZW_2	50	50 × 12	=	600
W_1A	ε	-----	--	-----
W_1C	70	70 × 7	=	140
W_2W_2	70	70 × 0	=	0
W_2A	80	80 × 8	=	160
W_2B	100	100 × 7	=	700
W_2D	100	100 × 12	=	1200
		Total Cost in Rs.		6,600

VAM:

(1)

	W_1	W_2	*A*	*B*	*C*	*D*	*Avail*	*ROC*
X	20	10	X	X	X	X	150	10
Y	15	10	X	X	X	X	160	5
Z	10 **70**	12	X	X	X	X	120	2
W_1	0	X	8	9	7	14	70	1
W_2	X	0	8	7	11	12	350	1
Req.	70	430	80	100	70	100	850	
COC	10	10	0	2	4	2		

(2)

	W_2	*A*	*B*	*C*	*D*	*Avail*	*ROC*
X	10	X	X	X	X	150	10
Y	10	X	X	X	X	160	10
Z	12	X	X	X	X	50	12
	50						
W_1	X	8	9	7	14	70	1
W_2	0	8	7	11	12	350	7
Req.	430	80	100	70	100	780	
COC	10	0	2	4	2		

(3)

	W_2	*A*	*B*	*C*	*D*	*Avail*	*ROC*
X	10	X	X	X	X	150	10
	150						
Y	10	X	X	X	X	160	10
W_1	X	8	9	7	14	70	1
W_2	0	8	7	11	12	350	7
Req.	380	80	100	70	100	730	
COC	10	0	2	4	2		

(4)

	W_2	*A*	*B*	*C*	*D*	*Avail*	*ROC*
Y	10 **160**	X	X	X	X	160	10
W_1	X	8	9	7	14	70	1
W_2	0	8	7	11	12	350	7
Req.	230	80	100	70	100	580	
COC	10	0	2	4	2		

(5)

	W_2	*A*	*B*	*C*	*D*	*Avail*	*ROC*
W_1	X	8	9	7	14	70	1
W_2	0 **70**	8	7	11	12	350	7
Req.	70	80	100	70	100	420	
COC	INF	0	2	4	2		

(6)

	A	*B*	*C*	*D*	*Avail*	*ROC*
W_1	8	9	7 **70**	14	70	1
W_2	8	7	11	12	280	1
Req.	80	100	70	100	350	
COC	0	2	4	2		

(7)

	A	*B*	*D*	*Avail*	*ROC*
W_2	8 **80**	7 **100**	12 **100**	280	
Req.	80	100	100	280	
COC					

REDUNDANCY IN TRANSPORTATION PROBLEMS

Some times, it may very rarely happen or while writing the alternate solution it may happen or during modifying the basic feasible solution it may happen that the number of occupied cells of basic feasible solution or some times the optimal solution may be greater than $m + n - 1$. This is called **redundancy in transportation problem**. This type of situation is very helpful to the manager who is looking about shipping of available loads to various destinations. This is as good as having more number of independent simultaneous equations than the number of unknowns. It may fail to give unique values of unknowns as far as mathematical principles are concerned. But for a transportation manager, it enables him to plan for more than one orthogonal path for an or several cells to evaluate penalty costs, which obviously will be different for different paths.

SENSITIVITY ANALYSIS

(a) Non - basic variables

While discussing MODI method for getting optimal solution, we have discussed significance of **implied cost**, which fixes the upper limit of cost of the empty cell to entertain the cell in the next programme. Now let us discuss the influence of variations in present parameters on the optimum solution i.e **sensitivity of optimal solution for the variations in the costs of empty cells and loaded cells**. If unit cost of transportation of a particular non-basic variable changes, at what value of the cost of present optimum will no longer remain optimum? To answer this question, in the first instance, it is obvious that as the empty cell is not in the solution, any increase in its unit transportation cost will to qualify it for entering variable. But if the unit cost of empty cells is reduced the chances of changing the optimum value may be examined. Let us take an optimum solution and examine the above statement.

	A	B	C	D	E	Avail.	u_i
X	10 –3	15 –3	17 –1	19 –27	16 50	50	0
Y	20 –23	12 40	16 30	18 –26	20 –4	70	0
Z	9 40	14 10	X	10 **30**	18 0	80	2
DR	0 –9	0 –4	0 30	0 –24	0 20	50	–16
Req.	40	50	60	30	70	250	
v_j	7	12	16	–8	16		

In the solution shown above as all the opportunity costs of empty cells are negative. Consider empty cell *XA*. Its opportunity cost is Rs. -3/- This means to say that the units cost of transportation of cell *XA* decreases by Rs.3/- or more *i.e* Rs.10/- the unit cost of transportation of the empty cell *XA* minus 3 = 7, or less than 7 the optimal solution changes, *i.e*. the cell *XA* will become eligible for entering into solution. Hence this cost, which shows the limit of the unit cost of empty cell, is known as implied cost in transportation problems. We can see that the opportunity cost of empty cell *ZE* is zero. This shows that the cell ZE is as good as a loaded cell and hence we can write alternate solutions by taking the cell *ZE* into consideration. **(Note: No unit cost of transportation is given for the cell *ZC*. Hence that cell should not be included in the programme. For this purpose, we can cross the cell or allocate very high unit cost of transportation for the cell. In case zero or any negative element is given as the unit cost of transportation for a cell, the value can be taken for further treatment.)**

(b) Basic variables

If unit cost of loaded cell *i.e.* basic variable is changed, it affects the opportunity costs of several cells. Now let us take the same solution shown above for our discussion. In case the unit cost of transportation for the cell *XE* is θ instead of 16, and other values remaining unchanged. Now let us workout the opportunity costs of other cells.

	A	B	C	D	E	Avail.	u_i
X	$\boxed{10}$ θ – 19	$\boxed{15}$ θ – 19	$\boxed{17}$ θ – 17	$\boxed{19}$ θ – 27	$\boxed{\theta}$ 50	50	0
Y	$\boxed{20}$ –13	$\boxed{12}$ 40	$\boxed{16}$ 30	$\boxed{18}$ –10	$\boxed{20}$	70	θ + 16
Z	$\boxed{9}$ 40	$\boxed{14}$ **10**	X	$\boxed{10}$ 30	$\boxed{18}$ 0	80	θ + 18
DR	$\boxed{0}$ –9	$\boxed{0}$ –4	$\boxed{0}$ 30	$\boxed{0}$ –8	$\boxed{0}$ 20	50	–θ
Req.	40	50	60	30	70	250	
v_j	θ.9	θ.4	θ	θ–8	θ		

Cells *XA* and *XB* is positive when θ is > than 19. Cell *XC* is positive when θ is > 17 and cell *XD* is positive when θ is > 27. Other cells are not influenced by θ .

If unit cost of transportation increase and becomes 17, the present optimum may change. In case the unit cost of transportation of the cell *XA* is reduced, the solution will still remain optimum, as our objective is to minimize the total transportation cost.

A point to note here is we have used Northwest corner method and Vogel's approximation method to get basic feasible solution. Also we have discussed the least cost method and there are some methods such as row minimum and column minimum methods. These methods attempt to optimize the sub-system and do not consider marginal trade-offs. Therefore, such methods have no merit to serve useful purpose.

SUMMARY

1. Read the statement of the problem. Confirm whether you have to maximize the objective function or minimize the objective function.
2. Construct the transportation matrix.
3. Check whether the given problem is balanced or not.
4. If balanced proceed further. If not balanced, balance the problem by opening a dummy row or a dummy column depending on the need. Let the unit cost of transportation of cells of dummy row or column be zero.

5. If the problem is maximization one convert that into a minimization problem by multiplying the matrix by –1 or by subtracting all the elements of the matrix form the highest element in the matrix.
6. Find the basic feasible solution. The characteristics of the basic feasible solution are it must have $(m + n – 1)$ allocations, where **m** is the number of rows and **n** is the number of columns.
7. The basic feasible solution may be obtained by (*a*) Northwest corner method, (*b*) Least Cost method or Matrix minimum method, or (*c*) Vogel's approximation method or Opportunity cost method.
8. If initial allocations are equal to $(m + m – 1)$ proceed to next step. If it is not equal to $(m + n$ **-** $1)$ it is known as degenerate solution.
9. To solve degeneracy, add a small and negligible element $\in$ to empty cells. Take care to see that the $\in$ loaded cell do not make closed loop with other loaded cells when lines are drawn from epsilon loaded cells to other loaded cells by travelling vertically and horizontally by taking turns at loaded cells.
10. Write allocations and calculate the total cost of transportation.
11. Give optimality test to the basic feasible solution. Optimality test can be given by (*a*) Stepping stone method or (*b*) Modified distributing method or **MODI** method.
13. The characteristic of optimal solution is the opportunity costs of all empty cells are either negatives or zeros.
14. Remember if any empty cell has zero as its opportunity cost, then we can write alternate optimal solutions.
15. Write the allocations and calculate total transportation cost.
16. In case, the unit cost of transportation of any cell is zero or negative elements, take the same into considerations for further calculations. Suppose nothing is given in the cell as the unit cost of transportation, then presume that the route connecting the origin and the destination through that cell is not existing and cancel that cell and do not consider it at all while solving the problem, or else allocate very high cost of unit cost of transportation (infinity or any number which is greater than all the elements in the matrix), so that that cell will not enter into programme. (In maximization problem allocate a negative profit or return to the cell).

Problem 4.13. A company has three factories *X*, *Y*, and *Z* and four warehouses *A*, *B*, *C*, and *D*. It is required to schedule factory production and shipments from factories to warehouses in such a manner so as to minimize total cost of shipment and production. Unit variable manufacturing costs (UVMC) and factory capacities and warehouse requirements are given below:

From	*UVMC*	*To warehouses*				*Capacity in units per month.*
Factories.	Rs.	Unit shipping costs in Rs.				
		A	B	C	D	
X	10	0	1	1	2	75
Y	11	1	2	3	1	32
Z	12	4	3	3	6	67
Requirement:		65	24	16	15	

Find the optimal production and transportation schedule.

Solution: We have to optimize production and shipment cost. Hence the transportation matrix elements are the total of manufacturing cost plus transportation cost. For example, the manufacturing cost of factory *X* is Rs. 10. Hence the transportation and shipment cost will be equal to 10 + 0, 10 +1, 10 +1 and 10 +2 respectively for warehouses *A*, *B*, *C* and *D* respectively. As the total available is 174 units and the total demand is 120 units we have to open a dummy column with requirement of 54 units. The production cum transportation matrix is given below:

Production cum transportation cost per unit in Rs.

	A	B	C	D	DC	Avail	u_i
X	10 + 35	11 24	11 − 16	11 −2	0 −6	75	0
Y	12 17	13 0	14 −1	12 15	0 −4	32	2
Z	16 13	15 2	15 + 2	18 −2	0 54	67	6
Req.	65	24	16	15	54	174	
v_j	10	10	11	10	−6		

(1) Initial basic feasible solution by VAM :

	A	*B*	*C*	*D*	*DC*	*Avail*	*Roc*
X	10	11	11	11	0	75	10
Y	12	13	14	12	0	32	12
Z	16	15	15	18	0 **5**	67	15 ←
Req	65	24	16	15	5	174	
Coc	2	2	3	1	0		

(2)

	A	*B*	*C*	*D*	*Avail*	*Roc*
X	10	11	11 **16**	11	75	0
Y	12	13	14	12	32	0
Z	16	15	15	18	13	0
Req	65	24	16	15	124	
Coc	2	2	3 ↑	0		

(3)

	A	*B*	*D*	*Avail*	*Roc*
X	10	11 **24**	11	75	0
Y	12	13	12	32	1
Z	16	15	18	13	0
Req	65	24	15	104	
Coc	2	2	0		

(4)

	A	*D*	*Avail*	*Roc*
X	10 **35**	11	35	2
Y	12	12	32	0
Z	16	18	13	2
Req	65	15	80	
Coc	2	0		

(5)

	A	*D*	*Avail*	*Roc*
Y	12	12 **15**	32	0
Z	16	18	13	2
Req	30	15	45	
Coc	4	6		

(6)

	A	*Avail*	*Roc*
Y	12 **17**	17	0
Z	16 **13**	13	2
Req	30	30	
Coc	4		

Production cum transportation cost per unit in Rs.

	A	B	C	D	DC	Avail	u_i
X	10 48	11 24	11 29	11 –6	0 –4	75	0
Y	12 **17**	13 0	14 –1	12 15	0 –2	32	2
Z	16 –2	15 0	15 13 2	18 –4	0 54	67	4
Req.	65	24	16	15	54	174	
v_j	10	11	11	10	–4		

As there are $m + n - 1$ allocations and all the opportunity costs of empty cells are negative, the solution is optimal.

The optimal allocations are:

Cell	Load	Cost in Rs.		Rs.
XA	48	48 × 10	=	480
XB	24	24 × 11	=	264
XC	29	29 × 11	=	319
YA	17	17 × 12	=	204
YB	15	15 × 12	=	180
ZC	13	13 × 15	=	195
Z DC	54	54 × 0	=	0
	Total cost in RS.		=	1642

QUESTIONS

1. Explain the process of solving a transportation problem.
2. List out the differences and similarities between Resource allocation model and Transportation model in linear programming.
3. Explain the procedure of getting basic feasible solution by using VAM.
4. Explain what are degeneracy and redundancy in transportation problem. How do you solve degeneracy in transportation problem? Distinguish between tie and degeneracy in linear programming problem.
5. Is transportation problem is of maximization type or minimization type problem? If it is one of the two, how do you solve the other version of the transportation model?

6. How do you say that a transportation model has an alternate solution? In case it has an alternate optimal solution, how do you arrive at alternate solution?
7. What is transshipment problem? In what way it differs from general transportation problem?
8. Explain the terms: (*a*) Opportunity cost, (*b*) Implied cost, (*c*) Row opportunity cost, (*d*) Column opportunity cost.
9. The DREAM - DRINK Company has to work out a minimum cost transportation schedule to distribute crates of drinks from three of its factories *X*, *Y*, and *Z* to its three warehouses *A*, *B*, and *C*. The required particulars are given below. Find the least cost transportation schedule.

Transportation cost in Rs per crate.

From / To	*A*	*B*	*C*	*Crates Available.*
X	75	50	50	1040
Y	50	25	75	975
Z	25	125	25	715
Crates required.	1300	910	520	2730

10. The demand pattern for a product at for consumer centers, *A*, *B*, *C* and *D* are 5000 units, 7000 units, 4000 units and 2000 units respectively. The supply for these centers is from three factories *X*, *Y* and *Z*. The capacities for the factories are 3000 units, 6000 units and 9000 units respectively. The unit transportation cost in rupees from a factory to consumer center is given below in the matrix. Develop an optimal transportation schedule and find the optimal cost.

From:	*To*			
	A	*B*	*C*	*D*
X	8	9	12	8
Y	3	4	3	2
Z	5	3	7	4

11. From three warehouses, *A*, *B*, and *C* orders for certain commodities are to be supplied to demand points *X*, *Y*, and *Z*. Find the least cost transportation schedule with relevant information given below:

From Warehouses	*To demand points (Transportation cost in Rs. per units).*			*Availability in units.*
	X	*Y*	*Z*	
A	5	10	2	100
B	3	7	5	25
C	6	8	4	75
Units demand	105	30	90	

12. From three warehouses *A*, *B*, and *C* orders for certain commodities are to be supplied to demand points 1, 2, 3, 4 and 5 monthly. The relevant information is given below:

Warehouses	*Demand points (Transportation cost in Rs per unit.*					*Availability in units.*
	1	2	3	4	5	
A	4	1	2	6	9	100
B	6	4	3	5	7	120
C	5	2	6	4	8	120
Units demand:	40	50	70	90	90	

During certain month a bridge on the road-connecting warehouse *B* to demand point 3 is closed for traffic. Modify the problem suitably and find the least cost transportation schedule. (The demand must be complied with).

13. A tin box company has four factories that supply to 5 warehouses. The variable cost of manufacturing and shipment of one ton of product from each factory to each warehouse are shown in the matrix given below, Factory capacities and warehouse requirements are shown in the margin. After several iterations the solution obtained is also shown.

Warehouses (Cost in Rs. per unit)

Factories	A	B	C	D	E	DMY	Capacity
W	17	9	14 **25**	10 **30**	14	0 **20**	75
X	13 **10**	6 **20**	11 **15**	11	12	0	45
Y	6 **30**	17	9	12	12	0	30
Z	15	20	11 **10**	14	6 **40**	0	50
Req	40	20	50	30	40	20	200

(a) Is this an optimal solution? How do you know?

(b) Is there an alternate solution? If so find it.

(c) Suppose some new equipment was installed that reduces the variable operation cost by Rs. 2/- per ton in factory *X*, is the shipping schedule remain optimum? If not what is the new optimum?

(d) Suppose the freight charges from *W* to A were reduced by Rs.2/- would this change the shipping schedule? If so what is the new optimum?

(e) How much would the manufacturing cost have to be reduced in W before production would be increased beyond 55 tons?

14. A company has a current shipping schedule, which is being questioned by the management as to whether or not it is optimal. The firm has three factories and five warehouses. The necessary data in terms of transportation costs in Rs. per unit from a factory to a destination and factory capacities and warehouse requirements are as follows:

Factories. (Transportation costs in Rs. per unit.)

Warehouses.	*X*	*Y*	*Z*	*Requirement of warehouses in units.*
A	5	4	8	400
B	8	7	4	400
C	6	7	6	500
D	6	6	6	400
E	3	5	4	800
Factory capacities.	800	600	1100	

Solve for an optimal shipping schedule in terms of lowest possible shipping costs.

15. Solve the following transportation problem.

Destination

Source	*A*	*B*	*C*	*D*	*E*	*Supply*
W	20	19	14	21	16	40
X	15	20	13	19	16	60
Y	18	15	18	20		70
Z	0	0	0	0	0	50
Demand.	30	40	50	40	60	

(Note: Nothing is given in cell *YE*. So you have to ignore it).

16. A manufacturing organization has 3 factories located at *X*, *Y* and *Z*. The centralized planning cell has to decide on allocation of 4 orders over the 3 factories with a view to minimizing the total cost to the organization, Demand and capacity and cost details are given as under:

Customer	*Demand per month in units.*
A	960
B	380
C	420
D	240

Capacities and Costs (Rs.).

Factories	*Capacity units per month*	*Overhead costs in Rs, per month*	*Direct cost in Rs. per unit.*
X	400	400	2.50
Y	900	720	3.00
Z	640	320	3.50

Shipping cost in Paise per unit dispatch.

To

From	*A*	*B*	*C*	*D*
X	50	70	40	35
Y	45	75	40	55
Z	70	65	60	75

It is also possible to produce 25% higher than the capacity in each factory by working overtime at 50% higher in direct costs.

(a) Build a transportation model so that the total demand is met with.

(b) Do the allocation of factory capacity by minimum cost allocation and check the solution for optimality.

16. In a transportation problem the distribution given in the table below was suggested as an optimal solution. The capacities and requirement are given. The number in bold are allocations. The transportation costs given in Rs, per unit from a source to a destination.

(a) Test whether the given distribution is optimal?

(b) If not optimal obtain all basic optimal solution.

Destination

Source	A	B	C	D	Capacity
X	9 **12**	8 **14**	12	10 **10**	36
Y	10	10 **16**	12 **28**	14	44
Z	8	9	11 **32**	12	32
Demand	12	30	60	10	

17. A department stores wishes to purchase 7,500 purses of which 2,500 are of style *X*, 2,500 are of style *Y* and 2,500 are of style *Z*. Four manufacturers *A*, *B*, *C* and *D* bid to supply not more than the following quantities, all styles combined. *A* = 1,000, *B* = 3,000, *C* = 2, 100 and *D* = 1,900. The following table gives the cost per purse of each style of the bidders in Rs. per purse.

MANUFACTURER.

Style	*A*	*B*	*C*	*D*
X	10	4	9	5
Y	6	7	8	7
Z	3	8	6	9

(a) How should orders to be placed by the department store to minimize the total cost?
(b) If the store were to introduce a new style W, which manufacturer can supply it? How many of W can he supply?

MULTIPLE CHOICE QUESTIONS

1. Transportation problem is basically a
   (a) Maximization model
   (b) Minimization model
   (c) Transshipment problem
   (d) Iconic model ( )
2. The column, which is introduced in the matrix to balance the rim requirements, is known as:
   (a) Key column
   (b) Idle column
   (c) Slack column
   (d) Dummy Column ( )
3. The row, which is introduced in the matrix to balance the rim requirement, is known as:
   (a) Key row
   (b) Idle row
   (c) Dummy row
   (d) Slack row ( )
4. One of the differences between the Resource allocation model and Transportation Model is:
   (a) The coefficients of problem variables in Resource allocation model may be any number and in transportation model it must be either zeros or ones.
   (b) The coefficients of problem variable in Resource allocation model must be either zeros or ones and in Transportation model they may be any number.
   (c) In both models they must be either zeros or ones only.
   (d) In both models they may be any number. ( )
5. To convert the transportation problem into a maximization model we have to
   (a) To write the inverse of the matrix
   (b) To multiply the rim requirements by –1
   (c) To multiply the matrix by –1
   (d) We cannot convert the transportation problem in to a maximization problem, as it is basically a minimization problem. ( )
6. In a transportation problem where the demand or requirement is equals to the available resource is known as
   (a) Balanced transportation problem,
   (b) Regular transportation problem,
   (c) Resource allocation transportation problem
   (d) Simple transportation model. ( )

7. The total number of allocation in a basic feasible solution of transportation problem of $m \times n$ size is equal to:
   (*a*) $m \times n$ (*b*) $(m / n) - 1$
   (*c*) $m + n + 1$ (*d*) $m + n - 1$ ( )

8. When the total allocations in a transportation model of $m \times n$ size is not equals to $m + n - 1$ the situation is known as:
   (a) Unbalanced situation (*b*) Tie situation
   (*c*) Degeneracy (*d*) None of the above ( )

9. The opportunity cost of a row in a transportation problem is obtained by:
   (a) Deducting the smallest element in the row from all other elements of the row,
   (b) Adding the smallest element in the row to all other elements of the row,
   (c) Deducting the smallest element in the row from the next highest element of the row
   (d) Deducting the smallest element in the row from the highest element in that row. ( )

10. In Northwest corner method the allocations are made
   (a) Starting from the left hand side top corner,
   (b) Starting from the right hand side top corner
   (c) Starting from the lowest cost cell
   (d) Starting from the lowest requirement and satisfying first. ( )

11. VAM stands for:
   (a) Value added method (*b*) Value assessment method
   (*c*) Vogel Adam method, (*d*) Vogel's approximation method. ( )

12. MODI stands for
   (a) Modern distribution, (*b*) Mendel's distribution method
   (*c*) Modified distribution method (*d*) Model index method ( )

13. In the optimal solution, more than one empty cell have their opportunity cost as zero, it indicates
   (a) The solution is not optimal (*b*) The problem has alternate solution
   (*c*) Something wrong in the solution (*d*) The problem will cycle. ( )

14. In case the cost elements of one or two cells are not given in the problem, it means:
   (a) The given problem is wrong
   (b) We can allocate zeros to those cells
   (c) Allocate very high cost element to those cells
   (d) To assume that the route connected by those cells are not available. ( )

15. To solve degeneracy in the transportation problem we have to:
   (a) Put allocation in one of the empty cell as zero
   (b) Put a small element epsilon in any one of the empty cell

(c) Allocate the smallest element epsilon in such a cell, which will not form a closed loop with other loaded cells.

(d) Allocate the smallest element epsilon in such a cell, which will form a closed loop with other loaded cells. ( )

16. A problem where the produce of a factory is stored in warehouses and then they are transported to various demand point as and when the demand arises is known as:
(a) Transshipment problem
(b) Warehouse problem
(c) Storing and transport problem
(d) None of the above ( )

17. Implied Cost in transportation problem sets (in the existing program):
(a) The lowest limit for the empty cell beyond which it is not advisable to include in the programme,
(b) The highest limit for the empty cell beyond which it is not advisable to include in the programme,
(c) The opportunity cost of the empty cell,
(d) None of the above. ( )

18. In transportation model, the opportunity cost is given by
(a) Implied cost + Actual cost of the cell
(b) Actual cost of the cell – Implied cost,
(c) Implied cost – Actual cost of the cell
(d) Implied cost × Actual cost of the cell ( )

19. If ui and v_j are row and column numbers respectively, then the implied cost is given by:
(a) $u_i + v_j$ (b) $u_i - v_j$
(c) $u_i \times v_j$ (d) u_i / v_j ( )

20. If a transportation problem has an alternate solution, then the other alternate solutions are derived by:
(Given that the two matricides of alternate solutions are A and B, and d is any positive fraction number)
(a) $A + (1 - d) \times B$ (b) $A(1 - d) + B$
(c) $dA + dB$ (d) $dA + (1 - d) \times B$ ( )

ANSWERS

1. (*b*)	2. (*d*)	3. (*d*)	4. (*c*)
5. (*a*)	6. (*c*)	7. (*a*)	8. (*d*)
9. (*c*)	10. (*a*)	11. (*d*)	12. (*a*)
13. (*b*)	14. (*d*)	15. (*c*)	16. (*a*)
17. (*b*)	18. (*b*)	19. (*a*)	20. (*a*)

CHAPTER – 5

Linear Programming : III
Assignment Model

INTRODUCTION

In earlier discussion in chapter 3 and 4, we have dealt with two types of linear programming problems, *i.e*. Resource allocation method and Transportation model. We have seen that though we can use simplex method for solving transportation model, we go for transportation algorithm for simplicity. We have also discussed that how a resource allocation model differ from transportation model and similarities between them. Now we have another model comes under the class of linear programming model, which looks alike with transportation model with an objective function of minimizing the time or cost of manufacturing the products by allocating one job to one machine or one machine to one job or one destination to one origin or one origin to one destination only. This type of problem is given the name **ASSIGNMENT MODEL**. Basically assignment model is a minimization model. If we want to maximize the objective function, then there are two methods. One is to subtract all the elements of the matrix from the highest element in the matrix or to multiply the entire matrix by –1 and continue with the procedure. For solving the assignment problem we use Assignment technique or Hungarian method or Flood's technique. All are one and the same. Above, it is mentioned that one origin is to be assigned to one destination. This feature implies the existence of two specific characteristics in linear programming problems, which when present, give rise to an assignment problem. The first one being **the pay of matrix for a given problem is a square matrix** and the second is the optimum solution (or any solution with given constraints) for the problem is such that there can be **one and only one assignment in a given row or column of the given payoff matrix.** The transportation model is a special case of linear programming model (Resource allocation model) and assignment problem is a special case of transportation model, therefore it is also a special case of linear programming model. Hence it must have all the properties of linear programming model. That is it must have: (*i*) an objective function, (*ii*) it must have structural constraints, (*iii*) It must have non-negativity constraint and (*iv*) The relationship between variables and constraints must have linear relationship. In our future discussion, we will see that the assignment problem has all the above properties.

The Problem

There are some types in assignment problem. They are:

(i) Assigning the jobs to machines when the problem has square matrix to minimize the time required to complete the jobs. Here the number of rows *i.e.* jobs are equals to the number of machines *i.e.* columns. The procedure of solving will be discussed in detail in this section.

(ii) The second type is maximization type of assignment problem. Here we have to assign certain jobs to certain facilities to maximize the returns or maximise the effectiveness. This is also discussed in problem number 5.2.

(iii) Assignment problem having non-square matrix. Here by adding a dummy row or dummy columns as the case may be, we can convert a non-square matrix into a square matrix and proceed further to solve the problem. This is done in problem number.5.9.

(iv) Assignment problem with restrictions. Here restrictions such as a job cannot be done on a certain machine or a job cannot be allocated to a certain facility may be specified. In such cases, we should neglect such cell or give a high penalty to that cell to avoid that cell to enter into the programme.

(v) Traveling sales man problem (cyclic type). Here a salesman must tour certain cities starting from his hometown and come back to his hometown after visiting all cities. This type of problem can be solved by Assignment technique and is solved in problem 5.14.

Let us take that there are 4 jobs, *W*, *X*, *Y* and *Z* which are to be assigned to four machines, *A*, *B*, *C* and *D*. Here all the jobs have got capacities to machine all the jobs. Say for example that the job *W* is to drill a half and inch hole in a Wooden plank, Job *X* is to drill one inch hole in an Aluminum plate and Job *Y* is to drill half an inch hole in a Steel plate and job *Z* is to drill half an inch hole in a Brass plate. The machine *A* is a Pillar type of drilling machine, the machine *B* is Bench type of drilling machine, Machine *C* is radial drilling machine and machine *D* is an automatic drilling machine. This gives an understanding that all machines can do all the jobs or all jobs can be done on any machine. The cost or time of doing the job on a particular machine will differ from that of another machine, because of overhead expenses and machining and tooling charges. The objective is to minimize the time or cost of manufacturing all the jobs by allocating one job to one machine. Because of this character, *i.e.* one to one allocation, the assignment matrix is always a square matrix. If it is not a square matrix, then the problem is unbalanced. Balance the problem, by opening a dummy row or dummy column with its cost or time coefficients as zero. Once the matrix is square, we can use assignment algorithm or Flood's technique or Hungarian method to solve the problem.

Jobs	Machines (Time in hours)				Availability
	A	*B*	*C*	*D*	
W	C_{11}	C_{12}	C_{13}	C_{14}	1
X	C_{21}	C_{22}	C_{23}	C_{24}	1
Y	C_{31}	C_{32}	C_{33}	C_{34}	1
Z	C_{41}	C_{42}	C_{43}	C_{44}	1
Requirement:	1	1	1	1	

Mathematical Model:

$$\textbf{Minimize Z} = \sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij} \times_{ij} \longrightarrow \textbf{Objective Constraint.}$$

Subject to: $X_{ij} = (X_{ij})^2$ **i and j = 1 to n**

$$\sum_{J=1}^{n} X_{ij} = 1\ (d_j) \text{ and } \sum_{i=1}^{n} X_{ij} = 1\ (b_i) \rightarrow \textbf{Structural Constraints.}$$

For *i* and *j* = 1 to *n*

(Each machine to one job only) (Each job to one machine only)

And

X_{ij} = 0 for all values of *j* and *i*. → Non-negativity constraint.

COMPARISION BETWEEN TRANSPORTATION PROBLEM AND ASSIGNMENT PROBLEM

Now let us see what are the similarities and differences between Transportation problem and Assignment Problem.

Similarities

1. Both are special types of linear programming problems.
2. Both have objective function, structural constraints, and non-negativity constraints. And the relationship between variables and constraints are linear.
3. The coefficients of variables in the solution will be either 1 or zero in both cases.
4. Both are basically minimization problems. For converting them into maximization problem same procedure is used.

Differences

Transportation Problem	*Assignment Problem.*
1. The problem may have rectangular matrix or square matrix.	1.The matrix of the problem must be a square matrix.
2.The rows and columns may have any number of allocations depending on the rim conditions.	2.The rows and columns must have one to one allocation. Because of this property, the matrix must be a square matrix.
3.The basic feasible solution is obtained by northwest corner method or matrix minimum method or VAM	3. The basic feasible solution is obtained by Hungarian method or Flood's technique or by Assignment algorithm.
4.The optimality test is given by stepping stone method or by MODI method.	4. Optimality test is given by drawing minimum number of horizontal and vertical lines to cover all the zeros in the matrix.
5.The basic feasible solution must have m + $n - 1$ allocations.	5. Every column and row must have at least one zero. And one machine is assigned to one job and vice versa.
6.The rim requirement may have any numbers (positive numbers).	6. The rim requirements are always 1 each for every row and one each for every column.
7.In transportation problem, the problem deals with one commodity being moved from various origins to various destinations.	7. Here row represents jobs or machines and columns represents machines or jobs.

APPROACH TO SOLUTION

Let us consider a simple example and try to understand the approach to solution and then discuss complicated problems.

1. Solution by visual method

In this method, first allocation is made to the cell having lowest element. (In case of maximization method, first allocation is made to the cell having highest element). If there is more than one cell having smallest element, tie exists and allocation may be made to any one of them first and then second one is selected. In such cases, there is a possibility of getting alternate solution to the problem. This method is suitable for a matrix of size 3 × 4 or 4 × 4. More than that, we may face difficulty in allocating.

Problem 5.1.

There are 3 jobs *A*, *B*, and *C* and three machines *X*, *Y*, and *Z*. All the jobs can be processed on all machines. The time required for processing job on a machine is given below in the form of matrix. Make allocation to minimize the total processing time.

Machines (time in hours)

Jobs	*X*	*Y*	*Z*
A	11	16	21
B	20	13	17
C	13	15	12

Allocation: *A* to *X*, *B* to *Y* and *C* to *Z* and the total time = 11 + 13 +12 = 36 hours. (Since 11 is least, Allocate *A* to *X*, 12 is the next least, Allocate *C* to *Z*)

2. Solving the assignment problem by enumeration

Let us take the same problem and workout the solution.

Machines (time in hours)

C	13	15	12
Jobs	X	Y	Z
A	11	16	21
B	20	13	17
C	13	15	12

S.No	Assignment	Total cost in Rs.
1	AX BY CZ	11 + 13 + 12 = 36
2	AX BZ CY	11 + 17 + 15 = 43
3	AY BX CZ	16 + 20 + 12 = 48
4	AY BZ CX	16 + 17 + 13 = 46
5	AZ BY CX	21 + 13 + 13 = 47
6	AZ BX CY	21 + 20 + 15 = 56

Like this we have to write all allocations and calculate the cost and select the lowest one. If more than one assignment has same lowest cost then the problem has alternate solutions.

3. Solution by Transportation method

Let us take the same example and get the solution and see the difference between transportation problem and assignment problem. The rim requirements are 1 each because of one to one allocation.

Machines (Time in hours)

Jobs	*X*	*Y*	*Z*	*Available*
A	11	16	21	1
B	20	13	17	1
C	13	15	12	1
Req	1	1	1	3

By using northwest corner method the assignments are:

Machines (Time in hours)

Jobs	*X*	*Y*	*Z*	*Available*
A	1	E		1
B		1	$\in$	1
C			1	1
Req	1	1	1	3

As the basic feasible solution must have $m + n - 1$ allocations, we have to add 2 epsilons. Next we have to apply optimality test by MODI to get the optimal answer.

This is a time consuming method. Hence it is better to go for assignment algorithm to get the solution for an assignment problem.

4. Hungarian Method / Flood's technique / Assignment algorithm: (opportunity cost method)

Let us once again take the same example to workout with assignment algorithm.

Machines (time in hours)

Jobs	*X*	*Y*	*Z*
A	11	16	21
B	20	13	17
C	13	15	12

Step 1. Deduct the smallest element in each row from the other elements of the row. The matrix thus got is known as **Row opportunity cost matrix (ROCM). The logic here is if we assign the job to any machine having higher cost or time, then we have to bear the penalty. If we subtract smallest element in the row or from all other element of the row, there will be at least one cell having zero, i.e zero opportunity cost or zero penalty. Hence that cell is more competent one for assignment.**

Step 2. Deduct the smallest element in each column from other elements of the column. The matrix thus got is known as **Column opportunity cost matrix (COCM). Here also by creating a zero by subtracting smallest element from all other elements we can see the penalty that one has to bear. Zero opportunity cell is more competent for assignment.**

Step 3. Add COCM and ROCM to get the **Total opportunity cost matrix (TOCM).**

Step 4. (modified): Total opportunity cost matrix can be got by simplify doing row operation on Column opportunity matrix or column operation on row opportunity cost matrix. This method is simple one and saves time. (Doing row operation on column opportunity matrix means: Deduct the smallest element in the row from all other elements in the row in column opportunity matrix and vice versa).

The property of total opportunity cost matrix is that it will have at least one zero in every row and column. All the cells, which have zero as the opportunity cost, are eligible for assignment.

Step 5. Once we get the total opportunity **cost matrix, cover all the zeros by MINIMUM NUMBER OF HORIZONTAL AND VERTICAL LINES**. (First cover row or column, which is having maximum number of zeros and then next row or column having next highest number of zeros and so on until all zeros are covered. Remember, only horizontal and vertical lines are to be drawn.

Step 6. If the lines thus drawn are equal to the number of rows or columns (because of square matrix), we can make assignment. If lines drawn are not equal to the number of rows or columns go to step 7.

Step 7. **To make assignment**: Search for a single zero either row wise or column wise. If you start row wise, proceed row by row in search of single zero. Once you find a single zero; assign that cell by enclosing the element of the cell by a **square**. Once all the rows are over, then start column wise and once you find single zero assign that cell and enclose the element of the one cell in a square. Once the assignment is made, then all the zeros in the row and column corresponding to the assigned cell should be cancelled. Continue this procedure until all assignments are made. Some times we may not find single zero and find more than one zero in a row or column. It indicates, that the problem has an alternate solution. We can write alternate solutions. (The situation is known as a **TIE** in assignment problem).

Step 8. If the lines drawn are less than the number of rows or columns, then we cannot make assignment. Hence the following procedure is to be followed:

The cells covered by the lines are known as **Covered cells**. The cells, which are not covered by lines, are known as **uncovered cells**. The cells at the intersection of horizontal line and vertical lines are known as **Crossed cells.**

(a) Identify the smallest element in the uncovered cells.
   (i) Subtract this element from the elements of all other uncovered cells.
   (ii) Add this element to the elements of the crossed cells.
   (iii) Do not alter the elements of covered cells.

(b) Once again cover all the zeros by **minimum number of horizontal and vertical lines.**

(c) Once the lines drawn are equal to the number of rows or columns, assignment can be made as said in step (6).

(d) If the lines are not equal to number of rows or columns, repeat the steps 7 (a) and 7 (b) until we get the number of horizontal and vertical lines drawn are equal to the number of rows or columns and make allocations as explained in step (6).

Note: For maximization same procedure is adopted, once we convert the maximization problem into minimization problem by multiplying the matrix by (-1) or by subtracting all the elements of the matrix from highest element in the matrix. Once we do this, the entries in the matrix gives us **relative costs**, hence the problem becomes minimisaton problem. Once we get the optimal assignment, the total value of the original pay off measure can be found by adding the individual original entries for those cells to which assignment have been made.

Now let us take the problem given above and solve.

Solution

Machines (time in hours)

Jobs ↓	*X*	*Y*	*Z*
A	11	16	21
B	20	13	17
C	13	15	12

Step1: To find ROCM.

Machines (time in hours)

Jobs ↓	*X*	*Y*	*Z*
A	0	5	10
B	7	0	4
C	1	3	0

Step 2. To find TOCM (do column operation in ROCM)

Machines (time in hours)

Jobs ↓	*X*	*Y*	*Z*
A	0	5	10
B	7	0	4
C	1	3	0

Because in each column, zero is the lowest element, the matrix remains unchanged, *i.e.* The COCM itself TOCM.

Step 3. To cover all the zeros by minimum number of horizontal and vertical lines.

Machines (time in hours)

Jobs ↓	*X*	*Y*	*Z*
A	0	5	10
B	7	0	4
C	1	3	0

Assignment is:

Machines (time in hours)

Jobs	X	Y	Z
A	**0**	5	10
B	7	**0**	4
C	1	3	**0**

Assignment	*Time in hours.*
A TO X	11
B TO Y	13
C TO Z	12
Total:	36 hours.

Problem 5.2.

A company has five jobs *V*, *W*, *X*, *Y* and *Z* and five machines *A*, *B*, *C*, *D* and E. The given matrix shows the return in Rs. of assigning a job to a machine. Assign the jobs to machines so as to maximize the total returns.

Machines.
Returns in Rs.

Jobs	*A*	*B*	*C*	*D*	*E*
V	5	11	10	12	4
W	2	4	6	3	5
X	3	12	5	14	6
Y	6	14	4	11	7
Z	7	9	8	12	5

Solution

As the objective function is to maximize the returns, we have to convert the given problem into minimization problem.

Method 1. Here highest element in the matrix is 14, hence subtract all the element form 14 and write the relative costs. (Transformed matrix).

Machines
Returns in Rs.

Jobs	*A*	*B*	*C*	*D*	*E*
V	9	3	4	2	10
W	12	10	8	11	9
X	11	2	9	0	8
Y	8	0	10	3	7
Z	7	5	6	2	9

ROCM:

Machines
Returns in Rs.

Jobs	*A*	*B*	*C*	*D*	*E*
V	7	1	2	0	8
W	4	2	0	3	1
X	11	2	9	0	8
Y	8	0	10	3	7
Z	5	3	4	0	7

By doing column operation on ROCM, we get the total opportunity cost matrix.
TOCM:

Machines
Returns in Rs.

Jobs	*A*	*B*	*C*	*D*	*E*
V	3	1	2	0	7
W	0	2	0	3	0
X	7	2	9	0	7
Y	4	0	10	3	6
Z	1	3	4	0	6

Only three lines are there. So we have to go to step 7. The lowest element in uncovered cell is 1, hence subtract 1 from all uncovered cells and add this element to crossed cells and write the matrix. The resultant matrix is:

Machines
Return in Rs.

Jobs	*A*	*B*	*C*	*D*	*E*
V	2	0	1	0	6
W	0	3	0	4	0
X	6	1	8	0	6
Y	4	0	10	4	6
Z	0	2	3	0	5

Only foor lines are there, hence repeat the step 7 until we get 5 lines.

Machines
Return in Rs.

Jobs	*A*	*B*	*C*	*D*	*E*
V	1	0	0	0	5
W	0	3	0	5	0
X	5	1	7	0	5
Y	3	0	9	4	5
Z	0	3	3	1	5

All zeros are covered by 5 lines, Hence assignment can be made. Start row wise or column wise and go on making assignment, until all assignments are over.

Machines
Return in Rs.

Jobs	*A*	*B*	*C*	*D*	*E*
V	2	1	**0**	x0	5
W	1	4	0x	5	**0**
X	6	2	7	**0**	5
Y	3	0	8	3	4
Z	**0**	3	2	0x	4

Job	*Machine*	*Return in Rs.*
V	C	10
W	E	5
X	D	14
Y	B	14
Z	A	7
Total in Rs.		50

Problem 5.3.

Five jobs are to be assigned to 5 machines to minimize the total time required to process the jobs on machines. The times in hours for processing each job on each machine are given in the matrix below. By using assignment algorithm make the assignment for minimizing the time of processing.

Machines (time in hours)

Jobs	*V*	*W*	*X*	*Y*	*Z*
A	2	4	3	5	4
B	7	4	6	8	4
C	2	9	8	10	4
D	8	6	12	7	4
E	2	8	5	8	8

Solution

Machines (time in hours)

Jobs	*V*	*W*	*X*	*Y*	*Z*
A	2	4	3	5	4
B	7	4	6	8	4
C	2	9	8	10	4
D	8	6	12	7	4
E	2	8	5	8	8

COCM

Machines (time in hours)

Jobs	*V*	*W*	*X*	*Y*	*Z*
A	0	0	0	0	0
B	5	0	3	3	0
C	0	5	5	5	0
D	6	2	9	2	0
E	0	4	2	3	4

As the COCM has at least one zero in every column and row, this itself can be considered as TOCM, because as the zero is the lowest number in each column, the matrix remains unchanged. If we cover all the zeros by drawing horizontal and vertical lines, we get only four lines. Applying step 7 we get the following matrix.

Machines (time in hours)

Jobs	*V*	*W*	*X*	*Y*	*Z*
A	2	0	0	0	2
B	7	0	3	3	2
C	0	3	3	3	0
D	6	0	7	0	0
E	0	2	0	1	4

As there are five lines that cover all zeros, we can make assignment.

Machines (time in hours)

Jobs	*V*	*W*	*X*	*Y*	*Z*
A	2	$\cancel{0}$	**0**	$\cancel{0}$	2
B	7	**0**	3	3	2
C	$\cancel{0}$	3	3	3	**0**
D	6	$\cancel{0}$	7	**0**	$\cancel{0}$
E	**0**	2	$\cancel{0}$	1	4

Alternate solution:

Machines (time in hours)

Jobs	*V*	*W*	*X*	*Y*	*Z*
A	2	$\cancel{0}$	$\cancel{0}$	**0**	2
B	7	**0**	3	3	2
C	**0**	3	3	3	$\cancel{0}$
D	6	$\cancel{0}$	7	$\cancel{0}$	**0**
E	$\cancel{0}$	2	**0**	1	4

First Solution: *A* to *X*, *B* to *W*, *C* to *Z*, *D* to *Y* and *E* to *V* Cost is: 3 + 4 + 4 + 7 + 2 = 20 hours.

Second Solution: *A* to*Y*, *B* to *W*, *C* to *V*, *D* to *Z* and *E* to *X*. Cost is: 5 + 4 + 2 + 4 + 5 = 20 Hours.

When there is a tie, make assignment arbitrarily first to one of the zeros and then proceed, we will get the assignment. When there is a tie, there exists an alternate solution.

Problem 5.4.

A manager has 4 jobs on hand to be assigned to 3 of his clerical staff. Clerical staff differs in efficiency. The efficiency is a measure of time taken by them to do various jobs. The manager wants to assign the duty to his staff, so that the total time taken by the staff should be minimum. The matrix given below shows the time taken by each person to do a particular job. Help the manager in assigning the jobs to the personnel.

Jobs.	*Men (time taken to do job in hours).*		
	X	Y	Z
A	10	27	16
B	14	28	7
C	36	21	16
D	19	31	21

Solution

The given matrix is unbalanced. To balance the matrix, open a dummy column with time coefficients as zero.

(DC = Dummy column).

Men (Time taken in hours)

	X	*Y*	*Z*	*DC*
A	10	27	16	0
B	14	28	7	0
C	36	21	16	0
D	19	31	21	0

As every row has a zero, we can consider it as ROCM and by doing column operation, we can write TOCM. Now apply step 7.

Men (Time taken in hours).

Jobs	*X*	*Y*	*Z*	*DC*
A	0	6	9	0
B	4	7	0	0
C	26	0	9	0
D	9	10	14	0

Men (Time taken in hours).

Jobs	*X*	*Y*	*Z*	*DC*
A	**0**	6	9	~~0~~
B	4	7	**0**	~~0~~
C	26	**0**	9	~~0~~
D	9	10	14	**0**

The assignment is: *A* to *X*, *B* to *Z*, and *C* to *Y* and *D* is not assigned.
Total time required is: 10 + 7 + 21 = 38 Hours.

Problem 5.5.

A company has four market segments open and four salesmen are to be assigned one to each segment to maximize the expected total sales. The salesmen differ in their ability and the segments also differ in their sales potential. The details regarding the expected sales in each segment by a typical salesman under most favourable condition are given below.

Segment *A* = Rs. 60,000, Segment *B* = Rs. 50,000, Segment *C* = Rs. 40,000 and Segment *D* = Rs. 30,000. It is estimated that working under same condition, the ability of salesmen in terms of proportional yearly sales would be as below:
Salesman *W* = 7, Salesman *X* = 5, Salesman *Y* = 5 and Salesman *Z* = 4.
Assign segments to salesmen for maximizing the total expected sales.

Solution

To simplify the calculations, let us consider sales of Rs.10, 000/- as one unit of sale, then salesman *W* 's annual sales in four segments are:

His proportionate sale is seven out of 21 (7 + 5 + 5 + 4 = 21). In case the annual sales is 6 units

(Rs.60, 000), then his proportional sales would be (7 / 21) × 6 = 42 / 21 similarly his sales in all the segments would be (7/21) × 6, (7/21) × 5, (7/21) × 5, and (7/21) × 4 *i.e.* 42/21, 35/21, 35/21 and 28/21. Like wise we can calculate the proportional sales of all salesmen and write the matrix showing the sales of each salesman in different market segments. The matrix is given below:

Market segments.

Sales (x1000)		*6*	*5*	*4*	*3*
Salesproportion	*Salesmen*	*W*	*X*	*Y*	*Z*
7	W	42/21	3 5/21	28/21	21/21
5	X	30/21	25/21	20/21	15/21
5	Y	30/21	25/21	20/21	15/21
4	Z	24/21	20/21	16/21	12/21

Multiply the matrix by21 to avoid the denominator. As the problem is maximization one, convert the problem into minimization problem by multiplying by (-1) (Second method). The resultant matrix is:

Market segments.

SalesMen ↓	*A*	*B*	*C*	*D*
W	–42	–35	–28	–21
X	–30	–25	–20	–15
Y	–30	–25	–20	–15
Z	–24	–20	–16	–12

ROCM:

Market segments.

SalesMen ↓	*A*	*B*	*C*	*D*
W	0	7	14	21
X	0	5	10	15
Y	0	5	10	15
Z	0	4	8	12

TOCM:

Market segments.

SalesMen ↓	*A*	*B*	*C*	*D*
W	0	3	6	9
X	0	1	2	3
Y	0	1	2	3
Z	0	0	0	0

TOCM:

Market segments.

SalesMen	*A*	*B*	*C*	*D*
W	0	2	5	8
X	0	0	1	2
Y	0	0	1	2
Z	1	0	0	0

TOCM:

Market segments.

SalesMen	*A*	*B*	*C*	*D*
W	0	2	4	7
X	0	0	0	1
Y	0	0	0	1
Z	2	1	0	0

Assignment (First solution)

Market segments.

SalesMen	*A*	*B*	*C*	*D*
W	**0**	2	4	7
X	0x	**0**	0x	1
Y	0x	0x	**0**	1
Z	2	1	0x	**0**

(Alternate Solution)

Market segments.

SalesMen	*A*	*B*	*C*	*D*
W	**0**	2	4	7
X	0x	0x	**0**	1
Y	0x	**0**	0x	1
Z	2	1	0x	0

Solution I: W to A, X to B, Y to C and Z to D. Sales: 42 + 25 + 20 + 12 = Rs. 99 x 10, 000
Solution II: W to A, X to C, Y to B and Z to D Sales: 42 + 20 + 25 + 12 = Rs. 99 x 10,000

Problem 5.6.

The city post office has five major counters namely, Registration (*R*), Savings (*S*), Money – Order (*M*), Postal stationary (*P*) and Insurance / license (*I*). The postmaster has to assign five counters to five clerks *A*, *B*, *C*, *D* and *E* one for each counter. Considering the experience and ability of these clerks he rates their suitability on a certain 10 - point scale of effectiveness of performance for accomplishing different counter duties, as listed below. Assign the counters to the clerks for maximum effective performance.

Clerks (effective performance)

Counters	*A*	*B*	*C*	*D*	*E*
R	6	6	4	6	7
S	5	4	3	6	8
M	7	6	3	5	5
P	7	5	6	8	8
I	4	3	6	7	6

Convert the problem into minimization problem. (We can deduct all other elements form highest element).

Note : As every row has a zero, we can consider it as Row Opportunity Cost Matrix.

ROCM

Clerks (effective performance)

Counters	*A*	*B*	*C*	*D*	*E*
R	1	1	3	1	0
S	3	4	5	2	0
M	0	1	4	2	2
P	1	3	2	0	0
I	3	4	1	0	1

TOCM:

Clerks (effective performance)

Counters	*A*	*B*	*C*	*D*	*E*
R	1	0	2	1	0
S	3	3	4	2	0
M	0	0	3	2	0
P	1	2	1	0	0
I	3	3	0	0	1

As five lines are there we can make assignment.

Clerks (effective performance)

Counters ↓	*A*	*B*	*C*	*D*	*E*
R	1	**0**	2	1	0x
S	3	3	4	2	**0**
M	**0**	0 x	3	2	2
P	1	2	1	**0**	0x
I	3	3	0	0x	1

Assignment: *R* to *B*, *S* to *E*, *M* to *A*, *P* to *D* and *I* to *C*. Total effectiveness: 6 + 8 + 7 + 8 + 6 = 35 points.

Problem 5.7.

There are 5 jobs namely, *A*, *B*, *C*, *D*, and *E*. These are to be assigned to 5 machines *P*, *Q*, *R*, *S* and *T* to minimize the cost of production. The cost matrix is given below. Assign the jobs to machine on one to one basis.

Jobs (Cost in Rs.)

Machines ↓	*A*	*B*	*C*	*D*	*E*
P	8	7	4	11	6
Q	10	5	5	13	7
R	6	9	8	7	12
S	6	7	2	3	2
T	7	8	8	1 0	5

ROCM:

Jobs (Cost in Rs.)

Machines ↓	A	B	C	D	E
P	4	3	0	7	2
Q	5	0	0	8	2
R	0	3	2	1	6
S	4	5	0	1	0
T	2	3	3	5	0

TOCM:

Jobs (Cost in Rs.)

Machines	*A*	*B*	*C*	*D*	*E*
P	4	3	0	6	2
Q	5	0	0	7	2
R	0	3	2	0	6
S	4	5	0	0	0
T	2	3	3	4	0

There are five lines and hence we can make assignment.

Jobs (Cost in Rs.)

Machines	*A*	*B*	*C*	*D*	*E*
P	4	3	**0**	6	2
Q	5	**0**	~~0~~	7	2
R	**0**	3	2	~~0~~	6
S	4	5	~~0~~	**0**	~~0~~
T	2	3	3	4	**0**

Assignment: *P* to *C*, *Q* to *B*, *R* to *A*, and *S* to *D* and T to *E*. Total cost = 4 + 5 + 6 + 3 + 5 = Rs.23/-

Problem 5.8.

Four different jobs are to be done on four machines, one job on each machine, as set up costs and times are too high to permit a job being worked on more than one machine. The matrix given below gives the times of producing jobs on different machines. Assign the jobs to machine so that total time of production is minimized.

Machines (time in hours)

Jobs	*A*	*B*	*C*	*D*
P	10	14	22	12
Q	16	10	18	12
R	8	14	20	14
S	20	8	16	6

Solution

ROCM:

Machines (time in hours)

Jobs ↓	*A*	*B*	*C*	*D*
P	0	4	12	2
Q	6	0	8	2
R	0	6	12	6
S	14	2	10	0

TOCM:

Machines (time in hours)

Jobs ↓	*A*	*B*	*C*	*D*
P	0	4	4	2
Q	6	0	0	2
R	0	6	4	6
S	14	2	2	0

TOCM:

Machines (time in hours)

Jobs ↓	*A*	*B*	*C*	*D*
P	0	4	4	2
Q	6	0	0	2
R	0	6	4	6
S	14	2	6	0

TOCM:

Machines (time in hours)

Jobs ↓	*A*	*B*	*C*	*D*
P	0	2	2	0
Q	8	0	0	2
R	0	4	2	4
S	16	2	4	0

TOCM:

Machines (time in hours)

Jobs	*A*	*B*	*C*	*D*
P	0	0	0	0
Q	10	0	0	4
R	0	2	0	0
S	16	0	0	0

Four lines are there hence we can make assignment. As there is a tie, we have more than one solution.

Solution I.

TOCM:

Machines (time in hours)

Jobs	*A*	*B*	*C*	*D*
P	**0**	~~0~~	~~0~~	~~0~~
Q	10	**0**	~~0~~	4
R	~~0~~	2	**0**	~~0~~
S	14	~~0~~	4	**0**

Assignment: *P* to *A*, *Q* to *B*, *R* to *C* and *S* to *D*.
Time: 10 + 10 + 20 + 06 = 46 hours.

Solution II.

TOCM:

Machines (time in hours)

Jobs	*A*	*B*	*C*	*D*
P	~~0~~	**0**	~~0~~	~~0~~
Q	10	~~0~~	**0**	4
R	**0**	2	~~0~~	~~0~~
S	14	~~0~~	4	**0**

Assignment: *P* to *B*, *Q* to *C*, *R* to *A* and *S* to *D*.
Time: 14 + 18 + 8 + 6 = 46 hours.

We can write many alternate solutions.

Problem 5.9.

On a given day District head quarter has the information that one ambulance van is stationed at each of the five locations *A*, *B*, *C*, *D* and *E*. The district quarter is to be issued for the ambulance van to reach 6 locations namely, *P*, *Q*, *R*, *S*, *T* and *U*, one each. The distances in Km. between present locations of ambulance vans and destinations are given in the matrix below. Decide the assignment of vans for minimum total distance, and also state which destination should not expect ambulance van to arrive.

To (distance in Km.)

From	*P*	*Q*	*R*	*S*	*T*	*U*
A	18	21	31	17	26	29
B	16	20	18	16	21	31
C	30	25	27	26	18	19
D	25	33	45	16	32	20
E	36	30	18	15	31	30

Solution

As the given matrix is not square matrix, balance the same by opening one dummy row (DR), with zero as the elements of the cells.

To (distance in Km.)

From	*P*	*Q*	*R*	*S*	*T*	*U*
A	18	21	31	17	26	29
B	16	20	18	16	21	31
C	30	25	27	26	18	19
D	25	33	45	16	32	20
E	36	30	18	15	31	30
DR	0	0	0	0	0	0

As every column has got one zero, we can take it as COCM. Now doing row operation on COCM, we get TOCM.

TOCM

	P	*Q*	*R*	*S*	*T*	*U*
A	1	4	14	0	9	12
B	0	4	2	0	5	15
C	12	7	9	8	0	1
D	9	17	29	0	16	4
E	21	15	3	0	16	15
DR	0	0	0	0	0	0

As there are only four lines, we cannot make assignment.

TOCM

	P	*Q*	*R*	*S*	*T*	*U*
A	0	3	13	0	8	11
B	0	4	2	0	5	15
C	12	7	9	9	0	1
D	8	16	28	0	15	3
E	20	14	2	0	15	14
DR	0	0	0	1	0	0

As there are four lines, we cannot make an assignment.

To (Distance in Km.)

	P	*Q*	*R*	*S*	*T*	*U*
A	0	1	11	0	6	9
B	0	2	0	1	3	13
C	14	7	9	11	0	1
D	8	14	26	0	13	1
E	20	12	0	0	13	12
DR	2	0	0	3	0	0

As there are only 5 lines we cannot make assignment.

	P	*Q*	*R*	*S*	*T*	*U*
A	0	0	11	0	5	8
B	0	1	0	1	2	12
C	15	7	10	12	0	1
D	8	13	26	0	12	0
E	20	11	0	0	12	11
DR	3	0	1	4	0	0

As there are 6 lines, we can make assignment. As there is a tie, we have alternate solutions.

To (Distance in Km.)

From	*P*	*Q*	*R*	*S*	*T*	*U*
A	**0**	$\cancel{0}$	11	$\cancel{0}$	5	8
B	$\cancel{0}$	1	**0**	1	2	12
C	15	7	10	12	**0**	1
D	8	13	26	$\cancel{0}$	12	**0**
E	20	11	$\cancel{0}$	**0**	12	11
DR	4	**0**	1	4	$\cancel{0}$	$\cancel{0}$

Assignment: *A* to *P*, *B* to *R*, *C* to *T*, *D* to *U*, *E* to *S* and DR to *Q* i.e the van at *Q* will not go to any destination.
Total Distance: 18 + 18 + 18 + 20 + 15 = 89 Km.
Other alternative assignments are:

From:	*A*	*B*	*C*	*D*	*E*	*DR*	*Station for which no van*	*Total Distance in Km.*
To:	*Q*	*P*	*T*	*S*	*U*	*R*	*R*	89
	S	*P*	*T*	*U*	*R*	*Q*	*Q*	89

Brain tonic:

a) In case the cost of dispatching an ambulance is 3 times the distance, determine the assignment of ambulances to destinations.

(*b*) In case the operating cost of a van is proportional to the square of the distance decide the assignment.

(Note: a) By multiplying the entire matrix by 3 we get the cost matrix. This does not have any effect on the final solution. Hence the same solution will hold good.

(*b*) We have to write the elements by squaring the elements of the original matrix and make fresh assignment.)

Problem 5.10.

A job order company has to work out the assignment of 5 different jobs on five different machines. The cost of machining per unit of job and set up cost of the job on a machine are as given in the matrix *A* and *B* given below. The jobs are to be made in bathe sizes show against them. Set up cost is independent of previous set up.

Matrix A. (Operating cost in Rs)

Jobs (machining cost in Rs)

Machines.	*A*	*B*	*C*	*D*	*E*
P	0.80	1.10	0.70	1.60	6.20
Q	1.20	0.90	1 .20	0.80	5.40
R	2.10	2.00	1.00	2.20	4.90
S	---	1.60	2.00	1.90	3.60
T	3.20	2.00	2.00	2.00	2.60
Batch size in units.	100	100	150	100	50

Matrix B (Set up cost in Rs)

Jobs (cost in Rs)

Machines	*A*	*B*	*C*	*D*	*E*
P	60	70	70	30	40
Q	40	50	50	20	80
R	30	40	40	40	100
S	---	90	60	50	60
T	80	100	80	60	60

Solution

Multiply the Matrix *A* by 100 and add it to the matrix *B* we get the matrix given below. For the element *SA* as nothing is given, we can eliminate it for further consideration or assign a very high cost for the element so as to avoid it from further calculations.

Jobs (combined setup and processing cost in Rs)

Machines ↓	*A*	*B*	*C*	*D*	*E*
P	140	180	175	190	350
Q	160	140	230	100	350
R	240	240	190	260	255
S	1000	250	360	240	240
T	400	300	380	260	190

ROCM:

Machines ↓	*A*	*B*	*C*	*D*	*E*
P	0	40	35	50	210
Q	60	40	130	0	250
R	50	50	0	70	65
S	760	10	120	0	0
T	210	110	190	70	0

TOCM:

Machines ↓	*A*	*B*	*C*	*D*	*E*
P	0	30	35	50	210
Q	60	30	130	**0**	250
R	50	40	**0**	70	65
S	760	**0**	120	0	0
T	210	100	190	70	**0**

Assignment: *P* to *A*, *Q* to *D*, *R* to *C*, *S* to *B* and *T* to *E*.
Total cost = 140 + 100 + 190 + 250 + 190 = Rs. 870/-

Problem 5.11.

There are five major projects namely, Fertiliser plants, Nuclear poser plants, Electronic park, Aircraft complex and Heavy machine tools. These five plants are to be assigned to six regions namely *A*, *B*, *C*, *D*, *E* and *F*, insisting on allocation of as many number of projects as possible in their region. The state department has evaluated the effectiveness of projects in different regions for (*a*) Employment potential, (*b*) Resource utilization potential, (*c*) Economic profitability and (*d*) Environmental degradation index as given below in

Tableau I. (The ranking is on a 20 point scale). Assign one project to one region depending on the maximum total effectiveness. (Plants are given serial numbers 1 to 5)

Tableau I.

	Local Employment Potential.					Resource Allocation Potential.					Economic Profitability Index.					Environmental Degradation index.				
Reg. long.	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
A	16	10	8	12	11	7	6	4	5	3	11	13	14	15	10	15	14	5	3	2
B	18	15	12	10	7	11	4	3	2	1	10	15	17	11	16	13	14	5	3	2
C	12	16	12	5	8	16	5	4	3	2	13	14	16	12	11	12	11	5	4	2
D	14	10	13	6	8	15	3	2	4	1	7	10	5	11	8	12	11	5	4	2
E	15	17	11	18	11	8	3	4	2	4	10	12	7	11	16	9	6	5	4	3
F	12	18	11	15	1 4	1 7	5	2	1	3	5	10	12	13	12	6	3	5	5	2

Solution

In this problem, for maximization of total effectiveness, the first three *i.e.* Employment potential, Resource utilization potential and economic profitability index are to be added and the environmental degradation is to be subtracted from the sum to get the total effectiveness. Once we get the effectiveness matrix, then the projects are to be assigned to the regions for maximization of total effectiveness.

The total effectiveness matrix: (Note: The matrix is of the order 5 × 6, hence it is to be balanced by opening a dummy column - *DC*). The first element of the matrix can be worked out as: 16 + 7 + 11 – 15 = 19. Other elements can be worked out similarly.

Total effectiveness matrix:

Plants.

Regions ↓	*1*	*2*	*3*	*4*	*5*	*DC*
A	19	15	21	29	22	0
B	26	20	27	20	22	0
C	29	24	27	16	19	0
D	24	12	15	17	15	0
E	24	26	17	27	28	0
F	28	30	20	24	27	0

As there is a dummy column the same matrix may be considered as ROCM. By deducting all the elements of a column from the highest element of the column, we get the Total Opportunity Cost Matrix.

TOCM:

Plants.

Regions ↓	1	2	3	4	5	DC
A	10	15	6	0	6	0x
B	3	10	0	9	6	0x
C	0	6	0x	13	11	0x
D	5	18	12	12	13	0
E	5	4	10	2	0	0x
F	2	0	7	5	1	0x

Allocation: Fertilizer: C, Nuclear Plant: F, Electronic Park: B,
Aircraft Complex: A, Heavy Machine Tools: E

SCHEDULING PROBLEM

Now let us work scheduling problem. This type of problems we can see in arranging air flights or bus transport or rail transport. The peculiarities of this type of problem is that one flight / train / bus leaves form a station with some flight number / train number / bus number. After reaching the destination, the same plane / train/bus leaves that place (destination) and reaches the hometown with different number. For example plane bearing flight number as 101 leaves Bangalore and reaches Bombay and leaves Bombay as flight number 202 and reaches Bangalore. Our problem here is how to arrange a limited number of planes with crew / trains with crew / bus with crew between two places to make the trips without inconvenience, by allowing required lay over time. Lay over time means the time allowed for crew to take rest before starting.

Problem 5. 12. (Scheduling Problem).

For the following Airline time table between Banglore and Mumbai it is required to pair to and for flights for the same crew, so as to minimize the lay over time of the crew on ground away from Head quarters. It is possible to assign Banglore or Bombay as the head quarter. Decide the pairing of flights and head quarters of the concerned crew. It is stipulated that the same crew cannot undertake next flight, within one hour of the arrival. That is one hour is the layover time.

Flight No.	*Departure Mumbai*	*Arrival Bangalore*	*Flight No.*	*Departure Bangalore*	*Arrival. Mumbai*
101	6-30 a.m	7.45 a.m	102	7.00 a.m	8.00 a.m
103	9.00 a.m.	10.30 a.m	104	11.00 a.m.	12.15 p.m
105	1.00 p.m.	2.15 p.m.	106	3.00 p.m.	4.15 p.m.
107	4.00 p.m.	5.30 p.m	108	5.45 p.m	7.15 p.m
109	8.00 p.m	9.30 p.m.	110	8.30 p.m.	9.45 p.m.

Solution

Now let us consider the layover times separately for crew based at Mumbai and crew based at Bangalore.

Let us consider one flight and discuss how to calculate layover time. For example, flight No. 101 leaves Mumbai at 6.30 a.m and reaches Bangalore at 7.45 a.m. Unless the crew takes one our rest, they cannot fly the airplane. So if the crew cannot leave Bangalore until 8.45 a.m. So there is no chance for the crew to go for flight No. 102. But they can go as flight Nos. 103, 106, 108 and 110. As we have to minimize the flyover time, we can take the nearest flight *i.e*. 103. The flight 103 leaves Bangalore at 11.00 a.m. By 11.00 a.m the crew might have spent time at Bangalore from 7.45 a.m to 11.00 a.m. That is it has spent 3 hours and 15 minutes. If we convert 3 hours and 15 minutes in terms of quarter hours, it will become 13-quarter hours. Similarly the flight 102 which arrives at Mumbai at 8.00 a.m. wants to leave as flight 101 at 6.30 a.m. it has to leave next day morning. Hence the layover time will be 22 hours and 30 minutes. Like wise, we can workout layover time for all flights and we can write two matrices, one for crew at Mumbai and other for crew at Bangalore.

Tableau I. Lay over time for Mumbai based crew:

Flight numbers. (Quarter hours)

Flight No.	*102*	*104*	*106*	*108*	*110*
101	23.25	3.25	7.25	10.00	11.75
103	18.50	24.50	4.50	7.25	10.00
105	16.75	20.75	24.75	3.50	5.25
107	13.50	17.50	21.50	24.25	3.00
109	9.50	13.50	17.50	20.25	23.00

Tableau II. Lay over time for Bangalore based crew:

Layover time in quarter hours.

Flight No.	*102*	*104*	*106*	*108*	*110*
101	23.50	18.25	14.25	11.75	8.75
103	1.00	20.75	16.75	13.75	13.25
105	5.00	24.75	20.75	17.75	15.25
107	8.00	3.75	23.75	20.75	18.25
109	12.00	7.75	7.75	24.25	22.25

The matrices can be multiplied by four to convert decimals into whole numbers for convenience of calculations.

Tableau II. Bombay based layover times

Flight No.	*102*	*104*	*106*	*108*	*110*
101	93	13	29	40	47
103	74	98	14	29	40
105	67	83	97	14	21
107	54	70	86	97	12
109	38	54	70	81	92

Layover time of crew stationed at Bangalore. (*)

Flight No.	*102*	*104*	*106*	*108*	*110*
101	90	73	57	67	35
103	4	83	67	55	53
105	20	97	83	71	61
107	32	15	95	83	73
109	48	31	15	97	89

Now let us select the minimum elements from both the matrices and write another matrix with these elements. As our objective is to minimize the total layover time, we are selecting the lowest element between the two matrices. Also, let us mark a * for the entries of the matrix showing layover time of the crew at Bangalore.

Matrix showing the lowest layover time

(The elements marked with * are from Bangalore matrix)

Flight No.	*102*	*104*	*106*	*108*	*110*
101	90*	13	29	40	35*
103	4*	83*	14	29	40
105	20*	83	83*	14	21
107	32*	15*	86	83*	12
109	38	31*	15*	81	89*

ROCM: As every column has got a zero, this may be considered as TOCM and assignment can be made. Note that all zeros in the matrix are in independent position we can make assignment.

Flight No.	*102*	*104*	*106*	*108*	*110*
101	77*	**0**	16	27	22*
103	**0** *	79*	10	25	36
105	6*	69	69*	**0**	7
107	20*	3*	74	71*	**0**
109	23	16*	**0** *	65	74*

Assignment and pairing:

Flight No.	*Leaves as*	*Crew based at.*
101	104	Bombay
103	102	Bangalore
105	108	Bombay
107	110	Bombay
109	106	Bangalore.

Total Layover time is: 3.25 + 1.00 + 3.50 + 3.0 + 17.50 = 28 hours and 15 minutes.

Problem 5.13.

An airline that operates seven days a week has the timetable shown below. Crews must have a minimum layover time 5 hours between flights. Obtain the pairing of flights that minimises layover time away from home. For any given pairing, the crew will be based at the city that results in the smaller layover. For each pair also mention the town where crew should be based.

Chennai - Bangalore			Bangalore - Chennai.		
FlightNo.	*Departure*	*Arrival*	*Flight No.*	*Departure*	*Arrival*
101	7.00 a.m	8.00 a.m	201	8.00 a.m	9.00 a.m
102	8.00 a.m	9.00 a.m	202	9.00 a.m	10.00 a.m
103	1.00 p.m	2.00 p.m	203	12.00 noon	1.00 p.m.
104	6.00 p.m.	7.00 p.m	204	8.00 p.m	9.00 p.m

Let us write two matrices one for layover time of Chennai based crew and other for Bangalore based crew.

As explained in the example 5.11 the departure of the crew once it reaches the destination, should be found after taking the minimum layover time given, *i.e.* 5 hours. After words, minimum elements from both the matrices are to be selected to get the matrix showing minimum layover times. Finally, we have to make assignment for minimum layover time.

Layover time for Chennai based crew in hours.

Tableau I.

FlightNo.	*201*	*202*	*203*	*203*
101	24	25	28	12
102	23	24	27	11
103	20	19	22	6
104	13	14	17	25

Layover time for Bangalore based crew in hours.
Tableau I.

FlightNo.	*201*	*202*	*203*	*203*
101	22	21	18	10
102	23	22	19	11
103	28	27	24	16
104	9	8	5	21

Minimum of the two matrices layover time. The Bangalore based times are marked with a (*).
Tableau I.

FlightNo.	*201*	*202*	*203*	*203*
101	22*	21*	18*	10*
102	23**	22*	19*	11**
103	20	19	22	6
104	9*	8*	5*	21*

The elements with two stars (**) appear in both the matrices.

ROCM

Tableau I.

FlightNo.	*201*	*202*	*203*	*203*
101	12	11	8	0
102	12	11	8	0
103	14	13	16	0
104	4	3	0	16

TOCM:

FlightNo.	*201*	*202*	*203*	*203*
101	8	8	8	0
102	8	8	8	0
103	6	10	16	0
104	0	0	0	16

FlightNo.	*201*	*202*	*203*	*203*
101	2	2	2	0
102	2	2	2	0
103	0	4	10	0
104	0	0	0	22

FlightNo.	*201*	*202*	*203*	*203*
101	0	0	0	0
102	0	0	0	0
103	0	4	10	2
104	0	0	0	24

FlightNo.	*201*	*202*	*203*	*204*
101	0	0	0	0*
102	0	0*	0	0
103	0	4	10	2
104	0	0	0*	24

Assignment:

Flight No.	*Leaves as*	*Based at*
101	204	Bangalore
102	202	Bangalore
103	201	Chennai
104	203	Bangalore.

Total layover time: 10 + 22 + 20 + 5 = 67 hours.

TRAVELING SALESMAN PROBLEM

Just consider how a postman delivers the post to the addressee. He arranges all the letters in an order and starts from the post office and goes from addressee to addressee and finally back to his post office. If he does not arrange the posts in an order he may have to travel a long distance to clear all the posts. Similarly, a traveling sales man has to plan his visits. Let us say, he starts from his head office and go round the branch offices and come back to his head office. While traveling he will not visit the branch already visited and he will not come back until he visits all the branches.

There are different types of traveling salesman's problems. One is **cyclic problem**. In this problem, he starts from his head quarters and after visiting all the branches, he will be back to his head quarters. The second one is **Acyclic problem**. In this case, the traveling salesman leaves his head quarters and after visiting the intermediate branches, finally reaches the last branch and stays there. The first type of the problem is solved by Hungarian method or Assignment technique. The second one is solved by Dynamic programming method.

Point to Note: The traveling salesman's problem, where we sequence the cities or branches he has to visit is a SEQUENCING PROBLEM. But the solution is got by Assignment technique. Hence basically, the traveling salesman problem is a SEQUENCING PROBLEM; the objective is to minimize the total distance traveled.

The mathematical statement of the problem is: Decide variable $x_{ij} = 1$ or 0 for all values of I and j so as to:

Minimise $$Z = \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} \quad \text{for all } i \text{ and } j = 1,2\ldots..n \text{ Subject to}$$

$$\sum_{J=1}^{n} X_{ij} = 1 \text{ for } i = 1,2,\ldots.n \text{ (Depart from a city once only)}$$

$$\sum_{i=1}^{n} X_{ij} = 1 \text{ for } j = 1,2,\ldots.n \text{ (Arrive at a city once only)}$$

And all $x_{ij} \geq 0$ for all i and j

This is indeed a statement of assignment problem, which may give to or more disconnected cycles in optimum solution. This is not permitted. That is salesman is not permitted to return to the origin of his tour before visiting all other cities in his itinerary. The mathematical formulation above does not take care of this point.

A restriction like $X_{ab} + X_{bc} + X_{ca} \leq 2$ will prevent sub-cycles of cities A, B, C and back to A. It is sufficient to state at this stage that all sub- cycles can be ruled out by particular specifications of linear constraints. This part, it is easy to see that a variable $x_{ij} = 1$, has no meaning. To exclude this from solution, we attribute very large cost to it *i.e.* infinity or big M, which is very larger than all the elements in the matrix.

In our solutions big M is used.

Problem 5.14.

A salesman stationed at city A has to decide his tour plan to visit cities B, C, D, E and back to city A I the order of his choice so that total distance traveled is minimum. No sub touring is permitted. He cannot travel from city A to city A itself. The distance between cities in Kilometers is given below:

Cities	*A*	*B*	*C*	*D*	*E*
A	M	16	18	13	20
B	21	M	16	27	14
C	12	14	M	15	21
D	11	18	19	M	21
E	16	14	17	12	M

Instead of big M we can use infinity also. Or any element, which is sufficiently larger than all the elements in the matrix, can be used.

Solution

COCM:

Cities	*A*	*B*	*C*	*D*	*E*
A	M	3	5	0	7
B	7	M	2	13	0
C	0	2	M	3	9
D	0	7	8	M	10
E	4	2	5	0	M

TOCM:

Cities	*A*	*B*	*C*	*D*	*E*
A	M	1	3	0	7
B	7	M	0	13	0
C	0	0	M	3	9
D	0	5	6	M	10
E	4	0	3	0	M

We can make only 4 assignments. Hence modify the matrix. Smallest element in the uncovered cells is 3, deduct this from all other uncovered cells and add this to the elements at the crossed cells. Do not alter the elements in cells covered by the line.

TOCM

Cities	*A*	*B*	*C*	*D*	*E*
A	M	1	3	0	7
B	7	M	0	13	0
C	0	0	M	3	9
D	0	5	6	M	10
E	4	0	3	0	M

We can make only 4 assignments. Hence once again modify the matrix.

Sequencing: *A* to *C*, *C* to *B*, *B* to *E*, *E* to *D*, and *D* to *A*. As there is a tie

TOCM:

Cities	*A*	*B*	*C*	*D*	*E*
A	M	1	**0**	0	4
B	10	M	0 x	16	**0**
C	0x	**0**	M	3	6
D	**0**	5	3	M	7
E	4	0x	0	**0**	M

Sequencing: *A* to *C*, *C* to *B*, *B* to *E*, *E* to *D* and *D* to *A*. as there is a tie between the zero cells, the problem has alternate solution. The total distance traveled by the salesman is: 18 + 14 + 14 + 11 + 12 = 69 Km.

A to *C* to *B* to *E* to *D* to *A*, the distance traveled is 69 Km.

Note: See that no city is visited twice by sales man.

Problem 5.15.

Given the set up costs below, show how to sequence the production so as to minimize the total setup cost per cycle.

Jobs	*A*	*B*	*C*	*D*	*E*
A	M	2	5	7	1
B	6	M	3	8	2
C	8	7	M	4	7
D	12	4	6	M	5
E	1	3	2	8	M

Solution

COCM:

Jobs	*A*	*B*	*C*	*D*	*E*
A	M	1	4	6	0
B	4	M	1	6	0
C	4	3	M	0	3
D	8	0	2	M	1
E	0	2	1	7	M

TOCM:

Jobs	*A*	*B*	*C*	*D*	*E*
A	M	1	3	6	**0**
B	4	M	**0**	6	0x
C	4	3	M	**0**	3
D	8	**0**	1	M	1
E	**0**	2	0x	7	M

We can draw five lines and make assignment. The assignment is:

From *A* to *E* and From *E* to *A* cycling starts, which is not allowed in salesman problem. Hence what we have to do is to select the next higher element than zero and make assignment with those elements. After assignment of next higher element is over, then come to zero for assignment. If we cannot finish the assignment with that higher element, then select next highest element and finish assigning those elements and come to next lower element and then to zero. Like this we have to finish all assignments. In this problem, the next highest element to zero is 1. Hence first assign all ones and then consider zero for assignment. Now we shall first assign all ones and then come to zero.

TOCM:

Jobs	*A*	*B*	*C*	*D*	*E*
A	M	**1**	3	6	0x
B	4	M	**0**	6	0x
C	4	3	M	**0**	3
D	8	~~0~~	~~1~~	M	**1**
E	**0**	2	0x	7	M

The assignment is *A* to *B*, *B* to *C*, *C* to *D* and *D* to *E* and *E* to *A*. (If we start with the element DC then cycling starts.

Now the total distance is 5 + 3 + 4 + 5 + 1 = 18 + 1 + 1 = 20 Km. The ones we have assigned are to be added as penalty for violating the assignment rule of assignment algorithm.

Problem 5.16.

Solve the traveling salesman problem by using the data given below:

$C_{12} = 20$, $C_{13} = 4$, $C_{14} = 10$, $C_{23} = 5$, $C_{34} = 6$, $C_{25} = 10$, $C_{35} = 6$, $C_{45} = 20$ and $C_{ij} = C_{ji}$. And there is no route between cities '*i*' and '*j*' if a value for C_{ij} is not given in the statement of the problem. (*i* and *j* are = 1,2,..5)

Solution

Cities	*1*	*2*	*3*	*4*	*5*
1	M	20	4	10	M
2	20	M	5	M	10
3	4	5	M	6	6
4	10	M	6	M	20
5	M	10	6	20	M

Now let us work out COCM/ROCM and TOCM, and then make the assignment.

TOCM:

Cities.	*1*	*2*	*3*	*4*	*5*
1	M	12	**0**	0x	M
2	11	M	0x	M	**0**
3	0x	1	M	**0**	1
4	**0**	M	0x	M	9
5	M	**0**	0x	8	M

The sequencing is: 1 to3, 3 to 4, 4 to 1 and 1 to 3 etc., Cycling starts. Hence we shall start assigning with 1 the next highest element and then assign zeros. Here also we will not get the sequencing. Next we have to take the highest element 8 then assign 1 and then come to zeros.

TOCM:

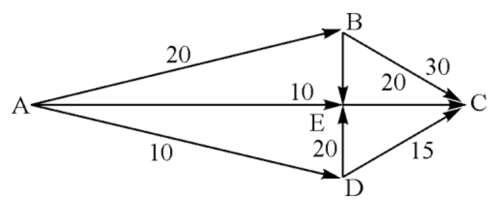
Cities.	*1*	*2*	*3*	*4*	*5*
1	M	12	**0**	0	M
2	11	M	0	M	**0**
3	0	**1**	M	0	1
4	**0**	M	0	M	9
5	M	0	0	8	M

Sequencing is: 1 to 3, 3 to 2, 2 to 5, 5 to 4 and 4 to 1.

The optimal distance is : 4 + 10 + 5 + 10 + 20 = 49 + 1 + 8 = 58 Km.

Problem 5.17.

A tourist organization is planning to arrange a tour to 5 historical places. Starting from the head office at A then going round *B*, *C*, *D* and *E* and then come back to *A*. Their objective is to minimize the total distance covered. Help them in sequencing the cities. *A*, *B*, *C*, *D* and E as the shown in the figure. The numbers on the arrows show the distances in Km.



Solution

The distance matrix is as given below:

Places	*A*	*B*	*C*	*D*	*E*
A	M	20	M	10	10
B	20	M	30	M	35
C	M	30	M	15	20
D	10	M	15	M	20
E	10	35	20	20	M

COCM

Places	*A*	*B*	*C*	*D*	*E*
A	M	10	M	0	0
B	0	M	10	M	15
C	M	15	M	0	5
D	0	M	5	M	10
E	0	25	10	10	M

TOCM:

Places	*A*	*B*	*C*	*D*	*E*
A	M	0	M	0	0
B	0	M	5	M	15
C	M	5	M	0	5
D	0	M	0	M	10
E	0	15	5	10	M

TOCM:

Places	*A*	*B*	*C*	*D*	*E*
A	M	0	M	5	0
B	0	M	5	M	10
C	M	0	M	0	0
D	0	M	0	M	5
E	0	10	5	10	M

TOCM:

Places	*A*	*B*	*C*	*D*	*E*
A	M	0	M	5	0
B	0	M	5	M	5
C	M	0	M	0	0
D	0	M	0	M	0
E	0	5	5	5	M

Places	*A*	*B*	*C*	*D*	*E*
A	M	**0**	M	5	0x
B	0x	M	**0**	M	0x
C	M	0x	M	**0**	0x
D	5	M	0x	M	**0**
E	0	0x	0x	0x	M

The sequencing is: *A* to *B*, *B* to *C*, *C* to *D*, *D* to *E* and *E* to *A*.
The total distance is: 20 +30 +15 + 20 + 10 = 95 Km.

SENSITIVITY ANALYSIS

In fact there is very little scope for sensitivity analysis in Assignment Problem because of the mathematical structure of the problem. If we want to avoid high cost assigning a facility (*i* th) to a job (*j* th), then we can do it by giving a cost of assignmat say infinity or Big M to that cell so that it will not enter into programme. In case of maximisaton model, we can allocate a negative element to that cell to avoid it entering the solution. Further, if one facility (man) can do two jobs *i.e.* 2 jobs are to be assigned to the facility, then this problem can be dealt with by repeating the man's or facility's column and introducing a dummy row to maintain the square matrix. Similarly, if two similar jobs are there, write two identical rows of the two jobs separately and then solve by making a square matrix. Besides these, the addition of a constant throughout any row or column does not affect the optimal solution of the assignment problem.

QUESTIONS

1. Four engineers are available to design four projects. Engineer 2 is not competent to design the project B. Given the following time estimates needed by each engineer to design a given project, find how should the engineers be assigned to projects so as to minimize the total design time of four projects.

Engineers.	*Projects*			
	A	*B*	*C*	*D*
1	12	10	10	8
2	14	NOT ELIGIBLE	15	11
3	6	10	16	4
4	8	10	9	7

2. (*a*) Explain the differences and similarities between Assignment problem and Transportation problem.

   (*b*) Explain why VAM or any other methods of getting basic feasible solution to a transportation problem is not used to get a solution to assignment problem. What difficulties you come across?

3. Explain briefly the procedure adopted in assignment algorithm.
4. Is traveling salesman problem is an assignment problem? If yes how? If not what are the differences between assignment problem and traveling salesman problem.
5. What do you mean by balancing an assignment problem? What steps you take to solve maximization case in assignment problem?Explain.
6. A Computer center has got three expert programmers. The center needs three application programmes to be developed. The head of the computer center, after studying carefully the programmes to be developed estimate the computer time in minutes required by the experts to the application programmes as given in the matrix below. Assign the programmers to the programmes in such a way that the total computer time is least.

Programmers.	Programme.		
	A	B	C
1	120	100	80
2	70	90	110
3	110	140	120

7. (*a*). A company is faced with the problem of assigning six different machines to five different jobs. The costs are estimated as follows in hundreds of rupees. Assign the jobs to machines to minimize the total cost.

(*b*) If the given matrix happens to be returns to the company by assigning a particular job to a machine, then what will be the assignment? Will the same assignment hold well? If not what will you do to get the new solution.

Jobs (hundreds of rupees)

Machines.	1	2	3	4	5
1	2.5	5	1	6	1
2	2	5	1.5	7	3
3	3	6.5	2	8	3
4	3.3	7	2	9	4.5
5	4	7	3	9	6
6	6	9	5	10	6

8. Miss *A*, *B*, *C*, *D*, *E*, *F* and *G* are seven girls in a 15-member college musical extravaganza team. M/S I, II, III, IV, V, VI, VII and VIII are the male members of the team and eligible bachelors except Mr. IV. The team decides thinking in terms of matrimonial bondage amongst them that the match - making should be such as to maximize the happiness of the entire group. Fortunately for Mr. IV is already married and he is asked to devise measure of happiness, collect data and decide the pairs.

Mr. IV collects on a 20 - point scale girl's liking for different boys and calls it as *X* - factor and the boy's liking for different girls as *Y*- factor. The matrix given below shows these to factors. The elements in the brackets are *Y* - factors. Mr. IV is baffled by the pattern of emotional linkages and variations in their intensities.

He decides on his own without consulting anyone concerned, to give more weightages to *X* - factor on account of intuitional soundness of girl's soundness of judgement and their emotional steadfastness, beside flexibility in adjustments. Therefore, he takes $K = 2X + B$ as the factor of pair's matrimonial happiness. Then coded matrix of *A*, *B*, *C*, *D*, *E*, *F*, and *G* against *K*, *L*, *M*, *N*, *P*, *Q*, and *R* is given to an Operations Research student to solve it as an assignment problem. (Girls form A to G and Boys form K to R). The solution was handed over to all the concerned. Make the assignment.

Points. Factor Y (Factor X)

Boys

Girls ↓	*K*	*L*	*M*	*N*	*P*	*Q*	*R*
A	11 (14)	14 (15)	15 (10)	13 (18)	16 (15)	17 (14)	12 (10)
B	16 (15)	13 (17)	18 (11)	15 (18)	16 (14)	17 (14)	14 (12)
C	10 (16)	10 (11)	10 (12)	10 (18)	10 (18)	10 (18)	10 (13)
D	16 (16)	11 (14)	10 (12)	10 (18)	11 (15)	12 (13)	14 (15)
E	7 (18)	5 (17)	12 (13)	14 (10)	6 (16)	17 (12)	15 (16)
F	9 (18)	16 (12)	12 (11)	13 (18)	12 (11)	15 (12)	16 (15)
G	15 (15)	17 (10)	16 (7)	17 (18)	15 (10)	17 (13)	16 (14)

9. Solve the traveling salesman problem given below for minimizing the total distance traveled. Distance in Km.

Cities	*A*	*B*	*C*	*D*	*E*
A	M	10	8	29	12
B	16	14	12	10	9
C	6	3	17	14	12
D	12	19	17	14	12
E	11	8	16	13	M

10. An airline that operates flights between Delhi and Bombay has the following timetable. Pair the flights, so as to minimize the total layover time for the crew. The plane, which reaches its destination, cannot leave that place before 4 hours of rest.

Flight No.	*Departure*	*Arrival*	*Flight No.*	*Departure*	*Arrival*
101	9.00 a.m	11.00 a.m	201	10.00 a.m	12.00 Nn.
102	10.00 a.m	12.00 Nn	202	12.00 Nn	2.00 p.m
103	4.00 p.m	6.00 p.m	203	3.00 p.m	5.00 p.m
104	7.00 p.m	9.00 p.m	204	8.00 p.m	10.p.m.

11. The productivity of operators *A, B, C, D,* and *E* on different machines *P, Q, R, S,* and *T* are given in the matrix below. Assign machine to operators of maximum productivity.

Productivity
Machines.

Operators ▼	*P*	*Q*	*R*	*S*	*T*
A	9	14	10	7	12
B	8	11	12	---	13
C	10	10	8	11	---
D	12	14	11	10	7
E	13	10	12	13	10

12. In the above problem, operating costs of machines / shift are Rs.6/-, Rs.7/- Rs.15/-, Rs. 11/- and Rs. 10/- respectively, and Daily wages are Rs. 25/- , Rs. 30/- , Rs. 28/-, Rs. 26/- and Rs.20/- respectively for machine a, *B*, c, *D* and *E*. And all the operators on piece - bonus, so that for every one piece above the basic production per shift the bonus is paid at the rates are as shown on next page on different machines along with basic production per shift. Find the cost of production and the cost per unit. Assign the machines to operators for minimum cost of production per piece.

Machines.

Particulars.	P	Q	R	S	T
Basic productionPieces per shift.	8	10	8	7	7
Incentive bonusPer piece in Rs.	1.0	1.0	1.6	2.0	2.0

MULTIPLE CHOICE QUESTIONS

1. Assignment Problem is basically a
   (a) Maximization Problem, (*b*) Minimization Problem, (*c*) Transportation Problem
   (*d*) Primal problem ( )
2. The Assignment Problem is solved by
   (a) Simplex method, (*b*) Graphical method, (*c*) Vector method, (*d*) Hungarian method ( )
3. In Index method of solving assignment problem
   (a) The whole matrix is divided by smallest element, (*b*) The smallest element is subtracted from whole matrix (*c*) Each row or column is divided by smallest element in that particular row or column, (*d*) The whole matrix is multiplied by – 1.
4. In Hungarian method of solving assignment problem, the row opportunity cost matrix is obtained by:
   (a) Dividing each row by the elements of the row above it,
   (b) By subtracting the elements of the row from the elements of the row above it.
   (c) By subtracting the smallest element from all other elements of the row.
   (d) By subtracting all the elements of the row from the highest element in the matrix. ( )
5. In Flood's technique of solving assignment problem the column opportunity cost matrix is obtained by:
   (a) Dividing each column by the elements of a column which is right side of the column
   (b) By subtracting the elements of a column from the elements of the column which is right side of the column
   (c) By subtracting the elements of the column from the highest element of the matrix.
   (d) By subtracting the smallest elements in the column from all other elements of the column. ( )
6. The property of total opportunity cost matrix is
   (a) It will have zero as elements of one diagonal,
   (b) It will have zero as the elements of both diagonals,
   (c) It will have at least one zero in each column and each row
   (d) It will not have zeros as its elements. ( )

7. The horizontal and vertical lines drawn to cover all zeros of total opportunity matrix must be:
   (a) Equal to each other,
   (b) Must be equal to $m \times$ n (where m and n are number of rows and columns)
   (c) $m + n$ (m and n are number of rows and columns)
   (d) Number of rows or columns. ( )
8. The assignment matrix is always is a
   (a) Rectangular matrix, (*b*) Square matrix (*c*) Identity matrix (*d*) None of the above. ( )
9. To balance the assignment matrix we have to:
   (a) Open a Dummy row,
   (b) Open a Dummy column,
   (c) Open either a dummy row or column depending on the situation,
   (d) You cannot balance the assignment matrix. ( )
10. In cyclic traveling salesman problem the elements of diagonal from left top to right bottom are
    (a) Zeros, (*b*) All negative elements, (*c*) All are infinity (*d*) all are ones. ( )
11. To convert the assignment problem into a maximization problem
    (a) Deduct smallest element in the matrix from all other elements.
    (b) All elements of the matrix are deducted from the highest element in the matrix.
    (c) Deduct smallest element in any row form all other elements of the row.
    (d) Deduct all elements of the row from highest element in that row. ( )
12. The similarity between Assignment Problem and Transportation problem is:
    (a) Both are rectangular matrices, (*b*) Both are square matrices,
    (*c*) Both can be solved by graphical method, (*d*) Both have objective function and non-negativity constraints. ( )
13. The following statement applies to both transportation model and assignment model
    (a) The inequalities of both problems are related to one type of resource.
    (b) Both use VAM for getting basic feasible solution
    (c) Both are tested by MODI method for optimality
    (d) Both have objective function, structural constraint and non-negativity constraints. ( )
14. To test whether allocations can be made or not (in assignment problem), minimum number of horizontal and vertical lines are drawn. In case the lines drawn is not equal to the number of rows (or columns), to get additional zeros, the following operation is done:
    (a) Add smallest element of the uncovered cells to the elements to the line
    (b) Subtract smallest element of uncovered rows from all other elements of uncovered cells.
    (c) Subtract the smallest element from the next highest number in the element.
    (d) Subtract the smallest element from the element at the intersection of horizontal and vertical line. ( )

15. The total opportunity cost matrix is obtained by doing:
    (a) Row operation on row opportunity cost matrix,
    (b) by doing column operation on row opportunity cost matrix,
    (c) By doing column operation on column opportunity cost matrix
    (d) None of the above ( )
16. Flood's technique is a method used for solving
    (a) Transportation problem, (*b*) Resource allocation model, (*c*) Assignment mode.
    (*d*) Sequencing model ( )
17. The assignment problem will have alternate solutions when total opportunity cost matrix has
    (a) At least one zero in each row and column,
    (b) When all rows have two zeros,
    (c) When there is a tie between zero opportunity cost cells,
    (d) If two diagonal elements are zeros. ( )
18. The following character dictates that assignment matrix is a square matrix:
    (a) The allocations in assignment problem are one to one
    (b) Because we find row opportunity cost matrix
    (c) Because we find column opportunity matrix
    (d) Because one to make allocations, one has to draw horizontal and veridical lines. ( )
19. When we try to solve assignment problem by transportation algorithm the following difficulty arises:
    (a) There will be a tie while making allocations
    (b) The problem will get alternate solutions,
    (c) The problem degenerate and we have to use epsilon to solve degeneracy
    (d) We cannot solve the assignment problem by transportation algorithm. ( )

ANSWERS

1. (*b*)	2. (*d*)	3. (*c*)	4. (*c*)
5. (*d*)	6. (*c*)	7. (*d*)	8. (*b*)
9. (*c*)	10. (*c*)	11. (*b*)	12. (*d*)
13. (*d*)	14. (*b*)	15. (*b*)	16. (*c*)
17. (*c*)	18. (*a*)	19. (*c*)	

***CHAPTER* – 6**

Sequencing

INTRODUCTION

In the previous chapters we have dealt with problems where two or more competing candidates are in race for using the same resources and how to decide which candidate (product) is to be selected so as to maximize the returns (or minimize the cost).

Now let us look to a problem, where we have to determine the order or sequence in which the jobs are to be processed through machines so as to minimize the total processing time. Here the total effectiveness, which may be the time or cost that is to be minimized is the function of the order of sequence. Such type of problem is known as **SEQUENCING PROBLEM.**

In case there are three or four jobs are to be processed on two machines, it may be done by trial and error method to decide the optimal sequence (*i.e.* by method of enumeration). In the method of enumeration for each sequence, we calculate the total time or cost and search for that sequence, which consumes the minimum time and select that sequence. This is possible when we have small number of jobs and machines. But if the number of jobs and machines increases, then the problem becomes complicated. It cannot be done by method of enumeration. Consider a problem, where we have 'n' machines and 'm' jobs then we have $(n!)^m$ theoretically possible sequences. For example, we take $n = 5$ and $m = 5$, then we have $(5!)^5$ sequences *i.e.* which works out to 25, 000,000,000 possible sequences. It is time consuming to find all the sequences and select optima among all the sequences. Hence we have to go for easier method of finding the optimal sequence. Let us discuss the method that is used to find the optimal sequence. Before we go for the method of solution, we shall define the sequencing problem and types of sequencing problem. The student has to remember that the sequencing problem is basically a **minimization problem or minimization model.**

THE PROBLEM:(DEFINITION)

A general sequencing problem may be defined as follows:

Let there be 'n' jobs (J_1, J_2, J_3 ………J_n) which are to be processed on 'm' machines (A, B, C, ………), where the order of processing on machines *i.e.* for example, *ABC* means first on machine A, second on machine *B* and third on machine *C* or *CBA* means first on machine *C*, second on machine *B* and third on machine *A* etc. and the processing time of jobs on machines (actual or expected) is known to us, then our job is to find the optimal sequence of processing jobs that minimizes the total processing time or cost. Hence our job is to find that sequence out of $(n!)^m$ sequences, which minimizes the total elapsed time (*i.e.*. time taken to process all the jobs). The usual notations used in this problem are:

A_i = Time taken by i th job on machine A where i = I, 2,3…n. Similarly we can interpret for machine B and C *i.e.* B_i and C_i etc.

T = Total elapsed time which includes the idle time of machines if any and set up time and transfer time.

Assumptions Made in Sequencing Problems

Principal assumptions made for convenience in solving the sequencing problems are as follows:

(a) The processing times A_i and B_i etc. are exactly known to us and they are independent of order of processing the job on the machine. That is whether job is done first on the machine, last on the machine, the time taken to process the job will not vary it remains constant.

(b) The time taken by the job from one machine to other after processing on the previous machine is negligible. (Or we assume that the processing time given also includes the transfer time and setup time).

(c) Each job once started on the machine, we should not stop the processing in the middle. It is to be processed completely before loading the next job.

(d) The job starts on the machine as soon as the job and the machine both become idle (vacant). This is written as **job is next to the machine and the machine is next to the job.** (This is exactly the meaning of transfer time is negligible).

(e) No machine may process more than one job simultaneously. (This means to say that the job once started on a machine, it should be done until completion of the processing on that machine).

(f) The cost of keeping the semi-finished job in inventory when next machine on which the job is to be processed is busy is assumed to be same for all jobs or it is assumed that it is too small and is negligible. That is in process inventory cost is negligible.

(g) While processing, no job is given priority *i.e.* the order of completion of jobs has no significance. The processing times are independent of sequence of jobs.

(h) There is only one machine of each type.

Applicability

The sequencing problem is very much common in Job workshops and Batch production shops. There will be number of jobs which are to be processed on a series of machine in a specified order depending on the physical changes required on the job. We can find the same situation in computer center where number of problems waiting for a solution. We can also see the same situation when number of critical patients waiting for treatment in a clinic and in Xerox centers, where number of jobs is in queue, which are to be processed on the Xerox machines. Like this we may find number of situations in real world.

Types of Sequencing Problems

There are various types of sequencing problems arise in real world. All sequencing problems cannot be solved. Though mathematicians and Operations Research scholars are working hard on the problem satisfactory method of solving problem is available for few cases only. The problems, which can be solved, are:

- **(a)** **'*n*' jobs are to be processed on two machines say machine *A* and machine *B* in the order *AB*. This means that the job is to be processed first on machine *A* and then on machine *B*.**
- **(b)** **'*n*' jobs are to be processed on three machines *A*,*B* and *C* in the order *ABC* *i.e.* first on machine *A*, second on machine B and third on machine *C*.**
- **(c)** **'*n*' jobs are to be processed on '*m*' machines in the given order**
- **(d)** **Two jobs are to be processed on '*m*' machines in the given order.**

SOLUTIONS FOR SEQUENCING PROBLEMS

Now let us take above mentioned types problems and discuss the solution methods.

'N' Jobs and Two Machines

If the problem given has two machines and two or three jobs, then it can be solved by using the Gantt chart. But if the numbers of jobs are more, then this method becomes less practical. (For understanding about the Gantt chart, the students are advised to refer to a book on Production and Operations Management (chapter on Scheduling).

Gantt chart consists of *X* -axis on which the time is noted and *Y*-axis on which jobs or machines are shown. For each machine a horizontal bar is drawn. On these bars the processing of jobs in given sequence is marked. Let us take a small example and see how Gantt chart can be used to solve the same.

Problem 6.1.

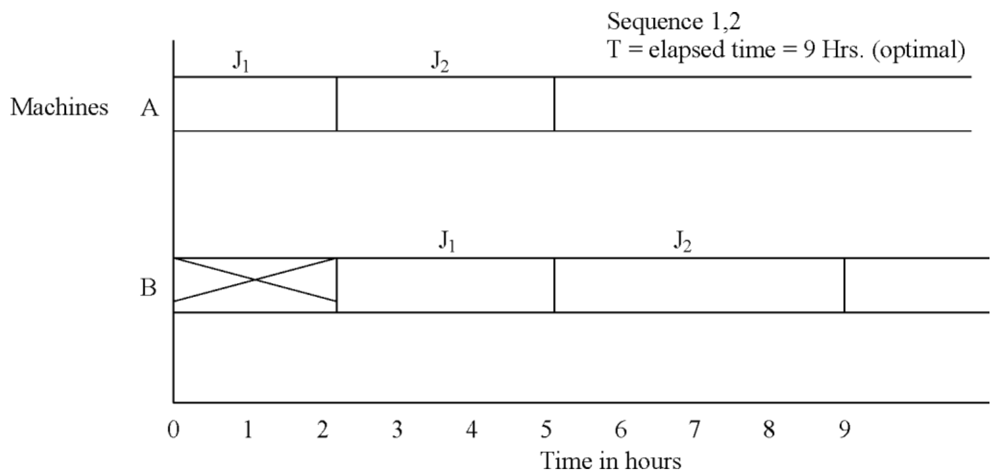
There are two jobs job 1 and job 2. They are to be processed on two machines, machine *A* and Machine *B* in the order *AB*. Job 1 takes 2 hours on machine *A* and 3 hours on machine *B*. Job 2 takes 3 hours on machine *A* and 4 hours on machine *B*. Find the optimal sequence which minimizes the total elapsed time by using Gantt chart.

Solution

Jobs.	*Machines (Time in hours)*	
	A	B
1	2	3
2	3	4

(a) Total elapsed time for sequence 1,2 *i.e.* first job 1 is processed on machine *A* and then on second machine and so on.

Draw X - axis and Y- axis, represent the time on X - axis and two machines by two bars on Y-axis. Then mark the times on the bars to show processing of each job on that machine.



Sequence 1,2
Total = elapsed time = 9 Hrs. (optimal sequence)

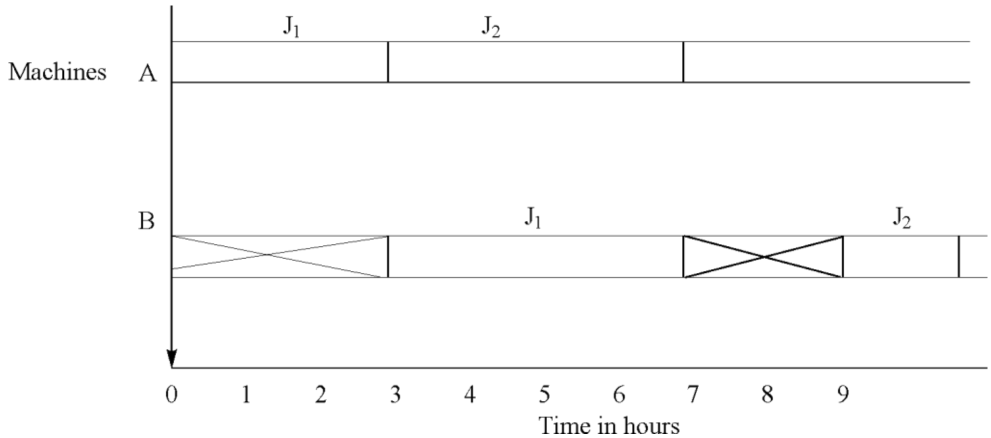


Figure 6.1 Gantt chart.

Both the sequences shows the elapsed time = 9 hours.

The draw back of this method is for all the sequences, we have to write the Gantt chart and find the total elapsed times and then identify the optimal solution. This is laborious and time consuming. If we have more jobs and more machines, then it is tedious work to draw the chart for all sequences. Hence we have to go for analytical methods to find the optimal solution without drawing charts.

Analytical Method

A method has been developed by **Johnson and Bellman** for simple problems to determine a sequence of jobs, which minimizes the total elapsed time. The method:

1. 'n' jobs are to be processed on two machines A and B in the order AB (*i.e*. each job is to be processed first on A and then on B) and passing is not allowed. That is which ever job is processed first on machine A is to be first processed on machine B also, Which ever job is processed second on machine A is to be processed second on machine B also and so on. That means each job will first go to machine A get processed and then go to machine B and get processed. ***This rule is known as no passing rule.***
2. Johnson and Bellman method concentrates on minimizing the idle time of machines. Johnson and Bellman have proved that optimal sequence of 'n' jobs which are to be processed on two machines A and B in the order AB necessarily involves the same ordering of jobs on each machine. This result also holds for three machines but does not necessarily hold for more than three machines. Thus total elapsed time is minimum when the sequence of jobs is same for both the machines.
3. Let the number of jobs be 1,2,3,… .... n

The processing time of jobs on machine A be A_1, A_2, A_3.......................... A_n
The processing time of jobs on machine B be B_1, B_2, B_3.......................... B_n

Jobs	*Machining time in hours.*		
	Machine A	*Machine B*	*(Order of processing is AB)*
1	A_1	B_1	
2	A_2	B_2	
3	A_3	B_3	
………	………	………	
I	A_I	B_I	
………	………	………	
S	A_S	B_S	
………	………	………	
………	………	………	
T	A_T	B_T	
………	………	………	
………	………	………	
N	A_N	B_N	

4. Johnson and Bellman algorithm for optimal sequence states *that* **identify the smallest element in the given matrix.** ***If the smallest element falls under column 1 i.e under machine I then do that job first***. As the job after processing on machine 1 goes to machine 2, it reduces the idle time or waiting time of machine 2. ***If the smallest element falls under column 2 i.e under machine 2 then do that job last***. This reduces the idle time of machine 1. *i.e*. if r th job is having smallest element in first column, then do the r^{th} job first. If s th job has the smallest element, which falls under second column, then do the s th job last. Hence the basis for Johnson and Bellman method is to keep the idle time of machines as low as possible. Continue the above process until all the jobs are over.

1	2	3		n–1	n
r					s

5. If there are 'n' jobs, first write 'n' number of rectangles as shown. When ever the smallest elements falls in column 1 then enter the job number in first rectangle. If it falls in second column, then write the job number in the last rectangle. Once the job number is entered, the second rectangle will become first rectangle and last but one rectangle will be the last rectangle.
6. Now calculate the total elapsed time as discussed. Write the table as shown. Let us assume that the first job starts at Zero th time. Then add the processing time of job (first in the optimal sequence) and write in out column under machine 1. This is the time when the first job in the optimal sequence leaves machine 1 and enters the machine 2. Now add processing time of job on machine 2. This is the time by which the processing of the job on two machines over. Next consider the job, which is in second place in optimal sequence. This job enters the machine 1 as soon the machine becomes vacant, i.e first job leaves to second machine. Hence enter the time in out column for first job under machine 1 as the starting time of job two on machine 1. Continue until all the jobs are over. Be careful to see that whether the machines are vacant before loading. Total elapsed time may be worked out by drawing Gantt chart for the optimal sequence.
7. Points to remember:

(*a*) If there is tie i.e we have smallest element of same value in both columns, then:

(i) Minimum of all the processing times is A_r which is equal to B_s *i.e.* Min $(A_i, B_i) = A_r = B_s$ then do the r th job first and s th job last.

(ii) If Min $(A_i, B_i) = A_r$ and also $A_r = A_k$ (say). Here tie occurs between the two jobs having same minimum element in the same column *i.e.* first column we can do either r th job or k th job first. There will be two solutions. When the ties occur due to element in the same column, then the problem will have alternate solution. If more number of jobs have the same minimum element in the same column, then the problem will have many alternative solutions. If we start writing all the solutions, it is a tedious job. Hence it is enough that the students can mention that the problem has alternate solutions. The same is true with B_i s also. If more number of jobs have same minimum element in second column, the problem will have alternate solutions.

Problem 6.2.

There are five jobs, which are to be processed on two machines A and B in the order AB. The processing times in hours for the jobs are given below. Find the optimal sequence and total elapsed time. ***(Students has to remember in sequencing problems if optimal sequence is asked, it is the duty of*** the student to find the total elapsed time also).

Jobs:	1	2	3	4	5
Machine A (Time in hrs.)	2	6	4	8	10
Machine B (Time in Hrs)	3	1	5	9	7

The smallest element is 1 it falls under machine *B* hence do this job last i.e in 5 th position. Cancel job 2 from the matrix. The next smallest element is 2, it falls under machine A hence do this job first, i.e in the first position. Cancel the job two from matrix. Then the next smallest element is 3 and it falls under machine *B*. Hence do this job in fourth position. Cancel the job one from the matrix. Proceed like this until all jobs are over.

1	3	4	5	2

Total elapsed time:

OPTIMAL	*MACHINE -A*		*MACHINE - B*		*MACHINE IDLE JOB IDLE*			*REMARKS*
SEQUENCE	*IN*	*OUT*	*IN*	*OUT*	*A*	*B*		
1	0	2	2	5		2		
3	2	6	6	11		1		As the Machine B Finishes Work at 5 Th hour will be Idle for 1 Hour.
4	6	14	14	23		3		-do- 3 hr.
5	14	24	24	31		1		-do- 1 hr.
2	24	30	31	32	1	2		1 hr as job finished early 1 hr idle.

Total elapsed time = 32 hours. (This includes idle time of job and idle time of machines).

The procedure: Let Job 1 is loaded on machine *A* first at zero th time. It takes two hours to process on the machine. Job 1 leaves the machine *A* at two hours and enters the machine 2 at 2-nd hour. Up to the time i.e first two hours, the machine *B* is idle. Then the job 1 is processed on machine *B* for 3 hours and it will be unloaded. As soon as the machine *A* becomes idle, *i.e*. at 2 nd hour then next job 3 is loaded on machine *A*. It takes 4 hours and the job leaves the machine at 6 th hour and enters the machine *B* and is processed for 6 hours and the job is completed by 11 th hour. (Remember if the job is completed early and the Machine *B* is still busy, then the job has to wait and the time is entered in job idle column. In case the machine B completes the previous job earlier, and the machine *A* is still processing the next job, the machine has to wait for the job. This will be shown as machine idle time for machine *B*.). Job 4 enters the machine *A* at 6 th hour and processed for 8 hours and leaves the machine at 14 th hour. As the machine *B* has finished the job 3 by 11 th hour, the machine has to wait for the next job (job 4) up to 14 th hour. Hence 3 hours is the idle time for the machine *B*. In this manner we have to calculate the total elapsed time until all the jobs are over.

Problem 6.3.

There are 6 jobs to be processed on Machine *A*. The time required by each job on machine *A* is given in hours. Find the optimal sequence and the total time elapsed.

Job:	1	2	3	4	5	6
Time in hours. Machine A	6	4	3	2	9	8

Solution

Here there is only one machine. Hence the jobs can be processed on the machine in any sequence depending on the convenience. The total time elapsed will be total of the times given in the problem. As soon as one job is over the other follows. The total time is 32 hours. The sequence may be any order. For example: 1,2,3,4,5,6 or 6,5,4,3,2,1, or 2, 4 6 1 3 5 and so on.

Problem 6.4.

A machine operator has to perform two operations, turning and threading, on a number of different jobs. The time required to perform these operations in minutes for each job is given. Determine the order in which the jobs should be processed in order to minimize the total time required to turn out all the jobs.

Jobs:	1	2	3	4	5	6
Time for turning (in min.)	3	12	5	2	9	11
Time for threading (in min).	8	10	9	6	3	1

Solution

The smallest element is 1 in the given matrix and falls under second operation. Hence do the 6 th job last. Next smallest element is 2 for the job 4 and falls under first operation hence do the fourth job first. Next smallest element is 3 for job 1 falls under first operation hence do the first job second. Like this go on proceed until all jobs are over. The optimal sequence is :

4	1	3	2	5	6

Optimal sequence.	*Turning operation*		*Threading operation*		*Job idle*	*Machine idle.*	
	In	*out*	*In*	*out*		*Turning*	*threading.*
4	0	2	2	8	------		2
1	2	5	8	16	3		
3	5	10	16	25	6		
2	10	22	25	35	3		
5	22	31	35	38	4		
6	31	42	42	43	--	1	----
	Total elapsed time:		**43minutes.**				

The Job idle time indicates that there must be enough space to store the in process inventory between two machines. This point is very important while planning the layout of machine shops.

Problem 6.5.

There are seven jobs, each of which has to be processed on machine *A* and then on Machine *B* (order of machining is *AB*). Processing time is given in hours. Find the optimal sequence in which the jobs are to be processed so as to minimize the total time elapsed.

JOB:	1	2	3	4	5	6	7
MACHINE: A (TIME IN HOURS).	3	12	15	6	10	11	9
MACHINE: B (TIME IN HOURS).	8	10	10	6	12	1	3

Solution

By Johnson and Bellman method the optimal sequence is:

1	4	5	3	2	7	6.

Optimal Sequence	Machine:A		Machine:B		Machine idle time		Job idle time	Remarks.
Squence	In	out	In	Out	A	B		
1	0	3	3	11		3	–	
4	3	9	11	17			2	Job finished early
5	9	19	19	31		2		Machine A take more time.
3	19	34	34	44		3		Machine A takes more time.
2	34	46	46	56		2		- do-
7	46	55	56	59			1	Job finished early.
6	55	66	66	67	1	7		Machine A takes more time. Last is finished on machine A at 66 th hour.
	Total		**Elapsed**		**Time = 67 hours.**			

Problem 6.6.

Find the optimal sequence that minimizes the total elapsed time required to complete the following tasks on two machines I and II in the order first on Machine I and then on Machine II.

Task:	A	B	C	D	E	F	G	H	I
Machine I (time in hours).	2	5	4	9	6	8	7	5	4
Machine II (time in hours).	6	8	7	4	3	9	3	8	11

Solution

By Johnson and Bellman method we get two sequences (this is because both machine B and H are having same processing times).

The two sequences are:

A	C	I	B	H	F	D	G	E.
A	C	I	H	B	F	D	G	E

Sequence	*Machine I*		*Machine II*		*Machine Idle*		*Job idle*	*Remarks.*
	In	*out*	*In*	*Out*	*I*	*II*		
A	O	2	2	8		2		
C	2	6	8	15			2	Job on machine I finished early.
I	6	10	15	26			5	Do
B	10	15	26	34			11	Do
H	15	20	34	42			14	Do
F	20	28	42	51			14	Do
D	28	37	51	55			14	Do
G	37	44	55	58			11	Do
E	44	50	58	61	11		8	Do.And machine I finishes its work at 50th hour.
	Total		**Elapsed time : 61 hours.**					

Problem 6.7.

A manufacturing company processes 6 different jobs on two machines A and B in the order AB. Number of units of each job and its processing times in minutes on A and B are given below. Find the optimal sequence and total elapsed time and idle time for each machine.

Job Number	*Number of units of each job.*	*Machine A: time in minutes.*	*Machine B: time in minutes.*
1	3	5	8
2	4	16	7
3	2	6	11
4	5	3	5
5	2	9	7.5
6	3	6	14

Solution

The optimal sequence by using Johnson and Bellman algorithm is

Sequence:	4	1	3	6	5	2
Number of units.	5	3	2	3	2	4

First do the 5 units of job 4, Second do the 3 units of job 1, third do the 2 units of job 3, fourth process 3 units of job 6, fifth process 2 units of job 5 and finally process 4 units of job 2.

Sequence of jobs	*Number. of units of job*	*Machine A Time in mins*		*Machine B Time in mins.*		*Idle time of machines*		*Job idle.*	*Remarks.*
		In	*out*	*In*	*out*	*A*	*B*		
4	1 st.	0	3	3	8	--	3	–	–
	2 nd	3	6	8	13				
	3 rd.	6	9	13	18				
	4 th	9	12	18	23				
	5th	12	15	23	28				
1	1 st	15	20	28	36			8	Machine B
	2 nd	20	25	36	44				BecomesVacant
	3rd	25	30	44	52				at 8th min.
3	1 st	30	36	52	63			16	Do (52 nd min.)
	2 nd.	36	42	63	74				
6	1 st.	42	48	74	88			26	Do (74 th min.)
	2 nd	48	54	88	102				
	3 rd	54	60	102	116				
5	1 st	60	69	116	123.5			47	Do (116 th min.)
	2 nd.	69	78	123.5	131				
2	1 st	78	94	131	138			37	Do (131 th min.)
	2 nd.	94	110	138	145				
	3 rd	110	126	145	152				
	4 th	126	142	152	159	17			
		Total Elapsed		**Time = 159 min**					

Total elapsed time = 159 mins. Idle time for Machine A = 17 mins. And that for machine B is 3 mins

SEQUENCING OF 'N' JOBS ON THREE MACHINES

When there are '*n*' jobs, which are to be processed on three machines say *A*, *B*, and *C* in the order *ABC* first on machine *A*, second on machine *B* and finally on machine *C*. We know processing times in time units. As such there is no direct method of sequencing of '*n*' jobs on three machines. Before solving, **a three-machine problem is to be converted into a two-machine problem**. The procedure for converting a three-machine problem into two-machine problem is:

(a) Identify the smallest time element in the first column, *i.e.* for machine 1 let it be A_r.
(b) Identify the smallest time element in the third column, *i.e.* for machine 3, let it be C_s
(c) Identify the highest time element in the second column, *i.e.* for the center machine, say machine 2, let it be B_i.
(d) Now minimum time on machine 1 *i.e.* A_r must be ≥ maximum time element on machine 2, *i.e.* B_i

OR

Minimum time on third machine *i.e.* C_s must be ≥ maximum time element on machine 2 *i.e.* B_i

OR

Both A_r and C_s must be $\geq B_i$

(e) If the above condition satisfies, then we have to workout the time elements for two hypothetical machines, namely machine *G* and machine *H*. The time elements for machine *G*, $G_i = A_i + B_i$.
The time element for machine *H*, is $H_i = B_i + C_i$
(f) Now the three-machine problem is converted into two-machine problem. We can find sequence by applying Johnson Bellman rule.
(g) All the assumption mentioned earlier will hold good in this case also.

Problem 6.8.

A machine operator has to perform three operations, namely plane turning, step turning and taper turning on a number of different jobs. The time required to perform these operations in minutes for each operating for each job is given in the matrix given below. Find the optimal sequence, which minimizes the time required.

Job.	*Time for plane turning In minutes*	*Time for step turning in minutes*	*Time for taper turning. in minutes.*
1	3	8	13
2	12	6	14
3	5	4	9
4	2	6	12
5	9	3	8
6	11	1	13

Solution

Here Minimum $A_i = 2$, Maximum $B_i = 8$ and Minimum $C_i = 8$.

As the maximum $B_i = 8$ = Minimum C_i, we can solve the problem by converting into two-machine problem.

Now the problem is:

Job	*Machine G* $(A_i + B_i)$ *Minutes.*	*Machine H* $(B_i + H_i)$ *Minutes.*
1	11	21
2	18	20
3	9	13
4	8	18
5	12	11
6	12	14

By applying Johnson and Bellman method, the optimal sequence is:

4	3	1	6	5	2

Now we can work out the Total elapsed time as we worked in previous problems.

Sequence	Plane turning Time in min.		Step turning Time in min.		Taper turning Time in Min.		Job Idle Time in Min.	Machine idle Time in Min.	Remarks.
	In	out	In	out	In	out		Tu StTu Tap Tu	
4	0	2	2	8	8	20		2 8	Until first Job comes 2nd and 3rd Operations idle.
3	2	7	8	12	20	29	1 + 8		
1	7	10	12	20	29	42	2 + 9		
6	10	21	21	22	42	55	20	1	
2	21	33	33	39	55	69	16	11	
5	33	42	42	45	69	77	14	3	
	Total		**Elapsed**		**Time: 77 min.**				

Problem 6.9.

There are 5 jobs each of which is to be processed on three machines *A*, *B*, and *C* in the order *ACB*. The time required to process in hours is given in the matrix below. Find the optimal sequence.

Job:	1	2	3	4	5
Machine A:	3	8	7	5	4
Machine B:	7	9	5	6	10
Machine C:	4	5	1	2	3.

Solution

Here the given order is *ACB*. *i.e.* first on machine *A*, second on Machine *C* and third on Machine *B*. Hence we have to rearrange the machines. Machine *C* will become second machine. Moreover optimal sequence is asked. But after finding the optimal sequence, we have to work out total elapsed time also. The procedure is first rearrange the machines and convert the problem into two-machine problem if it satisfies the required condition. Once it is converted, we can find the optimal sequence by applying Johnson and Bellman rule.

The problem is:

Job:	1	2	3	4	5
Machine A:	3	8	7	5	4
Machine C:	4	5	1	2	3
Machine B:	7	9	5	6	10

Max A_i = 8 Hrs. , Max B_i (third machine) = 5 Hrs. and minimum C_i = Middle machine = 5 Hrs. As Max B_i = Min C_i = 5, we can convert the problem into 2- machine problem.

Two-machine problem is:

Job:	1	2	3	4	5
Machine G: $(A + C)$	7	13	8	7	7
Machine H: $(C + B)$	11	14	6	8	13

By applying, Johnson and Bellman Rule, the optimal sequence is: We find that there are alternate solutions, as the elements 7 and 8 are appearing more than one time in the problem.

The solutions are:

4	1	5	2	3

4	5	1	2	3

1	4	5	2	3

5	1	4	2	3

5	4	1	2	3

Let us work out the total time elapsed for any one of the above sequences. Students may try for all the sequence and they find that the total elapsed time will be same for all sequences.

Sequence.	*Machine A Time in Hrs.*		*Machine C Time in Hrs.*		*Machine B Time in Hrs.*		*Job idle. Time in Hrs*	*Machine Idle. Time in Hrs.*		
	In	*out*	*In*	*out*	*In*	*out*		*A*	*C*	*B*
4	0	5	5	7	7	13			5	7
1	5	8	8	12	13	20	1		1	
5	8	12	12	15	20	30	5			
2	12	20	20	25	30	39	5		5	
3	20	27	27	28	39	44	11	17	2+16	
	Total		**Elapsed**		**Time:44 Hrs.**					

Total elapsed time = 44 hours. Idle time for Machine *A* is 17 hours. For machine *C* = 29 hrs and that for machine *B* is 7 hours.

Problem 6.10.

A ready-made dress company is manufacturing its 7 products through two stages *i.e.* cutting and Sewing. The time taken by the products in the cutting and sewing process in hours is given below:

Products:	1	2	3	4	5	6	7
Cutting:	5	7	3	4	6	7	12
Sewing:	2	6	7	5	9	5	8

(a) Find the optimal sequence that minimizes the total elapsed time.

(b) Suppose a third stage of production is added, namely Pressing and Packing, with processing time for these items as given below:

Product:	1	2	3	4	5	6	7
Pressing and Packing: (Time in hrs)	10	12	11	13	12	10	11

Find the optimal sequence that minimizes the total elapsed time considering all the three stages.

Solution

(*a*) Let us workout optimal sequence and total elapsed time for first two stages:

By Johnson and Bellman rule, the optimal sequence is:

3	4	5	7	2	6	1

Total Elapsed time:

Sequence	*Cuttin Dept. Time in Hrs.*		*Sewin dept. Time in Hrs.*		*Job idle Time in Hrs.*	*Machine idle. Time in Hrs.*		*Remarks.*
	In	*out*	*In*	*out*		*Cutting*	*Sewing.*	
3	0	3	3	10			3	Sewing starts after cutting.
4	3	7	10	15	3			
5	7	13	15	24	2			
7	13	25	25	33			1	
2	25	32	33	39	1			
6	32	39	39	44				
1	39	44	44	46		2		
	Total		**Elapsed**		**Time in Hrs. = 46 Hrs.**			

Total elapsed time is 46 Hrs. Idle time for cutting is 2 Hrs, and that for Sewing is 4 Hrs.

b) When the Pressing and Packing department is added to Cutting and Sewing, the problem becomes '*n*' jobs and 3-machine problem. We must check whether we can convert the problem into 2- machine problem.

The problem is

Products:	1	2	3	4	5	6	7
Cutting dept. (Hrs):	5	7	3	4	6	7	12
Sewing dept (Hrs);	2	6	7	5	9	5	8
Pressing and Packing dept. (Hrs.):	10	12	11	13	12	10	11

Minimum time element for first department is 3 Hrs. and that for third department is 10 Hrs. And maximum time element for second department i.e sewing department is 9 Hrs. As the minimum time element of third department is greater than that of minimum of second department, we can convert the problem into 2-machine problem.

Now 7 jobs and 2- machine problem is:

Product:	1	2	3	4	5	6	7
Department G (= Cutting + Sewing):	7	13	10	9	15	12	20
Department H (= Sewing + Packing):	12	18	18	18	21	15	19

By Johnson and Bellman rule the optimal sequence is:

1	4	3	6	2	5	7

Sequence	Cutting Dept. Time in Hrs.		Sewing Dept. Time in Hrs.		Packing dept. Time in Hrs.		Job idle. Time in Hrs.	Dept. Idle Time in Hrs..			Remarks.
	In	out	In	out	In	out		Cut	Sew	Pack.	
1	0	5	5	7	7	17			5	7	
4	5	9	9	14	17	30	3		2		
3	9	12	14	21	30	41	2 + 9				
6	12	19	21	26	41	51	2 + 15				
2	19	26	26	32	51	63	19				
5	26	32	32	41	63	75	22				
7	32	44	44	52	75	86		42	3+34		
	Total		**ElapsedTime**		**= 86 Hrs**.						

Total elapsed time = 86 Hrs. Idle time for Cutting dept. is 42 Hrs. Idle time for sewing dept, is 44 Hrs. and for packing dept. it is 7 hrs.

(Point to note: The Job idle time shows that enough place is to be provided for in process inventory and the machine or department idle time gives an indication to production planner that he can load the machine or department with any job work needs the service of the machine or department. Depending on the quantum of idle time he can schedule the job works to the machine or department).

Processing of 'N' Jobs on 'M' Machines: (Generalization of 'n' Jobs and 3 - machine problem)

Though we may not get accurate solution by generalizing the procedure of '*n*' jobs and 3- machine problem to '*n*' jobs and '*m*' machine problem, we may get a solution, which is nearer to the optimal solution. In many practical cases, it will work out. The procedure is :

A general sequencing problem of processing of '*n*' jobs through '*m*' machines M_1, M_2, M_3, ...M_{n-1} , M_n in the order M_1, M_2, M_3 ...... M_{n-1}, M_n can be solved by applying the following rules.

If a_{ij} where I = 1,2,3.....*n* and *j* = 1, 2, 3..........*m* is the processing time of *i th* job on *j th* machine, then find $\underset{I}{\text{Minimum}}\ a_{i1}$ and $\underset{i}{\text{Min.}}\ a_{im}$ (*i.e.* minimum time element in the first machine and minimum time element in last

Machine) and find $\underset{i}{\text{Maximum}}\ a_{ij}$ of intermediate machines *i.e* 2 nd machine to m-1 machine.

The problem can be solved by converting it into a two-machine problem if the following conditions are satisfied.

(a) Min $a_{i1} \geq$ Max. a_{ij} for all j = 1,2,3,…..m-1
i i

(b) Min $a_{im} \geq$ Max a_{ij} for all j = 1, 2,3 …….m-1
i i

At least one of the above must be satisfied. Or both may be satisfied. If satisfied, then the problem can be converted into 2- machine problem where Machine $G = a_{i1} + a_{i2} + a_{i3} + \ldots\ldots\ldots\ldots + a_{i\,m-1}$ and

Machine $G = a_{i2} + a_{i3} + \ldots\ldots\ldots + a_{im}$. Where i = 1,2,3,….n.

Once the problem is *a* 2- machine problem, then by applying Johnson Bellman algorithm we can find optimal sequence and then workout total elapsed time as usual.

(Point to remember: Suppose $a_{i2} + a_{i3} + \ldots\ a_{i\ m-1}$ = *a* constant number for all 'i' , we can consider two extreme machines *i.e.* machine 1 and machine -m as two machines and workout optimal sequence).

Problem 6.11.

There are 4 jobs *A*, *B*, *C* and *D*, which is to be, processed on machines M_1, M_2, M_3 and M_4 in the order $M_1\ M_2\ M_3\ M_4$ .The processing time in hours is given below. Find the optimal sequence.

Job	*Machine (Processing time in hours)*			
	M_1	M_2	M_3	M_4
	a_{i1}	a_{i2}	a_{i3}	a_{i4}
A	15	5	4	14
B	12	2	10	12
C	13	3	6	15
D	16	0	3	19

Solution

From the data given, Min a_{i1} is 12 and Min a_{i4} is 12.

Max a_{i2} = 5 and Max a_{i3} = 10.

As Min a_{i1} is > than both Min a_{i2} and Min a_{i3}, the problem can be converted into 2 – machine problem as discussed above. Two-machine problem is:

Jobs.	*Machines (Time in hours)*	
	G	*H*
A	15+5+4 = 29	5+4+14 = 23
B	12+2+10 = 24	2+10+12 = 24
C	13+3+6 = 22	3+6+15 = 24
D	16+0+3 = 19	0+3+19 = 22

Applying Johnson and Bellman rule, the optimal sequence is:

D	*C*	*B*	*A*

Total elapsed time:

Sequence	*Machine M_1 Time in hours*		*Machine M_2 Time in hours*		*Machine M_3 Time in hours*		*Machine M_4 Time in hours.*		*Job idle Time in hours.*	*Machine idle Time in hours.*			
	In	*out*	*In*	*out*	*in*	*out*	*In*	*out*		M_1	M_2	M_3	M_4
D	0	16	16	16	16	19	19	38				16	19
C	19	29	29	32	32	38	38	53			29	13	
B	29	41	41	43	43	53	53	65			9	5	
A	41	56	56	61	61	65	65	79		23	18	14	
	Total		**Elapsed**		**Time= 79 hrs**								

Total Elapsed time = 79 hours.

Problem 6.12.

In a maintenance shop mechanics has to reassemble the machine parts after yearly maintenance in the order *PQRST* on four machines *A*, *B*, *C* and *D*. The time required to assemble in hours is given in the matrix below. Find the optimal sequence.

Machine.	*Parts (Time in hours to assemble)*				
	P	Q	R	S	T
A	7	5	2	3	9
B	6	6	4	5	10
C	5	4	5	6	8
D	8	3	3	2	6

Solution

Minimum assembling time for component *P* = 5 hours. Minimum assembling time for component *T* = 6 hours. And Maximum assembling time for components *Q*, *R*, *S* are 6 hrs, 5 hrs and 6 hours respectively.

This satisfies the condition required for converting the problem into 2 - machine problem. The two-machine problem is:

Machine	*Component G*	*Component H*	*(Condition: Minimum P_i > Maximum Q_i, R_i,*
	(Time in hours)		*and Si. OR*
	(P+Q+R+S)	*(Q+R+S+T)*	*Minimum T_i > Maximum Q_i, R_i, and S_i.*
A	17	19	
B	21	25	
C	20	23	
D	16	14	

The optimal sequence by applying Johnson and Bellman rule is:

A	C	B	D

Total Elapsed Time:

Sequence	*Component P Time in hours*		*Component Q Time in hours*		*Component R Time in hours.*		*Component S Time in hours*		*Component T Time in hours.*		*Men idle Hrs P Q,R,S,T*	*Job idle Hrs.*
	In	*out*	*In*	*out*	*In*	*out*	*In*	*out*	*In*	*out.*		
A	0	7	7	12	12	14	14	17	17	26	– 7, 12, 14, 17	
C	7	12	12	16	16	21	21	27	27	35	- 2, 4, 1	
B	12	18	18	24	24	28	28	33	35	45	- 2, 3, 1,	2
D	18	26	26	29	29	32	33	35	45	51	25, 2	1, 1, 10
	Total		**Elapsed**		**Time**		**= 51 hours.**					

Total elapsed time is 51 hours.

Idle time for various workmen is:

P: 51 – 26 = 25 hrs.

Q: 7+(18-16)+ (26 - 24) + (51 - 29) = 33 hrs.

R: 12 + (16 – 14) + (24 – 21) + (29 – 28) + (51 – 32) = 37 hrs.

S: 14 + (21 – 17) + (28 – 27) + 51 – 35) = 35 hrs.

T: 17 + 27 – 26) = 18 hrs.

The waiting time for machines is:

A: No waiting time. The machine will finish it work by 26 th hour.

B: 12 + 35 – 33) = 14 hrs. The assembling will over by 45 th hour.

C: 7 hours. The assembling will over by 35 th hour.

D: 18 + 33 – 32) + (45 – 35) = 29 hrs. The assembling will over by 51-st hour.

Problem 6.13.

Solve the sequencing problem given below to give an optimal solution, when passing is not allowed.

Machines (Processing time in hours)

Jobs	*P*	*Q*	*R*	*S*
A	11	4	6	15
B	13	3	7	8
C	9	5	5	13
D	16	2	8	9
E	17	6	4	11

Solution

Minimum time element under machine *P* and *S* are 9 hours and 8 hours respectively. Maximum time element under machines *Q* and *R* are 6 hours and 8 hours respectively. As minimum time elements in first and last machines are > than the maximum time element in the intermediate machines, the problem can be converted into two machine, *n* jobs problem.

See that sum of the time elements in intermediate machines (*i.e.* machines Q and R is equals to 10, hence we can take first and last machines as two machines and by application of Johnson and Bellman principle, we can get the optimal solution. The optimal sequence is:

Two-machine problem is:

Job:	*A*	*B*	*C*	*D*	*E*
Machine G (Hrs)	11	13	9	16	17
Machine H (Hrs)	15	8	13	9	11

Optimal sequence:

C	A	E	D	B

Total elapsed time:

Sequence	*Machine P Time in Hrs.*		*Machine Q Time in Hrs.*		*Machine R Time in Hrs.*		*Machine S Time in Hrs.*		*Job idle Time in Hrs*	*Machine idle. Time in Hrs.*			
	In	*out*	*In*	*out*	*In*	*out*	*In*	*out*		*P*	*Q*	*R*	*S*
C	0	9	9	14	14	19	19	32		-	9	14	19
A	9	20	20	24	24	30	32	45	2		6	5	
E	20	37	37	43	43	47	47	58	13+2			13	
D	37	52	52	54	54	62	62	66			9	7	4
B	52	65	65	68	68	75	75	83		18,	9+21,	8+9	-
	Total		**Elapsed**		**Time in Hrs.**		**= 83 hrs.**						

Total elapsed time is 83 hours.

PROCESSING OF 2 - JOBS ON 'M' MACHINES

There are two methods of solving the problem. (*a*) By enumerative method and (*b*) Graphical method.

Graphical method is most widely used. Let us discuss the graphical method by taking an example.

Graphical Method

This method is applicable to solve the problems involving 2 jobs to be processed on 'm' machines in the given order of machining for each job. In this method the procedure is:

(a) Represent Job 1 on X- axis and Job 2 on Y-axis. We have to layout the jobs in the order of machining showing the processing times.

(b) The **horizontal line** on the graph shows the **processing time of Job 1** and **idle time of Job 2.** Similarly, **a vertical line** on the graph shows **processing time of job 2** and **idle time of job 1.** Any inclined line shows the processing of two jobs simultaneously.

(c) Draw horizontal and vertical lines from points on X- axis and Y- axis to construct the blocks and hatch the blocks. (Pairing of same machines).

(d) Our job is to find the minimum time required to finish both the jobs in the given order of machining. Hence we have to follow inclined path, preferably a line inclined at 45 degrees (in a square the line joining the opposite coroners will be at 45 degrees).

(e) While drawing the inclined line, care must be taken to see that it will not pass through the region indicating the machining of other job. That is the inclined line should not pass through blocks constructed in step (c).

(f) After drawing the line, the total time taken is equals to Time required for processing + idle time for the job.

The sum of processing time + idle time for both jobs must be same.

Problem 6.14.

Use graphical method to minimize the time needed to process the following jobs on the machines as shown. For each machine find which job is to be loaded first. Calculate the total time required to process the jobs. The time given is in hours. The machining order for job 1 is *ABCDE* and takes 3, 4, 2, 6, 2 hours respectively on the machines. The order of machining for job 2 is *BCADE* and takes 5, 4, 3, 2, 6 hours respectively for processing.

Solution

The given problem is:

Job 1	Sequence:	A	B	C	D	E
	Time in Hrs.	3	4	2	6	2
Job2	Sequence:	B	C	A	D	E
	Time in Hrs.	5	4	3	2	6

To find the sequence of jobs, *i.e.* which job is to be loaded on which machine first and then which job is to be loaded second, we have to follow the inclined line starting from the origin to the opposite corner. First let us start from origin. As Job 2 is first on machine *B* and Job 1 is first on machine *A*, job 1 is to be processed first on machine *A* and job 2 is to be processed on machine B first. If we proceed further, we see that job 2 is to be processed on machine *C* first, then comes job 2 first on *D* and job 2 first on machine *E*. Hence the optimal sequence is: (Refer figure 6.2)

Job 1 before 2 on machine A,

Job 2 before 1 on machine B,

Job 2 before 1 on machine C,

Job 2 before 1 on machine D, and

Job 2 before 1 on machine E.

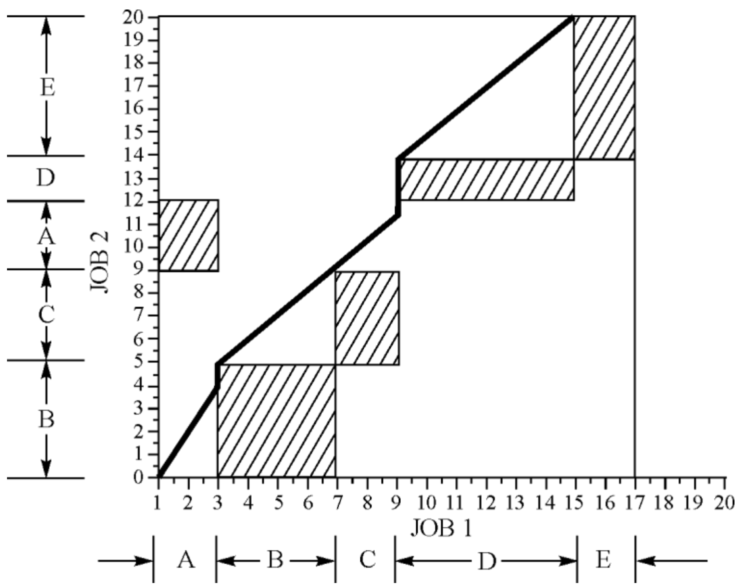


Figure 6.2

The processing time for Job 1 = 17 hours processing + 5 hours idle time (Vertical distance) = 22 hours.

The processing time for Job 2 = 20 hours processing time + 2 hours idle time (horizontal distance) = 22 hours.

Both the times are same. Hence total Minimum processing time for two jobs is 22 hours.

Problem 6.15.

Two jobs are to be processed on four machines *A*, *B*, *C* and *D*. The technological order for these two jobs is: Job 1 in the order *ABCD* and Job 2 in the order DBAC. The time taken for processing the jobs on machine is:

Machine:	*A*	*B*	*C*	*D*
Job 1:	4	6	7	3
Job 2:	5	7	8	4

Solution

Processing time for jobs are: Job 1 = 4 + 6 + 7 + 3 = 20 hours.

Job 2 = 5 + 7 + 8 + 4 = 24 hours.

The graph is shown in figure 6.3. The line at 45 degrees is drawn from origin to opposite corner.

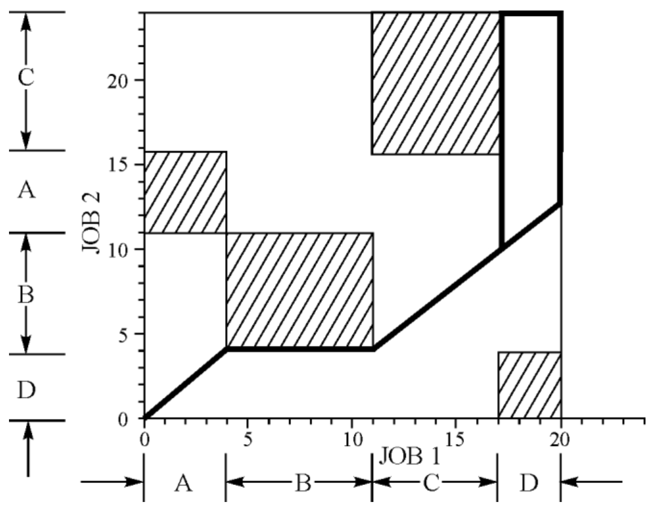


Figure 6.3.

The total elapsed time for job 1 = Processing time + idle time (horizontal travel) = 20 + 10 = 30 hours.

The same for job 2 = Processing time + Idle time (vertical travel) = 24 + 6 = 30 hours. Both are same hence the solution. To find the sequence, let us follow inclined line.

Job 1 first on *A* and job 2 second on *A*, Job 1 first on *B* and job 2 second on *B* Job *C* first on *C* and job 2 second on *C*, Job 2 first on *D* and job 1 second on *D*.

Problem 6.16.

Find the optimal sequence of two jobs on 4 machines with the data given below:

Job 1	Order of machining:	*A*	*B*	*C*	*D*
	Time in hours:	2	3	3	4
Job 2	Order of machining:	*D*	*C*	*B*	*A*
	Time in hours:	4	3	3	2

Solution

Job 1 is scaled on *X* - axis and Job 2 is scaled on *Y* - axis. 45° line is drawn. The total elapsed time for two jobs is:

Job 1: Processing time + idle time = 12 + 2 = 14 hours.

Job 2: Processing time + idle time = 12 + 2 = 14 hours. Both are same and hence the solution;

Job 1 first on machine *A* and *B* and job 2 second on *A* and *B*. Job 2 first on *C* and job 1 second on *C*. Job 2 first on *D* and job 1 second on *D*.

Problem 6.17.

Find the sequence of job 1 and 2 on four machines for the given technological order.

Job 1.	Order of machining:	*A*	*B*	*C*	*D*
	Time in hours.	2	3	3	4
Job2.	Order of machining	*A*	*B*	*C*	*D*
	Time in hours.	2	3	3	4

Solution

From the graph figure 6.4 the total elapsed time for job 1 = 12 + 4 = 16 hours. Elapsed time for Job 2 = 12 + 4 = 16 hours.

The sequence is Job 1 first on *A*, *B*, *C*, and *D* and then the job 2 is second on *A*, *B*, *C* and *D*. OR we can also do Job 2 first on *A*, *B*, *C*, *D* and job 1 second on *A*, *B*, *C*, *D*. When technological order is same this is how jobs are to be processed.

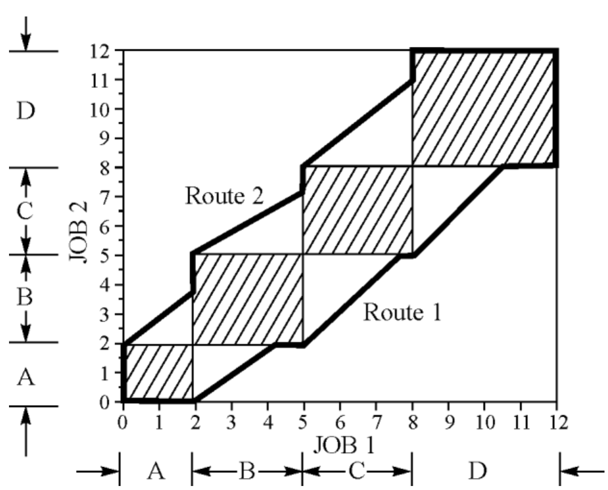


Figure 6.4

Problem 6.18

Find the optimal sequence for the given two jobs, which are to be processed on four machines in the given technological order.

Job1	Technological order:	*A*	*B*	*C*	*D*
	Time in hours.	2	3	3	4
Job2	Technological order:	*D*	*C*	*B*	*A*
	Time in hours.	2	3	3	4

Solution

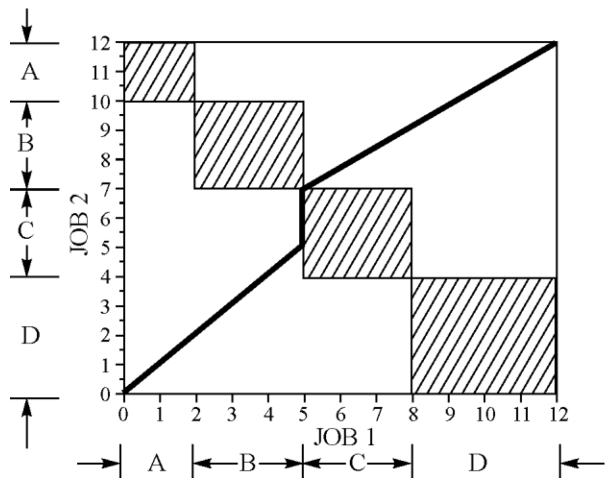


Figure 6.5

(Note: Students can try these problems and see how the graph appears:

Job 1:	Technological order:	*A*	*B*	*C*	*D*
	Time in hours:	2	2	2	2
Job 2:	Technological order:	*A*	*B*	*C*	*D*
	Time in hours:	2	2	2	2

AND

Job 1	Technological order:	*A*	*B*	*C*	*D*
	Time in hours:	2	2	2	2
Job 2	Technological order:	*D*	*C*	*B*	*A*
	Time in hours.	2	2	2	2

TRAVELING SALESMAN PROBLEM: (RELATED PROBLEMS)

Just consider how a postman delivers the post to the addressee. He arranges all the letters in an order and starts from the post office and goes from addressee to addressee and finally back to his post office. If he does not arrange the posts in an order he may have to travel a long distance to clear all the posts. Similarly, a traveling sales man has to plan his visits. Let us say, he starts from his head office and go round the branch offices and come back to his head office. While traveling he will not visit the branch already visited and he will not come back until he visits all the branches.

There are different types of traveling salesman‘s problems. One is **cyclic problem**. In this problem, he starts from his head quarters and after visiting all the branches, he will be back to his head quarters. The second one is **Acyclic problem**. In this case, the traveling salesman leaves his head quarters and after visiting the intermediate branches, finally reaches the last branch and stays there. The first type of the problem is solved by Hungarian method or Assignment technique. The second one is solved by Dynamic programming method.

Point to Note: The traveling salesman‘s problem, where we sequence the cities or branches he has to visit is a SEQUENCING PROBLEM. But the solution is got by Assignment technique. Hence basically, the traveling salesman problem is a SEQUENCING PROBLEM; the objective is to minimize the total distance traveled.

The mathematical statement of the problem is: Decide variable $x_{ij} = 1$ or 0 for all values of I and j so as to:

$$\text{Minimise } Z = \sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij} \quad \text{for all } i \text{ and } j = 1,2\ldots..n \text{ Subject to}$$

$$\sum_{j=1}^{n} x_{ij} = 1 \text{ for } i = 1,2,\ldots.n \text{ (Depart from a city once only)}$$

$$\sum_{i=1}^{n} x_{ij} = 1 \text{ for } j = 1,2,\ldots.n \text{ (Arrive at a city once only)}$$

And all $x_{ij} \geq 0$ for all i and j

This is indeed a statement of assignment problem, which may give two or more disconnected cycles in optimum solution. This is not permitted. That is salesman is not permitted to return to the origin of his tour before visiting all other cities in his itinerary. The mathematical formulation above does not take care of this point.

A restriction like $X_{ab} + X_{bc} + X_{ca} \leq 2$ will prevent sub-cycles of cities A, B, C and back to A. It is sufficient to state at this stage that all sub - cycles can be ruled out by particular specifications of linear constraints. This part, it is easy to see that a variable $x_{ij} = 1$, has no meaning. To exclude this from solution, we attribute very large cost to it i.e infinity or big M, which is very larger than all the elements in the matrix.

In our solutions big M is used.

Problem 6.17.

A salesman stationed at city A has to decide his tour plan to visit cities B, C, D, E and back to city A in the order of his choice so that total distance traveled is minimum. No sub touring is permitted. He cannot travel from city A to city A itself. The distance between cities in Kilometers is given below:

Cities	*A*	*B*	*C*	*D*	*E*
A	M	16	18	13	20
B	21	M	16	27	14
C	12	14	M	15	21
D	11	18	19	M	21
E	16	14	17	12	M

Instead of big *M* we can use infinity also. Or any element, which is sufficiently larger than all the elements in the matrix, can be used.

Solution

COCM:

Cities	*A*	*B*	*C*	*D*	*E*
A	M	3	5	0	7
B	7	M	2	13	0
C	0	2	M	3	9
D	0	7	8	M	10
E	4	2	5	0	M

TOCM:

Cities	*A*	*B*	*C*	*D*	*E*
A	M	1	3	0	7
B	7	M	0	13	0
C	0	0	M	3	9
D	0	5	6	M	10
E	4	0	3	0	M

We can make only 4 assignments. Hence modify the matrix. Smallest element in the uncovered cells is 3, deduct this from all other uncovered cells and add this to the elements at the crossed cells. Do not alter the elements in cells covered by the line.

TOCM

Cities	*A*	*B*	*C*	*D*	*E*
A	M	1	3	0	7
B	7	M	0	13	0
C	0	0	M	3	9
D	0	5	6	M	10
E	4	0	3	0	M

We can make only 4 assignments. Hence once again modify the matrix.
Sequencing: *A* to *C*, *C* to *B*, *B* to *E*, *E* to *D*, and *D* to *A*. As there is a tie
TOCM:

Cities	*A*	*B*	*C*	*D*	*E*
A	M	1	**0**	0	4
B	10	M	0x	16	**0**
C	0x	**0**	M	3	6
D	**0**	5	3	M	7
E	4	0x	0	**0**	M

Sequencing: *A* to *C*, *C* to *B*, *B* to *E*, *E* to *D* and *D* to *A*. as there is a tie between the zero cells, the problem has alternate solution. The total distance traveled by the salesman is: 18 + 14 + 14 + 11 + 12 = 69 Km.

A to *C* to *B* to *E* to *D* to *A*, the distance traveled is 69 Km.

Note: See that twice sales man visits no city.

Problem 6.18.

Given the set up costs below, show how to sequence the production so as to minimize the total setup cost per cycle.

Jobs	*A*	*B*	*C*	*D*	*E*
A	M	2	5	7	1
B	6	M	3	8	2
C	8	7	M	4	7
D	12	4	6	M	5
E	1	3	2	8	M

Solution

COCM:

Jobs	*A*	*B*	*C*	*D*	*E*
A	M	1	4	6	0
B	4	M	1	6	0
C	4	3	M	0	3
D	8	0	2	M	1
E	0	2	1	7	M

TOCM:

Jobs	*A*	*B*	*C*	*D*	*E*
A	M	1	3	6	**0**
B	4	M	0	6	0x
C	4	3	M	**0**	3
D	8	**0**	1	M	1
E	**0**	2	0x	7	M

We can draw five lines and make assignment. The assignment is:

From *A* to *E* and From *E* to *A* cycling starts, which is not allowed in salesman problem. Hence what we have to do is select the next higher element than zero and make assignment with those elements. After assignment of next higher element is over, then come to zero for assignment. If we cannot finish the assignment with that higher element, then select next highest element and finish assigning those elements and come to next lower element and then to zero. Like this we have to finish all assignments. In this problem, the next highest element to zero is 1. Hence first assign all ones and then consider zero for assignment. Now we shall first assign all ones and then come to zero.

TOCM:

Jobs	*A*	*B*	*C*	*D*	*E*
A	M	**1**	3	6	0x
B	4	M	**0**	6	0x
C	4	3	M	**0**	3
D	8	0x	1x	M	**1**
E	**0**	2	0x	7	M

The assignment is *A* to *B*, *B* to *C*, *C* to *D* and *D* to *E* and *E* to *A*. (If we start with the element DC then cycling starts.

Now the total distance is 5 + 3 + 4 + 5 + 1 = 18 + 1 + 1 = 20 Km. The ones we have assigned are to be added as penalty for violating the assignment rule of assignment algorithm.

Problem 6.19.

Solve the traveling salesman problem by using the data given below:

$C_{12} = 20$, $C_{13} = 4$, $C_{14} = 10$, $C_{23} = 5$, $C_{34} = 6$, $C_{25} = 10$, $C_{35} = 6$, $C_{45} = 20$ and $C_{ij} = C_{ji}$. And there is no route between cities '*i*' and '*j*' if a value for C_{ij} is not given in the statement of the problem. (*i* and *j* are = 1,2,..5)

Solution

Cities	*1*	*2*	*3*	*4*	*5*
1	M	20	4	10	M
2	20	M	5	M	10
3	4	5	M	6	6
4	10	M	6	M	20
5	M	10	6	20	M

Now let us work out COCM/ROCM and TOCM, and then make the assignment.

TOCM:

Cities.	*1*	*2*	*3*	*4*	*5*
1	M	12	**0**	0x	M
2	11	M	0x	M	**0**
3	0x	1	M	**0**	1
4	**0**	M	0x	M	9
5	M	**0**	0x	8	M

The sequencing is: 1 to 3, 3 to 4, 4 to 1 and 1 to 3 etc., Cycling starts. Hence we shall start assigning with 1 the next highest element and then assign zeros. Here also we will not get the sequencing. Next we have to take the highest element 8 then assign 1 and then come to zeros.

TOCM:

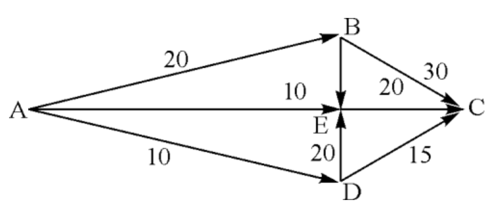
Cities.	*1*	*2*	*3*	*4*	*5*
1	M	12	**0**	0	M
2	11	M	0	M	**0**
3	0	**1**	M	0	1
4	**0**	M	0	M	9
5	M	0	0	**8**	M

Sequencing is: 1 to 3, 3 to 2, 2 to 5, 5 to 4 and 4 to 1.

The optimal distance is: 4 + 10 + 5 + 10 + 20 = 49 + 1 + 8 = 58 Km.

Problem 5.18.

A tourist organization is planning to arrange a tour to 5 historical places. Starting from the head office at *A* then going round *B*, *C*, *D* and *E* and then come back to *A*. Their objective is to minimize the total distance covered. Help them in sequencing the cities. *A*, *B*, *C*, *D* and *E* as shown in figure. The numbers on the arrows show the distances in Km.



Solution

The distance matrix is as given belows:

Places	*A*	*B*	*C*	*D*	*E*
A	M	20	M	10	10
B	20	M	30	M	35
C	M	30	M	15	20
D	10	M	15	M	20
E	10	35	20	20	M

COCM

Places	*A*	*B*	*C*	*D*	*E*
A	M	10	M	0	0
B	0	M	10	M	15
C	M	15	M	0	5
D	0	M	5	M	10
E	0	25	10	10	M

TOCM:

Places	*A*	*B*	*C*	*D*	*E*
A	M	0	M	0	0
B	0	M	5	M	15
C	M	5	M	0	5
D	0	M	0	M	10
E	0	15	5	10	M

TOCM:

Places	*A*	*B*	*C*	*D*	*E*
A	M	0	M	5	0
B	0	M	5	M	10
C	M	0	M	0	0
D	0	M	0	M	5
E	0	10	5	10	M

TOCM:

Places	*A*	*B*	*C*	*D*	*E*
A	M	0	M	5	0
B	0	M	5	M	5
C	M	0	M	0	0
D	0	M	0	M	0
E	0	5	5	5	M

Places	*A*	*B*	*C*	*D*	*E*
A	M	**0**	M	5	0x
B	0x	M	**0**	M	0x
C	M	0x	M	**0**	0x
D	5	M	0x	M	**0**
E	0	0x	0x	0x	M

The sequencing is: *A* to *B*, *B* to *C*, *C* to *D*, *D* to *E* and *E* to *A*.
The total distance is: 20 +30 +15 + 20 + 10 = 95 Km.

QUESTIONS

1. A bookbinder has one printing press, one binding machine and the manuscripts of a number of different books. The times required to perform printing and binding operations for ach book are known. Determine the order in which the books should be processed in order to minimize the total time required to process all the books. Find also the total time required processing all the books.

Printing time in minutes.

BOOK:	*A*	*B*	*C*	*D*	*E*
Printing time:	40	90	80	60	50
Binding Time:	50	60	20	30	40

Suppose that an additional operation, finishing is added to the process described above, and the time in minutes for finishing operation is as given below what will be the optimal sequence and the elapsed time.

BOOK:	*A*	*B*	*C*	*D*	*E*
Finishing time (min):	80	100	60	70	110

(Answer for two processes: sequence is *ABEDC*and the elapsed time is 340 min.
For three processes: the optimal sequence is: *DAEBC* and the total elapsed time is 510 min.)

2. A ready-made garments manufacturer has to process 7 items through two stages of production, *i.e*. Cutting and Sewing. The time taken for each of these items at different stages are given in hours below, find the optimal sequence and total elapsed time.

Item:	1	2	3	4	5	6	7
Cutting time in Hrs.:	5	7	3	4	6	7	12
Sewing time in Hrs:	2	6	7	5	9	5	8

Suppose a third stage of production is added, say pressing and packing with processing time in hours as given below, find the optimal sequence and elapsed time.

Pressing time (Hrs)	10	12	11	13	12	10	11

(Answer: For two stages the sequence is 3457261 and the time is 46 hours. For three stages the sequence is 1436257 and the time is 86 hours.)

3. Find the optimal sequence for the following sequencing problem of four jobs and five machines when passing is not allowed. The processing time given is in hours.

JOBS:		1	2	3	4
	A	6	5	4	7
	B	4	5	3	2
Machines:	*C*	1	3	4	2
	D	2	4	5	1
	E	8	9	7	5

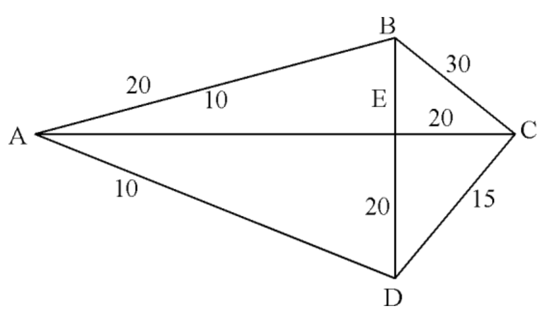
(Ans: Sequence: 1324, Time: 43 hours).

4. Find the optimal sequence and total elapsed time for processing two jobs on 5 machines by graphical method.

Job 1:	Time in hours:	2	3	4	6	2
	Order of machining:	*A*	*B*	*C*	*D*	*E*
Job2:	Time in hours:	4	5	3	2	6
	Order of machining:	*B*	*C*	*A*	*D*	*E*

(Answer: 1,2 for *A*, 1,2 for *B*, 2,1, for *C*, 2,1, for *D* and 2,1for E and the time is 20 hours).

5. The tourist bureau of India wants to find the optima tour policy of five cities *A*, *B*, *C*, *D* and *E* starting from city *A* and finally returning to city *A* after visiting all cities. The cost of travel in rupees is given below. Find the optimal policy.



(Answer: Sequence: ABCDEA, The cost is Rs. 95/-

6. Seven jobs are to be processed on three machines *X*, *Y* and *Z* in the order *XYZ*. The time required for processing in hours is given below. Find the optima sequence and the time elapsed. State clearly the conditions to be satisfied to convert three machines problem into two-machine problem.

JOBS (Time in hours)

		A	*B*	*C*	*D*	*E*	*F*	*G*
MACHINES:	*X*	3	8	7	4	9	8	7
	Y	4	3	2	5	1	4	3
	Z	6	7	5	11	5	6	10

7. Explain the application of sequencing model. Mention different types of sequencing problem you come across.
8. Explain the methodology of Johnson and Bellman method to solve sequencing problem.
9. Explain the assumption made in solving sequencing problem.
10. Explain the conditions required to satisfy when you want to convert a 3-machine problem into 2- machine problem.
11. Find the optimal sequence and the total time elapsed for sequencing 6 jobs on two machines *A* and *B* in the order *AB*. Time given is in hours.

JOBS:	1	2	3	4	5	6
Time in hours:						
Machine A:	2	5	4	3	2	1
Machine B:	6	8	1	2	3	5

(Answer: Sequence: 615243 or 651243 and the time elapsed is 26 hours.)

MULTIPLE CHOICE QUESTIONS

1. The objective of sequencing problem is:
   (a) To find the order in which jobs are to be made
   (b) To find the time required for completing all the jobs on hand.
   (c) To find the sequence in which jobs on hand are to be processed to minimize the total time required for processing the jobs.
   (d) To maximize the effectiveness. ( )

2. The time required for printing of four books *A*, *B*, *C* and *D* is 5, 8, 10 and 7 hours. While its data entry requires 7, 4, 3 and 6 hours respectively. The sequence time that minimizes total elapsed time is
   (a) ACBD, (*b*) ABCD (*c*) ADCB (*d*) CBDA ( )
3. If there are '*n*' jobs and '*m*' machines, there will be --------------- sequences of doing the jobs.
   (*a*) $n \times m$, (*b*) $m \times n$, (*c*) n^m (*d*) $(n!)^m$ ( )
4. In general sequencing problem will be solved by using …………
   (a) Hungarian Method.
   (b) Simplex method.
   (c) Johnson and Bellman method,
   (d) Flood's technique. ( )
5. In solving 2 machine and '*n*' jobs, the following assumption is wrong:
   (a) No passing is allowed
   (b) Processing times are known,
   (c) Handling time is negligible,
   (d) The time of processing depends on the order of machining.
6. The following is the assumption made in processing of '*n*' jobs on 2 machines:
   (a) The processing time of jobs is exactly known and is independent of order of processing.
   (b) The processing times are known and they depend on the order of processing the job.
   (c) The processing time of a job is unknown and it is to be worked out after finding the sequence.
   (d) The sequence of doing jobs and processing times are inversely proportional. ( )
7. The following is one of the assumptions made while sequencing '*n*' jobs on 2 machines.
   (a) Two jobs must be loaded at a time on any machine.
   (b) Jobs are to be done alternatively on each machine.
   (c) The order of completing the jobs has high significance.
   (d) Each job once started on a machine is to be performed up to completion on that machine. ( )
8. This is not allowed in sequencing of '*n*' jobs on two machines:
   (a) Passing, (*b*) loading (*c*) Repeating the job
   (*d*) Once loaded on the machine it should be completed before removing from the machine. ( )
9. Write the sequence of performing the jobs for the problem given below:

Jobs:	*A*	*B*	*C*	*D*	*E*
Time of machining On Machine X:	6	8	5	9	1

(a) They can be processed in any order.
(b) As there is only one machine, sequencing cannot be done.
(c) This is not a sequencing problem.
(d) None of the above. ( )

10. Johnson Bellman rule states that:
    (a) If smallest processing time occurs under first machine, do that job first.
    (b) If the smallest processing time occurs under the second machine, do that job first.
    (c) If the smallest processing time occurs under first machine, do that job last.
    (d) If the smallest processing time occurs under second machine keep the processing pending. ( )

12. To convert '*n*' jobs and 3-machine problem into '*n*' jobs and 2-machine problem, the following rule must be satisfied.
    (a) All the processing time on second machine must be same.
    (b) The maximum processing time of 2nd machine must be ≤ to minimum processing times of first and third machine.
    (c) The maximum processing time of 1st machine must be ≤ to minimum processing time of other two machines.
    (d) The minimum processing time of 2nd machine must be ≤ to minimum processing times of first and third machine. ( )

13. If two jobs J_1 and J_2 have same minimum process time under first machine but processing time of J_1 is less than that of J_2 under second machine, then J_1 occupies:
    (a) First available place from the left.
    (b) Second available place from left,
    (c) First available place from right,
    (d) Second available place from right. ( )

14. If Job A and B have same processing times under machine I and Machine II, then prefer
    (a) Job A, (*b*) Job B
    (*c*) Both A and B (*d*) Either A or B ( )

15. The given sequencing problem will have multiple optimal solutions when the two jobs have same processing times under:
    (a) First Machine,
    (b) Under both machines,
    (c) Under second machine,
    (d) None of the above. ( )

16. If a job is having minimum processing time under both the machines, then the job is placed in:
    (a) Any one (first or last) position,
    (b) Available last position,
    (c) Available first position,
    (d) Both first and last position. ( )

17. FIFO is most applicable to sequencing of
    (a) One machine and ‘*n*’ jobs,
    (b) 2 machines and ‘*n*’ jobs,
    (c) 3 machine ‘*n*’ jobs,
    (d) ‘*n*’ machines and 2 jobs. ( )
18. At a petrol Bunk, when ‘*n*’ vehicle are waiting for service then this service rule is used:
    (a) LIFO
    (b) FIFO
    (c) Service in Random Order
    (d) Service by highest profit rule. ( )
19. Consider the following sequencing problem, and write the optimal sequence:

Jobs:		1	2	3	4	5
Processing	M/C X	1	5	3	10	7
Time in Hrs.						
	M/C Y	6	2	8	4	9

    (*a*) 1 2 3 4 5 (*b*) 1 3 5 4 2
    (*c*) 5 4 3 2 1 (*d*) 1 4 3 5 2 ( )
20. In a 3 machine and 5 jobs problem, the least of processing times on machine *A*, *B* and *C* are 5, 1, and 3 hours and the highest processing times are 9, 5, and 7 respectively, then Johnson and Bellman rule is applicable if order of the machine is:
    (a) B-A-C, (*b*) A-B-C
    (*c*) C - B - A (*d*) Any order. ( )
21. In maximization case of sequencing problem of 2 machines and ‘*n*’ jobs, the job is placed at available left first position if it has --------------- process time under machine---------------.
    (a) Least, first, (*b*) highest, first,
    (*c*) least, second, (*d*) highest, second. ( )
22. The fundamental assumption of Johnson‘s method of sequencing is:
    (a) No Passing rule,
    (b) Passing rule,
    (c) Same type of machines are to be used,
    (d) Non zero process time. ( )
23. If a job has zero process time for any machine, the job must be
    (a) Possess first position only,
    (b) Possess last position only,
    (c) Possess extreme position,
    (d) Be deleted from the sequencing. ( )
24. The assumption made in sequencing problems *i.e.* No passing rule means:
    (a) A job once loaded on a machine should not be removed until it is completed,

(b) A job cannot be processed on second machine unless it is processed on first machine
(c) A machine should not be started unless the other is ready to start,
(d) No job should be processed unless all other are kept ready to start. ( )

25. The technological order of machine to be operated is fixed in a problem having:
(a) 1 machine and '*n*' jobs,
(b) 2 machines and '*n*' jobs,
(c) 3 machine and '*n*' jobs,
(d) '*n*' machines and 2 jobs. ( )

26. A sequencing problem is infeasible in case of:
(a) 1 machine and '*n*' jobs,
(b) 2 machines and '*n*' jobs,
(c) 3 machines and '*n*' jobs,
(d) 2 jobs and '*n*' machines. ( )

27. In a 2 jobs and '*n*' machine problem a lie at 45° represents:
(a) Job 2 is idle,
(b) Job 1 is idle,
(c) Both jobs are idle,
(d) both jobs are under processing. ( )

28. In a 2 jobs and '*n*' machine problem, the elapsed time for job 1 is calculated as (Job 1 is represented on X -axis).
(a) Process time for Job 1 + Total length of vertical line on graph.
(b) Process time for Job 2 + Idle time for Job 1
(c) Process time for job 1 + Total length of horizontal line on graph,
(d) Process time for job 2 – Idle time for job 1 ( )

29. In a 2 jobs and '*n*' machine-sequencing problem the horizontal line on a graph indicates:
(a) Processing time of Job I,
(b) Idle time of Job I,
(c) Idle time of both jobs,
(d) Processing time of both jobs. ( )

30. In a 2 job, '*n*' machine sequencing problem, the vertical line on the graph indicates:
(a) Processing time of Job 1,
(b) Processing time of Job 2,
(c) Idle time of Job 2,
(d) Idle time of both jobs. ( )

31. In a 2 job and '*n*' machine sequencing problem we find that:
(a) Sum of processing time of both the jobs is same,
(b) Sum of idle time of both the jobs is same,
(c) Sum of processing time and idle time of both the jobs is same,
(*c*) Sum of processing time and idle time of both the jobs is different. ( )

ANSWERS

1. (*c*)	2. (*d*)	3. (*d*)	4. (*c*)
5. (*d*)	6. (*c*)	7. (*d*)	8. (*a*)
9. (*a*)	10. (*a*)	11. (*a*)	12. (*b*)
13. (*b*)	14. (*d*)	15. (*c*)	16. (*a*)
17. (*a*)	18. (*a*)	19. (*b*)	20. (*b*)
21. (*b*)	22. (*b*)	23. (*c*)	24. (*b*)
25. (*d*)	26. (*c*)	27. (*d*)	28. (*a*)
29. (*a*)	30. (*b*)	31. (*c*)	

CHAPTER – 7

Replacement Model

INTRODUCTION

The problem of replacement arises when any one of the components of productive resources, such as machinery, building and men deteriorates due to time or usage. The examples are:

(a) A machine, which is purchased and installed in a production system, due to usage some of its components wear out and its efficiency is reduced.

(b) A building in which production activities are carried out, may leave cracks in walls, roof etc, and needs repair.

(c) A worker, when he is young, will work efficiently, as the time passes becomes old and his work efficiency falls down and after some time he will become unable to work.

In general, when any production facility is new, it works with full operating efficiency and due to usage or of time, it may become old and some of its components wear out and the operating efficiency of the facility falls down. To regain the efficiency, a remedy by, namely ***maintenance*** is to be attended. The act of maintenance consists of replacing the worn out part, or oiling or overhauling, or repair etc. In modern industrial scene, the presence of highly sophisticated machinery in manufacturing system will bother the manager, when any one of the facilities goes out of order or breakdown. Because of the breakdown of one of the facility, the entire production system may be affected. This is particularly true in batch manufacturing system and continuous manufacturing system. The loss of production hours is more expensive factor for the manufacturing unit. Hence, the management must take interest in maintaing the production facility properly, so that facility's available time will be more than the down time. All the production facilities are subjected to deterioration due to their use and exposure to the environmental conditions. The process of deterioration, if unchecked or neglected, it culminates and makes the facility useless. Hence, the management has to check the facilities periodically, and keep all the facilities in operating condition. Once the maintenance is attended, the efficiency may not be regained to previous level but a bit less than that of previous level. For example, if the operating efficiency is 95 percent and due to deterioration, the efficiency reduces to 90 percent, after maintenance, it may regain to the level of 93 percent. Once again due to usage the efficiency falls down and the maintenance is to be attended. This is an ongoing business of the management. After some time, the efficiency reduces to such a level, the maintenance cost will become very high and due to less efficiency the unit production cost will be very high and this is the time the management has to think of replacing the facility. This may be well explained by means of a figure. Referring to figure 7.1, the operating efficiency at the beginning is

95%. When first maintenance is attended, it is reduced to 93%. In the second maintenance it is reduced to 80 percent. Like this the facility deteriorates, and finally the operating efficiency reduces to 50 percent, where the it is not economical to use the facility for further production, as the maintenance cost will be very high, and the unit production cost also increases, hence the replacement of the facility is due at this stage. In this chapter, we will discuss the mathematical models used for finding the optimal replacement time of facilities.

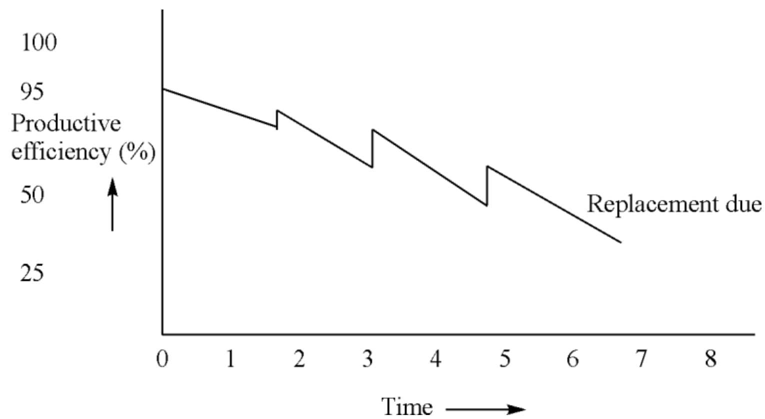


Fig. 7.1

Thus the problem of replacement is experienced in systems where machines, individuals or capital assets are the main production or job performing units. The characteristics of these units is that their level of performance or efficiency decreases with time or usage and one has to formulate some suitable replacement policy regarding these units to keep the system up to some desired level of performance. We may have to take different type of decision such as:

(a) We may decide whether to wait for complete failure of the item (which may result in some losses due to deterioration or to replace earlier at the expense of higher cost of the item,

(b) The expensive item may be considered individually to decide whether we should replace now or, if not, when it should be reconsidered for replacement,

(c) Whether the item is to be replaced by similar type of item or by different type for example item with latest technology

The problem of replacement is encountered in the case of both men and machines. Using probability, it is possible to estimate the chance of death or failure at various ages. ***The main objective of replacement is to help the organization for maximizing its profit or to minimize the cost.***

FAILURE MECHANISIM OF ITEMS

The word failure has got a wider meaning in **industrial maintenance** than what it has in our daily life. We can categorize the failure in two classes. They are (*i*) ***Gradual failure*** and (*ii*) ***Sudden failure.*** Once again the sudden failure may be classified as: (*a*) ***Progressive failure,*** (*b*) ***Retrogressive failure*** and (*c*) ***Random failure***.

(i) Gradual failure:

In this class as the life of the machine increases or due continuous usage, due to wear and tear of components of the facility, its efficiency deteriorates due to which the management can experience:

(*a*) Progressive Increase in maintenance expenditure or operating costs, (*b*) Decreased productivity of the equipment and (*c*) decrease in the value of the equipment *i.e.* resale value of the equipment/facility decreases.

Examples of this category are: Automobiles, Machine tools, etc.

(ii) Sudden failure:

In this case, the items ultimately fail after a period of time. The life of the equipment cannot be predicted and is some sort of random variable. The period between installation and failure is not constant for any particular type of equipment but will follow some frequency distribution, which may be:

(a) Progressive failure: In this case probability of failure increases with the increase in life of an item. The best example is electrical bulbs and computer components. It can be shown as in figure 7.2 (*a*).

(b) Retrogressive failure: Some items will have higher probability of failure in the beginning of their life, and as the time passes chances of failure becomes less. That is the ability of the item to survive in the initial period of life increases its expected life. The examples are newly installed machines in production systems, new vehicles, and infant baby (The probability of survival is very less in infant age, but once the baby get accustomed to nature, the probability of failure decreases). This can be shown as in figure 7.2 (*b*).

(c) Random failure: In this class, constant probability of failure is associated with items that fail from random causes such as physical shocks, not related to age. In such cases all items fail before aging has any effect. This can be shown as in figure 7.2. (*c*). Example is vacuum tubes.

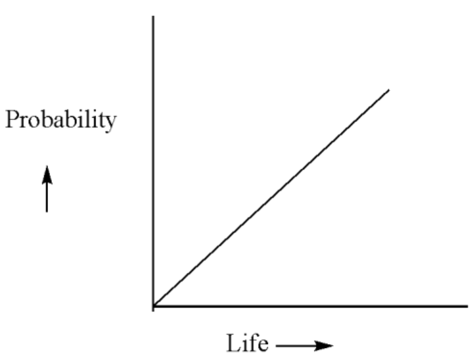


Figure 7.2(a) Progressive failure
Probability of failure increases with life of the item

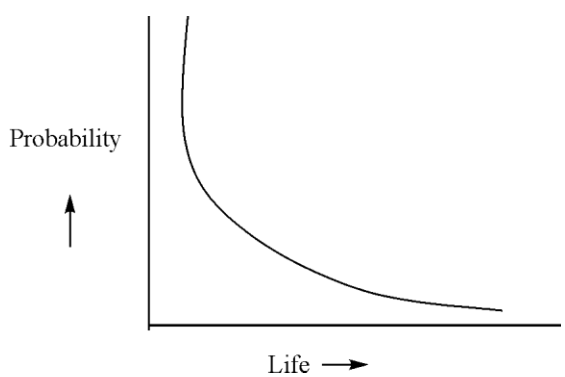


Figure 7.2(b) Retrogressive failure
Probability of failure is more in the nore in early life of the item and then chance of failure decreases.

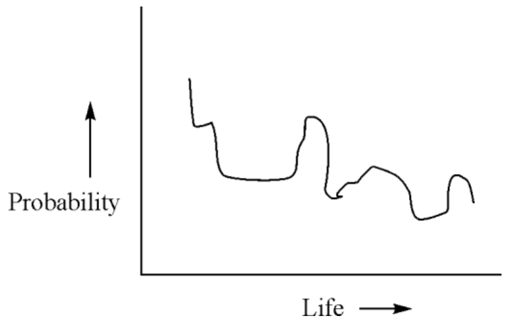


Figure 7.2(c) Random failure

Item fail due to some random cause but not due to age

The above may be shown as in figure 7.3

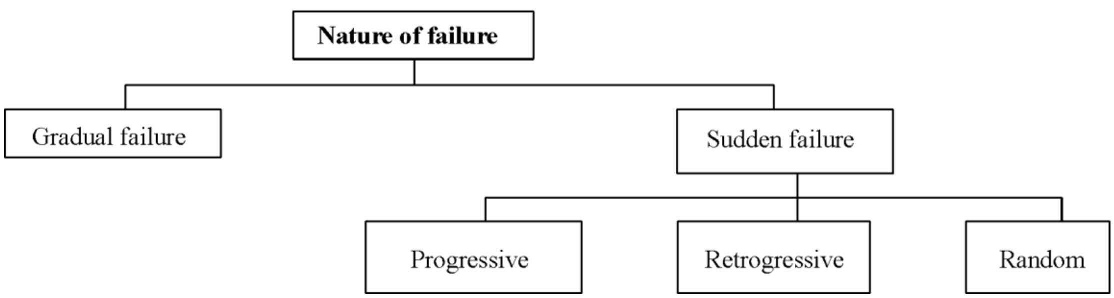


Figure 7.3

Bathtub Curve

Machine or equipment or facility life can be classified into three stages. They are ***Infant stage, Youth stage or Youth phase and Old age stage or Old age phase.***

(a) Infant stage or phase

This is also known as early failure stage. When new equipment is purchased and installed in the existing system, it has to cope up with the operating efficiency of the existing system. Also it has to accustom to operating skill of the operator. Perhaps when a new vehicle is purchased, due to mechanical conditions of the machine and the operating skill of the person, the vehicle may give trouble in the early stages. The owner may have to visit the mechanic many times. It is like a baby, which has come out from mother's womb. The baby until birth, it was in a controlled atmosphere and when the birth takes place, it has to get accustom to outside atmosphere. Hence it cries. Sometimes there is a danger of failure of life. Hence we see more failures in infant stage. Once an age of 10 to 15 years is reached, the death rate or failure rates will reduce. In this stage the safety of machine is covered by the guarantee period given by the manufacturer. This is represented by a curve on the left hand side of the **Bathtub curve.**

(b) Youth stage or random failure stage

In this stage the equipment or machine is accustomed to the system in which it is installed and works at designed operating efficiency. Regular maintenance such as overhauling, oiling, greasing, cleaning keep the machine working. Now and then due to wear and tear of components or heavy load or electrical voltage fluctuations, breakdown may occur, which can be taken care of by repair maintenance. The machine or equipment works for longer periods without any trouble. This is like youth stage in human life that is full of vigor and energy and the person will be healthy and work for longer periods without any diseases. This is shown as a horizontal line in bathtub curve. Here repair maintenance; preventive maintenance or other maintenance techniques are used to keep the machine or equipment in working condition.

(c) Old age stage or Old age phase or Wear out failures

Due to continuous usage and age of the machine, there will be wear and tear of various components. Not only this, during youth stage, some of the components might have been replaced due to wear and tear. These replaced components may not suit well in the system if they are not from original manufacturer. As the manufacturers are changing the design, one has to go for spares available in the market. All this may reduce to operating efficiency of the machine or equipment and the management has to face frequent failures. This is very similar to old age in human life. Due to old age, people will get diseases and old age weakness and many a time they have to go to hospitals for treatment before the life fails. This is shown on the right side of the bath- tub curve. Here one will think of replacement of the equipment or machine. When all the above three curves are assembled, we get a curve which is in the form of a **bathtub** and is known as **bathtub curve,** figure 7.4.

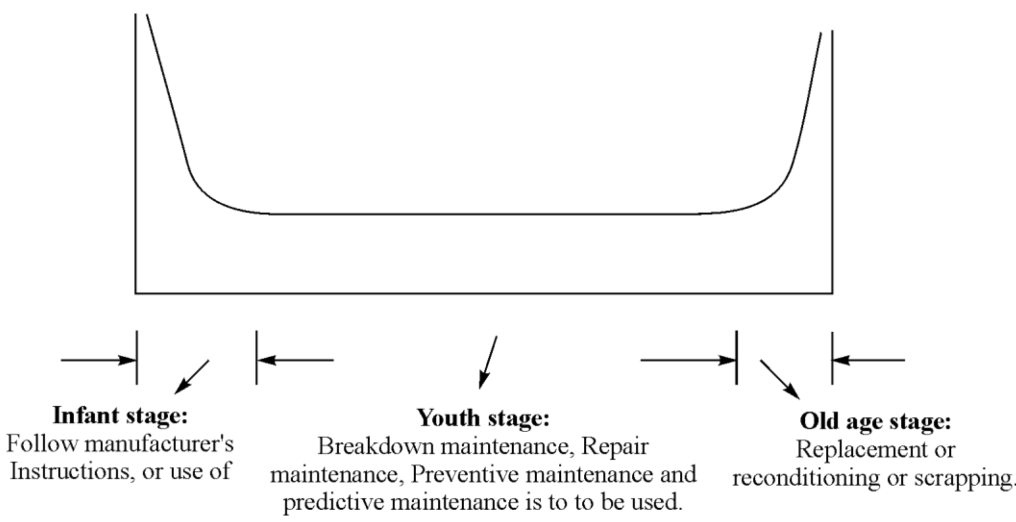


Figure 7.4 Bathtub curve.

All the above-discussed points may be summarized as:

Table 7.1. Summary of three- stages of maintenance

Phase	Type of Failure	Failure Rate	Causes of failure	Cost of Failure	Suitable maintenance Policy
Infant phase	Early Failures	Decreeing Trend.	Faulty design, Erratic Operation, Installation Errors, environmental Problems.	Medium to high	Warrantee / guarantee by manufacturer.
Youth phase	Random or Chanced Failures or Rare event Failures.	Constant	Operational errors, Heavy load, over run	Low to medium	Break down, Predictive Preventive, Repair Maintenance etc.
Old age phase	Wear out or Age failures due to wear and tear..	Increasing	Wear, tear, Creep, Fatigue etc.	Low High	Operate to fail and Corrective maintenance Reconditioning or Replacement.

Costs Associated with Maintenance

Our main aim in this chapter is to find optimal replacement period so as to minimize the maintenance cost. Hence we are very much interested in the various cost associated with maintenance. Various costs to be discussed are:

(a) Purchase cost or Capital cost: (C)

This cost is independent of the age of the machine or usage of the machine. This is incurred at the beginning of the life of the machine, *i.e.* at the time of purchasing the machine or equipment. But the interest on the invested money is an important factor to be considered.

(b) Salvage value / Scrap value / Resale value / Depreciation: (S)

As the age of the machine increases, the resale value decreases as its operating efficiency decreases and the maintenance costs increases. It depends on the operating conditions of the machine and life of the machine.

(c) Running costs including maintenance, Repair and Operating costs:

These costs are the functions of age of the machine and usage of the machine. As the usage increases or the age increases, due to wear and tear, many components fail to work and they are to be replaced. As the age increases, failures also increase and the maintenance costs goes on increasing. At some period the maintenance costs are so high, which will indicate that the replacement of the machine or equipment is essential.

These costs can be shown by means of a curve as in figure 7.5.

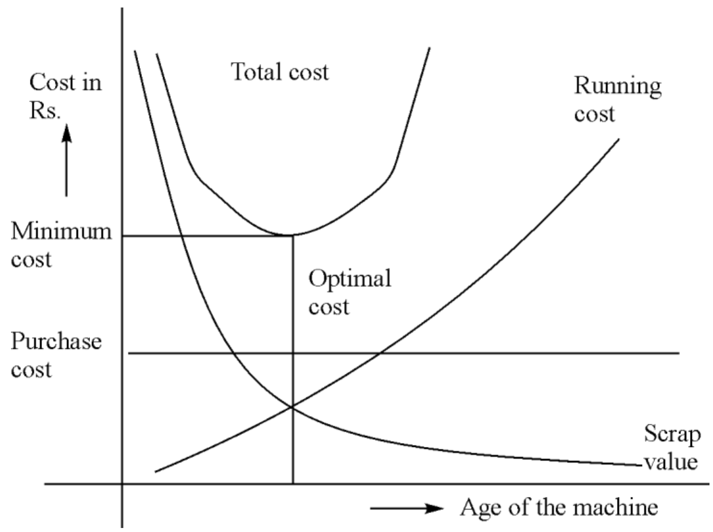


Figure 7.5.

TYPES OF REPLACEMET PROBLEMS

One must remember that the study of **Replacement of items** is a field of application rather than a method of analysis. The study involves, the comparison of alternative replacement policies. Various types of replacement problems we come across in this chapter is:

(a) Replacement of **Capital equipment,** which looses its operating efficiency due to aging (passage of time), or due to continuous usage (due to wear and tear of components). Examples are: Machine tools, Transport and other vehicles, etc., Here the system can maintain the level of performance by installing a new unit at the beginning of some unit of time (year, month or week) and decide to keep it up to some suitable period so as to minimize the operating and maintenance costs. In this case the deterioration process is predictable and is represented by an increased maintenance cost and decreased in scrap cost and increased production cost per unit. In such cases the optimum life of the item is determined on the assumption that increased age reduces efficiency. Deterministic models explain the problem and they are very much similar to that of inventory models where deterioration corresponds to demand against the desired level of efficiency (level of inventory). The cost of new item is similar to cost of replenishment of inventory and maintenance cost corresponds to cost of holding inventory.

These types of problems are solved by two methods. They are:

(i) By calculating the cost per unit of time, without considering the money value. Here we calculate the total cost up to the period and divide by time unit (years, months, weeks etc.,) to find the average cost to decide the period of replacement.

(ii) By taking the money value into consideration using ***present value concept*** to compare on a one number basis.

(b) Replacement of **items that fail completely** all in a sudden in a random nature. We use **Group replacement or Preventive maintenance technique** for these items and these are expensive to replace individually. Examples are: Electric bulbs, Transistors, Electronic components etc., Here replacement of items are done in anticipation of failure, which is known as ***preventive maintenance.*** We assume that the items will have relatively constant efficiency until they fail or die. These models require the knowledge of statistics and stochastic process involving probability of failure. The replacement policy is formulated to balance the wasted life of items replaced before failure against the costs incurred when items fail in service.

(c) Replacement of **human beings in organizations,** known as **Staffing problem,** or known as **Human resource planning** or **Mortality and Staffing problem.** This problem requires the knowledge of life distribution for service of staff in a system.

(d) Miscellaneous problems such as replacement of existing units due to availability of more effective and new and advanced technology. In these problems replacement will become necessary due to research of new and advanced and more effective technology and old technology becomes out of date.

GENERAL APPROACH TO SOLUTION TO REPLACEMENT PROBLEM

Though it is not possible or it is difficult to predict the time of failure of an item exactly, likely failure pattern could be established by observation. Generating the probability distribution for the given situation and then using them in conjunction with relevant cost information we can formulate the optimum replacement policy. The information necessary to formulate optimum replacement policy is:

(i) Objective assessment of the probability of the item failing at a particular point of time

(ii) Assessments of the cost of replacement in terms of:

- (a) Actual cost of the item,
- (b) Direct costs of labour involved in replacement,
- (c) Costs of disruption in terms of lost production, lost orders etc.,

REPLACEMENT OF ITEMS WHOSE EFFICIENCY REDUCES OR MAINTENCNCE COST INCREASES WITH TIME OR DUE TO AGE AND MONEY VALUE IS NOT CONSIDERED

Costs to be considered: Various cost items to be considered in replacement decisions are the costs that depend upon the choice or age of item or equipment. The costs those do not change with the age of the machine or item need not be taken into consideration. The replacement of items whose efficiency reduces with time is justified when the average cost per time period goes on reducing longer the replacement is postponed. However, there will come an age at which the rate of increase of running costs more than compensates the saving in average capital costs. At this age the replacement is justified.

In the case of replacement of items whose efficiency deteriorates with time, the most important criteria to be considered is the measurement of efficiency. Consider a machine, in this case, the maintenance cost always increases with time and usage and a time comes when the maintenance cost becomes large enough, which indicates that it is better and economical to replace the machine with a new one. When we want to replace the machine, we may come across various alternative choices, where we have to compare the various cost elements such as running costs and maintenance costs to select optimal choice. The various techniques we may come across to analyze the situation are:

(a) Replacement of items whose maintenance cost increases with time and ***value of money remains same*** during the period,

(b) Replacement of items whose maintenance cost increases with time and the ***value of money also changes*** with time, and

(c) To compare alternative choices, use of ***concept of present value.***

Replacement of Items whose Maintenance Cost Increases with Time and the Value of Money Remains Same During the Period

Let C = Purchase cost or Capital cost of the item,

S = Scrap value or resale value of the item, it is assumed that this cost will remain constant over time.

Case 1:

Here we assume that the time **'*t*' is a continuous variate.** Let $u(t)$ be the maintenance or running cost at the time 't'. If the item is used in the system for a period 'y', then the total maintenance cost or cumulative running cost incurred during the period 'y' will be:

$$M(\mathbf{y}) = \int_0^y u\,(t)\ dt \qquad \text{...(7.1)}$$

The total cost incurred on the item during period 'y' =

Capital cost + total maintenance cost in the period 'y' - Scrap value. = $C + M(y) - S$

Hence average cost per unit of time incurred during the period 'y' on the item is given by:

$G(y) = \{C + M(y) - S\}/y$, to find the value of '$y$' for which $G(y)$ is minimum the first derivative of $G(y)$ with respect to 'y' is equated to zero.

$dG/dy = \{(C - S)/y^2\} + \{u(y)/y\} - (1/y^2)\int_o^y u(t)dt = 0$, substituting from 7.1 we get,

$\{(C\text{-}S)/y^2\} + \{u(y)/y\} - (1/y^2)M(y) = 0$, **OR this is written as:**

$u(y) = \{C - S + M(y)\}/y = G(y)$ ...(7.2)

So, replace the item when the average annual cost reaches at the minimum that will always occur at a time when the average cost becomes equal to the current maintenance cost. Point to note: If time is measured continuously then the average annual cost will be minimized by replacing the machine or item when the average cost to date becomes equal to the current maintenance cost.

Case 2.

Here time 't' is considered as a discrete variable. In this case, the time period is taken as one year and 't' can take the values of 1, 2, 3 …etc., then,

$$M(y) = \sum_{t=0}^{y} u(t) = \text{Total running cost of '}y\text{' years.}$$

Total cost incurred on the item during 'y' years is $T(y) = C + M(y) - S = C - S + \sum_{t=1}^{y} u(t)$

Average annual cost incurred during 'y' years is

$G(y) = \{T(y) / y\} = \{C + M(y) - S\} / y$

$G(y)$ will be minimum for that value of 'y', for which $G(y+1) > G(y)$ and $G(y-1) > G(y)$ or say that

$$\mathbf{G(y+1) - G(y) > 0 \text{ and } G(y-1) - G(y) > 0} \qquad ...(7.3)$$

This will exist at:

$$\mathbf{G(y+1) - G(y) > 0 \text{ if } u(y+1) > G(y) \text{ and } G(y-1) - G(y) > 0 \text{ if } u(y) < G(y-1)} \qquad ...(7.4)$$

This show that do not replace, if the next years running cost is less than the previous years average total cost but replace at the end of 'y' years if the next year's {i.e. (y + 1) th year} running cost is more than the average cost of 'y' th year.

The above statement shows that when the time is measured in discrete units, replacing the machine when the next period's maintenance cost becomes greater than the current average cost will minimize then the average annual cost.

Points to remember

(a) If time is measured continuously, then the average annual costs will be minimized by replacing the machine or item, when the average cost to date becomes equal to the current maintenance cost.

(b) If time is measured in discrete units, then the average annual cost will be minimized by replacing the machine or item when the next period's maintenance cost becomes greater than the current average cost.

Problem 7.1.

A firm is thinking of replacing a particular machine whose cost price is Rs. 12,200. The scrap value of the machine is Rs. 200/-. The maintenance costs are found to be as follows:

Year	1	2	3	4	5	6	7	8
Maintenance Cost in Rs.	220	500	800	1200	1800	2500	3200	4000

Determine when the firm should get the machine replaced.

Solution

Year (t) Y	*u (t) Maintenance Cost. (Rs)*	*M (y) = $\sum_{t=1}^{y} u(t)$*	*C = Capital Cost in Rs.*	*Scrap Cost (S) In Rs.*	*T (y) = C - S + M (y)*	*Average Cost : G (y) = T (y) / y*
1	2	3	4	5	6	7
1	2	34		5	6 = 4-5 +3	7 = 6/1
1	220	220	12200	200	12220	12220
2	500	720	12200	200	12720	6360
3	800	1520	12200	200	13520	4506.67
4	1200	2720	12200	200	14720	3680
5	1800	4520	12200	200	16520	3304
6	2500	7020	12200	200	19020	**3170**
7	3200	10220	12200	200	22220	3174.29
8	4000	14220	12200	200	26220	3277.5

Replace the machine at the end of 6th year when the average annual maintenance cost is minimum.

Problem 7.2.

The initial cost of a machine is Rs. 6100/- and its scrap value is Rs.100/-. The maintenance costs found from experience are as follows:

Year:	1	2	3	4	5	6	7	8
Annual maintenance cost in Rs.:	100	250	400	600	900	1200	1600	2000

When should the machine be replaced?

Solution

The time period is discrete. We have to find the period when the average maintenance cost will be minimum.

Years 't' = y	*u (t) Rs*	*M (y) = $\sum_{t=1}^{y} u(t)$*	*T (y) = C-S + M (y) Rs.*	*G (y) = T (y) / y Rs.*
1	100	100	6100	6100/1 = 6100
2	250	350	6350	6350 / 2 = 3175
3	400	750	6750	6750 / 3 = 2250
4	600	1350	7350	7350 / 4 = 1837.50
5	900	2250	8250	8250 / 5 = 1650
6	1200	3450	9450	**6450 / 6 = 1575**
7	1600	5050	11050	11050 / 7 = 1578.57
8	200	7050	13050	13050 / 8 = 1631.25

The annual average maintenance cost is minimum at the end of 6th year and it goes on increasing from 7th year. Hence the machine is to be replaced at the end of 6th year.

Problem 7.3.

The maintenance cost and resale value per year of a machine whose purchase price is Rs. 7000/- is given below:

Year:	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
Maintenance cost in Rs.:	900	1200	1600	2100	2800	3700	4700	5900
Resale value in Rs.:	4000	2000	1200	600	500	400	400	400

When should the machine be replaced?

Solution

Years (t) = y	*Running Cost u (y) In Rs.*	*Cumulative Running cost M (Y) in Rs.*	*Resale Value S (y) In Rs.*	*C - S (y) In Rs. C = 7000/-*	*T (y) = C – S (y) + M (y) In Rs.*	*T (y) / y = G (y) Average cost in Rs.*
1	900	900	4000	3000	3900	3900
2	1200	2100	2000	5000	7100	3550
3	1600	3700	1200	5800	9500	3166.67
4	2100	5800	600	6400	12200	3050
5	2800	8600	500	6500	15100	**3020**
6	3700	12300	400	6600	18900	3150
7	4700	17000	400	6600	23600	3371.43
8	5900	2 2900	400	6600	29500	3687.50

From the table we can see that the average cost is minimum at the end of the 5th year. Hence the machine may be replaced at the end of the 5th year.

Problem 7.4

A fleet owner finds form his past records that the cost per year of running a truck and resale values whose purchase price is Rs. 6000/- are given as under. At what stage the replacement is due?

Year:	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
Running cost in Rs.	1000	1200	1400	1800	2300	2800	3400	4000
Resale value in Rs.	3000	1500	750	375	200	200	200	200

Solution

Let C = Capital cost = Rs. 6000/-, S (y) = Scrap value changes yearly, G (y) Average yearly cost, u (t) = Annual maintenance cost.

Years (t) = y	*Running Cost u (y) In Rs.*	*Comulative Running cost M (Y) in Rs.*	*Resale Value S (y) In Rs.*	*C - S (y) In Rs.*	*T (y) = C - S (y) + In Rs.*	*T (y) / y = G (y) Average cost in Rs.*
1	1000	1000	3000	3000	4000	4000
2	1200	2200	1500	4500	6700	3350
3	1400	3600	750	5250	8850	2950
4	1800	5400	375	5625	11025	2756
5	2300	7700	200	5800	13500	**2700**
6	2800	10500	200	5800	16300	2717
7	3400	13900	200	5800	19700	2814
8	4000	17900	200	5800	23700	2962

From the table above we can see that the G (y) is minimum at the end of 5th year. Hence the truck is to be replaced at the end of 5th year.

Problem 7.5.

The initial cost of a vehicle is Rs. 3,800/- and the trade in value drops as time passes until it reaches Rs. 600/-. The maintenance costs are as shown below. Find when the replacement is due?

Year of service:	*1*	*2*	*3*	*4*	*5*
Year-end trade - in value in Rs.:	2000	1200	800	700	600
Annual operating cost in Rs.:	1600	1900	2200	2500	2800
Annual Maintenance cost in Rs.	400	500	700	900	1100

Solution

C = Capital cost in Rs. = 3800/-, S (y) =, Scrap value or trade in value, u (t) = Annual maintenance cost in Rs.

Here we can add operating cost and annual maintenance cost and put it together.

Years (t) = y	*Running Cost u (y) In Rs.*	*Cumulative Running cost M (Y) In Rs.*	*Resale Value S (y) In Rs.*	*C – S (y) In Rs.*	*T (y) = C – S (y) + M (y) In Rs.*	*T (y) / y = G (y) Average cost in Rs.*
1	2000	2000	20 00	1800	3800	3800
2	2400	4400	1200	2600	7000	3500
3	2900	7300	800	3000	10300	**3433**
4	3400	10700	700	3100	13800	3450
5	3900	14600	600	3200	17800	3560

The optimal replacement period is at the end of third year. And the minimum annual average cost is Rs. 3433/-

Problem 7.6.

A Plant manager is considering the replacement policy for a new machine. He estimates the following costs in Rupees. Find an optimal replacement policy and corresponding minimum cost.

Year:	*1*	*2*	*3*	*4*	*5*	*6*
Replacement cost at the beginning of the year. (Rs)	100	110	125	140	160	190
Salvage value at the end of the year: (Rs)	60	50	40	25	10	0
Operating costs (Rs.)	25	30	40	50	65	80

Solution

In this problem Replacement cost at the beginning and the salvage value at the end of the year is given. If we subtract salvage value from the replacement cost we get the **net value,** *i.e*. *C* (capital cost) - *S* (resale value). Rest of the problem is worked as usual.

Years (t) = y	*Running Cost u (y) In Rs.*	*Cumulative Running cost M (Y) in Rs.*	*Resale Value S (y) In Rs.*	*C – S (y) In Rs.*	*T (y) = C – S (y) + M (y) In Rs.*	*T (y)/y = G (y) Average cost in Rs.*
1	25	25	60	100 – 60 = 40	65	65
2	30	55	50	110 – 50 = 60	115	57.50
3	40	95	40	125 – 40 = 85	180	60
4	50	145	25	140 – 25 = 115	260	65
5	65	210	10	160 – 10 = 150	360	72
6	80	290	0	190 – 0 = 190	480	80

From the table we see that the average annual maintenance cost is minimum at the end of 2nd year and is Rs. 57.50. Hence the machine is to be replaced at the end of 2nd year.

Problem 7.7.

A fleet owner finds form his past records that the cost per year of running a vehicle whose purchase price is Rs. 50000/- are as under:

Year:	*1*	*2*	*3*	*4*	*5*	*6*	*7*
Running cost in Rs.:	5000	6000	7000	9000	21500	18000	18000
Resale value in Rs.:	30000	15000	7500	3750	2000	2000	2000

Thereafter running cost increases by Rs.2000/- per year but resale value remains constant at Rs. 2000/-. At what stage the replacement is due?

Solution

Years (t) = y	*Running Cost u (y) In Rs.*	*Cumulative Running cost M (Y) In Rs.*	*Resale Value S (y) In Rs.*	*C – S (y) In Rs.*	*T (y) = C – S (y) + M (y) In Rs.*	*T (y) / y = G (y) Average cost in Rs.*
1	5000	5000	30000	20000	25000	25000
2	6000	11000	15000	35000	46000	23000
3	7000	18000	7500	42500	60500	20166.50
4	9000	27000	3750	46250	73250	**18312.50**
5	21500	48500	2000	48000	96500	19300
6	16000	64500	2000	48000	1,12,500	18750
7	18000	82500	2000	48000	1,30,500	**18642.80**
8	20000	1,02,500	2000	48000	1,50,500	18812.50
9	22000	1,24,500	2000	48000	1,70,500	18944.40

In this problem, the running cost increases from first year (Rs.5000) to Rs. 21, 500 in the 5th year and then it reduces in 6th year and then it increase year wise. Hence there are two minimum annual maintenance cost *i.e*.

Rs. 18,312.50 at the end of 4th year and Rs. 18,642.80 at the end of 7th year. Hence we can conclude that the machine is to be replaced at the end of 4th year. Due to any financial constraint if it is not replaced at the end of 4th year, it must be replaced at the end of 7th year.

Problem 7.8.

Machine *A* costs Rs. 45,000/- and the operating costs are estimated at Rs. 1000/- for the first year, increasing by Rs. 10,000/- per year in the second and subsequent years. Machine *B* costs Rs.50000/- and operating costs are Rs. 2000/- for the first year, increasing by Rs. 4000/- in the second and subsequent years. If we now have a machine of type *A*, should we replace it by *B*? If so when? Assume both machines have no resale value and future costs are not discounted.

Solution

Let us now calculate the average annual running cost for machine *A* and *B* in the tales given below: (*S* = Rs.0/-)

Machine A

Year (y)	*Running Cost (Rs) u (t)*	*Cumulative Running Cost in Rs, $\sum u(t) = M(y)$*	*Depreciation C – S*	*Total cost TC = C – S + M (y)*	*Average cost F (y) = TC/ y*
1	1000	1000	45000	46000	46000
2	11000	12000	45000	57000	28500
3	21000	33000	45000	78000	**26000**
4	31000	64000	45000	1,09,000	27200
5	41000	1,05,000	45000	1,50.000	30000
6	51,000	1,56,000	45000	2.01,000	33500

As the annual maintenance cost is minimum at the end of 3rd year, the machine A is to be replaced at the end of 3rd year.

Machine B

Year (y)	*Running Cost (Rs) u (t)*	*Cumulative Running Cost in Rs, $\sum u(t) = M(y)$*	*Depreciation C – S*	*Total cost TC = C – S + M (y)*	*Average cost F (y) = TC/ y*
1	2000	2000	50000	52000	52000
2	6000	8000	50000	58000	29000
3	10000	18000	50000	68000	22667
4	14000	32000	50000	82000	20500
5	18000	50000	50000	1,00,000	**20000**
6	22000	72000	50000	1,22,000	20333

As the average annual maintenance cost is minimum at the end of 5th year, the machine is to be replaced at the end of 5th year.

When we compare machine *A* and machine *B*, the average annual maintenance cost of Machine *B* is less than that of *B*, the machine *A* should be replaced by machine *A*.

Now to find the time of replacement of machine *A* by machine *B*, the total cost of the machine *A* in the successive years is computed as given below:

Year:	*1*	*2*	*3*	*4*
Total cost incurred in Rs.:	46000	57000 – 46000	78000 – 57000	1,09,000 – 78000
		= 11000	= 21000	= 31000

The criterion is to replace machine *A* by machine *B* at the age when its running cost for the next year exceeds the lowest average running cost *i.e*. Rs. 20000 per year of machine *B*. From the calculations we can see that running cost of machine is in the third year, *i.e*. Rs. 21000/- is more than the lowest average running cost per year of machine *B i.e*. Rs. 20000/- at the end of fifth year. **Hence the machine *A* should be replaced by machine *B* after two years.**

Problem 7.9.

A taxi owner estimates from his past records that the costs per year for operating taxi whose purchase price when new is Rs.60000/- are as given below:

Age (year):	*1*	*2*	*3*	*4*	*5*
Operating cost in Rs.:	10000	12000	15000	18000	20000

After 5 years, the operating cost is Rs. $6000 \times k$ Where $k = 6, 7, 8, 9, 10$, *i.e*. 'k' denotes years. If the resale value decreases by 10% of purchase price each year, what is the best replacement policy? Cost of money is zero.

Solution

Capital cost is C = Rs. 60000/-.

Resale value decreases by 10% of capital cost. Hence it reduces by $60000 \times (10/100)$ = Rs. 6000/-

This means $C - S$ increase by Rs. 6000/- every year.

Average annual maintenance cost of machine is:

Year	*Annual Maintenance Cost Rs. u (t)*	*Σ u (t) Rs. M (y)*	*S = Resale Value Rs.*	*C – S Rs.*	*TC = Total Cost C – S – M (y)*	*Average annual Cost G (y) = TC / y*
1	10000	10000	54000	6000	16000	**16000**
2	12000	22000	48000	12000	34000	17000
3	15000	37000	42000	18000	55000	18333
4	18000	55000	36000	24000	79000	19750
5	20000	75000	30000	30000	1,05,000	21000
6	36000	1,11,000	24000	36000	1,47,000	24500
7	42000	1,53,000	18000	42000	1,95,000	27857
8	48000	2,01,000	12000	48000	2,49,000	31125
9	54000	2,55,000	6000	54000	3,09,000	34333
10	60000	3,15,000	0	60000	3,75,000	37500

As the average annual cost is minimum in the first year itself, the machine is to be replaced every year. We can interpret the situation as: The taxi owner's estimate of operating cost may be wrong or the taxi is of low quality as it is to be replaced every year.

Problem 7.10

(a) A machine A costs Rs.9000/-. Annual operating costs are Rs. 200/- for the first year and then increases by Rs.2000/- every year. Determine the best age at which the machine *A* is to be replaced? If the optimum replacement policy is followed, what will be the average yearly cost of owning and operating the machine? Assume machine has no resale value when replaced and that future costs are not discounted.

(b) Machine *B* costs Rs. 10000/-. Annual operating costs are Rs. 400/- for the first year and then increases by Rs. 800/- every year. You have now a machine of type *A*, which is of one year old. Should you replace it with *B*, and if so, when?

Solution

Given that resale value is Rs. zero. Purchase price for machine *A* is Rs. 9000/- and purchase price for machine *B* is Rs. 10,000. Hence for machine *A*, $C - S$ = Rs. 9000/- and that for *B* is Rs. 10000/-.

Years (y) T	*Annual Maintenance Cost Rs. u (t)*	$\Sigma\, u(t)$ *Rs.* $= M(y)$	$C - S$ *Rs.*	*T.C =* $C - S + M(y)$ *Rs.*	$G(y) =$ *T.C / y Rs.*
1	200	200	9000	9200	9200
2	2200	2400	9000	11400	5700
3	4200	6600	9000	15600	**5200**
4	6200	12800	9000	21800	5450
5	8200	21000	9000	30000	6000

The minimum annual maintenance cost occurs at the end of 3 rd year and it is Rs. 5200/-. Hence the machine *A* is to be replaced at the end of 3 rd year.

Machine B

Years (y) T	*Annual Maintenance Cost Rs. u (t)*	$\Sigma\, u(t)$ *Rs.* $= M(y)$	$C - S$ *Rs.*	*T.C =* $C - S + M(y)$ *Rs.*	$G(y) =$ *T.C / y Rs.*
1	400	400	10000	10400	10400
2	1200	1600	10000	11600	5800
3	2000	3600	10000	13600	4533.33
4	2800	6400	10000	16400	4100
5	3600	10000	10000	20000	**4000**
6	4400	14400	10000	24400	4066.67

As the average annual maintenance cost is minimum *i.e.* Rs. 4000/- at the end of 5 th year, the machine *B* is to be replaced at the end of 5 th year. As the minimum average yearly maintenance cost of machine *B*.

(Rs. 4000/-) is less than that of machine *A i.e.* Rs. 5200, Machine *A* is replaced by machine *B*.

Now we have to workout as when machine *A* is to be replaced by machine *B*? Machine *A* should be replaced when the cost for next year of running this machine becomes more than the average yearly cost for machine *B*.

Total cost of machine *A* in the first year is Rs. 9200/-.

Total cost of machine *A* in the second year is Rs. 11400 – Rs. 9200/- = Rs. 4200/-, (= Total cost of present year – Total cost of previous year) Similarly, the total cost of machine in third year is Rs. 4200/- and in fourth year is Rs.6200/-.

As the cost of running machine *A* in third year (Rs. 4200/-) is more than the average yearly cost for machine *B* (Rs.4000/-), machine *A* should be replaced at the end of second year. Since machine *A* is one year old, it should run for one year more and then it should be replaced.

Problem 7.11.

(a) An auto rickshaw owner finds from his previous records that the cost per year of running an auto rickshaw whose purchase cost is Rs. 7000/- is as given below:

Year:	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
Running Cost in Rs.:	1100	1300	1500	1900	2400	2900	3500	4100
Resale value in Rs.:	3100	1600	850	475	300	300	300	300

At what age the replacement is due?

(b) Another person has three auto rickshaws of the same purchase price and cost of running each in part (a). Two of these rickshaws are of two years old and the third one is one year old. He is considering a new type of auto rickshaw with 50% more capacity than one of the old ones and at a unit price of Rs. 9000/- He estimates that the running costs and resale price of the new vehicle will be as follows:

Year:	1	2	3	4	5	6	7	8
Running cost (Rs.):	1300	1600	1900	2500	3200	4100	5100	6200
Resale price (Rs.):	4100	2100	1100	600	400	400	400	400

Assuming that the loss of flexibility due to fewer vehicles is of no importance, and that he will continue to have sufficient work for three of the old vehicles, what should be his policy?

Solution

The average annual cost is calculated as under. *C* = Rs. 7000/-.

Years (y) T	*Annual Maintenance cost Rs. u (t)*	*M (y) = ∑ u (t) Rs.*	*S = Resale Value Rs.*	*C – S*	*T.C. = C– S + M (y)*	*G (y) = T.C / y*
1	1100	1100	3100	3900	5000	5000
2	1300	2400	1600	5400	7800	3900
3	1500	3900	850	6150	10050	3350
4	1900	5800	475	6525	12325	3081
5	2400	8200	300	6700	14900	2980
6	2900	11100	300	6700	17800	**2967**
7	3500	14600	300	6700	21300	3043
8	4100	18700	300	6700	25400	3175

The auto rickshaw is to be replaced at the end of 6 th year as the average annual maintenance cost is low at the end of 6th year.

Average annual maintenance cost of larger rickshaw: C = Rs. 9000/-.

Years (y) T	*Annual Maintenance Rs. u (t)*	*M (y) = ∑ u (t) Rs.*	*S = Resale Value Rs.*	*C – S*	*T.C. = C– S + M (y)*	*G (y) = T.C / y*
1	1300	1300	4100	4900	6200	6200
2	1600	2900	2100	6900	9800	4900
3	1900	4800	1100	7900	12700	4233
4	2500	7300	600	8400	15700	3925
5	3200	10500	400	8600	19100	**3820**
6	4100	14600	400	8600	23200	3867
7	5100	19700	400	8600	28300	4043
8	6200	25900	400	8600	34500	4312

As the auto rickshaw has 50 % more capacity than the old one, the minimum average annual cost of Rs. 3820/- for the former rickshaw is equivalent to Rs. 3820 × (2/3) = Rs.2546.66/- (approximately Rs. 2547/-) for the latter. Since this amount is less than Rs. 2967/- for it, the new auto rickshaw will replace the latter.

Now we have decided to replace the old vehicle by the new one. Now let us find when it should be replaced? Let us assume that two new larger ones will replace all the three old auto rickshaws. ***The new vehicles will be purchased when the cost for the next year of running the three old vehicles becomes more than the average annual cost of the two new ones.***

Total annual cost of one smaller auto rickshaw during the first year: Rs. 5000/-.

Annual cost of one smaller auto rickshaw during the second year is: Rs. 7800 – Rs. 5000/- = Rs. 2800/-.(= Present total cost – previous year total cost).

Annual cost of one smaller auto rickshaw during the third year is: Rs. 10800 – Rs. 7800 = Rs. 2250

Annual cost of one smaller auto rickshaw during the fourth year is: Rs. 12325 – 10050 = Rs.2275/
Annual cost of one smaller auto rickshaw during the fifth year is: Rs. 14900 – Rs. 12325/- = Rs. 2575/
Annual cost of one smaller auto rickshaw during the sixth year is: Rs. 17800/- – Rs. 14900/- = Rs. 2900/-

Do do do seventh year is: Rs. 21300 – Rs.17800/- = Rs. 3500

Do do do eighth year is: Rs. 25400 – Rs. 21300 = Rs. 4100/-

Therefore, total cost during one year hence for two smaller vehicles aged two years and one vehicle aged one year is: 2 × 2250 + 2800 = Rs. 7300/-.

Similarly for two years the cost is 2 × 2275 + 2250 = Rs. 6800/-

For three years: Rs. 2 × 2575 + 2275 = Rs. 7425/-

For four years: Rs. 2 × 2900 + 2575 = Rs. 8375/-

Here minimum average cost for two new vehicles is = 2 × 3820 = Rs. 7640/-

As the total cost of old vehicles at the end of three years is less than the minimum average cost of the new vehicles and increases after four years, the old auto rickshaws are to be replaced at the end of three years by new larger ones.

Replacement of Items whose Maintenance Costs Increases with Time and Value of Money also Changes with Time

7.5.2.1 Present worth factor: Before dealing with this model, it is better to have the concept of **Present value**. Consider replacement of items which involve huge expenditure, both initial value (Purchase price) and maintenance expenses. For a decision maker, there may be number of alternatives for replacement. But he always chooses the alternative, which minimizes the annual average cost. Here manager uses the concept of present value of money to select the alternatives. ***The present value of number of expenditures incurred over different periods of time represents their value at the current time. It is based on the fact that, one can invest money at an interest rate 'r' to produce the same amount of money at the end of certain time period or if an amount is to be spent in different years what is the worth of total expected expenses or its worth today?*** We can also think in another way. Suppose a businessman borrows money for his initial investment and working capital from various sources, he has to pay interest for the money he has borrowed. The amount of interest he has to pay depends on the rate of interest and the period for which he has borrowed (that is the period in which he has repaid the amount borrowed). The borrowed money is known as **Principal** (*P*), and the excess amount he has paid is known as **Interest** (*i*). The sum of both principal and the interest is known as **Amount (*A*).**

If *P* is the principal, '*i*' is the rate of interest, and A is the amount, then the amount at the end of '*n*' years with compound interest is:

$$A = P(1+i)^n \quad \text{OR} \quad P = (A) / (1+i)^n \quad \text{OR} \quad P = A \times Pwf \qquad ...(7.5)$$

Where *Pwf* is Single payment present worth factor. It is represented by '*v*' and is also known as *discount rate*. Discount rate is always less than one. In other words we can say that '*v*' or the present worth factor (*pwf*) is present value of one rupee spent after '*n*' years from now. Hence *P* is known as the present worth of an amount *A* paid in '*n*' years at interest rate '*i*'. For calculation purpose present worth factor tables are available.

Similarly, if *R* denotes the uniform amount spent at the end of each year and *S* is the total expenditure at the end of '*n*' years, then

$$S = R \{ (1 + i)^n - 1\} / i \quad \text{OR}$$
$$R = (S \times i) / (1 + i)^n - 1 = \{P (1 + i)^n \times i\} / \{(1 + i)^n - 1\} \quad \text{OR}$$
$$P = R \times \{P (1 + i)^n - 1\} / \{i (1 + i)^n\} = R \times (Pwfs) \quad ...(7.6)$$

Pwfs is known as uniform annual series present worth factor. In other words, suppose if we want Rs. 50,000/- for investment after 10 years, how much we save yearly, so that at the end of 10 years, we will have Rs. 50,000/- ready for investment. The discount rate to find this amount is known as Pwfs.

Replacement of Items whose Maintenance Cost Increases with Time and Money Value also Changes

This problem is complicated as the money value changes with time. This can be dealt under two different conditions:

(a) The maintenance cost goes on increasing with usage or age or time and then we have to find out optimum time of replacing the item. Here the value of money **decreases** with a constant rate which is known as its **depreciation ratio** or **discounted factor** and is represented by 'd'.

Here the money value changes can be understood as follows: Suppose a person borrows Rs. 1000/- at an interest rate of 10 % per year. After one year from now, he has to pay back Rs. 1100/-. That means to say today's Rs. 1000/- is equivalent to Rs. 1100/- after one year. OR Rs. 1100/- after one year is equivalent to

Rs. 1000/- today. That is Re.1/- after one-year from now is equal to 1000 / 1100 = $(1.1)^{-1}$ at present. This is known as **present value.**

To generalize, if the interest on Re.1/- is 'i' per year then the present value or present worth of Re. 1/- after one year from now is **1 / (1 + i),** this is the **depreciation ratio, represented by 'd'.**
Similarly, the present worth of Re.1/- to be spent after 'n' years from now (the rate of interest is 'i') is

$$\frac{1}{(1 + i)^n}$$

(b) If a businessman takes a loan for a certain period at a given interest rate and agrees to pay it in a number of instalments, then we have to find the most suitable period during which the loan would be repaid.

Case: I

Let the equipment cost be Rs. A and the maintenance costs be Rs. C_1, C_2, C_3 …….C_n (where C_{n+1} is > C_n) during the first, second and third years respectively up to 'n' years. If 'd' is the depreciation value per unit of money during a year then to find the optimum replacement policy, which minimizes the total of all future discounted costs.

It is assumed that the expenditure incurred at the beginning of the year and the resale value of the item is zero. Finding the total expenditure incurred on the equipment and its maintenance during the desired period and its present value solves the problem. The criterion is the period for which the total discounted cost is minimum will be the best period for replacement.

Let us assume that the equipment will be replaced by new equipment after every 'X' years of service. We have to calculate the expenditure made in different years and their present value, as shown below:

Year (i)	*Capital cost*	*Maintenance Cost ©*	*Total cost in the year*	*Present value of total expenditure.*
1	A	C_1	$A + C_1$	$A + C_1$
2	--	C_2	C_2	dC_2
3	--	C_3	C_3	d^2C_3
…	….	…..	…..	……
…..	…..	…..	…..	…..
X		C_x	C_x	$d^{x-1} C_x$
X + 1	A (item replaced By the same type Of item after X Years.	C_1	$A + C_1$	$d^x (A + C_1)$
X + 2	--	C_2	C_2	$d^{x+1} C_2$
…	…..	…..	…..	……
…..	……	…..	…..	……
2X	--	C_x	C_x	$d^{2x-1} C_x$
….	…..	….	…..	…… and so on

For derivation of the formula students are advised to refer to Operations Research books with mathematical approach or higher-level mathematics books.

The Weighted average expenditure is given by:

$$\mathbf{G\ (X)} = \left\{ A + \sum_{i=1}^{x} C_i d^{i-1} \right\} \Big/ \left(\sum_{i=1}^{x} d^{i-1} \right)$$

Where, X is the period, i = year, d = discount factor and A = Capital expenditure.

The criterion is:

(i) **Do not replace the item if the operating cost of next period is less than the weighted average of previous costs, where weights are 1, d, d^2….. d^n .**

(ii) **Replace the item if operating costs of next period is greater than the weighted average of the previous costs.**

$$C_X < \quad G\ (X) \quad < C_{X+1}$$

Let us understand the procedure by working a problem.

Problem 7.12.

The yearly cost of two machines *A* and *B*, when money value is neglected is shown in the table given below. Find their cost pattern if money value is 10% per year and hence find which machine is more economical.

Year	*1*	*2*	*3*
Machine A (Rs.):	1800	1200	1400
Machine B (Rs.);	2800	200	1400

Solution

When the value of money is 10% per year, the discount rate is: $d = (1 / 1 + i = 1 / 1 + 0.10 = 1 / 1.1 = 0.9091$

The discounted cost pattern of machines *A* and *B* are as below:

Year	*1*	*2*	*3*	*Total Cost (Rs.)*
Machine A (Rs.)	1800	1200×0.9091	1400×0.9091^2	4047.94
Discounted cost.		= 1090.90	= 1157.04	
Machine B (Rs.)	2800	200×0.9091	1400×0.9091^2	4138.86
Discounted cost.		= 181.82	= 1157.04	

The table shows that the total cost of machine *A* is less than that of machine *B*. Hence machine *A* is more economical when money value is changing.

Problem 7.13

Value of the money is assumed to be 10 % per year and suppose that machine *A* is replaced after every three years whereas machine *B* is replaced every 6 years. Their yearly costs are given as under:

Year:	*1*	*2*	*3*	*4*	*5*	*6*
Machine A (Rs.):	1000	200	400	1000	200	400
Machine B (Rs.):	1700	100	200	300	400	500

Find which machine is to be purchased?

Solution

The present worth of expenditure of machine *A* for three years:

Year	*Cost (Rs.)*	*Discount factor (d)At 10%*	*Present worth (Rs.)*
1	1000	1.0000	1000.00
2	200	0.9091	$200 \times 0.9091 = 181.82$
3	400	0.8264	$400 \times 0.8264 = 330.56$
		Total cost (Rs)	Rs. 1512.38

The present worth of expenditure of machine *B* for 6 years:

Year	*Cost (Rs.)*	*Discount factor (d)At 10%*	*Present worth (Rs.)*
1	1700	1.0000	1700.00
2	100	0.9091	100 × 0.9091 = 90.91
3	200	0.8264	200 × 0.8264 = 165.28
4	300	0.7513	300 × 0.7513 = 225.39
5	400	0.6830	400 × 0.6830 = 273.20
6	500	0.6209	500 × 0.6209 = 310.45
		Total cost:	Rs. 2,765.23

To compare the two machines, we have to find the average yearly cost for which the total cost is to be divided by the number of years.

Average yearly cost of Machine *A* is Rs. 1512.38 / 3 = Rs. 504.13

Average yearly cost of machine *B* is Rs. 2765.23 / 6 = Rs. 460.87

This shows that the apparent advantage is with machine B. But the comparison is unfair since the periods for which the costs are considered are different. So, if we consider 6 years period for both machines then total present worth of Machine *A* is:

Rs.{1000 + 200 × 0.9091 + 400 × 0.8264 +1000 × 0.7513 + 200 × 0.6830 + 400 × 0.6209} =

{1000 + 181.82 + 330.56 + 751.30 + 136.60 + 248.36} = Rs. 2,648.64

Average yearly cost is: 2648.64 / 6 = Rs. 441.44

As average yearly cost is less than that of *B*, the machine *A* is more economical, hence machine *A* is to be purchased.

Problem 7.14

A machine costs Rs.500/-. Operation and maintenance costs are zero for the first year and increase by Rs. 100/-every year. If money is worth 5 % every year, determine the best age at which the machine should be replaced. The resale value of the machine is negligibly small. What is the weighted average cost of owning and operating the machine?

Solution

Discount rate = d = 1 / 1 + i = 1 / 1 + 0.05 = 0.9524

Weighted average cost:

Year of Service X	*Maintenance Cost C_i (Rs)*	*Discount Factor d^{i-1}*	*Discounted Maintenance Cost $C_i \times d^{i-1}$ Rs.*	*Total cost $A + \sum_{i=1}^{X} C_i \times d^{i-1}$ Rs.*	*Cumulative Discount Factor. $\sum_{i=1}^{X} d^{i-1}$*	*Weighted average Annual cost (Rs) $G(X) = \dfrac{A + \sum_{i=1}^{X} C_i \times d^{i-1}}{\sum_{i=1}^{X} d^{i-1}}$*
1	0	1.0000	0.0000	0.0000	1.0000	500.00
2	100	0.9524	95.2400	595.24	1.9524	304.88
3	**200**	0.9070	181.4000	776.64	2.8594	**217.61 (Replace)**
4	**300**	0.8638	259.14	1035.78	3.7232	278.20
5	400	0.8277	320.08	1364.86	4.5459	300.25

200 < 217.61 < 300 *i.e.* G_3 = Rs. 217.61 > C_3 = Rs. 200.00 and G_3 = Rs. 217.61 < C_4 = Rs. 300/ Rs. 200 the running cost of 3rd year is less than weighted average of 3rd year and running cost of 4th year is greater than the weighted average of 3rd year, hence the machine is to be replaced at the end of 3rd year. The weighted average cost of running the machine is Rs. 217.61.

Problem 7.15.

A machine costs Rs. 10,000. Operating costs are Rs. 500/- for the first five years. In the sixth and succeeding years operating cost increases by Rs. 100/- per year. Assuming a 10 % value of money per year find the optimum length of time to hold the machine before we replace it.

Solution

The value of money is 10 % *i.e*. 0.1. Hence the discount factor $d = 1 / 1 + 0.1 = 1 / 1.1 = 0.9091$.

The purchase price A = Rs. 10,000/-. The weighted average is calculated as under:

Year of Service X	*Maintenance Cost C_i (Rs)*	*Discount Factor d^{i-1}*	*Discounted Maintenance Cost $C_i \times d^{i-1}$ Rs.*	*Total cost* $A + \sum_{i=1}^{x} C_i \times d^{i-1}$	*Cumulative Discount Factor.*	*Weighted average Annual cost (Rs) G (X) =* $\frac{A + \sum_{i=1}^{x} C_i \times d^{i-1}}{\sum_{i=1}^{X} d^{i-1}}$
1	500	1.0000	500	10,500	1.0000	10,500
2	500	0.9091	456	10,956	1.9091	5738.80
3	500	0.8264	413	11,369	2.7355	4156.30
4	500	0.7513	376	11,745	3.4868	3368.40
5	500	0.6830	342	12,087	4.1698	2898.70
6	600	0.6209	373	12,460	4.7907	2600.80
7	700	0.5645	395	12,855	5.3552	2400.40
8	800	0.5132	411	13,266	5.8684	2260.40
9	900	0.4665	420	13,686	6.3349	2160.40
10	1000	0.4241	424	14,110	6.7590	2087.50
11	1100	0.3856	424	14,534	7.1446	2034.20
12	1200	0.3506	421	14,955	7.4952	1995.20
13	1300	0.3187	414	15,369	7.8139	1966.80
14	1400	0.2897	406	15,775	8.1036	1946.60
15	1500	0.2637	396	16,171	8.3673	1932.60
16	1600	0.2397	384	16,555	8.6070	1923.40
17	1700	0.2197	370	16,925	8.8249	1917.80
18	1800	0.1983	357	17,282	9.0230	1915.30
19	**1900**	0.1801	342	17,624	9.2031	**1915.00 Replace**
20	**2000**	0.1637	327	17,951	9.3668	1916.40

C_{20} = Rs.2000/- is greater than C_{19} = Rs. 1900/-

C_{19} = Rs. 1900/- is less than G_{19}= Rs. 1915.30. G_{19} = Rs. 1915/- < C_{20} = Rs. 2000/-. Hence replace the machine at the end of 19 the year.

Problem 7.16.

Assume that present value of one rupee to be spent in a year's time is Re.0.90 and the purchase price A = Rs. 3000/-. The running cost of the equipment is given in the table below. When should the machine be replaced?

Year:	*1*	*2*	*3*	*4*	*5*	*6*	*7*
Running cost in Rs.:	500	600	800	1000	1300	1600	2000

Solution

Given that *A* = Rs. 3000/-, Discount factor = *d* = 0.9

Year of Service X	*Maintenance Cost C_i (Rs)*	*Discount Factor d^{i-1}*	*Discounted Maintenance Cost $C_i \times d^{i-1}$ Rs.*	*Total cost* $A + \sum_{i=1}^{x} C_i \times d^{i-1}$ *Rs.*	*Cumulative Discount Factor.* $\sum_{1}^{X} d^{i-1}$	*Weighted average Annual cost (Rs) G (X) =* $\frac{A + \sum_{i=1}^{x} C_i \times d^{i-1}}{\sum_{1}^{X} d^{i-1}}$
1	500	1.00	500	3500	1.0000	3500.00
2	600	0.90	540	4040	1.90	2126.32
3	800	0.81	648	4688	2.71	1729.89
4	**1000**	0.73	730	5418	3.44	1575.00
5	**1300**	0.66	858	6276	4.10	**1530.73 Replace**
6	1600	0.59	944	7220	4.69	1539.45
7	2000	0.53	1060	8280	5.22	1586.21

C_5 = Rs.1300/- < G_5 = Rs. 1530.73 < C_6 = Rs. 1600/-. The machine is to be replaced at the end of 5 th year.

Problem 7.17.

A manufacturer is offered two machines *A* and *B*. A has the cost price of Rs. 2,500/- its running cost is

Rs. 400 for each of the first 5 years and increase by Rs.100/- every subsequent year. Machine B having the same capacity as *A*. and costs Rs. 1250/-, has running cost of Rs.600/- for first 6 years, increasing thereby Rs. 100/- per year. Which machine should be purchased? Scrap value of both machines is negligible. Money value is 10% per year.

Solution

$d = 1 / 1 + i = 1 / 1 + 0.10 = 1 / 1.1 = 0.9091.$

Present worth of machine A is:

Years Service X	*Maintenance Cost C_i (Rs)*	*Discount Factor d^{i-1}*	*Discounted Maintenance Cost $C_i \times d^{i-1}$ Rs.*	*Total cost* $A + \sum_{i=1}^{x} C_i \times d^{i-1}$ *Rs*	*Cumulative Discount Factor.* $\sum_{i=1}^{x} d^{i-1}$	*Weighted average Annual cost (Rs) G (X) =* $\dfrac{A + \sum_{i=1}^{x} C_i \times d^{i-1}}{\sum_{i=1}^{X} d^{i-1}}$
1	400	1.0000	400.00	2900.00	1.0000	2900.00
2	400	0.9091	363.64	3263.64	1.9091	1709.45
3	400	0.8264	330.56	3594.20	2.7355	1313.84
4	400	0.7513	300.52	3894.72	3.4868	1116.90
5	400	0.6830	273.20	4167.92	4.1698	999.50
6	500	0.6209	310.45	4478.37	4.7907	934.80
7	600	0.5645	338.70	4817.07	5.3552	889.92
8	700	0.5132	359.24	5176.31	58684	881.92
9	**800**	0.4665	372.20	5549.51	6.3349	**875.86 (Replace)**
10	**900**	0.4241	381.69	5931.20	6.7590	877.35

As C_9 = Rs. 800 < G_9 = Rs. 875.86 < C_{10} = Rs.900, it is replaced at the end of 9 th year.

Replacement period for Machine B:

Year of Service X	*Maintenance Cost C_i (Rs)*	*Discount Factor d^{i-1}*	*Discounted Maintenance Cost $C_i \times d^{i-1}$ Rs.*	*Total cost* $A + \sum_{i=1}^{x} C_i \times d^{i-1}$ *Rs.*	*Cumulative Discount Factor.* $\sum_{i=1}^{x} d^{i-1}$	*Weighted average Annual cost (Rs) G (X) =* $\dfrac{A + \sum_{i=1}^{x} C_i \times d^{i-1}}{\sum_{i=1}^{x} d^{i-1}}$
1	600	1.0000	600.00	1850.00	1.0000	1850.00
2	600	0.9091	545.46	2395.46	1.9091	1254.75
3	600	0.8264	495.84	2891.30	2.7355	1056.95
4	600	0.7513	450.78	3342.08	3.4868	958.49
5	600	0.6830	409.80	3751.88	4.1698	899.77
6	600	0.6209	372.54	4124.42	4.7907	860.92
7	700	0.5645	395.15	4519.57	5.3552	843.96
8	**800**	0.5132	410.56	4930.13	5.8684	**840.11 (Replace)**
9	**900**	0.4665	419.85	5349.98	6.3349	844.52
10	1000	0.4241	424.10	5774.08	6.7590	854.28

C_8 = Rs. 800 < G_8 = Rs. 840.11 < C_9 = Rs.900/-. The machine *B* is to be replaced at the end of 8 th year.

Problem 7.18.

The cost of a new machine is Rs. 5000/-. The maintenance cost during the *n* th year is given by u_n = Rs. 500 (*n*-1), where *n* = 1, 2, 3, …., *n*. If the discount rate per year is 0.05, after how many years will it be economical to replace the machine by a new one?

Solution

Since the discount rate is 0.05, *d* = 1 / 1 + 0.05 = 1 / 1.05 = 0.9523. The replacement period for the machine is:

Year of Service X	*Maintenance Cost C_i (Rs)*	*Discount Factor d^{i-1}*	*Discounted Maintenance Cost $C_i \times d^{i-1}$ Rs.*	*Total cost $A + \sum_{i=1}^{x} C_i \times d^{i-1}$ Rs.*	*Cumulative Discount Factor. $\sum_{i=1}^{x} d^{i-1}$*	*Weighted average Annual cost (Rs) G (X) = $\dfrac{A + \sum_{i=1}^{x} C_i \times d^{i-1}}{\sum_{i=1}^{x} d^{i-1}}$*
1	0.00	1.0000	0.00	5000	1.0000	5000.00
2	500	0.9523	476	5476	1.9523	2805.00
3	1000	0.9070	907	6383	2.8593	2232.00
4	1500	0.8638	1296	7679	3.7231	2063.00
5	**2000**	0.8227	1645	9324	4.5458	**2051.00 (Replace)**
6	**2500**	0.7835	1959	11283	5.3293	2117.00

C_5 = Rs. 2000/- $< G_5 <$ Rs. 2051.00 $< C_6$ = Rs. 2500.00. Hence the machine is to be replaced at the end of 5 th year.

Problem 7.19.

A new vehicle costs Rs. 6000/-. The running cost and the salvage value at the end of the year is given below. If the interest rate is 10 % per year and the running costs are assumed to have occurred at mid of the year, find when the vehicle is to be replaced by new one.

Year:	*1*	*2*	*3*	*4*	*5*	*6*	*7*
Running cost (Rs):	1200	1400	1600	1800	2000	2400	3000
Salvage value (Rs):	4000	2666	2000	1500	1000	600	600

Solution

As interest rate is 10 % the discount factor = $d = 1 / 1 + i$ = 1 / 1 + 0.10 = 1/ 1.1 = 0.9091. In this problem it is given that the running costs are assumed to be occurr in the middle of the year. Hence to discount them at the start of the year, we have to multiply $d^{½} = \sqrt{0.9091} = 0.95346$. Rest of the calculations is made as we have done in previous problems.

Replacement period of the vehicle:

Yr Y(t)	*Salvage Value (S) Rs.*	*Runn-ing cost u(t) Rs.*	*Running Cost at The start of the Period d ½ u (t) Rs.*	d^{i-1}	*d*	*discounted Running cost u (t)* d^{i-1} *Rs.*	*Discoun-ted Salvage Value* $S\,d = S_1$	$A + \sum u$ *(t)* d^{i-1} *= M (y) Rs.*	*M(y)* S_1	$\sum d$	G_x
1	4000	1200	1144.20	1.0000	0.9091	1144.20	3636.4	7144.20	3507.80	0.9091	3859.00
2	2666	1400	1334.80	0.9091	0.8264	1211.60	2203.20	8335.80	6152.60	1.7355	3544.20
3	2000	1600	1525.60	0.8264	0.7513	1260.80	1502.60	9616.60	8114.00	2.4868	3262.60
4	1500	1800	1716.20	0.7513	0.6830	1289.20	1024.50	10905.80	9881.30	3.1698	3117.20
5	1000	2000	1907.00	0.6830	0.6209	1302.40	620.90	12208.20	11587.30	3.7907	3056.60
6	600	**2400**	2288.40	0.6209	0.5645	1420.80	338.70	13629.00	13290.00	4.3552	3051.60 **(Replace)**
7	600	3000	2860.40	0.6545	0.5132	1614.70	307.9	15243.70	14935.80	4.8684	3067.90

C_6 = Rs. 2400/- < G_6 = Rs. 3051.60. Here one condition is satisfied. The vehicle may be replaced at the end of 6th year.

Problem 7.20.

A manufacturer is offered 2 machines *A* and *B*. *A* is priced at Rs. 5000/- and running costs are estimated at Rs. 800/- for each of the first five years and increasing there by Rs. 200/- per year in the sixth and subsequent years. Machine *B*, which has the same capacity as *A* with Rs. 2500/- but will have running costs of Rs.1200 per year for 6 years, and increasing by Rs. 200/- per year thereafter. If money is worth 10 % per year, which machine should be purchased? Assume that the scrap value for both machines is negligible.

Solution

Machine A: As i = 10%, d = 1/1.1 = 0.9091 and A = Rs. 5000/-

Year of Service X	*Maintenance Cost C_i (Rs)*	*Discount Factor d^{i-1}*	*Discounted Maintenance Cost $C_i \times d^{i-1}$ Rs.*	*Total cost $A + \sum_{i=1}^{x} C_i \times d^{i-1}$ Rs.*	*Cumulative Discount Factor. $\sum_{i=1}^{x} d^{i-1}$*	*Weighted average Annual cost (Rs) G (X) =* $\dfrac{A + \sum_{i=1}^{x} C_i \times d^{i-1}}{\sum_{i=1}^{x} d^{i-1}}$
1	800	1.0000	800	5800	1.0000	5800.00
2	800	0.9091	727	6527	1.9091	3418.88
3	800	0.8264	661	7188	2.7355	2627.67
4	800	0.7513	601	7789	3.4868	2233.85
5	800	0.6830	546	8335	4.1698	1998.89
6	1000	0.6209	621	8956	4.7907	1869.45
7	1200	0.5645	677	9633	5.3552	1798.81
8	1400	0.5132	718	10351	5.8684	1763.85
9	**1600**	0.4665	746	11097	6.3349	**1751.72 (Replace)**
10	**1800**	0.4241	763	11860	6.7590	1754.70

C_9 = Rs. 1600 < G_9 = Rs. 1751.72 < C_{10} = 1800. The machine A is to be replaced at the end of 9th year.

Machine B: A = Rs. 2500/-, d = 0.9091

Year of Service X	Maintenance Cost C_i (Rs)	Discount Factor d^{i-1}	Discounted Maintenance Cost $C \times d^{i-1}$ Rs.	Total cost $A + \sum_{i=1}^{x} C_i \times d^{i-1}$ Rs.	Cumulative Discount Factor. $\sum^{x} d^{i-1}$	Weighted average Annual cost (Rs) G (X) = $\dfrac{A + \sum_{i=1}^{x} C_i \times d^{i-1}}{\sum^{x} d^{i-1}}$
1	1200	1.0000	1200.00	3700.00	1.0000	3700.00
2	1200	0.9091	1090.91	4790.91	1.9091	2509.51
3	1200	0.8264	991.98	5782.59	2.7355	2113.91
4	1200	0.7513	901.56	6684.15	3.4868	1916.99
5	1200	0.6830	819.60	7503.75	4.1698	1799.55
6	1200	0.6209	745.08	8248.83	4.7907	1721.84
7	1400	0.5645	790.30	9039.13	5.3552	1687.92
8	**1600**	0.5132	821.12	9860.25	5.8684	**1680.23 (Replace)**
9	**1800**	0.4665	839.70	10699.95	6.3349	1689.25
10	2000	0.4241	848.20	11548.15	6.7590	1708.56

C_8 = Rs. 1600/- < G_8 = Rs. 1680.23 < C_9 = Rs. 1800/- Replace the machine B at the end of 8 the year.

The fixed annual payment for machine A is $(1 - d) / (1 - d^n)$ × total cost of 9th year =

$\{(1 - 0.9091) / (1 - 0.9091^9)\} \times 11097$ = Rs. 1847/-

The fixed annual payment for machine B is = $\{(1- 0.9091) / 1- 0.9091)^8\}$ = Rs. 1680/-

As annual fixed payment for machine B is less than that of A, purchase machine B instead of machine A. Alternatively weighted average cost I 9 years for machine A is Rs. 1781.72 and that for B in 8 years is

Rs. 1680.23, which is lowest, hence to purchase, machine B.

COMPARING OF REPLACEMENT ALTERNATIVES BY USING CRITERIA OF PRESENT VALUE

Some times a business man may come across a problem, when he want to purchase a machine or a vehicle for his business. He may have different models with comparatively equal price and with different maintenance and other expenses. In such cases, he can use **present worth concept** to select the right type of machine or a vehicle. **Here the present value of all future expenditures and revenues is calculated for each alternative and the one for which the present value is minimum is preferred.** Let

Q = Annual cost or purchase price in Rs.

i = Annual interest rate.

t = Period in years.

P = Principal amount in Rs.

S = Scrap value or salvage value in Rs.

The present value of total cost is given by:

P + Q (Pwfs for i % interest rate for n years) – S (*Pwf* for i % interest for n years)

Similarly, if the annual operating costs vary for different years then the present value of these costs will be calculated on the basis of time period for which these expenditures are made. *i.e.* say for example, if Q_1, Q_2, Q_3……Q_n are the operating costs for different years, then the present value of the operating cost during 'n' years will be given by:

Q_1 (Pwfs at i % interest for 1 year) + Q_2 (*Pwfs* at i % for 2 years) ....+ Q_n (*Pwfs* at i % interest for n years).

Now present value of total cost will be:

P + Q_1 (Pwfs at i % interest for 1 year) + Q_2 (Pwfs at i % interest rate for 2 years) + Q_3 (*Pwfs* at i % rate of interest for 3 years) + ...+ Q_n (*Pwfs* at i% interest for n years) – S (*Pwf* at i % interest for n years).

Problem 7.21.

An entrepreneur is considering purchasing a machine for his factory. The related data for alternative machines are as follows:

	Machine A	*Machine B*	*Machine C*
Present investment in Rs.:	10000	12000	15000
Total annual cost in Rs.:	2000	1500	1200
Life of machine in years:	10	10	10
Salvage value in Rs.:	500	1000	1200

As an advisor of the company, you have been asked to select the best machine considering 12 % normal rate of return per year. Given that:

Present worth factor series @ 12 % for 10 years is 5.650

Present worth factor @ 12 % for 10 th year is 0.322

Solution

	Machine A	*Machine B*	*Machine C*
1. Present investment in Rs:	10000	12000	15000
2. Total annual cost in Rs.	2000 × 5.650= 11300	1500 × 5.650= 8475	1200 × 5.650= 6780
3. Present values of Salvage value in Rs.	500 × 0.322= 161	1000 × 0.322 = 322	1200 × 0.322= 386.40
4. Total cost = 1 + 2 – 3	10000 + 11300 – 162 = Rs. 21139.00	12000 + 8475 – 322 = Rs. **20153.00**	15000 + 6780 – 386.40 = Rs. 21393.60

Machine B is having less present value of total cost. Hence to purchase the machine B.

Problem 7.22.

A company is considering purchasing a new grinder, which will cost Rs. 10000/-. The economic life of the machine is expected to be 6 years. The salvage value of the machine will be Rs. 2000/-. The average operating and maintenance costs are estimated to be Rs. 5000/- per annum.

(a) Assuming an interest rate of 10 %, determine the present value of future cost of the proposed grinder.

(b) Compare this grinder with the presently owned grinder that has an annual operating cost of Rs. 4000/- per annum and expected maintenance cost of Rs. 2000/-in the second year with an annual increase of Rs. 1000/- thereafter.

Solution

(a) Present value of annual operating costs = Rs. 5000 × Pwfs at 10 % interest for 6 years.
= Rs. 5000 × 4.355 = Rs. 21775/-
Present value of the salvage value = Rs. 2000 × Pwf at 10 % for 6 years = 2000 × 0.5646 = Rs. 1129/Present value of total future costs = Rs. 21775/- – Rs. 1129/- = Rs. 20646/-

(b) As the annual operating and maintenance costs vary with time. The present value is calculated as follows:

Year 1	*Operating Cost in Rs.* 2	*Maintenance Cost in Rs.* 3	*Total of Operating* And maintenance Costs. 4	*Pwf for single Payment at 10%* Rate. 5	*Present value in Rs.* 6 = 4 × 5
1	4000	--	4000	0.9091	3636
2	4000	2000	6000	0.8264	4956
3	4000	3000	7000	0.7513	5257
4	4000	4000	8000	0.6830	5464
5	4000	5000	9000	0.6209	5589
6	4000	6000	10000	0.5645	5640
				Total: Rs.	30542

Total present value is Rs. 30542/-. Now we can see that the present value of a new grinder is Rs. 20646/-

(Refer part a). Hence the cost saving if a new grinder is purchased is = Rs. 30542/- – Rs. 20646/- = Rs. 9896/-

The cost of the grinder is Rs. 10000/-. But the cost savings is only Rs. 9896/-. The management is advised not to purchase the new grinder.

Problem 7.23.

A manual stamping machine currently valued at Rs. 1000/- is expected to last 2 years and costs Rs 4000/- per year to operate. An automatic stamping machine, which can be purchased for Rs. 3000/-, which last for 4 years and can be operated at an annual cost of Rs. 3000/-. If money carries the rate of interest 10 % per annum, determine which stamping machine is to be purchased?

Solution

Present worth factor = d = 100 / (100 + 10) = 0.9091.

The given stamping machines have different expected lives. So, we shall consider a span of four years during which we have to purchase either two manual stamping machines (the second one is purchased after three years, *i.e*. at the beginning of third year) or one automatic stamping machine.

The present worth of investments of the two manual stamping machines used in 4 years is approximately,

$1000 \times (1 + d^2) + (1 + d + d^2 + d^3) = 1000\ \{(1 + (0.9091)^2\} + 4000 + 4000\ \{0.9091 + 9091)^2 + 9091^3\} =$

= 1000 (1 + 0.8264) + 4000 + 4000 (0.9091 + 0.8264 + 0.7513) = Rs. 1926 + 13947 =

Rs. 15773/-

And the present worth of investments on the automatic stamping machine for the next four years = Rs. 3000 + 3000 $(1 + d + d^2 + d^3)$ = Rs. 3000 + 19469 = Rs. 13460/-

Since the present worth of future costs for the automatic stamping machine is less than that of manual stamping machines, management may be advised to purchase an automatic stamping machine.

Problem 7. 24.

A pipeline is due for repairs. It will cost Rs. 10000/- and lasts for 3 years. Alternatively, a new pipeline can be laid at a cost of Rs. 30000/- and lasts for 10 years. Assuming the cost of capital to be 10 % and ignoring salvage value, which alternative should be chosen?

Solution

The present worth factor is d = 10 / (10 + 1) = 10 / 11 = 0.9091

Considering the 10 year replacement of new pipeline and 3 year replacement of existing pipeline, if M_1 is discounted value of all future costs associated with a policy of replacing the equipment after 'n' years then initial outlay is (taking C as annual maintenance cost) as $T_n = C + d^n C + d^{2n} C + d^{3n} C + \ldots..$

$= C\ (1 + d^n + d^{2n} + d^{3n} + \ldots\ldots. = C / (1 - d^n)$

Now substituting the values of d, n, and C for two types of pipelines; the discounted value for the existing pipeline is $T_3 = 10000 / \{1 - (0.9091)^3\}$ = Rs. 4021/-.

For new pipeline $T_{10} = 30000 / \{1 - (0.9091)^{10} = 30000 / (1 - 0.3855)$ = Rs. 48820/-

As T_3 is < T_{10} the existing pipeline may be continued.

REPLACEMENT OF ITEMS THAT FAIL COMPLETELY AND SUDDENLY AND ARE EXPENSIVE TO BE REPLACED

There are certain items or systems or products, whose probability of failure increases with time. They may work with designed efficiency throughout their life and if they fail to act they fail suddenly. The nature of these items is they are costly to replace at the same time and their failure affect the functioning of entire system. For example, resistors, components of air conditioning unit and certain electrical components. If we do not replace the item immediately, then loss of production, idle labour; idle raw materials, etc are the results. It is evident failure of such items causes heavy losses to the organization. Such situations demand the formulation of a policy, which will help the organization to avoid losses.

sometimes we find it is better to replace the item before it fails so that the expected losses due to failure can be avoided. The following courses of action can be followed:

(a) Individual replacement policy

This policy states that replace the item soon after its failure. Here the cost of replacement will be somewhat greater as the item is to be purchased individually from the seller as and when it fails. From the time of failure to the replacement, the system remains idle. More than that, as the item is purchased individually, the cost of the item may be more. In case, the component or the item is not available in the local market, we have to get it from other places, where the procurement cost may be higher for individual purchase. If the management wants to adopt this policy, it may have to waste its time and money also the losses due to failure.

(b) Group replacement policy

If the organization has got the statistics of failure of the item, it can calculate the average life of the item and replace the item before it fails, so that the system can work without break. In this case, all the items, even they are in good working condition, are replaced at a stipulated period as calculated by the organization by using the group replacement policy. One thing we have to remember is that, in case any item fails, before the calculated group replacement period, it is replaced individually immediately after failure. ***Hence this policy utilizes the strategy of both individual replacement and group replacement.***

The probability distribution of the failure of the item in a system can be determined by mortality tables for life testing techniques. Let us try to understand what a mortality table is.

Mortality Tables

The ***mortality theorem*** states that a large population is subjected to a given mortality law for a very long period of time. All deaths are immediately replaced by births and there are no other entries or exits. Here age distribution ultimately becomes stable and that the number of deaths per unit of time becomes constant, which is equal to the size of the total population divided by the mean age at death.

If we consider the problem of human population, no group of people ever existed under the conditions that:

(a) That all deaths are immediately replaced by births.

(b) That there are no other entries or exists.

These two assumptions help to analyze the situation more easily, by keeping virtual human population in mind. When we consider an industrial problem, deaths refer to item failure of items or components and birth refers to replacement by a new component.

Mortality table for any item can be used to derive the probability distribution of life span. If $M(t)$ represents the number of survivors at any time 't' and $M(t-1)$ is the number of survivors at the time $(t-1)$, then the probability that any item will fail in this time interval will be:

$$\{M(t-1) - M(t)\} / N, \qquad \ldots(1)$$

where N is the number of items in the system.

Conditional probability that any item survived up to age $(t-1)$ will die in next period, will be given by:

$$\{M(t-1) - M(t)\} / M(t-1) \qquad \ldots(2)$$

Problem 7.25.

Calculate the probability of failure of an item in good condition in each month from the following survival table:

Month Number: (t): Original number	0	1	2	3	4	5	6	7	8	9	10
Of items working At the end of each Year:	1000	940	820	580	400	280	190	130	70	30	0

Solution

Here 't' is the number of month; $M(t)$ is the number of items *i.e.* items in good condition at the end of t^{th} month.

The probability of failure in each month is calculated as under:

Year (t)	*Items in good Condition M (t)*	*Probability of items that fail in t th year =* $\{M(t-1) - M(t)\}/N$
0	1000	----
1	940	(1000 – 940) / 1000 = 0.06
2	820	(940 – 820) / 1000 = 0.12
3	580	(820 – 580) / 1000 = 0.24
4	400	(580 – 400) / 1000 = 0.18
5	280	(400 – 280) / 1000 = 0.12
6	190	(280 – 190) / 1000 = 0.09
7	130	(190 – 130) / 1000 = 0.06
8	70	(130 – 70) / 1000 = 0.06
9	30	(70 – 30) / 1000 = 0.04
10	0	(30 – 00) / 1000 = 0.03

Group Replacement of Items

A Group replacement policy consists of two steps. Firstly, it consists of individual replacement at the time of failure of any item in the system and there is group replacement of existing live units at some suitable time. Here the individual replacement at the time of failure ensures running of the system, whereas group replacement after some time interval will reduce the probability of failure of the system. The application of such type of policy has to take into consideration the following points: (*a*) The rate of individual replacement during the period and (*b*) The total cost incurred due to individual and group replacement during the period chosen. This policy is favors the group replacement, when the total cost is minimum and the period of replacement is known as optimal period of replacement. The information required to formulate this policy is: (*a*) Probability of failure,

(*b*) Losses due to these failures, (*c*) Cost of individual replacement, and (*d*) Cost of group replacement. The procedure is explained in the worked examples.

The group replacement policy states that: Group replacement should be made at the end of 'i' th period, if the cost of individual replacement for 'i' th period is greater than average cost per period by the end of the period 't' and one should not adopt a group replacement policy if the cost of individual replacement at the end of (t - 1) th period is not less than the average cost per period through time (t – 1).

Problem 7.26.

A system consists of 10000 electric bulbs. When any bulb fails, it is replaced immediately and the cost of replacing a bulb individually is Re.1/- only. If all the bulbs are replaced at the same time, the cost per bulb will be Rs. 0.35. The percent surviving *i.e.* $S(t)$ at the end of month 't' and $P(t)$ the probability of failure during the month 't' are as given below. Find the optimum replacement policy.

t in months:	*0*	*1*	*2*	*3*	*4*	*5*	*6*
$S(t)$:	100	97	90	70	30	15	0
$P(t)$:	---	0.03	0.07	0.20	0.40	0.15	0.15

Solution

The problem is to be solved in two stages: (*i*) Policy of individual replacement and (*ii*) Policy of group replacement.

As per the given data, we can see that no bulb will survive for more than 6 months. That is a bulb, which has survived for 5 months, is sure to fail during the sixth month. Though we replace the failed bulb immediately, it is assumed that the bulb fails during the month will be replaced just at the end of the month.

Let N_i = Number of bulbs replaced at the end of i th month, then we can calculate different values of N_i

N_0 = Number of bulbs at the beginning = 10000

N_1 = Number of bulbs replaced at end of first month = Number of bulbs at the beginning × Probability that a bulb fails during 1st month of installation = 10000 × 0.03 = **300**

N_2 = Number of bulbs to be replaced at the end of second month = (Number of bulbs at the beginning × probability of failure during the second month) + (Number of bulbs replaced at the end of second month × Probability of failure during the second month) = $N_0P_2 + N_1P_1$ = (10000 × 0.07) + (300 × 0.03) = **709.** Similarly,

$N_3 = N_0P_3 + N_1P_2 + N_2P_1$ = 10000 × 0.20 + 300 × 0.07 + 709 × 0.03 = **2042.**

$N_4 = N_0P_4 + N_1P_3 + N_2P_2 + N_3P_1$ = 10000 × 0.40 + 300 × 0.20 = 709 × 0.07 + 2042 × 0.03 = **4171.**

$N_5 = N_0P_5 + N_1P_4 + N_2P_3 + N_3P_2 + N_4P_1$ = 10000 × 0.15 + 300 × 0.40 + 709 × 0.20 + 2042 × 0.07 + 4171 × 0.03 = **2030.**

$N_6 = N_1P_5 + N_2P_4 + N_3P_3 + N_4P_2 + N_5P_1$ = 10000 × 0.15 + 300 × 0.15 + 709 × 0.40 + 2042 × 0.20 + 4171 × 0.07 + 2030 × 0.03 = **2590.**

From the above we can see that the failures increases from 300 to 4171 in 4th month and then decreases. It is very much common in any system that failure rate increases and then decreases and finally after certain period, it stabilizes, when the system attains steady state.

Now let us work out the expected life of each bulb which is = $\Sigma\, x_i\, P_i$, where x_i is the month and P_i is corresponding probability of failure.

$\Sigma\, x_i\, P_i = 1 \times 0.03 + 2 \times 0.07 + 3 \times 0.20 \times 4 \times 0.40 + 5 \times 0.15 + 6 \times 0.15 =$ **4.02 months.**

If the average life of a bulb is 4.02 months, the average number of replacements every month = Number of bulbs in the system / average life of the bulb. = 10000 / 4.02 = **2488 bulbs**. As the cost of individual replacement cost is Re.1/- per bulb, on an average, the organization has to spend 2488 × Re.1/- = Rs. 2488/- per month.

Now let us work the cost of group replacement:

End of the Period	*Total cost of group replacement in Rs. Individual replacement cost + group Replacement cost.*	*Cost per monthTotal cost/period. Rs./ month.*
1	300 × 1 + 10000 × 0.35 = 3800	3800 / 1 = 3800
2	(300 + 709) × 1 + 10000 × 0.35 = 4509	4509 / 2 = 2254.50
3	(300 + 709 + 2042) × 1 + 10000 × 0.35 = 6551	6551 / 3 = **2183.66**
4	(300 + 709 + 2042 + 4771) × 1 + 10000 × 0.35 = 10722	10722 / 4 = 2680.50
5	(300+ 709 + 2042 + 4771 + 2030) × 1 + 10000 × 0.35 = 12752	12752 / 5 = 2550.40
6	(300 + 709 + 2042 + 4771 + 2030 + 2590) × 1 + 10000 × 0.35 = 15342	15342 / 6 = 2557.00

In the above table in column No. 2 it is shown the cost of individual replacement at the end of month plus group replacement of all the bulbs at the end of month.

The minimum cost of group replacement *i.e.* Rs. 2183.66 is at the end of **third month.** This is compared with individual replacement cost per month, which is Rs. 2488/- per month. Hence replacement of all the bulbs at the end of third month is more beneficial to the organization. **Hence optimal replacement policy is replace all the bulbs at the end of third month.**

Problem 7.27.

Truck tyres, which fail in service, can cause expensive accidents. It is estimated that a failure in serviced results in an average cost of Rs.1000/- exclusive of the cost of replacing the burst tyre. New tyres cost Rs. 400/- each and are subject to mortality as in table on next page. If the tyres are to be replaced after a certain fixed mileage or on failure (which ever occurs first), determine the replacement policy that minimizes the average cost per mile. Mention the assumptions you made to arrive at the solution.

Table showing the Truck Tyre Mortality.

Age of tyre at failure (Miles)	*Proportion of tyre.*
≤ 10000	0.000
10001 – 12000	0.020
12001 – 14000	0.035
14001 – 16000	0.063
16001 – 18000	0.100
18001 – 20000	0.220
20001 – 22000	0.345
22001 – 24000	0.205
24001 – 26000	0.012
Total	1.000

Solution

Assumptions: (*i*) The failure occurs at the midpoint of the range given in the table *i.e.* exactly at 11000, 13000, 15000 … etc.

(ii) Initially let there be 1000 tyres.

(iii) Up to the age of 10000 miles proportion of tyres fails = 0. The proportion of failure from 11000 to 13000 miles is 0.030. Assume that in this period the average cost of maintaining is Rs. 1000 /-. If in this period a tyre bursts, the cost will be Rs. 1400/-. Thus the cost of individual replacement will be Rs, 1400/-. The cost of group replacement is given as Rs. 400/- per tyre. Hence the numbers of tyres fail and are to be replaced is:

$N_0 = 1000.$

$N_1 = N_0 P_1 = 1000 \times 0.020 =$ **20.**

$N_2 = N_0 \times P_2 + N_1 P_1 = 1000 \times 0.035 + 20 \times 0.020 = 35 + 0.04 = 35.04 =$ Approximately **35.**

$N_3 = N_0 \times P_3 + N_1 P_2 + N_2 P_1 = 1000 \times 0.063 + 20 \times 0.035 + 35 \times 0.02 = 63 + 0.7 + 0.7 = 64.4 =$ **64**

$N_4 = N_0 \times P_4 + N_1 \times P_3 + N_2 \times P_2 + N_3 \times P_1 = 1000 \times 0.1 + 20 \times 0.063 + 35 \times 0.035 + 64 \times 0.02 = 100 + 1.26 + 1.23 + 1.28 = 103.73 =$ **104**

$N_5 = N_0 \times P_5 + N_1 \times P_4 + N_2 \times P_3 + N_3 P_2 + N_4 \times P_1 = 1000 \times 0.220 + 20 \times 0.100 + 35 \times 0.063 + 64 \times 0.035 + 104 \times 0.020 = 220 + 2 + 2.205 + 2.24 + 2.08 = 224.205 =$ **App 224**

$N_6 = N_0 \times P_6 + N_1 \times P_5 + N_2 \times P_4 + N_3 \times P_3 + N_2 + P_2 + N_1 \times P_1 = 1000 \times 0.345 + 20 \times 0.220 + 35 \times 0.1 + 64 \times 0.063 + 104 \times 0.02 = 345 + 4.4 + 3.5 + 4.032 + 3.64 + 4.48 = 714.45 =$ **App. 714.**

$N_7 = N_0 \times P_7 + N_1 \times P_6 + N_2 \times P_5 + N_3 \times P_4 + N_4 \times P_3 + N_5 \times P_2 + N_6 \times P_1 = 1000 \times 0.205 + 20 \times 0.345 + 35 \times 0.220 + 64 \times 0.1 + 104 \times 0.063 + 224 \times 0.035 + 714 \times 0.02 = 205 + 7.3 + 7.7 + 6.4 + 6.24 + 7.84 + 14.28 = 254.76 =$ **App. 255.**

$N_8 = N_0 \times P_8 + N_1 \times P_7 + N_2 \times P_6 + N_3 \times P_5 + N_4 \times P_4 + N_5 \times P_3 + N_6 \times P_2 + N_7 \times P_1 = 1000 \times 0.012 + 20 \times 0.205 + 35 \times 0.345 + 64 \times 0.220 + 104 \times 0.10 + 224 \times 0.063 + 714 \times 0.035 + 255 \times 0.02 = 12 + 4.1 + 12.075 + 14.08 + 10.4 + 14.112 + 24.99 + 5.1 = 96.857 =$ **App. 97**

Expected average life of a tyre is: $\sum_{i=1}^{i=\inf} i\, p_i$ i = time period, p is the probability of failure.

$\Sigma\, i\, p_i$ = 1 × 0.020 + 2 × 0.035 + 3 × 0.063 + 4 × 0.100 + 5 × 0.220 + 6 × 0.345 + 7 × 0.205 + 8 × 0.012 =

= 0.02 + 0.07 + 0.189 + 0.40 + 1.1 + 2.07 + 1.435 + 0.096 = **5.38**

Average number of failures = (1000 / 5.38) = 185.87 = App. 186 tyres.

Cost of individual replacement per period = 186 × 1400 = 260400/-

Cost of individual replacement per mile = (234 × 1400) / 2000 = 260400 / 2000 = **Rs. 130.20**

Replacement by Group replacement policy:

End of the Period in miles	*Total cost of group replacement = Individual replacement + group replacement (Rs)*	*Average cost per Period in miles. (Rs)*
11000 - 13000 (1)	1000 × 400 + 20 × 1400 = 428000	428000
13000 - 15000 (2)	(1000 + 20) × 400 + 35 × 1400 = 408000 + 49000 = 457000	457000/2 = 228500
15000 - 17000 (3)	(1000 + 20 + 35) × 400 + 64 × 1400 = 422000 + 89600= 511600	511600/3 = 170533
17000 - 19000 (4)	(1000 + 20 + 35 + 64) × 400 + 104 × 1400 = 447600 + 145600 = 593200	593200 / 4 = 148300
19000 - 21000 (5)	(1000 + 20 + 35 + 64 + 104) × 400 + 224 x 1400 =489200 + 313600 = 802800	802800 / 5 = 160560
21000 - 23000 (6)	(1000 + 20 + 35 + 64 + 104 + 224) × 400 + 714 × 1400 = 578800 + 996600 = 1598400	1598400 / 6 = 263066
23000 - 25000 (7)	(1000 + 20 + 35 + 64 + 104 + 224 + 714) × 400 + 255 × 1400 =864400 + 357000 = 1221400	1221400 / 7 = 174485.70
25000 - 27000 (8)	(1000 + 20 + 35 + 64 + 104 + 224 + 714 +255) × 400 + 97 × 1400 =966400 + 135800 = 1102200	1102200 / 8 = 137775

Now we know that the individual replacement cost is Rs. 130.20. Where as even at the end of 8 the period, the average cost per period in miles is 137775 / 1000 = Rs. 137.775 which is higher than Rs. 130.02. Hence it is better to stick to individual replacement policy.

Problem 7.28.

The following failure rates have been observed for a certain type of light bulb.

Week:	1	2	3	4	5
Percent failing					
By the end of					
The week:	10	25	50	80	100

There are 1000 bulbs in use, and it costs Rs.2/- to replace an individual bulb, which is burnt out. If all bulbs were replaced simultaneously it would cost 50 paise per bulb. It is proposed to replace all bulbs at fixed intervals of time, whether or not they burnt out, and to continue replacing burnt out bulbs as and when they fail. At what intervals all the bulbs should be replaced? At what group replacement price per bulb would a policy of strictly individual replacement become preferable to the adopted policy?

Solution

Let us first work out the probability of failures. Let p_i is the probability that a light bulb which was new when placed in position for use, fails during the i th period of its life. Then,

$p_1 = 10/100 = 0.01$, $p_2 = (25 – 10) / 100 = 0.15$, $p_3 = (50 - 25) / 100 = 0.25$, $p_4 = (80 – 50) / 100 = 0.30$ and $p_5 = (100 – 80) / 100 = 0.20$.

Sum of probabilities = 0.10 + 0.15 + 0.25 + 0.30 + 0.20 = 1.00

A bulb, which has worked for four weeks, has to fail in the fifth week.

It is assumed that the bulbs that fail during the week are replaced just before the end of that week and the actual percentage of failures during a week for a sub population of bulbs with the same age is the same as the expected percentage of failures during the week for that sub population.

(i) Individual replacement policy:

Mean age of bulbs is = $1 \times p_1 + 2 \times p_2 + 3 \times p_3 + 4 \times p_4 + 5 \times p_5 =$

$1 \times 0.10 + 2 \times 0.15 + 3 \times 0.25 + 4 \times 0.30 + 5 \times 0.20 = 3.35$ weeks.

The number of failures in each week in steady state is given by 1000 / 3.35 = 299

Hence cost of replacing failed bulbs individually is Rs. 2/- × 299 = Rs. 598/- per week.

(ii) Group replacement policy:

Now we will work to find out the cost of replacing all the bulbs at a time (at a cost of Rs.0.50 per bulb) and at the same time replacing the individual bulbs (replacing at a cost of Rs.2/- per bulb) as and when they fail.

End of The week	*Cost of Individual Replacement in Rs.*	*Cost of group replacement in Rs.*	*Average cost per week in Rs.*
1	100 × 2 = 200	1000 × 2 + 100 × 2 = 500 + 200 = 700.00	700 / 1 = 700.00
2	160 × 2 = 320	1000 × 2 + (100 + 160) × 2 = 500 + 520 = 1020.00	**1020 / 2 = 510.00**
3	281 × 2 = **562**	1000 × 0.50 + (100 + 160 + 281) × 2 = 500 + 1082 = 1582.00	1582 / 3 = 527.33
4	377 × 2 = 754	1000 × 0.50 + (100 + 160 + 281 + 377) × 2 = 500 + 1836 = 2336.00	2336 / 3 = 778.66
5	350 × 2 = 700	1000 × 0.50 + (100 + 160 + 281 + 377 + 350) × 2 = 500 + 2536 = 3036.00	3936 / 4 = 504.00
6	230 × 2 = 460	1000 × 0.50 + (100 + 160 + 281 + 377 + 350 + 230) × 2 = 500 + 2316 = 2816.00	2810 / 5 = 563.20
7	286 × 2 = 572	1000 × 0.50 + (100 + 160 + 281 + 377 + 350 + 230 + 286) × 2 = 500 + 3568 = 4068.00	4068 / 6 = 678.00

As the weekly average cost is minimum at Rs. 510.00, replace all the bulbs at the end of 2-nd week. This is also less than the individual replacement cost *i.e.* Rs. 598/-.

Problem 7.29.

Find the cost per period of individual replacement policy of an installation of 300 bulbs, given the following:

(i) Cost of individual replacement of bulb is Rs. 2/- per bulb.

(ii) Conditional probability of failure of bulbs is as follows:

Weekend:	*0*	*1*	*2*	*3*	*4*
Probability of failure:	0	0.1	0.3	0.7	1.0

Solution

If p_i is the probability of failure of bulbs then:

$p_0 = 0$, $p_1 = 0.1$, $p_2 = 0.3 – 0.1 = 0.20$, $p_3 = 0.7 – 0.3 = 0.4$, and $p_4 = 1 – 0.7 = 0.3$

Since sum of probabilities is unity, all probability higher than p_4 must be zero, *i.e.* a bulb that has been survived up to 4 th week, is sure to fail at the end of fourth week.

Let us find the average life of a bulb, which is given by $\sum_{i=1} i \; x \; p_i$ =

$1 \times 0.1 + 2 \times 0.2 + 3 \times 0.4 + 4 \times 0.3 = 2.9$ weeks.

Average number of failures per week is 300 / 2.9 = 103. 448 = App. 103

Cost of individual replacement is Rs. 2 × 103 = Rs. 206/-

Number of bulbs to be replaced at the end of every week is:

$N_0 = 300$

$N_1 = N_0 \times p_1 = 300 \times 0.1 = 30.$

$N_2 = N_0 \times p_2 + N_1 \times p_1 = 300 \times 0.2 + 30 \times 0.1 = 60 + 3 = 63$

$N_3 = N_0 \times p_3 + N_1 \times p_2 + N_2 \times p_1 = 300 \times 0.4 + 30 \times 0.2 + 63 \times 0.1 = 120 + 6 + 6.3 = 132.3$ = App. 132

$N_4 = N_0 \times p_4 + N_1 \times p_3 + N_2 \times p_2 + N_3 \times p_1 = 300 \times 0.3 + 30 \times 0.4 + 63 \times 0.2 + 132 \times 0.1 =$
$90 + 12 + 12.6 + 13.2 = 127.8$ = App. 128.

The number failures increase up to 3 rd week and then it reduces. This is very much common in all the systems that after some time the system reach steady state.

Problem 7.30.

A typing pool of a large organization employs 100-copy typists. The distribution of length of service is given below:

Duration of employment in years:	1	2	3	4	5 or more.
Proportion of employees that] av in that year of employment.	30%	40%	20%	10%	0%

Assuming that an employee leaving is replaced by another at the end of the year, determine:

(a) The number of staff who leaves in each of the first 8 years of the department's existence, assuming it stared with 100 employees and this total number does not change.

(b) The number leaving each year when the steady state situation is reached, and

(c) The total annual cost of recruiting staff in the steady state if replacement of each new copy typist costs Rs. 200/-

Solution

Note that there are 100 copy - typists in the beginning and no person stays more than five years. Hence let us calculate the number of persons leaving the organization and new employees employed every year.

Year	*Number of employees leaving at the end of the year*	*No.of new Employees Employed.*
0	Nil	$N_0 = 100$
1	$N_0 \times p_1 = 100 \times 0.30 =$	$N_1 = 30$
2	$N_0 \times p_2 + n_1 \times p_1 = 100 \times 04 + 30 \times 0.30 =$	$N_2 = 40 + 9 = 49$
3	$N_0 \times p_3 + N_1 \times p_2 + n_2 \times p_1 = 100 \times 0.2 + 30 \times 0.4 + 49 \times 0.3 =$	$N_3 = 20 + 12 + 14.7 = 46.7$
4	$N_0 \times p_4 + N_1 \times p_3 + N_2 \times p_2 + N_3 \times p_1 = 100 \times 0.1 + 30 \times 0.2 + 49 \times 0.4 + 46.7 \times 0.3 = 10 + 6 + 19.6 + 14.1$	$N_4 = 49.7$
5	$N_0 \times p_5 + N_1 \times p_4 + N_2 \times p_3 + N_3 \times p_2 + N_4 \times p_1 = 100 \times 0 + 30 \times 0.1 + 49 \times 0.2 + 46.7 \times 0.4 + 49.6 \times 0.3 = 0 + 3 + 9.8 + 18.68 + 14.88 = 46.36$	$N_5 = 46.36 = 46.4$
6	$N_0 \times p_6 + N_1 \times p_5 + N_2 \times p_4 + N_3 \times p_3 + N_4 \times p_2 + N_5 \times p_1 = 100 \times 0 + 30 \times 0 + 49 \times 0.1 + 46.7 \times 0.2 + 49.6 \times 0.4 + 46.4 \times 0.3 = 0 + 0 + 4.9 + 9.34 + 19.84 + 13.92 = 48$	$N_6 = 48$
7	$N_0 \times p_7 + N_1 \times p_6 + N_2 \times p_5 + N_3 \times p_4 + N_4 \times p_3 + N_5 \times p_2 + N_6 \times p_1 = 100 \times 0 + 30 \times 0 + 49 \times 0 + 46.7 \times 0.1 + 49.6 \times 0.2 + 46.4 \times 0.4 + 48 \times 0.3 = 0 + 0 + 0 + 4.67 + 9.92 + 18.56 + 14.4 = 47.55$	$N_7 = 47.6$
8	$N_0 \times p_8 + N_1 \times p_7 + N_2 \times p_6 + N_3 \times p_5 + N_4 \times p_4 + N_5 \times p_3 + N_6 \times p_2 + N_7 \times p_1 = 100 \times 0 + 30 \times 0 + 49 \times 0 + 46.7 \times 0 + 49.6 \times 0.1 + 46.4 \times 0.2 + 48 \times 0.4 + 47.06 \times 0.3 = 0 + 0 + 0 + 0 + 4.96 + 9.28 + 19.2 + 14.12 = 47.56$	$N_8 = 47.6$

Expected length of service of a copy typist in the organization = $\Sigma' x_i \times p_i =$

$1 \times 0.3 + 2 \times 0.4 + 3 \times 0.2 + 4 \times 0.1 = 0.3 + 0.8 + 0.6 + 0.4 = 2.1$ years. Hence average number of employees leaving the organization at the end of the year = 100 / 2.1 = 47.62 employees.

Annual cost of replacing a copy typist in steady state = Cost of replacement × average number replaced = Rs. 200 × 47.62 = Rs. 9524/-.

Problem 7.30

The following mortality tables have been observed for a certain type of light bulbs:

End of the week:	*1*	*2*	*3*	*4*	*5*	*6*
Probability of failure due to date:	0.09	0.25	0.49	0.85	0.97	1.00

There are a large number of such bulbs, which are to be kept in working order. If a bulb fails in service, it costs Rs. 3/- to replace but if all bulbs are replaced in the same operation it can be done for only Rs. 0.70 a bulb. It is proposed to replace all bulbs at fixed intervals, whether or not they have burnt out and to continue replacing burnt out bulbs as they fail.

(a) What is the best interval between group replacements?

(b) At what group replacement price per bulb, would a policy of strictly individual replacement become preferable to the adopted policy?

Solution

Now the probability of failure of a bulb be p_i is as given below:

$p_1 = 0.09$, $p_2 = 0.25 – 0.09 = 0.16$, $p_3 = 0.49 – 0.25 = 0.24$, $p_4 = 0.85 – 0.49 = 0.36$, $p_5 = 0.97 – 0.85 = 0.12$

$p_6 = 1.00 – 0.97 = 0.03$.

As the sum of the probabilities of failures is unity all probabilities > p_6 are 0, this says that the bulb survived up to 6 weeks, will definitely fail at the end of 6th week. As per the conditions given in the problem, the bulbs that fail during the week are assumed to be replaced at the end of the week for simplicity, though they are replaced immediately after failure. Let us take that total bulbs in the system is 1000.

Week	*Number of failures per week*	*Number replaced.*
0	$N_0 = 1000$	$N_0 = 1000$
1	$N_1 = N_0 \times p_1 = 1000 \times 0.09 = 90$	$N_1 = 90$
2	$N_2 = N_0 \times p_2 + N_1 \times p_1 = 1000 \times 0.16 + 90 \times 0.09 = 160 + 8 = 168$	$N_2 = 168$
3	$N_3 = N_0 \times p_3 + N_1 \times p_2 + N_2 \times p_1 = 1000 \times 0.24 + 90 \times 0.16 + 168 \times 0.09 = 240 + 14.4 + 15.12 = 269.52$	$N_3 = 270$
4	$N_4 = N_0 \times p_4 + N_1 \times p_3 + N_2 \times p_2 + N_3 \times p_1 = 1000 \times 0.36 + 90 \times 0.24 + 168 \times 0.16 + 270 \times 0.09 = 360 + 21.6 + 26.88 + 24.3 = 432.78$	$N_4 = 433$
5	$N_5 = N_0 \times p_5 + N_1 \times p_4 + N_2 \times p_3 + N_3 \times p_2 + N_4 \times p_1 = 1000 \times 0.12 + 90 \times 0.36 + 168 \times 0.24 + 270 \times 0.16 + 433 \times 0.09 = 120 + 32.4 + 40.32 + 43.20 + 38.97 = 274.89$	$N_5 = 275$
6	$N_6 = N_0 \times p_6 + N_1 \times p_5 + N_2 \times p_4 + N_3 \times p_3 + N_4 \times p_2 + N_5 \times p_1 + N_6 \times p_0$ $=1000 \times 0.03 + 90 \times 0.12 + 168 \times 0.36 + 270 \times 0.24 + 433 \times 0.16 + 275 \times 0.09 =30 + 10.8 + 60.48 + 64.8 + 69.28 + 24.75 = 260.11$	$N_6 = 260$
7		

Expected life of the bulb is equals to sum of the product of period × probability.

$= 1 \times 0.09 + 2 \times 0.16 + 3 \times 0.24 + 4 \times 0.36 + 5 \times 0.12 + 6 \times 0.03 = 0.09 + 0.32 + 0.72 + 1.44 + 0.6 + 0.18 = 3.35$ weeks

Average number of failure per week = 1000 / 3.35 = 299 bulbs.

Cost of individual replacement is Rs.3/- × 299 = **Rs. 897/-**

Cost of group replacement :

End of The Week	*Cost of group replacement in Rs.*	*Average cost per Week.*
1	1000 × 0.70 = 700	700/1 = 700.00
2	1000 × 0.70 + 90 × 3 = 700 + 270 = 970	970 / 2 = **485.00**
3	1000 × 0.70 + (90 × 168) × 3 = 700 + 774 = 1474	1474 / 3 = 491.33
4	1000 × 0.70 + (90 + 168 + 270) × 3 = 2281	2281 / 4 = 570.25
5	1000 × 0.70 + (90 + 168 + 270 + 433) × 3 = 3583	3583 / 5 = 716.60
6	1000 × 0.70 + (90 + 168 + 270 + 433 + 275) × 3 = 4408	4408 / 6 = 734.66

(*a*) We see that the group replacement cost at the end of 2econd week is minimum and is Rs. 485/-. This is also less than the individual replacement cost of Rs. 897/- Hence Group replacement at the end of second week is recommended.

(*c*) Let Rs. c/- be the group replacement price per bulb. Then the individual replacement cost of Rs. 897/- must be < (1000 × *c* + 3 × 90) / 2.

By simplifying the value of c is Rs. 1. 52. At price per bulb RS. 1.52 the policy of replacing all the bulbs at the end of second week will become strictly individual replacement policy.

Problem 7. 31.

A unit of electrical equipment is subjected to failure. The probability of distribution of the age at failure is as follows:

Age at failure (weeks):	*2*	*3*	*4*	*5*
Probability:	0.2	0.4	0.3	0.1

Initially 10000 new units are installed and a new unit replaces any unit, which fails, at the end of the week in which it fails.

(a) Calculate the expected number of units to be replaced in each of weeks 1 to 7. What rate of failure can be expected in the long run?

(b) Among the 10000 installed units at the start of week 8, how many can be expected to be aged zero week, 1 week, 2 weeks, 3 weeks or 4 weeks? Compare this with the expected frequency distribution in long run.

(c) Replacement of individual units on failure costs Rs. 0.05 each. An alternative policy is to replace all units after a fixed number of weeks at a cost of Rs. 300/- and to replace any unit failing before the replacement week at the individual cost of 5 paise each. Would this preventive policy be adopted? If so, after how many weeks should all units be replaced?

Solution

End of week	*Failures*	*Number replaced.*
1	—	0
2	0.2 × 10000	2000
3	0.4 × 10000	4000
4	0.3 × 10000 + 0.2 × 2000	3000 + 400 = 3400
5	0.1 × 10000 + 0.2 × 4000 + 0.4 × 2000	1000 + 800 + 800 = 2600
6	0.2 × 3400 + 0.4 × 4000 + 0.3 × 2000	680 + 1600 + 600 = 2880
7	0.2 × 2600 + 0.4 × 3400 + 0.3 × 4000 + 0.1 × 2000	520 + 1360 + 1200 + 200 = 3280

Mean life at failure is given by week × probability = 2 × 0.2 + 3 × 0.4 + 4 × 0.3 + 5 × 0.1 = 3.3 weeks.

Hence average rate of failure in the long run = 10000 / 3.3 = 3030 units per week.

(*b*) Expected frequency distribution of ages at the beginning of 8th week:

Age in weeks		*Number of items.*
0		3280
1		2880
2	0.8 × 2600	2080
3	{1 – (0.2 + 0.4)} × 3400 =	1360
4	{1 – (0.2 + 0.3 + 0.4)} × 4000	400
	Total =	10000

Now, as 3030 units are replaced on the average each week, the expected number of units at any time having age 0 to one week is 3030 each. The long run expected age distribution is, therefore, given by:

Age in weeks		*Number of items.*
0		3030
1		3030
2	(1 – 0.2) × 3030 =	2424
3	{1– (0.2 + 0.4)} × 3030	1213
4	{1 – (0.2 + 0.3 + 0.4)} × 3030	303
	Total:	10000

(*d*) With individual replacement, the average replacement cost is:

Rs. (3030 × 0.05) = Rs. 151.50 per week.

Group replacement policy:

If there is a group replacement once every two weeks, there will be no individual replacements, and the weekly average replacement cost is Rs. 300 / 2 = Rs. 150/-. Therefore, it will be worth to adopt a group replacement policy as shown below:

Once in every three weeks: Individual replacement cost = 2000 × Rs. 0.05 = Rs. 100/-Average cost = {(Rs. 100 + Rs. 300)} / 3 = Rs. 133.33 per week.

Once in every 4 weeks: Individual replacement cost = (2000 + 4000) × Rs. 0.05 = Rs. 300/- Average cost = (*s*. 300 + Rs. 300) / 4 = Rs. 150 /- per week.

Therefore, the minimum cost replacement policy is group replacement every three weeks at a cost of Rs. 133.33 per week.

STAFFING PROBLEM

The replacement model may be well applied to **manpower planning**, where one can plan well in advance the requirement of different types of staff personnel or skilled / unskilled personnel. Here personnel are also considered as elements replaced for some reason or the other. Any organization requires at various period of time different types of personnel due to retirement, persons quitting the job in search of better jobs, vacancies arising due to death of personnel, termination, resignation etc. Therefore to maintain suitable strength of staff members in a system there is a need to formulate some useful recruitment policy. In this case we assume that the life distribution for the service of staff in a system is known.

Problem 7.32.

A research team is planed to raise its strength to 50 chemists and then to remain at that level. The wastage of recruits depends on their length of service, which is as follows:

Year	*Total percentage who have left up to the end of the year.*	*Year*	*Total percentage who have left up to the end of the year.*
1	5	6	73
2	36	7	79
3	56	8	87
4	63	9	97
4	68	10	100

What is the recruitment per year necessary to maintain the strength? There are 8 senior posts for which the length of service is the main criterion for promotion. What is the average length of service after which new entrant can expect his promotion to one of these posts.

Solution

Year (1)	*Number of persons who leave at the end of the year (2)*	*Number of persons in service at the end of the year (3) 100 - (2)*	*Probability of leaving at the end of the year (4) (2) / 100*	*Probability at the in service at the end of the year (5) 1 - (4) or (3) / 100*
0	0	100	0	1.00
1	5	95	0,05	0.95
3	36	64	0.36	0.64
4	56	44	0.56	0.44
5	63	37	0.63	0.37
6	73	27	0.73	0.27
7	79	21	0.70	0.21
8	87	13	0.87	0.13
9	97	3	0.97	0.03
100	0	0	1.00	0.00
	Total	**436**		

From the given data, we can find the probability of a chemist leaving during a certain year. The person who is joining the organization will not continue after 10 years. And we know that mortality table for any item can be used to derive the probability distribution of life span by $\{M(t-1) - M(t)\}\ N$.

The required probabilities are calculated in the table above. The column (3) shows that a recruitment policy 9 of 100 every year, the total number of chemists serving in the organization would have been 436. Hence, to maintain strength of 50 chemists, then the recruitment should be:

= (100 × 50) / 436 = 11.5 or approximately 12 chemists per year. As per life distribution of service 12 chemists are to be recruited every year, to maintain strength of 50 chemists. Now referring to column (5) of the table above, we find that number of survivals after each year. This is given by multiplying the various values in column (5) by 12 as shown in the table given below:

Number of years	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
Number of Chemists in service.	12	11	7	5	4	4	3	2	2	0	0

As there are 8 senior posts, from the table we find that there are 3 persons in service during the 6th year, 2 in 7th year, and 2 in 8th year. Hence the promotion for new recruits will start from the end of fifth year and will continue up to sixth year.

(OR it can be done in this way: if we recruit 12 persons every year, then we want 8 seniors. Suppose we recruit 100 every year, then we shall require (8 × 100) / 12 = 66.4 or approximately 64 seniors. It is seen from the first table above that required number of persons would be available, if we promote them at the end of fifth year.)

Problem 7.33.

An Automobile unit requires 200 junior engineers, 300 Assistant Engineers and 50 Executives. Trainees are recruited at the age of 21 years, if still in service; retire at the age of 60. Given the following life table, determine (*i*) how many Engineers should be recruited each year? (*ii*) At what ages promotions should take place?

Age	*No. in Service*	*Age*	*No. in Service*	*Age*	*No. in Service*	*Age*	*No in Service*	*Age*	*No. in Service*
21	1000'	22	600	23	400	24	384	25	307
26	261	27	228	28	206	29	190	30	181
31	173	32	167	33	161	34	155	35	159
36	146	37	144	38	136	39	131	40	125
41	119	42	113	43	106	44	99	45	93
46	87	47	80	48	73	49	66	50	59
51	53	52	46	53	39	54	33	55	27
56	22	57	18	58	14	59	11	60	0

Solution

If a policy of recruiting 1000 Engineers every year is followed, then the total number of Engineers in service between the age of 21 and 59 years will be equal to sum of the number in service *i.e.* = 6480. But we want 200 junior engineers + 300 Assistant engineers + 50 Executives = 550 engineers in all in the organization.

To maintain strength of 550 Engineers, we should recruit (1000 × 550) / 6480 = 84.87 = App.85 Engineers every year.

If junior engineers are promoted at the age of '*y*'years then up to age (*y*–1) we require 200 junior engineers. Out of a strength of 550 there 200 junior engineers. Hence out of strength of 1000 there will be:

(200 ×1000) / 550 = 364 junior engineers.

From the given data the strength 364 is available up to 24 years. Hence the promotion of junior engineers will take place in 25th year.

Again out of 550 staff, we require 300 Assistant engineers. If we recruit 1000 engineers, then we require:

(300 × 1000) / 550 = 545 assistant engineers.

Hence the number of junior engineers and assistant engineers in a recruitment of 1000 will be 364 + 545 = 909. *i.e.* we require 91 executives, whereas at the age of 46 only 87 will survive. Hence promotion of assistant engineers will take place in 46th year.

Problem 7.34.

An airline requires 250 assistant hostesses, 350 hostesses and 50 supervisors. Girls are recruited at the age of 21 and if in service, they retire at age of the 60 years. The table given below show the life pattern, determine:

(a) How many girls should be recruited each year?

(b) At what age promotions should take place?

Age	*No. in Service*	*Age*	*No. in Service*	*Age*	*No. in Service*	*Age*	*No. in Service*
21	1000	31	170	41	120	51	53
22	700	32	165	42	112	52	45
23	500	33	160	43	105	53	40
24	400	34	155	44	100	54	32
25	300	35	150	45	92	55	26
26	260	36	145	46	88	56	20
27	230	37	140	47	80	57	18
28	210	38	135	48	72	58	15
29	195	39	130	49	65	59	10
30	180	40	125	50	60	60	00

Solution

If 1000 girls are recruited every year for the past 39 years (21 to 59th year), the total number of them serving up to the age of 59 years is the sum of survivals *i.e.* 6,603 persons. Total number of girls required in airline is 250 assistant hostesses + 350 hostesses +50 supervisors = 650 girls.

(i) Number of girls recruited every year in order to maintain strength of 650 = (1000 × 650) / 6603 = 98.440 = 98 approximately.

(ii) Let the assistant hostesses be promoted at the age of 'y'. Then up to age (y – 1) year, number of assistant hostesses required = 250 members. Now out of 650 girls, 250 are assistant hostesses; therefore out of 1000, their number is (1000 × 250) / 650 = 384.615 = 385 approximately. This number occurs in the given data up to the age of 24th year. Therefore, the promotion assistant hostesses is due in the 25th year.

Now, out of 650 girls, 350 are hostesses. Therefore, if we recruit 1000 girls, the number of hostesses will be (350 × 1000) / 650 = app. 538

Therefore, total number of assistant hostesses and hostesses in a recruitment of 1000 = 385 + 538 = 923.

Therefore, number of supervisors required is 1000 – 923 = 77

From the given data, this number 77 is available up to the age of 47 years. Hence promotion is due in the 48th year.

Problem 7.35.

A faculty in a college is planned to rise to strength of 50 staff members and then to remain at that level. The wastage of recruits depends upon their length of service and is as follows:

Year:	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
Total percentage who left up to the end of the year:	5	35	56	65	70	76	80	86	95	100

(i) Find the number of staff members to be recruited every year.

(ii) If there are seven posts of Head of Departments for which length of service is the only criterion of promotion, what will be average length of service after which a new entrant should expect promotion?

Solution

Let us assume that the recruitment is 100 per year. Then, the 100 who join in the first year will become zero in 10th year, the 100 who join in the 2nd year will become 5 at the end of the 10th year (serve for 9 years), and the 100 who join in the 3rd year will become 15 at the end of 10th year (serve for 8 years), and so on. Thus when the equilibrium is attained, the distribution of length of service of the staff members will be as follows:

Year:	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
No. of Staff Members:	100	95	65	44	35	30	24	20	14	5	0

(i) Thus if 100 staff members are recruited every year, the total number of staff members after 10 years of service is equal to sum of the staff members shown above which is = 432.

To maintain strength of 50, the number to be recruited every year = (100 × 50) / 432 = 11.6 or app = 12 members.

It is assumed that those staff members who completed 'x' years of service but left before x + 1 years of service, actually left immediately before completing x + 1 years. If it is assumed that they left immediately after completing x years service, the total number will become (432 – 100) = 332 and the required intake will be (50 × 100) / 332 = 15. In actual practice they may leave at any time in the year so that reasonable number of recruitments per year will be (11.6 + 15) / 2 = app. 13.

(ii) If the college recruits 13 persons every year, then the college needs 7 seniors. Hence if the college recruit100 persons every year then the requirement is = (7 × 100) / 13 = App. 54 seniors. It is seen from the given data that 54 seniors will be available if the college promote them during 6th year of their service.

i.e. 0 + 5 + 14 + 20 + 24 = 63 which is > 54). Therefore, the promotion of a newly recruited staffed member will be done after completing 5 years and before putting in 6 years of service.

EXERCISE

1. What is replacement? Explain by means real world examples.
2. Explain different types of replacement problems by giving examples.
3. (*a*) Write a brief note on replacement.

   (c)The cost of maintenance of equipment is given by a function of increasing with time and its scrap value is constant. Show that replacing the equipment when the average cost to date becomes equal to the current maintenance cost will minimize the average annual cost.

4. A firm is considering when to replace its machine whose price is Rs. 12,200/-. The scrap value of the machine is Rs. 200/- only. From past experience maintenance cost of machine is as under.

Year:	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
Maintenance Cost in Rs.	200	500	800	1200	1800	2500	3200	4000

Find when the new machine should be installed. (Ans: 7th year)

5. The following is the cost of running a particular car to date and the forecast into the future. Assume that a similar car will replace the car, when is the best time to replace it and what will be the average yearly running cost?

Year	*Resale value at the end of the year. Rs.*	*Petrol and Tax during the year. Rs.*	*All other running cost During the year. Rs.*
0	700	—	—
1	625	90	10
2	575	90	30
3	550	90	50
4	500	90	70
5	450	90	90
6	450	90	110
7	350	90	130
8	300	90	150

(Ans. 3rd year).

(6) Machine *B* costs Rs. 10,000/-. Annual operating costs are Rs. 400/- for the first year and they increase by Rs. 800/- each year. Machine, *A* which is one year old, costs Rs. 9000/- and the annual operating costs are Rs. 200/- for the first year and they increase by Rs. 2000/- every year. Determine at what time is it profitable to replace machine *A* with machine *B*. (Assume that machines have no resale value and the future costs are not discounted).

(7) A firm pays Rs. 10,000/- for its automobiles. Their operating and maintenance costs are about Rs. 2,500/- per year for the first two years and then go up by Rs. 1500/- approximately per year. When should such vehicles be replaced? The discount rate is 0.9.

(8) The cost of new machine is Rs. 4000/-. The maintenance cost of 'n' th year is given by $R_n = 500\ (n - 1)$ where $n = 1,2,3\ldots n$. Suppose that the discount rate per year is 0.05. After how many years will it be economical to replace the machine by a new one?

(Ans: After 4 years)

(9) If you wish to have a return of 10% per annum on your investment, which of the following plans would you prefer?

	Plan A	*Plan B*
1st Cost in Rs.	2,00,000	2,50,000
Scrap value after 15 years in Rs.	1,50,000	1,80,000
Excess of annual revenue over annual disbursement in Rs.:	25,000	30,000

(Ans: Plan A).

(10) The following mortality rates have been observed for a certain type of light bulbs:

Week:	*1*	*2*	*3*	*4*	*5*
Percent failing by the weekend:	10	25	50	80	100

There are 1000 bulbs in use and it costs Rs.2/- to replace an individual bulb, which has burnt out. If all bulbs were replaced simultaneously, it would cost 50 paise per bulb. It is proposed to replace all the bulbs at fixed intervals, whether or not they have burnt out, and to continue replacing burnt out bulbs as they fail. At what intervals should all the bulbs be replaced?

(Ans: At the end of 2 weeks).

(11) The probability p_n of failure just before age 'n' years is shown below. If individual replacement cost is Rs.1.25 and group replacement cost is Re. 0.50 per item, find the optima group replacement policy.

n:	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*
p_n	0.01	0.03	0.05	0.07	0.10	0.15	0.20	0.15	0.11	0.08	0.05

(Ans: After every 6 weeks)

(11) A fleet owner finds from his past records that the costs per year of running a truck whose purchase price is Rs. 6,000/- are as follows:

Year:	*1*	*2*	*3*	*4*	*5*	*6*	*7*
Running cost in Rs.	1000	1200	1400	1800	2300	2800	3400
Resale value in Rs.:	3000	1500	750	375	200	200	200

(Ans: At the end of 5th year)

12. The following mortality rates have been found for a certain type of coal cutter motor:

Weeks:	*10*	*20*	*30*	*40*	*50*
Total % failure up to end of 10 weeks period:	5	15	35	65	100

If the motors are replaced over the week and the total cost is Rs. 200/-. If they fail during the week the total cost is Rs. 100/- per failure. Is it better to replace the motors before failure and if so when?

(Ans: Motors should be replaced every 20 weeks)

MULTIPLE CHOICE QUESTIONS

Replacement Model- Quiz

1. Contractual maintenance or agreement maintenance with manufacturer is suitable for equipment, which is
   (a) In its infant state, (*b*) When machine is old one,
   (*c*) Scrapped, (*d*) None of the above. ( )
2. When money value changes with time at 10 %, then PWF for first year is :
   (*a*) 1. (*b*) 0.909
   (*c*) 0.852 (*d*) 0.9 ( )
3. Which of the following maintenance policy is not used in old age stage of a machine:
   (a) Operate up to failure and do corrective maintenance
   (b) Reconditioning,
   (c) Replacement,
   (d) Scheduled preventive maintenance. ( )
4. When money value changes with time at 20%, the discount factor for 2 nd year is:
   (*a*) 1 (*b*) 0.833
   (*c*) 0 (*d*) 0.6955 ( )
5. Which of the following replacement policy is considered to be dynamic in nature?
   (a) Time is continuous variable and the money value does not change with time.
   (b) When money value does not changes with time and time is a discrete variable.
   (c) When money value changes with time.
   (d) When money value remains constant for some time and then goes on changing with time. ( )
6. When the probability of failure reduces gradually, the failure mode is said to be:
   (a) Regressive, (*b*) Retrogressive
   (*c*) Progressive (*d*) Recursive. ( )
7. The following replacement model is said to be probabilistic model:
   (a) When money value does not change with time and time is a continuous variable,
   (b) When money value changes with time,
   (c) When money value does not change with time and time is discrete variable
   (d) Preventive maintenance policy. ( )
8. A machine is replaced with average running cost
   (a) Is not equal to current running cost.
   (b) Till current period is greater than that of next period
   (c) Of current period is greater than that of next period,
   (d) Of current period is less than that of next period. ( )

9. The curve used to interpret machine life cycle is
   (a) Bath tub curve (*b*) Time curve
   (*c*) Product life cycle (*d*) Ogive curve. ( )
10. Decreasing failure rate is usually observed in ....stage of the machine
    (a) Infant (*b*) Youth
    (*c*) Old age (*d*) Any time in its life. ( )
11. Which cost of the following is irrelevant to replacement analysis?
    (a) Purchase cost of the machine,
    (b) Operating cost of the machine,
    (c) Maintenance cost of the machine
    (d) Machine hour rate of the machine. ( )
12. The type of failure that usually occurs in old age of the machine is
    (a) Random failure (*b*) Early failure
    (*c*) Chance failure (*d*) Wear - out failure ( )
13. Group replacement policy is most suitable for:
    (a) Trucks (*b*) Infant machines
    (*c*) Street light bulbs (*d*) New cars. ( )
14. The chance failure that occur on a machine are commonly found on a graph of time Vs Failure rate (on X and Y axis respectively as
    (a) Parabolic (*b*) Hyperbolic
    (*c*) Line nearly parallel to X axis (*d*) Line nearly parallel to Y-axis. ( )
15. Replacement of an item will become necessary when
    (a) Old item becomes too expensive to operate or maintain
    (b) When your operator desires to work on a new machine.
    (c) When your opponent changes his machine in his unit.
    (d) When company has surplus funds to spend. ( )
16. The production manager will not recommend group replacement policy in case of
    (a) When large number of identical items is to be replaced
    (b) Low cost items are to be replaced, where record keeping is a problem.
    (c) For items that fail completely,
    (d) Repairable items. ( )
17. In replacement analysis the maintenance cost is a function of:
    (a) Time (*b*) Function
    (*c*) Initial investment (*c*) Resale value ( )
18. Which of the following is the correct assumption for replacement policy when money value does not change with time
    (a) No Capital cost, (*b*) No scrap value
    (*c*) Constant scrap value (*d*) zero maintenance cost. ( )

19. Which one of the following does not match the group.
    (a) Present Worth Factor (PWF) (*b*) Discounted rate (DR)
    (*c*) Depreciation value (DV) (*d*) Mortality Tables (MT) ( )
20. Reliability of an item is
    (a) Failure Probability. (*b*) 1 / Failure probability
    (*c*) 1 - failure probability (*d*) Life period / Failure rate. ( )
21. The following is not discussed in-group replacement policy:
    (a) Failure Probability, (*b*) Cost of individual replacement,
    (*c*) Loss due to failure (*d*) Present worth factor series. ( )
22. It is assumed that maintenance cost mostly depends on:
    (a) Calendar age (*b*) Manufacturing date
    (*c*) Running age (*d*) User's age ( )
23. Group replacement policy applies to:
    (a) Irreparable items, (*b*) Repairable items.
    (*c*) Items that fail partially (*d*) Items that fail completely. ( )
24. If a machine becomes old, then the failure rate expected will be:
    (a) Constant (*b*) Increasing
    (*c*) decreasing (*d*) we cannot say. ( )
25. Replacement is said to be necessary if
    (a) Failure rate is increasing (*b*) Failure cost is increasing
    (*c*) Failure probability is increasing (*d*) Any of the above. ( )
26. In this stage, the machine operates at highest efficiency and its production rate will be high.
    (a) Infant stage (*b*) Youth stage,
    (*c*) Old age, (*d*) None of the above. ( )
27. Replacement decision is very much common in this stage:
    (a) Infant stage, (*b*) Old age,
    (*c*) Youth, (*d*) In all of the above. ( )
28. The replacement policy that is imposed on an item irrespective of its failure is
    (a) Group replacement (*b*) Individual replacement,
    (*c*) Repair spare replacement, (*d*) Successive replacement. ( )
29. When certain symptoms indicate that a machine is going to fail and to avoid failure if maintenance is done it is known as:
    (a) Symptoms maintenance, (*b*) Predictive maintenance
    (*c*) Repair maintenance (*d*) Scheduled maintenance. ( )
30. In retrogressive failures, the failure probability------------------- with time.
    (a) Increases, (*b*) Remains constant,
    (*c*) Decreases (*d*) None of the above. ( )

***CHAPTER* – 8**

Inventory Control

INTRODUCTION

One of the basic functions of management is to employ capital efficiently so as to yield the maximum returns. This can be done in either of two ways or by both, *i.e.* (a) By maximizing the margin of profit; or (b) By maximizing the production with a given amount of capital, *i.e.* to increase the productivity of capital. This means that the management should try to make its capital work hard as possible. However, this is all too often neglected and much time and ingenuity are devoted to make only labour work harder. In the process, the capital turnover and hence the productivity of capital is often totally neglected. Several new techniques have been developed and employed by modern management to remedy this deficiency. Among these **Materials Management** has become one of the most effective. In Materials Management, **Inventory Control** play vital role in increasing the productivity of capital.

Inventory management or Inventory Control is one of the techniques of Materials Management which helps the management to improve the productivity of capital by reducing the material costs, preventing the large amounts of capital being locked up for long periods, and improving the capital - turn over ratio. The techniques of inventory control were evolved and developed during and after the Second World War and have helped the more industrially developed countries to make spectacular progress in improving their productivity.

The importance of materials management/inventory control arises from the fact that materials account for 60 to 65 percent of the sales value of a product, that is to say, from every rupee of the sales revenue, 65 paise are spent on materials. Hence, small change in material costs can result in large sums of money saved or lost. Inventory control should, therefore, be considered as a function of prime importance for our industrial economy.

Inventory control provides tools and techniques, most of which are very simple to reduce/control the materials cost substantially. A large portion of revenue (65 percent) is exposed to the techniques, correspondingly large savings result when they are applied than when attempts are made to saver on other items of expenditure like wages and salaries which are about 16 percent or overheads which may be 20 percent. By careful financial analysis, it is shown that a 5 percent reduction in material costs will result in increased profits equivalent to a 36 percent increase in sales.

DEFINITION OF INVENTORY AND INVENTORY CONTROL

The word *inventory* means a physical stock of material or goods or commodities or other economic resources that are stored or reserved or kept in stock or in hand for smooth and efficient running of future affairs of an organization at the minimum cost of funds or capital blocked in the form of materials or goods (Inventories).

The function of directing the movement of goods through the entire manufacturing cycle from the requisitioning of raw materials to the inventory of finished goods in an orderly manner to meet the objectives of maximum customer service with minimum investment and efficient (low cost) plant operation is termed as *inventory control.*

Classification of Inventories

Inventories may be classified as those which play direct role during manufacture or which can be identified on the product and the second one are those which are required for manufacturing but not as a part of production or cannot be identified on the product. The first type is labeled as ***direct inventories*** and the second are labeled as ***indirect inventories***.

Further classification of direct and indirect inventories is as follows:

(A) Direct inventories

(i) ***Raw material inventories*****:** The inventory of raw materials is the materials used in the manufacture of product and can be identified on the product. In inventory control manager can concentrate on the

(a) Bulk purchase of materials to save the investment,

(b) To meet the changes in production rate,

(c) To plan for buffer stock or safety stock to serve against the delay in delivery of inventory against orders placed and also against seasonal fluctuations.

(ii) ***Work-in-process inventories or in process inventories:*** These inventories are of semi-finished type, which are accumulated between operations or facilities. As far as possible, holding of materials between operations to be minimized if not avoided. This is because; as we process the materials the economic value (added labour cost) and use value are added to the raw material, which is drawn from stores. Hence if we hold these semi finished material for a long time the inventory carrying cost goes on increasing, which is not advisable in inventory control. These inventories serves the following purpose:

(a) Provide economical lot production,

(b) Cater to the variety of products,

(c) Replacement of wastages,

(d) To maintain uniform production even if sales varies.

(iii) ***Finished goods inventories:*** After finishing the production process and packing, the finished products are stocked in stock room. These are known as finished goods inventory. These are maintained to:

(a) To ensure the adequate supply to the customers,

(b) To allow stabilization of the production level and

(c) To help sales promotion programme.

(iv) ***Spare parts inventories:*** Any product sold to the customer, will be subjected to wear and tear due to usage and the customer has to replace the worn-out part. Hence the manufacturers always calculate the life of the various components of his product and try to supply the spare components to the market to help after sales service. The use of such spare parts inventory is:

(a) To provide after sales service to the customer,

(b) To utilize the product fully and economically by the customer.

(*iv*) ***Scrap or waste inventory:*** While processing the materials, we may come across certain wastages and certain bad components (scrap), which are of no use. These may be used by some other industries as raw material. These are to be collected and kept in a place away from main stores and are disposed periodically by auctioning.

(B) Indirect Inventories

Inventories or materials like oils, grease, lubricants, cotton waste and such other materials are required during the production process. But we cannot identify them on the product. These are known as indirect inventories. In our discussion of inventories, in this chapter, we only discuss about the direct inventories.

Inventories may also be classified depending their nature of use. They are:

(i) ***Fluctuation Inventories:*** These inventories are carried out to safeguard the fluctuation in demand, non-delivery of material in time due to extended lead-time. These are some times called as Safety stock or reserves. In real world inventory situations, the material may not be received in time as expected due to trouble in transport system or some times, the demand for a certain material may increase unexpectedly. To safeguard such situations, safety stocks are maintained. The level of this stock will fluctuate depending on the demand and lead-time etc.

(ii) ***Anticipation inventory:*** When there is an indication that the demand for company's product is going to be increased in the coming season, a large stock of material is stored in anticipation. Some times in anticipation of raising prices, the material is stocked. Such inventories, which are stocked in anticipation of raising demand or raising rises, are known as anticipation inventories.

(iii) ***Lot size inventory or Cycle inventories :***This situation happens in batch production system. In this system products are produced in economic batch quantities. It some time happens that the materials are procured in quantities larger than the economic quantities to meet the fluctuation in demand. In such cases the excess materials are stocked, which are known as lot size or cycle inventories.

(iv) ***Transportation Inventories:*** When an item is ordered and purchased they are to be received from the supplier, who is at a far of distance. The materials are shipped or loaded to a transport vehicle and it will be in the vehicle until it is delivered to the receiver. Similarly, when a finished product is sent to the customer by a transport vehicle it cannot be used by the purchaser until he receives it. Such inventories, which are in transit, are known as Transportation inventories.

(v) ***Decoupling inventories:*** These inventories are stocked in the manufacturing plant as a precaution, in case the semi finished from one machine does not come to the next machine, this stock is used to continue a production. Such items are known as decoupling inventories.

COSTS ASSOCIATED WITH INVENTORY

While maintaining the inventories, we will come across certain costs associated with inventory, which are known as **economic parameters**. Most important of them are discussed below:

(A) Inventory Carrying Charges, or Inventory Carrying Cost or Holding Cost or Storage Cost (C_1) or (i%):

This cost arises due to holding of stock of material in stock. This cost includes the cost of maintaining the inventory and is proportional to the quantity of material held in stock and the time for which the material is maintained in stock. The components of inventory carrying cost are:

(i) Rent for the building in which the stock is maintained if it is a rented building. In case it is own building, depreciation cost of the building is taken into consideration. Sometimes for own buildings, the nominal rent is calculated depending on the local rate of rent and is taken into consideration.

(ii) It includes the cost of equipment if any and cost of racks and any special facilities used in the stores.

(iii) Interest on the money locked in the form of inventory or on the money invested in purchasing the inventory.

(iv) The cost of stationery used for maintaining the inventory.

(v) The wages of personnel working in the stores.

(vi) Cost of depreciation, insurance.

(vii) Cost of deterioration due to evaporation, spoilage of material etc.

(viii) Cost of obsolescence due to change in requirement of material or changed in process or change in design and item stored as a result of becomes old stock and become useless.

(ix) Cost of theft and pilferage *i.e.* indenting for the material in excess of requirement.

This is generally represented by C_1 rupees per unit quantity per unit of time for production model. That is manufacturing of items model. For purchase models it is represented by i% of average inventory cost.

If we take practical situation into consideration, many a time we see that the inventory carrying cost (some of the components of the cost) cannot be taken proportional to the quantity of stock on hand. For example, take rent of the stores building. As and when the stock is consumed, it is very difficult to calculate proportion of rent in proportion to the stock in the stores as the rent will not vary day to day due to change in inventory level. Another logic is that the money invested in inventory may be invested in other business or may be deposited in the bank to earn interest. As the money is in the form of inventory, we cannot earn interest but loosing the expected interest on the money. This cost of money invested, is generally compared to the interest rate i% and is taken as the inventory carrying cost. Hence the value of 'i' will be a fraction of a rupee and will be $0 < i < 1$. In many instances, the bank rate of interest is somewhere between 16 to 20 % and other components like salary, insurance, depreciation etc may work out to 3 to 5 %. Hence, the total of all components will be around 22 to 25 % and this is taken as the cost of inventory carrying cost and is expressed as i % of average inventory cost.

(B) Shortage cost or Stock - out - cost- (C_2)

Some times it so happens that the material may not be available when needed or when the demand arises. In such cases the production has to be stopped until the procurement of the material, which may lead to miss the delivery dates or delayed production. When the organization could not meet the delivery promises, it has to pay penalty to the customer. If the situation of stock out will occur very often, then the customer may not come to the organization to place orders, that is the organization is loosing the customers. In other words, the organization is loosing the goodwill of the customers. The cost of good will cannot be estimated. In some cases it will be very heavy to such extent that the organization has to forego its business. Here to avoid the stock out situation, if the organization stocks more material, inventory carrying cost increases and to take care of inventory cost, if the organization purchase just sufficient or less quantity, then the stock out position may arise. Hence the inventory manager must have sound knowledge of various factors that are related to inventory carrying cost and stock out cost and estimate the quantity of material to be purchased or else he must have effective strategies to face grave situations. The cost is generally represented as so many rupees and is represented by C_2.

(C) Set up cost or Ordering cost or Replenishment Cost (C_3)

For purchase models, the cost is termed as ordering cost or procurement cost and for manufacturing cost it is termed as set up cost and is represented by C_3.

(i) ***Set up cost:*** The term set up cost is used for production or manufacturing models. Whenever a job is to be produced, the machine is to set to produce the job. That is the tool is to be set and the material is to be fixed in the jobholder. This consumes some time. During this time the machine will be idle and the labour is working. The cost of idle machine and cost of labour charges are to be added to the cost of production. If we produce only one job in one set up, the entire set up cost is to be charged to one job only. In case we produce '*n*' number of jobs in one set up, the set up cost is shared by '*n*' jobs. In case of certain machines like N.C machines, or Jig boarding machine, the set up time may be 15 to 20 hours. The idle cost of the machine and labour charges may work out to few thousands of rupees. Once the machine set up is over, the entire production can be completed in few hours. If we produce more number of products in one set up the set up cost is allocated to all the jobs equally. This reduces the production cost of the product. For example let us assume that the set up cost is Rs. 1000/-. If we produce 10 jobs in one set up, each job is charged with Rs. 100/- towards the set up cost. In case, if we produce 100 jobs, the set up cost per job will be Rs. 10/-. If we produce, 1000 jobs in one set up, the set up cost per job will be Re. 1/- only. This can be shown by means of a graph as shown in figure 8.1.

(ii) ***Ordering Cost or Replenishment Cost :***The term Ordering cost or Replenishment cost is used in purchase models. Whenever any material is to be procured by an organization, it has to place an order with the supplier. The cost of stationary used for placing the order, the cost of salary of officials involved in preparing the order and the postal expenses and after placing the order enquiry charges all put together, is known as Ordering cost. In Small Scale Units, this may be around Rs. 25/- to Rs. 30/- per order. In Larger Scale Industries, it will be around Rs, 150 to Rs. 200 /- per order. In Government organizations, it may work out to Rs. 500/- and above per order. If the organization purchases more items per order, all the items share the ordering cost. Hence the materials manager must decide how much to purchase per order so as to keep the ordering cost per item at minimum. One point we have to remember here, to reduce the ordering cost per item, if we purchase more items, the inventory carrying cost increases. To keep inventory carrying cost under control, if we purchase less quantity, the ordering cost increase. Hence one must be careful enough to decide how much to purchase? The nature of ordering cost can also be shown by a graph as shown in figure 8.1. If the ordering cost is C_3 per order (can be equally applied to set up cost) and the quantity ordered / produced is 'q' then the ordering cost or set up cost per unit will be C_3/q is inversely proportional to the quantity ordered, *i.e*. decreased with the increase in 'q' as shown in the graph 8.1.

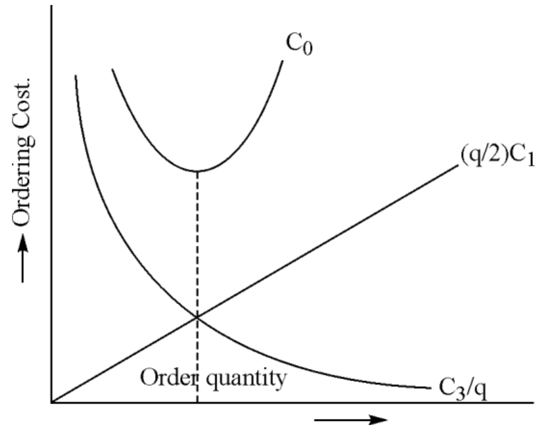


Figure 8.1 Nature of ordering cost.

(iii) ***Procurement Cost :*** These costs are very much similar to the ordering cost / set up cost. This cost includes cost of inspection of materials, cost of returning the low quality materials, transportation cost from the source of material to the purchaser‘s site. This is proportional to the quantity of materials involved. This cost is generally represented by '*b*' and is expressed as so many rupees per unit of material. For convenience, it always taken as a part of ordering cost and many a time it is included in the ordering cost / set up cost.

(D) Purchase price or direct production cost

This is the actual purchase price of the material or the direct production cost of the product. It is represented by '*p*'. *i.e*. the cost of material is Rs. '*p*' per unit. This may be constant or variable. Say for example the cost of an item is Rs. 10/- item if we purchase 1 to 10 units. In case we purchase more than 10 units, 10 percent discount is allowed. *i.e*. the cost of item will be Rs.9/- per unit. The purchase manager can take advantage of discount allowed by purchasing more. But this will increase the inventory carrying charges. As we are purchasing more per order, ordering cost is reduced and because of discount, material cost is reduced. Materials manager has to take into consideration these cost – quantity relationship and decide how much to purchase to keep the inventory cost at low level.

Points to be remembered

(i) Inventory cost increases with the quantity purchased.

(ii) If we purchase more items per order or produce more items per set up ordering cost or set up cost per item decreases, stock out situation reduces and inventory-carrying cost increases and if discount is allowed on quantity purchased the material cost also reduces.

(iii) If we purchase less items per order or produce less items per set up ordering cost per item or set up cost per item increases, stock out position may increase which increases stock out costs, and inventory-carrying cost decreases. Quantity discounts may not be available.

PURPOSE OF MAINTAINING INVENTORY OR OBJECTIVE OF INVENTORY COST CONTROL

The purpose of maintaining the inventory or controlling the cost of inventory is to use the available capital optimally (efficiently) so that inventory cost per item of material will be as minimum as possible. For this the materials manager has to strike a balance between the interrelated inventory costs. In the process of balancing the interrelated costs *i.e*. Inventory carrying cost, ordering cost or set up cost, stock out cost and the actual material cost. Hence we can say that **the objective of controlling the inventories is to enable the materials manager to place and order at right time with the right source at right price to purchase right quantity.**

The benefits derived from efficient inventory control are:

(i) It ensures adequate supply of goods to the customer or adequate of quantity of raw materials to the manufacturing department so that the situation of stock out may be reduced or avoided.

(ii) By proper inventory cost control, the available capital may be used efficiently or optimally, by avoiding the unnecessary expenditure on inventory.

(iii) In production models, while estimating the cost of the product the material cost is to be Added. The manager has to decide whether he has to take the actual purchase price of the material or the current market price of the material. The current market price may be less than or greater than the purchase price of the material which has been purchased some period back. Proper inventory control reduces such risks.

(iv) It ensures smooth and efficient running of an organization and provides safety against late delivery times to the customer due to uncontrollable factors.

(v) A careful materials manager may take advantage of price discounts and make bulk purchase at the same time he can keep the inventory cost at minimum.

(vi) It enables a manager to select a proper transportation mode to reduce the cost of transportation.

(vi) Avoids the chances of duplicate ordering.

(vii) It avoids losses due to deterioration and obsolescence etc.

(viii) Causes of surplus stock may be controlled or totally avoided.

(ix) Proper inventory control will ensure the availability of the required material in required quantity at required time with the minimum inventory cost.

Though many managers consider inventory as an enemy as it locks up the available capital, but by proper inventory control they can enjoy the benefits of inventory control and then they can realize that the inventory is a real friend of a manager in utilizing the available capital efficiently.

OTHER FACTORS TO BE CONSIDERED IN INVENTORY CONTROL

There are many factors, which have influence on the inventory, which draws the attention of an inventory manager, they are:

(i) Demand

The demand for raw material or components for production or demand of goods to satisfy the needs of the customer, can be assessed from the past consumption/supply pattern of material or goods. We find that the demand may be deterministic in nature *i.e.*, we can specify that the demand for the item is so many units for example say ‘q’ units per unit of time. Some times we find that the demand for the item may be probabilistic in nature *i.e.* we have to express in terms of expected quantity of material required for the period. Also the demand may be static, *i.e*. it means constant for each time period (uniform over equal period of times). Further, the demand may follow several patterns and so why it is uncontrolled variable, such as it may be uniformly distributed over period or instantaneous at the beginning of the period or it may be large in the beginning and less in the end etc. These patterns directly affect the total carrying cost of inventory.

(ii) Production of goods or Supply of goods to the inventory

The supply of inventory to the stock may deterministic or probabilistic (stochastic) in nature and many a times it is uncontrollable, because, the rate of production depends on the production, which is once again depends on so many factors which are uncontrollable / controllable factors. Similarly supply of inventory depends on the type of supplier, mode of supply, mode of transformation etc. The properties of supply mode have its effect in the level of inventory maintained and inventory costs.

(iii) Lead time or Delivery Lags or Procurement time

Lead-time is the time between placing the order and receipt of material to the stock. In production models, it is the time between the decision made to take up the order and starting of production. This time in purchase models depends on many uncontrollable factors like transport mode, transport route, agitations etc. It may vary from few days to few months depending on the nature of delay. The materials manager has to refer to the past records and approximately estimate the lead period and estimate the quantity of safety stock to be maintained. In production models, it may depend on the labour absenteeism, arrival of material to the stores, power supply, etc.

(iv) Type of goods

The inventory items may be discrete or continuous. Some times the discrete items are to be considered as continuous items for the sake of convenience.

(v) Time horizon

The time period for which the optimal policy is to be formulated or the inventory cost is to be optimized is generally termed as the Inventory planning period or Time horizon. This time is represented on X - axis while drawing graphs. This time may be finite or infinite.

(vi) Safety stock or Buffer stock

Whatever care taken by the materials manager, one cannot avoid the stock out situation due to many factors. To avoid the stock out position the manager some times maintains some extra stock, which is generally known as Buffer Stock, or Safety Stock. The level of this stock depends on the demand pattern and the lead-time. This should be judiciously calculated because, if we stock more the inventory carrying cost increases and there is chance of pilferage or theft. If we maintain less stock, we may have to face stock out position. The buffer stock or safety stock is generally the consumption at the maximum rate during the time interval equal to the difference between the maximum lead time and the normal (average) lead time or say the maximum, demand during lead time minus the average demand during lead time.

Depending on the characteristics above discussed terms, different types of inventory models may be formulated. These models may be deterministic models or probabilistic model depending on the demand pattern.

In any inventory model, we try to seek answers for the following questions:

(a) **When should the inventory be purchased for replenishment?** For example, the inventory should be replenished after a period 't' or when the level of the inventory is q_o

(b) **How much quantity must be purchased or ordered or produced at the time of replenishment so as to minimize the inventory costs?** For example, the inventory must be purchased with the supplier who is supplying at a cost of Rs. *p*/- per unit.

In addition to the above depending on the data available, we can also decide from which source we have to purchase and what price we have to purchase? But in general time and quantity are the two variables, we can control separately or in combination.

INVENTORY CONTROL SYSTEMS

There are various methods of controlling inventory. In this section, let us consider some of the important methods of controlling the inventory. They are listed below:

(a) *p* - System or Fixed Period System,
(b) *q* - System or Fixed quantity system, } These are also known as perpetual inventory control Systems.
(c) *pq* - System,
(d) *ABC* Analysis,
(e) *VED* Analysis,
(*f*) *XYZ* Analysis,
(g) *FNSD* analysis,
(h) Economic Order Quantity. (In manufacturing models, this is known as Economic Batch Quantity.)

(a) p - System or Fixed Period System

In this system inventory is replenished at fixed intervals, say for example every first of the month or every15th of the month and so on. The quantity we order depends on rate of consumption in that period. For example if 20 units are consumed in the first period, we place order for 20 pieces, if 40 pieces are consumed in the 2^{nd} cycle, order will be placed for 40 units and so on. Here period of ordering is constant and the quantity ordered per order will differ. Hence it is known as **fixed period system**. This is shown in figure 8.2.

(b) *q* - System or Fixed Quantity System

Against to *p* - system, here the quantity ordered per order is constant but the period of placing order will differ. Every time we place the order for the same quantity. This system is also known as **Two-Bin System**. A bin means a container. There will be two containers of same capacity; in which the material is stored. Once the material in one of the bin is consumed completely, then order is placed for the quantity consumed (*i.e*. capacity of the bin). The time required to consume all the material in the bin depends on the rate of demand. Depending on the demand to empty the bin, it may take 15 days or 20 days or any number of days. Here the order is placed for the material as soon as one of the bins becomes empty. This depends on the rate of demand. Depending on the rate of consumption, the time of placing will defer, but each time the order is placed for the same quantity (*i.e*. the capacity of the bin). Many a time we find that there will be no bins, but the order quantity is marked on the bin cards. As soon as the level of the inventory reaches the order quantity, the order is placed for the material. The principle is shown in figure 8.3. In figure, q_o is the order quantity and it is placed at periods t_1,t_2, and t_3 depending on the rate of demand. This system is recommended for high consumption '*A*' class items. The items in this class are few and it is worthwhile to have a continuous scrutiny of inventory. Continuous monitoring through computer is necessary for this as orders for replenishment are placed as soon as physical stock reaches the re-order level. q – system requires a continuous review of the inventory. It requires the maintenance of kardex system for stocks and timely entries of receipts and issues.

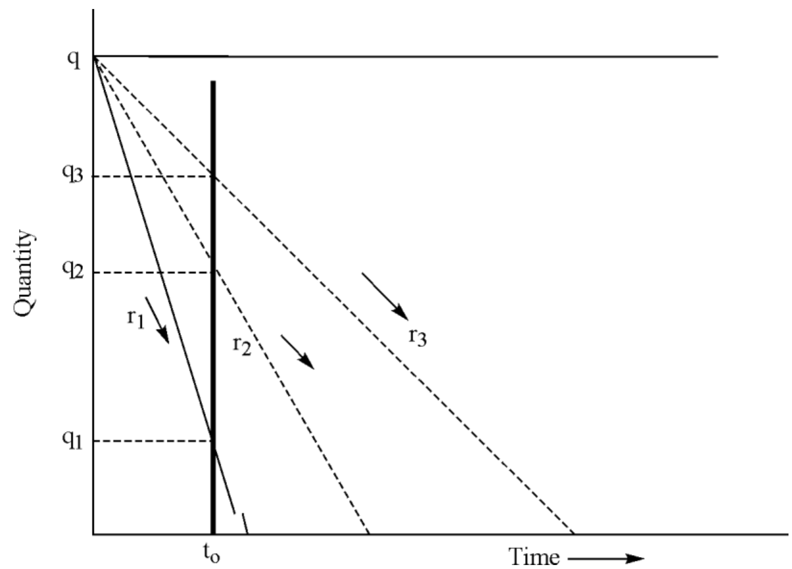


Figure 8.2. Fixed Period System.

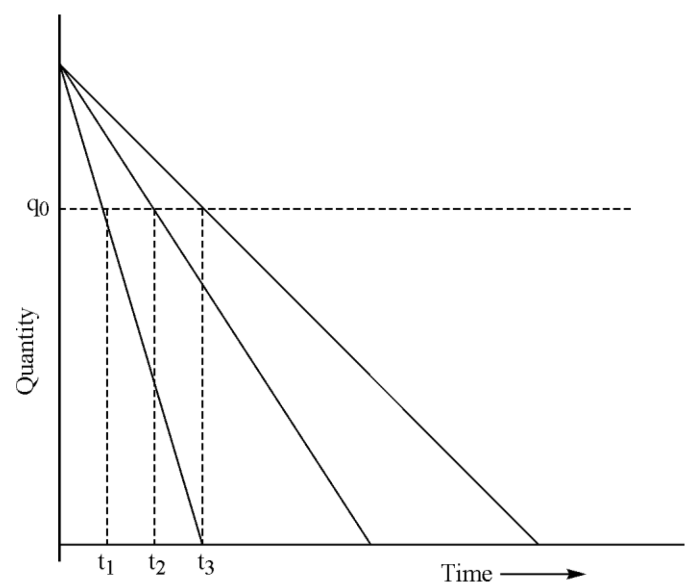


Figure 8.3 Fixed order quantity.

In figure 8.2 q_1, q_2 and q_3 are the different quantities to be ordered at period to depending on the demand rate r_1, r_2 and r_3. This system is not recommended for '*A*' class items but it is very useful in controlling the inventory of '*B*' and '*C*' class items.

(c) *pq* - System or / Optional Replenishment System

In some situations the cost of reviewing the inventory such as stock of certain chemicals where expert surveying is necessary to assess the stocks is high. Further, in some other context the cost of ordering is very significant. In such cases, the Optimal Replenishment model can be applied. When the stock on hand and stock on order falls below certain level (say '*s*') then an order is placed enough to bring the stock up to a level '*S*'. Here '*s*' represents re-order level and '*S*' denotes the desired inventory level. The review time also influences the order level '*s*'. In such situations, we can apply a combination of '*p*' and '*q*' system which is known as '*pq* system or Optional Replenishment System. This system is useful in case of bulk chemicals, pig iron etc. In fact no company will follow one particular system. Depending on the type of material, and need, they use either *p* – system or *q*- system or a combination of '*p*' and '*q*' system.

(d) *ABC* Analysis of Inventory

This is sometimes known as Always Better Control. This system of control is also known as **Selective Approach System.** In *ABC* system of inventory control, the materials are classified depending on their turnover and **annual consumption cost.**

A - Class Items

These items are less in number, but consumes large portion of the total inventory investment. Here annual consumption cost is important than the unit cost of the material. For example let us consider, two materials Material *X* and material *Y*. The unit cost of material *X* is Re.1/- and annual consumption is 1000 units. The unit cost of material *Y* is Rs.200 and the annual consumption is 3 units. Then annual consumption cost of material *X* is Rs.1000/- and that of *Y* is Rs. 600/-. Here Material *X* is considered as high consumption cost material than *Y*. Like that in any industry, we may find that there will be certain Items which are few in number but they consume nearly 70 % of inventory cost. Such items are classified as **'*A*' - class items.**

There will be certain materials, whose total annual consumption cost will be somewhere in-between 20 to 25 % of total inventory investment. These items are labeled as **'*B*' - class items**. These items will form 60 percent of number of items stored.

The last class of items which are labeled as **'*C*' - class items,** will be large in number may be 30 to 35 % of total number of items stored, but consumes only 5 to 10 % total inventory investment.

Hence we can say that *A* - Class items are less in number and consumes more money, *B* - Class items are medium in number and consumes 20 to 25 % inventory investment and *C* - Class items are large in number and consumes only 5 to 10 percent of inventory investment. This can be shown by means of a graph as shown in figure 8.4.

In fact *ABC* analysis cannot be restricted to inventory only. This *ABC* analysis is an extension to Pareto's 80 - 20 rule. The 80 - 20 rules states that 80 % countries economy is controlled by 20% of people. Let us take for example the monthly bill of an organization. Let us say it will workout to Rs. 10,00,000/- If we classify according to *ABC* rule, we see that 70 percent of the bill *i.e.* 7,00,000/- will be the bill of few people say some 10 percent of the officers. Next 2,00,000/- belongs to 40 percent of the people. And the balance of Rs. 1,00,000 belongs to rest 50% of the workers.

Similarly all the unrest in any organization or in a county is due to only 20% percent of the staff or population, which once again supports Pareto rule. Hence this rule *i.e.* Pareto rule or *ABC* rule can be applied to any situation where selective classification is possible. **The point to remember here is *ABC* analysis depends on annual consumption cost and not on unit cost of material.**

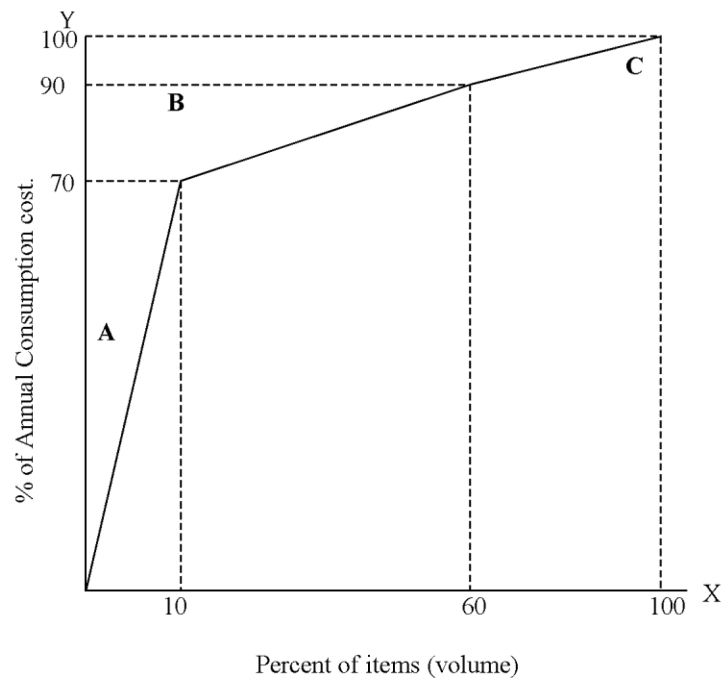


Figure 8.3 ABC Graph.

'*A*' class items needs the attention of higher officials and demand extreme control regarding the cost. As they consume 70% of money even 10 % saving through bargaining or inventory control techniques, the savings will be worthwhile. '*B*' Class items require the attention of middle level managers as they consume 20 to 25 % of investment on inventory. Where as '*C*' class items is left to the control of lower officials.

Procedure for *ABC* analysis

1. List out all items in stores along with their unit price and annual consumption.
2. Calculate the annual consumption cost of each item, which is given by multiplying the quantity consumed in the time period and the unit cost. If 'q' is the quantity consumed in the time period and 'p' is the unit price then annual consumption value = $q \times p = qp$.
3. Rearrange the list in the descending order of the annual consumption cost. *i.e.* highest cost at the top and next highest is the second and so on and the last item is the lowest consumption value item.

4. Calculate the cumulative total of annual consumption value.
5. Find the parentage of each cumulative value with respect to the total cost of inventory.
6. Mark a line at 70%, 90% and at 100%. All the items covered by 70% line are '*A*' class items, those which are covered between 70% line and 90% line are '*B*' class items and those are covered by 90% and 100 % are '*C*' class items.

Problem 8.1.

The details of material stocked in a company are given below with the unit cost and the annual consumption in Rs. Classify the material in to *A* class, *B* class and *C* class by *ABC* analysis.

S.No.	*Item Code No.*	*Annual consumption in pieces*	*Unit price in Paise.*
1	501	30,000	10
2	502	2,80,000	15
3	503	3,000	10
4	504	1,10,000	05
5	505	4,000	05
6	506	2,20,000	10
7	507	15,000	05
8	508	80,000	05
9	509	60,000	15
10	510	8,000	10

Solution

First let us find the annual usage value for each item (unit price × annual usage) and rank them in descending order.

S.No.	*Item Code No. (A)*	*Annual consumption in pieces (B)*	*Unit price in paise (C)*	*Annual usage value* $D = B \times C$	*Rank.*
1	501	30,000	10	3,000	6
2	502	2,80,000	15	42,000	1
3	503	3,000	10	300	9
4	504	1,10,000	05	5,500	4
5	505	4,000	05	200	10
6	506	2,20,000	10	22,000	2
7	507	15,000	05	750	8
8	508	80,000	05	4,000	5
9	509	60,000	15	9,000	3
10	510	8,000	10	800	7

List the items in their descending order of annual consumption value, find the cumulative value of annual consumption value and find the percentage of cumulative value with respect to total inventory value. Draw lines at 70%, 90% and at 100%.

Rank.	*Item No.*	*Annual Usage Rs.*	*Cumulative Annual usage*	*Cumulative Annual usage Percentage.*	*Percentage of items.*	*Category.*
1	502	42,000	42,000	48	10	A
2	506	22,000	64,000	73	20	A
3	509	9,000	73,000	83	30	B
4	508	5,500	78,500	90	40	B
5	504	4,000	82,500	94	50	B
6	501	3,000	85,500	98	60	B
7	510	800	86,300	98.6	70	C
8	507	750	87,050	99.4	80	C
9	503	300	87,350	99.6	90	C
10	505	200	87,550	100	100	C

Graph for the problem:

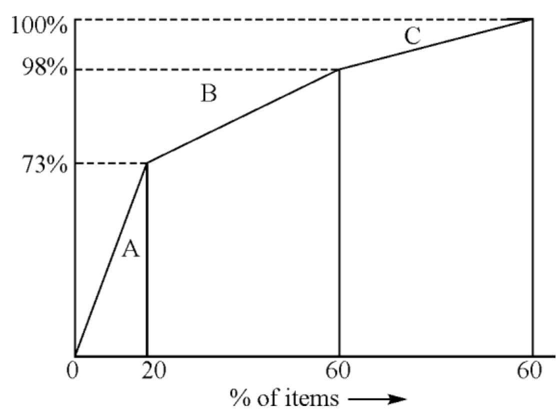


Figure 8.4. ABC graph for the problem.

In *A* class we have 2 items consuming 73% of the amount and in *B* class, we have 4 items consuming 25% of the amount and in *C* class, we have 4 items consuming about 2% of the inventory investment.

Problem 8.2

A sample of inventory details is given below. Conduct *ABC* analysis and classify them into three categories.

Item	*Annual consumption*	*Price per unit in paise.*
A	300	10
B	2,800	15
C	30	10
D	1,100	05
E	40	05
F	220	100
G	1,500	05
H	800	05
I	600	15
J	80	10

Solution

Item	*Usage Value in Descending Order (Unit price × Annual consumption) Rs.*	*Cumulative Number of Items*	*% of number or items*	*Cumulative Usage value*	*% Cumulative value.*
1	2	3	4	5	6
B	420	1	10	420	44.50
F	220	2	20	640	67.79
I	90	3	30	730	77.41
G	75	4	40	805	85.37
D	55	5	50	860	91.20
H	40	6	60	900	95.44
A	30	7	70	930	98.62
J	8	8	80	938	99.47
C	3	9	90	941	99.79
E	2	10	100	943	100.00

The above data is plotted on a graph (Figure 8.5).

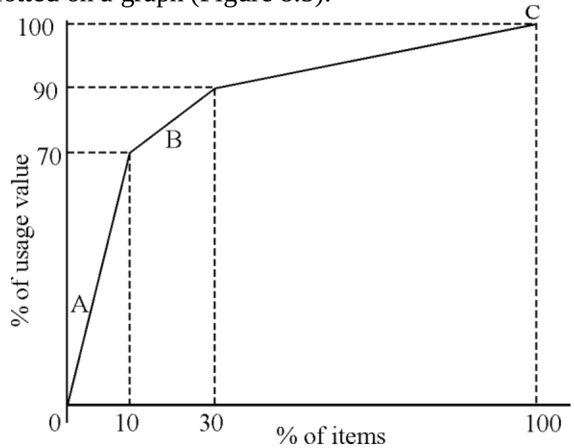


Figure 8.5 ABC Graph.

Class of items	*Name of the item*	*% of usage value*	*% of items.*
A	B and F	67.49	20
B	I and G	17.58	20
C	D, B, A, J, C, E.	14.63	60

Problem 8.3.

Classify the following materials into *A*, *B*, and *C* groups.

Item No:	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
Annual Usage in Rs. (x 1000)	36	14	75	37	11	16	32	08	95	04

Solution

Item No.	*Annual Usage In Rs.*	*Accumulated Usage in Rs.*	*Cumulative Percentage Usage (%)*	*Cumulative Percentage Of items (%)*	*Group*
9	95,000	95,000	28.96	10	A
3	75,000	1,70,000	51.82	20	A
4	37,000	2,07,000	63.10	30	A
1	36,000	2,43,000	74.08	40	A
7	32,000	2,75,000	83.84	50	B
6	16,000	2,91,000	88.71	60	B
2	14,000	3,05,000	92.98	70	B
5	11,000	3,16,000	96.34	80	B
8	8,000	3,24,000	98.78	90	C
10	4,000	3,28,000	100.00	100	C

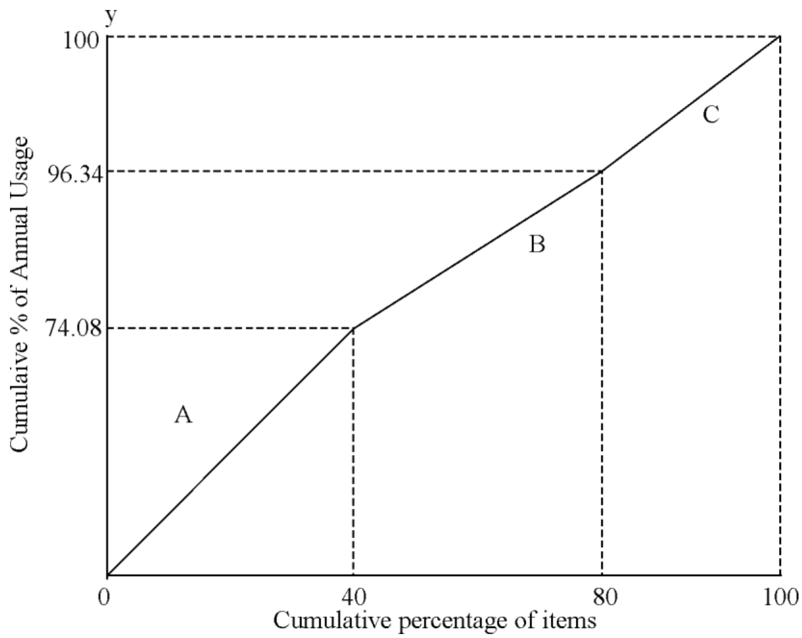


Figure 8.6 ABC Graph.

Problem 8.4.

From the data given below classify the items into ABC groups:

Item No.	*No. of units*	*Unit price in Rs.*	*Usage value in Rs.*
1	7,000	5.00	35,000
2	24,000	3.00	72,000
3	1,500	10.00	15,000
4	600	22.00	13,200
5	38,000	1.50	57,000
6	40.000	0.50	20,000
7	60,000	0.20	12,000
8	3,000	3.50	10,500
9	300	8.00	2,400
10	29,000	0.40	11,600
11	11,500	7.10	81,650
12	4,100	6.20	25,420

Solution

From the given data, we can work out Annual usage values and Cumulative annual usage values of the items and percentage of items and mark in which each item falls.

Item No.	*Cumulative % of items*	*Usage Value in Rs.*	*Cumulative Usage value In Rs.*	*Cumulative Percentage*	*Group in Which item Falls.*
11	8.3	81,650	81,650	23.0	A
2	16.6	72,000	1,53,650	43.2	A
5	25.0	57,000	2,10,650	59.2	A
1	33.3	35,000	2,45,650	69.0	A
12	41.6	25,420	2,71.070	76.2	B
6	50.0	20,000	2,91,070	81.8	B
3	58.3	15,000	3,06,070	86.0	B
4	66.6	13,200	3,19,270	89.7	B
7	75.0	12,000	3,31,270	93.1	C
10	83.3	11,600	3,42,870	96.4	C
8	91.6	10,500	3,53,370	99.3	C
9	100.0	2,400	3,55,770	100.00	C

The ABC Graph is shown in Figure 8.7.

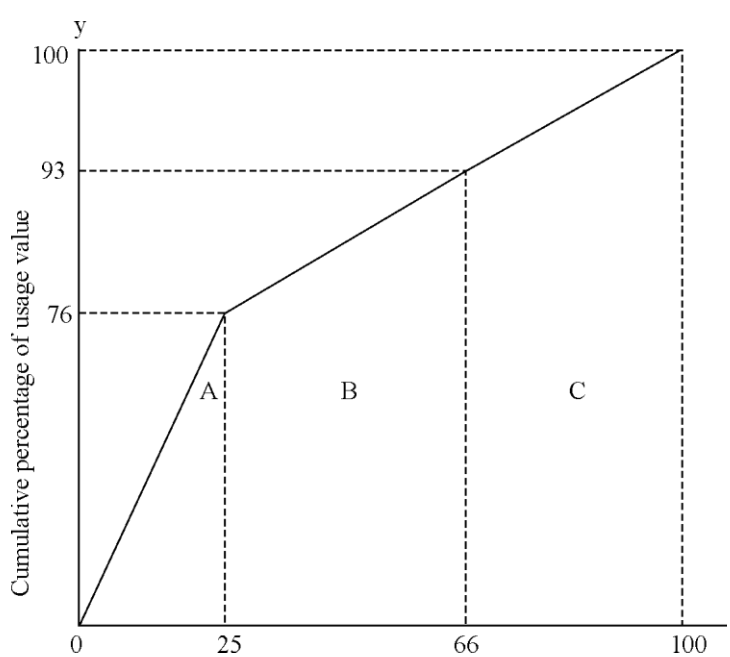


Figure 8.7 ABC Graph.

(c) VED analysis (Vital, Essential, Desirable Analysis)

We have seen that *ABC* Analysis depends on the annual consumption value but not on the unit price of the item. In *VED* analysis the **criticality** of the item is most important than the cost factor of the item. Here ***V*** stands for **Vital items**, *E* stands for **Essential items** and ***D*** stands for **Desirable items**. The criticality may be of two types *i.e*. (*a*) Technical criticality and (*b*) Environmental criticality.

(i) Vital items

V items are more critical in nature, that is, without which the system cannot run. In absence of critical items the organization has to come to stand still and it cannot keep up delivery promises. The idle cost and the penalty for not meeting the delivery promises may be a very big loss to the organization. Say for example in an automobile, a clutch wire, spare tyre, are critical items. It is because while we are on road, if clutch wire fails, then it is very difficult to drive the vehicle and we have to stop the vehicle until it is replaced. Similarly if any one tyre punctures, unless it is repaired we cannot run the vehicle, hence by replacing the punctured tyre by spare tyre we can drive the vehicle. One more example of vital items say for example for an officer who is working in forest area, the scorpion medicine or medicine for snake bite is more critical and he must have a stock of it for the use in emergency. But for a person who is living in a multistoried building in a posh locality of a city it is not vital item. So criticality of item depends on the nature of requirement. Another example of this is for a man who is suffering from heart problem, the medicine required for heart attack is so vital that he must always have it in his pocket to avoid the casuality due to non-availability of the medicine. For a healthy man it is not vital to have a stock of the same pills. Hence the **Vital situation** varies from industry-to-industry, person-to-person and situation-to-situation.

(ii) Essential items

Such items, which when demand arises are not available, they may not stop the operation of the system, but they reduces the efficiency of the system. Say for example, for a automobile vehicle, horn, head light bulb are essential item. If they are not there, the vehicle still can be run but with risk. For a family the pain balms, headache medicine, are essential items. Even without them they can work but with less efficiency. If they are available, they can apply the balm or take medicine and get relieved of the pain and work efficiently.

(iii) Desirable items

These items are of the nature, if they are not available, they will not stop the system from working nor they reduce the efficiency of the system. But it is better to have them in stock to run the system without any difficulty.

The *VED* analysis as said above depends on the criticality of the item and not on the cost – either unit cost or annual consumption value. Depending on the criticality and demand of the item one has to decide how much the stores manager has to stock the material. This is particularly important in capital-intensive process industries and in case of stock controlling of spare parts required for maintenance. This analysis also helpful in stocking of raw materials which are rarely available and which have demand in manufacturing the products.

Any materials manager has to consider both the cost factor and criticality of item while deciding how much to stock. Especially while dealing with spare parts for maintenance, the service level of different class of spares depending on the cost and criticality can be understood from the matrix given below: The matrix shows that vital and A class items must have 90% service level i.e. 90% of the time they must be available.

	V	*E*	*D*
A	90%	80%	70%
B	95%	85%	75%
C	99%	90%	80%

C class and desirable items must be available 80% of time. The other way of presenting the same thing is as given below:

	V	*E*	*D*
A	Regular stock with constant control	Medium stock	No Stock.
B	Medium Stock	Medium Stock	Very Low stock
C	High stock	Medium stock	Low stock

(d) *XYZ* Analysis based on the inventory value

In *ABC* analysis we have seen the analysis depends on the annual consumption value of the item. In *XYZ* - Analysis classification is made on the **closing Inventory value** of the item. By wrong purchase policy there might be an excess stock at the year ending stock verification. This shows that unnecessarily inventory is lying in the stores *i.e.* money is simply locked in the form of inventory, without any use. If we combine *ABC* analysis with *XYZ* analysis, we can get more benefits and unnecessary stock may be reduced.

	X	*Y*	*Z*
A	Attempt to reduce the stock	Attempt to convert Z items.	Items are with in control
B	Review stock and consumption More often	Items are with in control	Review bi-annually.
C	Dispose of the surplus items	Check and maintain the Control.	Review annually.

(e) *FNSD* - Based on usage rate of items

This classification of items depends on the usage rate of the items or movement of the items. Here ***F*** stands for **Fast moving items, *N* for Normal moving items, *S* for slow moving items and *D* for Dead items.**

This analysis is useful in optimal utilization of storage area or space available for storing the materials. This also helps in saving the issue time of material. This analysis is useful to combat obsolete items. While classifying the items the demand and issue pattern studied carefully. The items, which have high demand and frequently indented, are kept very nearer to storekeeper, so that the handling time is reduced. Slow moving items or item, which have low demand, can be kept at a distance so that they will not cause inconvenience for the movement of store personnel. *D* class items are moved to disposal cell, to dispose by auction. We can combine *FNSD* analysis with *XYZ* analysis to get more benefits.

All the above analysis techniques are termed as **selective control technique**. On next page, given is the summary of the selective control techniques.

S.No	*Selective control technique*	*Basis of classification*	*Main Use.*
1	ABC	Annual consumption value	Controlling raw material components and work in process inventory.
2	VED	Criticality of item.	Determining the inventory levels of spare parts.
3	XYZ	Value of items in storage	Reviewing the inventories and other uses
4	FNSD	Consumption rate or Movement of items.	Controlling obsolescence.

Before discussing Economic Order Quantity (*EOQ*) model let us discuss certain aspects, which are important to understand the model.

The inventory system may be classified depending on the nature of variables. The variables are various costs, such as Carrying cost (C_1), Shortage cost (C_2), Ordering Cost (C_3), demand, Lead time, Reorder cycle time, Input rate and shortages.

The cost elements C_1, C_2, and C_3 per time period and the unit price of the item may be constant or variable in an inventory system.

Demand may be known and constant (static) or known and variable (dynamic) or it may be estimated one in an inventory system.

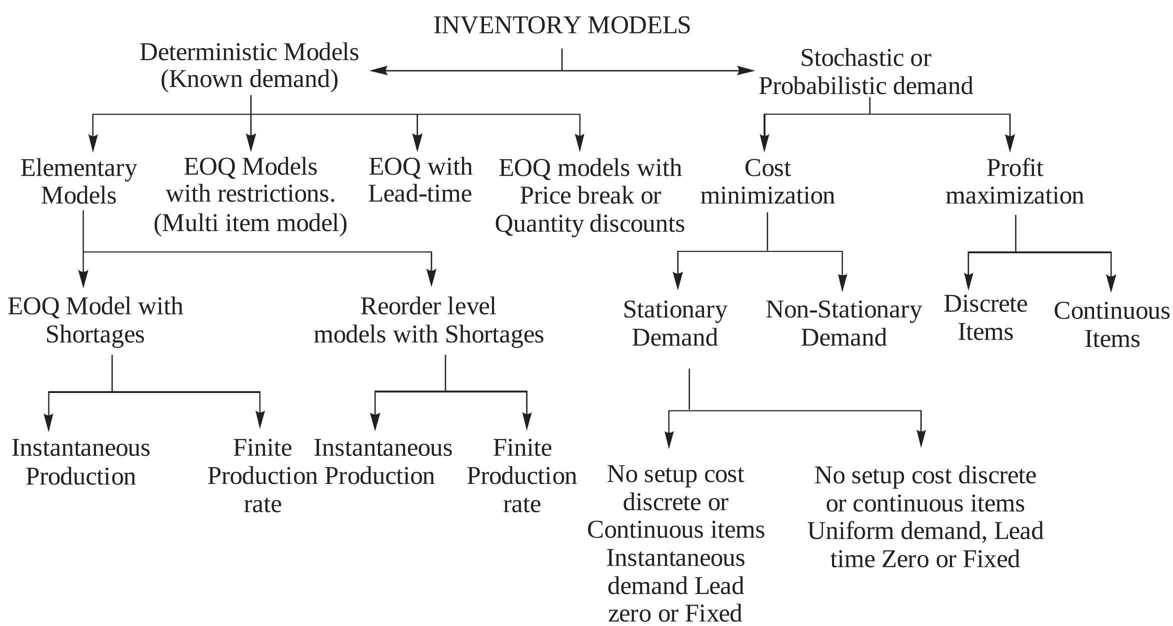
Lead-time may be zero or it may be known or it may be estimated one.

Re-order cycle time may be known constant or known and variable or it may be estimated one.

Input rate may be instantaneous or it may be finite.

Shortages may be allowed or not allowed. If allowed, it may be back logged, or lost sales.

Inventory models may also be classified as follows:



Notations used in the models

q = Lot size for one time interval for purchase models and for one run or cycle for manufacturing model.

r = Rate of demand or quantity required for one unit of time.

k = The rate of production or rate of supply of items to the inventory or rate of replenishment of inventory.

S = Level of inventory.

z = A level of inventory of short items *i.e.* unsatisfied demand.

t = Time interval between two consecutive replacements of inventory.

C (***q***) = Total inventory cost per unit of time as a function of level of inventory, *q*.

T = Time period in units for which the optimal policy is to be determined or Time horizon.

R = The total replenishment for the time *T*.

p (***r***) = Probability density function for '*r*', in case of discrete items of quantity.

f (***r***) = Probability density function for '*r*', in case of continuous units of quantity.

q_0, t_0 , S_0 = Optimal values of *q*, *t*, *S* respectively, *i.e*. the value for which the cost is minimum.

INVENTORY MODELS: DETERMINISTIC MODELS

Economic Lot Size Models or Economic Order Quantity models (EOQ models) - with uniform rate of demand

F. Harries first developed the Economic Order Quantity concept in the year 1916. The idea behind the concept is that the management is confronted with a set of opposing costs like ordering cost and inventory carrying costs. As the lot size '*q*' increases, the carrying cost 'C_1' also increases while the ordering cost 'C_3' decreases and vice versa. Hence, Economic Ordering Quantity – *EOQ* - is that size of order that minimizes the total annual (or desired time period) cost of carrying inventory and cost of ordering under the assumed conditions of certainty and that annual demands are known.

Economic Order Quantity by Trial and Error Method

Let us try to workout Economic Order Quantity formula by trial and error method to understand the average inventory concept. The steps involved are:

1. Select the number of possible lot sizes to purchase.
2. Determine total cost for each lot size chosen.
3. Calculate and select the order quantity that minimizes total cost.

While working the problems, we will consider ***Average inventory*** concept. This is because, the inventory carrying cost which is the cost of holding the inventory in the stock, cannot be calculated day to day as and when the inventory level goes on decreasing due to consumption or increases due to replenishment. For example, let us say the rent for the storeroom is Rs.500/- and we have an inventory worth Rs. 1000/-. Due to daily demand or periodical demand the level may vary and it is practically difficult to calculate the rent depending on the level of inventory of the day. Hence what we do is we use average inventory concept. This means that at the beginning of the cycle the level of inventory is Worth Rs. 1000/- and at the end of the cycle, the level is zero. Hence we can take the average of this two *i.e*. (0 + 1000) / 2 = 500. Let us take a simple example and see how this will work out.

Demand for the item: 8000 units. (q)

Unit cost is Re.1/- (p)

Ordering cost is Rs. 12.50 per order, (C_3)

Carrying cost is 20% of average inventory cost. (C_1)

Number or Orders Per year	*Lot size q*	*Average Inventory q / 2*	*Carrying Charges $C_1 = 0.20$ (Rs)*	*Ordering Cost C_3 (Rs)*	*Total cost (Rs.)*
1	8000	4000	800	12.50	812.50
2	4000	2000	400	25.00	425.00
4	2000	1000	200	50.00	250.00
8	**1000**	**500**	**100**	**100.00**	**200.00**
12	667	323	66	150.00	216.00
16	500	250	50	200.00	250.00
32	50	125	25	400.00	425.00

Observe the last column. The total cost goes on reducing and reaches the minimum of Rs. 200/- and then it increases. Also as lot size goes on decreasing, the carrying cost decreases and the ordering cost goes on increasing. Hence we can say the optimal order quantity is 1000 units and optima number of orders is 8. See at the optimal order quantity of 1000 units, both ordering cost and inventory costs are same. Hence we can say ***that the optimal order quantity occurs when ordering cost is equal to the inventory carrying cost.*** This we can prove mathematically and illustrate by a graph. This will be shown in the coming discussion.

It is not always easy to work for economic order quantity by trial and error method as it is difficult to get exact quantity and hence we may not get that ordering cost and inventory carrying costs equal. Hence it is better to go for mathematical approach.

Economic Lot Size (for manufacturing model) or Economic Order Quantity (EOQ for purchase models) without shortage and deterministic Uniform demand

When we consider a manufacturing problem, we call the formula as **Economic Lot Size (ELS) or Economic Batch Quantity (EBQ)**. Here the quantity manufactured per batch is lot size (order quantity in manufacturing model), fixed charges or set up cost per batch, which is shared by all the components manufactured in that batch is known as **Set up cost** (similar to ordering cost, as the cost of order is shared by the items purchased in that order), the cost of maintaining the in process inventory is the inventory carrying charges. Here a formula for economic lot size 'q' per cycle (production run) of a single product is derived so as to minimize the total average variable cost per unit time.

Assumptions made:

1. Demand is uniform at a rate of 'r' quantity units per unit of time.
2. Lead time or time of replenishment is zero (some times known exactly).
3. Production rate is infinite, *i.e.* production is instantaneous.
4. Shortages are not allowed. (*i.e*. stock out cost is zero).
5. Holding cost is Rs. C_1 per quantity unit per unit of time.
6. Set up cost is Rs. C_3 per run or per set up.

By trial and error method we have seen that economic quantity exists at a point where both ordering cost and inventory carrying cost are equal. This is the basis of algebraic method of derivation of formula.The figure 8.8 shows the lot size '*q*', uniform demand '*r*' and the pattern of inventory cycle.

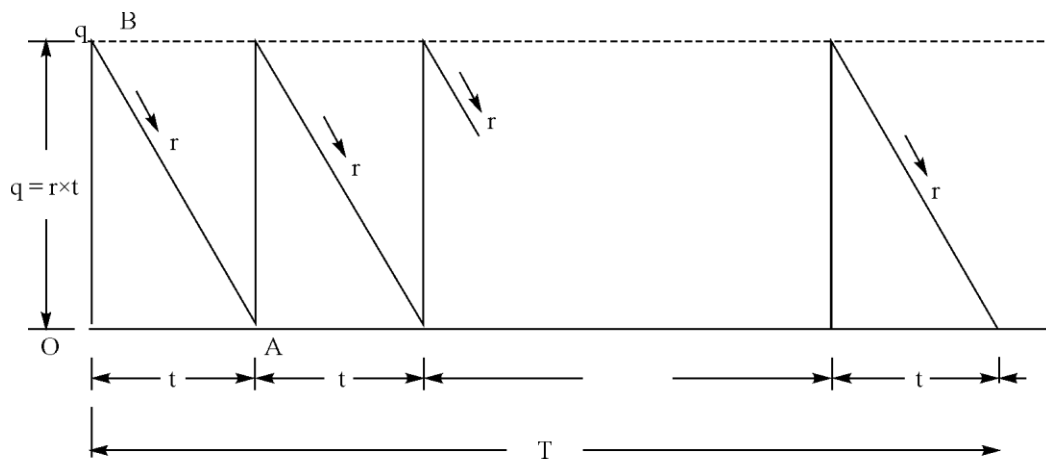


Figure 8.8. Deterministic uniform demand with no shortages.

Total inventory in one cycle *i.e.* for one unit of time = Area or triangle *OAB* = ½ the base (t) × altitude (q) =

$\frac{1}{2} \times q \times t = \frac{1}{2} qt$

{This can also be done mathematically by using calculus. At any time '*t*' from the beginning of the cycle (where '*t*' does not represent the time of one cycle), the inventory = $(q - rt)$. Hence the total inventory in the small time interval t to $(t + \delta t)$ is q- $rt) \times \delta t$. Summing over the period '*t*' of one cycle, the total inventory in one cycle is:

$$= \int_0^t (q - rt)\, dt == [qt - rt^2 / 2] \quad = [qt - \tfrac{1}{2} rt^2] = qt - \tfrac{1}{2} t \times rt$$

= As we know $q = rt$, we can write as $qt - \frac{1}{2} qt = \frac{1}{2} qt$.}

Carrying cost for '*t*' units of time = $\frac{1}{2} qt \times C_1$

Set up cost for one cycle is C_3

Hence total cost for one unit of time = carrying cost + ordering cost = $\frac{1}{2} qt\, C_1 + C_3$

Total cost per unit of time = $C_q = \{\frac{1}{2} qtC_1 + C_3\} / t = \frac{1}{2} q\, C_1 + C_3/t$

(We know that $q = rt$ hence $t = q/r$), substituting this for t in the above equation, we get

$C_q = \frac{1}{2} q\, C_1 + C_3 r/q$ - this is known as COST EQUATION.

(Note: For any inventory model, first we have to get this cost equation and then we have to optimize)

The optimum value of 'q', which minimizes C_q, is obtained by equating the first derivative of C_q with respect to 'q' to zero.

$dC_q/d_q = ½\, C_1 – C_3r/q^2 = 0$

$½\, C_1 = C_3r/q^2$ or $q^2C_1/2 = C_3r$ or $q^2 = 2\, C_3r / C_1$ or $\mathbf{q_o} = \sqrt{2C_3r/C_1}$

This is the formula for economic lot size or economic batch size. This is also known as **Harri's** formula or **Wilson** formula or **square root** formula. q_0 in manufacturing model is abbreviated as **EBQ, Economic Batch Quantity.**

> Note: Here we can show that *EBQ* exists at a point where carrying charges are equal to ordering cost.
>
> From the above derivation we have:
>
> $½\, C_1 = C_3r / q^2$
>
> This can be written as $½\, q\, C_1 = C_3$ x (r/q). = Average inventory × C_1 = ordering cost x number of orders.
>
> Now r is the demand and the q is the lot size. Hence r/q gives us number of batches in manufacturing model and number of orders in purchase models. Ordering cost C_3 x number of batches or set ups gives us total set up cost in manufacturing model. (In purchase model, number of orders × ordering cost C_3 gives us total ordering cost).
>
> $½\, q$ is the average inventory and multiplied by carrying cost C_1 gives us total inventory carrying cost. **Hence it is concluded that the Economic Batch Size (or Economic Order Quantity in purchase model) exists at a point where *inventory-carrying cost is equal to ordering cost.***

We know that $q = rt$, *i.e.* $t = q/r$ hence $q_0 = t_0 /r$ or $t_0 = q_0 \times r$

Therefore, by multiplying the Squre root formula by 'r' we get $t_0 = \sqrt{2C_3 / C_1r}$ To find optimal batch quantity the variable 'r' will be in the numerator of *EBQ* formula and when we want to find Optimal time of starting the batch in manufacturing model or optimal order time in purchase model, the variable 'r' will be in denominator of *EBQ* formula.

Similarly, we can find the optimal cost $C_0 = \{½ \sqrt{2C_3r / C_1\}} \times C_1 + rC_3 \times \sqrt{\{C_1 / 2C_3r}$

$= ½ \sqrt{(2C_1C_3r)} + ½ \sqrt{(2C_1C_3r)} = C_0 = \sqrt{(2C_1C_3 \times r)}$ give us optimal cost. If we want to find the total cost we have to add material cost which is equal to q × unit price (= p) = $q \times p$.

Total cost = Inventory carrying cost + material cost.

Graphical representation of Total cost curve:

(i) Behaviour of inventory carrying cost: As the level of inventory goes on increasing, the inventory carrying cost goes on increasing as it solely depends on the size of the inventoy.

(ii) The ordering cost or set up cost per unit reduces with the increase in the number of orders.

(iii) Total cost first goes on reducing and after reaching the minimum it goes on increasing. In the first part, *i.e.* while it decreases, it has the influence of ordering cost and in the latter part, *i.e.* while it is increasing, it has the influence of inventory carrying cost.

(iv) When curves are drawn, both carrying cost curve and ordering cost curve will intersect at a point. This point lies exactly where the lowest total cost appears on the graph.This is shown in the figure 8.9.

The figure 8.9 shows the cost curve. It consists of carrying cost curve, which is a straight line, ordering cost line, which is hyperbolic, and the total cost curve drawn with the sum of carrying cost and ordering cost. We can see that the curve is not pointed at minima, but it is flat. This shows that optimal order quantity varies over the significant range of flat curve (near $q = q_0$). One more point of importance is that changes in carrying and setup costs will give a small change in optimal lot size 'q'. Hence, we can conclude that using the approximate carrying cost we can obtain the *EBQ* or *EOQ*. Because of this fact, one can use approximate values of the costs and estimate the order quantity or batch quantity.

Summary of formulae

1. **Economic Batch quantity or Economic Order Quantity =** $\sqrt{(2C_3 \times r)/C_1}$ **=**
   $\sqrt{(2 \times \text{ordering cost} \times \text{demand rate})}$ / Carrying cost.

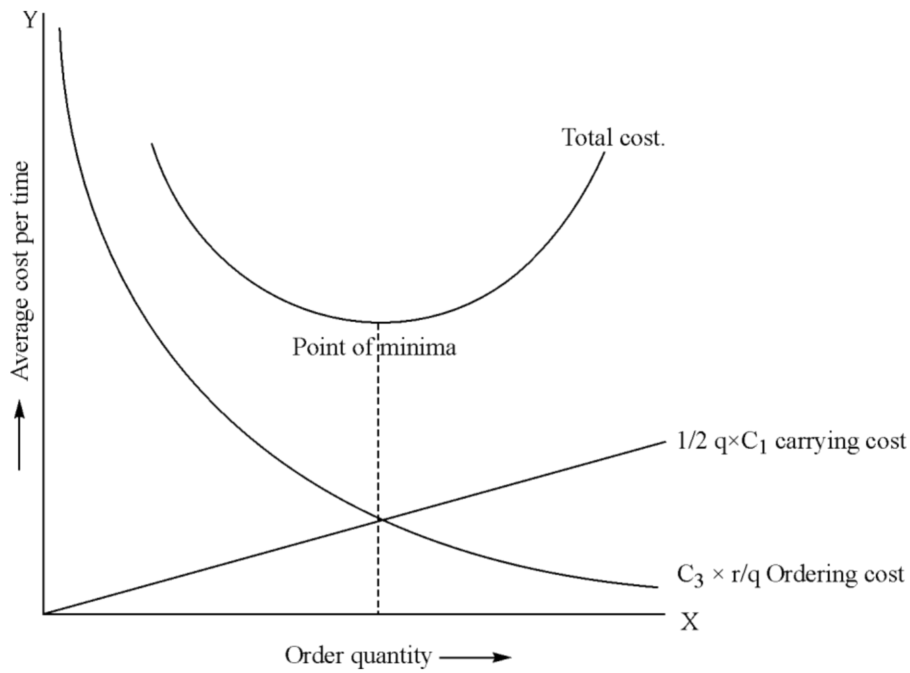


Figure 8.9 Cost curves.

2. **Optimal inventory cost =** $\sqrt{2 \times C_1 \times C_3 \times r}$ **=**
   $\sqrt{(2 \times \text{carrying cost} \times \text{ordering cost} \times \text{demand rate.})}$
3. **Optimal order time =** $\sqrt{(2 \times C_3)/C_1 \times r}$ **=** $\sqrt{(2 \times \text{ordering cost})}$ **/ (carrying cost × demand rate).**
4. **Optimum number of orders: $N_0 = \lambda / q_0$ = Annual demand / optimal order quantity.**

The number of day's supply per optimum order is obtained by = $d = 365 / N_0$
= 365 / optimum number of orders.

Problem 8.5.

The demand for an item is 8000 units per annum and the unit cost is Re.1/-. Inventory carrying charges of 20% of average inventory cost and ordering cost is Rs. 12.50 per order. Calculate optimal order quantity, optimal order time, optimal inventory cost and number of orders.

Solution

Data: λ = 8000 units, p = Re.1/-, C_1 = 20 % of average inventory or 0.20, Ordering cost = Rs. 12.50 per order.

$q_0 = \sqrt{(2 \times 12.50 \times 8000)}$ / (1.00 × 0.20) = $\sqrt{(16000 \times 12.50)}$ / 0.20 = $\sqrt{2{,}00{,}000}$ / 0.20 = 1000 units.

$C_0 = \sqrt{2 \times C_1 \times C_3 \times \lambda} = \sqrt{2 \times 0.02 \times 1.00 \times 12.50 \times 8000} = \sqrt{0.04 \times 12.50 \times 8000} = \sqrt{4000}$ = Rs.200/-

Inventory carrying cost = $(q/2) \times p \times C_1$ = (1000 /2) × 1.00 × 0.20 = Rs. 100/-

Total ordering cost = Number of orders × ordering cost = (Demand / q_0) × C_3 = (8000 / 1000) × 12.50 = Rs. 100/-

Total inventory cost = Carrying cost + ordering cost = Rs. 100 + Rs. 100 = Rs. 200/- (This is same as obtained by application of formula for total cost.

Optimal number of orders = Annual demand / optimal order quantity = λ / q_0 = 8000 / 1000 = 8 orders.

Optimal order period = $t_0 = q_0 / r$ = Optimal order quantity / demand rate. = 1000 / 8000 = 1/8 of a year.

= 365 / 8 = 45.6 days = app 46 days.

Total cost including material cost = Inventory cost + material cost = Rs. 200 + Rs. 8000 = Rs. 8200/-

Problem 8.6.

For an item the production is instantaneous. The storage cost of one item is Re.1/- per month and the set up cost is Rs. 25/- per run. If the demand for the item is 200 units per month, find the optimal size of the batch and the best time for the replenishment of inventory.

Solution

Here we take one month as one unit of time. (Note: Care must be taken to see that all the data given in the problem must have same time base *i.e*. year / month/week etc. If they are different, *e.g.* the carrying cost is given per year and the demand is given per month, then both of them should be taken on same time base.). Hence it is better to write date given in the problem first with units and then proceed to solve.

Data: Storage cost: Re.1/- per month = C_1, Set up cost per run = Rs. 25/- per run, Demand = 200 units per month.

Optima batch quantity = Economic Batch Quantity = $EBQ = \sqrt{(2C_3 r)}$ / C_1 = $\sqrt{(2 \times 25 \times 200)}$ / 1 = $\sqrt{10{,}000}$

= 100 units.

Optimal time of replenishment = $T_0 = q_0 / r$ or $\sqrt{(2C_3)}$ / $C_1 \times r$ = 100 /200 = ½ month = 15 days.

Optimal cost = C_0 = $\sqrt{2C_1C_3r}$ = $\sqrt{2 \times 1 \times 25 \times 200}$ = Rs. 100/- **OR**

It can also be found by Total cost = Carrying cost + Ordering cost = $(q/2) \times C_1 + C_3 \times r/q$ =
(100/2) × 1 + 25 × 200 / 100 = 50 + 50 =
Rs. 100/-

Problem 8.7.

A producer has to supply 12,000 units of a product per year to his customer. The demand is fixed and known and backlogs are not allowed. The inventory holding cost is Rs.0.20 per unit per month and the set up cost per run is Rs. 350/- per run. Determine (*a*) the optimal lot size, (*b*) Optimum scheduling period,

(*c*) Minimum total expected yearly cost.

Solution

Data: λ = Demand per year 12,000 units, C_1 Rs. 0.20 per unit per month, C_3 = Rs. 350/- per run.
r = demand per month 12,000/12 = 1000 units.

q_0 = $\sqrt{(2C_3 \times r)}$ / C_1 = $\sqrt{(2 \times 350 \times 1000)}$ / 0.02 = 1870 units per batch.

$t_0 = q_0 / r$ = 1870 / 1000 = 1.87 month. = 8.1 weeks.

C_0 = $\sqrt{(2C_3C_1r)}$ = $\sqrt{(2 \times 0.20 \times 350 \times 1000)}$ = $\sqrt{1{,}40{,}000}$ = Rs.374.16 = App Rs. 374/

Problem 8.8.

A particular item has a demand of 9,000 units per year. The cost of one procurement is Rs. 100/- and the holding cost per unit is Rs. 2.40 per year. The replacement is instantaneous and no shortages are allowed. Determine: (*a*) Economic lot size, (*b*) The number of orders per year, (*c*) The time between orders, and (*d*) the total cost per year if the cost of one units is Re.1/-.

Solution

Data: λ = 9,000 units per year, C_1 = Rs. 2.40 per year per unit, C_3 = Rs. 100/- per procurement.

(*a*) q_0 = $\sqrt{(2C_3\lambda)}$ / C_1 = $\sqrt{(2 \times 100 \times 9000)}$ / 2.40 = 866 units per procurement.

(*b*) $N = (1/t_0)$ = $\sqrt{(C_1 \times \lambda)}$ / 2 C_3 = $\sqrt{(2.40 \times 9{,}000)}$ / 2 x 100 = $\sqrt{108}$ = 10.4 orders per year.

This can also be found by (λ / q_0) = 9000 / 866 = 10.39 = 10.4 orders per year.

(*c*) $t_0 = 1 / N$ = 1 / 10.4 = 0.0962 years between orders. OR $t_0 = q_0 / \lambda$ = 866 / 9000 = 0.0962 year between orders. (= 35.12 days = App. 35 days.)

(*d*) C_0 = $\sqrt{(2C_1C_3\lambda)}$ = $\sqrt{(2 \times 2.40 \times 100 \times 9000)}$ = Rs. 2,080/-

Total cost including material cost = 9000 × 1 + 2,080 = Rs. 11, 080/- per year.

Problem 8.9.

A precision engineering company consumes 50,000 units of a component per year. The ordering, receiving and handling costs are Rs.3/- per order, while the trucking cost are Rs. 12/- per order. Further details are as follows:

Interest cost Rs. 0.06 per units per year. Deterioration and obsolescence cost Rs.0.004 per unit per year. Storage cost Rs. 1000/- per year for 50,000 units. Calculate the economic order quantity, Total inventory carrying cost and optimal replacement period.

Solution

Data: λ = 50,000 units per year.

C_3 = Rs. 3/- + Rs. 12/- = Rs. 15/- per order.

C_1 = Rs. 0.06 + 0.004 + 1000 / 50,000 per unit = Rs. 0.084 / unit. Hence,

$q_0 = \sqrt{(2C_3\lambda)} / C_1 = \sqrt{2 \times 15 \times 50000)}$ / 0.084 = 4226 units.

$t_0 = \lambda / q_0$ = 50000 / 4226 = 11.83 years.

$C_0 = \sqrt{2 \times C_3 \times C_1 \times \lambda} = \sqrt{2 \times 15 \times 0.084 \times 50{,}000)} = \sqrt{1{,}26{,}000}$ = Rs. 355/-

Problem 8.10.

You have to supply your customer 100 units of certain product every Monday and only on Monday. You obtain the product from a local supplier at Rs/ 60/- per units. The cost of ordering and transportation from the supplier are Rs. 150/- per order. The cost of carrying inventory is estimated at 15% per year of the cost of the product carried. Determine the economic lot size and the optimal cost.

Solution

Data: r = 100 units per week, C_3 = Rs. 150/- per order, C_1 = (15/100) × 60 per year Rs.9/ year. Hence

Rs. 9/52 per week.

$q_0 = \sqrt{(2C_3 \times r)} / C_1 = \sqrt{(2 \times 150 \times 100 \times 52)}$ / 9 = 416 units.

$C_0 = \sqrt{(2 \times C_3 \times C_1 \times r)} = \sqrt{\{2 \times (9/52) \times 150 \times 100\}}$ = Rs. 72/-

Including material cost (60 × 100) + 72 = Rs. 6072 per year.

Problem 8.11.

A stockiest has to supply 400 units of a product every Monday to his customers. He gets the product at Rs. 50/- per unit from the manufacturer. The cost of ordering and transportation from the manufacturer is Rs. 75 per order. The cost of carrying inventory is 7.5% per year of the cost of the product. Find

(*i*) Economic lot size, (*ii*) The total optimal cost (including the capital cost).

Solution

Data: r = 400 units per week, C_3 = Rs. 75/- per order, p = Rs. 50 per unit.

C_1 = 7.5% per year of the cost of the product. = Rs. (7.5 /100) × 50 per unit per year. = Rs. (7.5/ 100) × (50 / 52) per week. Rs. 3.75 / 52 per week = Rs. 0.072 per week.

$q_0 = \sqrt{(2C_3 \times r)} / C_1 = \sqrt{(2 \times 75 \times 400)}$ / 0.072 = 912 units per order.

$C_0 = \sqrt{2 \times C_3 \times C_1 \times r} = \sqrt{(2 \times 75 \times 0.072 \times 400)}$ = Rs. 65.80

Total cost including material cost = 400 × 50 = 65.80 = 40,000 = 65.80 = Rs. 20,065.80 per week.

Economic order Quantity for purchase model

All the assumptions made in the Economic Batch Quantity model will remain same but we will take annual demand (λ) and price per unit (*i.e.* Material cost) = p and inventory carrying charges are expressed is 'i' % of average inventory value. Hence, we take inventory carrying charges $C_1 = ip$.

Let us work Economic Order Quantity (*EOQ*) formula for purchase model.

Average inventory = $q/2$

Inventory carrying charges = ip

Therefore inventory carrying charges = $(q/2) \times ip$ or = $ipq/2$

As the annual demand = λ and the order quantity = q, Number of orders = $\lambda / q = N$

Hence ordering cost = $C_3 \times (\lambda /q)$

Total cost = $C_q = (ip) \times q/2 + (\lambda /q) \times C_3$

The minimum of C_q will get when first derivative is equals to zero.

$DC/dq = ½\, ip - C_3 \lambda / q^2 = 0$ *i.e.* $½\, ip = C_3 \lambda / q^2$, Simplifying, we get

$q_0 = \sqrt{(2C_3\lambda)} / ip$, This is known as Economic Order Quantity or *EOQ*.

Similarly, t_0 = **optimal order time** = $\sqrt{(2C_3)} / (ip \times \lambda)$

Optimal cost = $C_{q0} = \sqrt{(2C_3 \times ip \times \lambda)}$

Total cost including material cost is given by: Ordering cost + carrying cost + material cost.

$= (\lambda /q) \times C_3 + (q /2) \times ip + p\lambda$ OR = $\sqrt{(2C_3 \times ip \times \lambda)} + \lambda p$

Problem 8.12.

A stockiest purchases an item at the rate of Rs. 40/- per piece from a manufacturer. 2000 units are required per year. What should be the order quantity per order if the cost per order is Rs. 15/ and the inventory charges per year are 20 percent?

Solution

Data: p = Rs. 40/ - per item, Demand = λ = 2000 units per year, C_3 = Rs. 15/- per order, i = 20% = 0.20

$q_0 = \sqrt{(2C_3\lambda)} / ip = \sqrt{(2 \times 15 \times 2000)} / (0.20 \times 40)$ = 87 units per order.

$t_0 = q_0 / \lambda$ = 87 / 2000 = 0.044 of an year. = 15,87 days = app. 16 days,

$C_q = \sqrt{(2C_3 \times ip \times \lambda)} + \lambda \times p = \sqrt{(2 \times 15 \times 40 \times 0.02 \times 2000)} + 2000 \times 40$ = Rs. 692.8 + 80,000 = Rs. 80,693/-

Problem 8. 13.

A shopkeeper has a uniform demand of an item at the rate of 50 units per month. He buys from supplier at the cost of Rs. 6/- per item and the cost of ordering is Rs. 10/- each time. If the stock holding costs are 20% per year of stock value, how frequently should he replenish his stocks? What is the optimal cost of inventory and what is the total cost?

Solution

Data: Monthly demand r = 50 units, hence yearly demand λ = 600 units.

C_3 = Rs. 10/- per order, i = 20% of stock value, p = Rs. 6 per item.

$q_0 = \sqrt{(2C_3\lambda)}$ / ip = $\sqrt{(2 \times 10 \times 600)}$ / (0.20 × 6) = $\sqrt{10000}$ = 100 items

$t_0 = q_0 / \lambda$ = 100 / 600 = 1/6 of an year = 2 months. He should replenish every two months.

$C_{q0} = \sqrt{(2 \times C_3 \times ip \times \lambda)} = \sqrt{(2 \times 10 \times 0.20 \times 6 \times 600)}$ = Rs. 120/-

Material cost = 600 × Rs.6/- = Rs. 3600/- . Hence total cost = Rs. 3600 + 120 = Rs. 3720/-

Problem 8.14.

A company uses annually 24,000 units of raw material, which costs Rs. 1.25 per units. Placing each order cost Rs. 22.50, and the carrying cost is 5.4%of the average inventory. Find the economic lot size and the total inventory cost including material cost.

Solution

Data: λ = 24,000 units. C_3 = Rs. 22.50 per order, i = 5.4% of average inventory, p = Rs. 1.25 per unit.

$q_0 = \sqrt{(2C_3 \times \lambda)}$ / ip = $\sqrt{(2 \times 22.50 \times 24{,}000)}$ / (.054 × 1.25) = 4000 units.

Total cost: Total cost can be found in two ways.

(i) $C_0 = \sqrt{(2 \times C_3 \times ip \times \lambda)} = \sqrt{2 \times 2.50 \times 0.54 \times 1.25 \times 24{,}000}$ = Rs. 270/-

(ii) Ordering cost = Number of orders × ordering cost = (λ / q_0) × 22.50 = (24,000 / 4000) × 22.50 = 6 × 22.50 = Rs. 135/-

Inventory carrying cost = (q_0/2) × 0.054 × 1.25 = (4000/2) × 0.054 × 1.25 = Rs. 135/-

Hence total cost = Rs. 135/- + Rs. 135/- = Rs. 270/-

Material cost = 24,000 × 1.25 = Rs. 30,000/-

Total cost = Rs. 30,000 + 270 = Rs. 30,270/-

Problem 8.15.

ABC manufacturing company purchase 9,000 parts of a machine for its annual requirement, ordering one month's usage at a time. Each part costs Rs. 20/-. The ordering cost per order is Rs. 15/ - and the inventory carrying charges are 15% of the average inventory per year. You have been asked to suggest a more economical purchasing policy for the company. What advice would you offer and how much would it save the company per year.

Solution

Data: λ = 9,000 units, C_3 = Rs. 15/- i = 0.15, p = Rs. 20/- per unit. Other data = purchasing monthly requirement. Hence the number of orders = 12.

$r = \lambda$ / number of orders = 9000 / 12 = 750 units per order.

Carrying cost = (q/2) × ip = (750 / 2) × 0.15 × 20 = Rs. 1,125/-

Ordering cost = Number of orders × C_3 = 12 × 15 = Rs. 180/-

Total cost = Rs. 1,125 + 180 = Rs. 1,305.

Suggestion: To purchase q_0 Economic order quantity.

$q_0 = \sqrt{2 \times C_3 \times \lambda)}$ / ip = $\sqrt{2 \times 15 \times 9000)}$ / 0.15 × 20 = 300 Units.

$C_{qo} = \sqrt{2 \times C_3 \times ip \times \lambda)} = \sqrt{2 \times 15 \times 0.15 \times 20 \times 9000)}$ = Rs. 900/-

Annual savings by the company by purchasing EOQ instead monthly requirement is:
= Rs. 1305 – Rs. 900 = Rs. 405/- a year.

Problem 8.16

Calculate *EOQ* in units and total variable cost for the following items, assuming an ordering cost of Rs.5/- and a holding cost is 10% of average inventory cost. Compute *EOQ* in Rupees as well as in years of supply. Also calculate *EOQ* frequency for the items.

Item	*Annual demand* = λ *units.*	*Unit price in Rs.* = p
A	800	0.02
B	400	1.00
C	392	8.00
D	13,800	0.20

Solution

Data: C_3 = Rs. 5/- per order, i = 0.10, p = Rs. 0.02, 1.00, 8.00, 0.20, λ = 800, 400, 392, 13,800 units.

Item	λ Units.	C_3 *in* Rs.	i	p *in* Rs.	q_0 *in units.* = $\sqrt{(2C_3\lambda)/ip}$ $C_0 = \sqrt{2C_3\, ip.\lambda}$	q_0 *in Rupees* = $q_0 \times p$	*Year of* supply = q_o / λ	*EOQ* frequency = 1/ (years of Supply) No of orders Per year.
A.	800	5	0.10	0.02	$\sqrt{(2 \times 5 \times 800)}$ / 0. 10 × 0.02 = 2000 units. $C_0 = \sqrt{2 \times 5} \times 0.10 \times 0.02 \times 800$ = Rs. 4/-	2000 × 0.02 = Rs. 40/-	2000/800 = 2.5 years.	1/2.5 = 0.4
B.	400	5	0.10	1.00	$\sqrt{(2 \times 5 \times 400)}$ /0.10 × 1 = 200 units. $C_0 = \sqrt{2 \times 5} \times 0.10 \times 1 \times 400$ = Rs. 20	200 × 1 = Rs. 200/-	200 / 400 = ½ year	1/0.5 = 2 orders.
C.	392	5	0.10	1.00	$\sqrt{(2 \times 5 \times 392)}$ /0.10 × 8 = 70 units. $C_0 = \sqrt{2 \times 5} \times 0.10 \times 8 \times 392$ = Rs. 56/-	70 ×8 = Rs. 560/-	70/392 = 0.18 year	1/0.18 = 5.56 orders.
D.	13,800	5	0.10	0.20	$\sqrt{(2 \times 5 \times 13{,}800)}$ / 0.10 × 0.20 = 2,627 units.	2,627 × 0.20 = Rs.525.40	2,627/ 13800 = 0.19 year	1/0.19 = 5.26 Orders.

Problem 8.17

(a) Compute the *EOQ* and the total variable cost for the data given below:

Annual demand = λ = 25 units, Unit price = p = Rs. 2.50, Cost per order = Rs. 4/-, Storage rate = 1% Interest rate = 12%, Obsolescence rate = 7%.

(b) Compute the order quantity and the total variable cost that would result if an incorrect price of Rs. 1.60 were used for the item.

Solution

(*a*) $C_1 = \{(1 + 12 + 7)/100\} \times 2.50$ = Rs. 0.50 per unit per year.

$q_0 = \sqrt{(2 \times 4 \times 25)}$ / 0.50 = 20 units.

$C_{q0} = \sqrt{(2 \times 4 \times 25 \times 0.50)}$ = Rs. 10/-

(c) $q_0 = \sqrt{2 \times 4 \times 25)}$ / {(20/100) × 1.60 = 25 units.

Ordering cost = $(C_3 \times \lambda) / q_0 = (4 \times 25) / 25$ = Rs. 4/-

Carrying cost = $(q_0/2) \times C_1 = \{(20/100) \times 2.50\} \times 25$ = Rs. 6.25

(Here, for calculating carrying cost, correct price is used instead incorrect price of Rs. 1.60).

Total variable cost per year = Rs. 4/- + Rs. 6.25 = Rs. 10.25.

Problem 8.18

An aircraft company uses rivets at an approximate customer rate of 2,500 Kg. per year. Each unit costs Rs, 30/- per Kg. The company personnel estimate that it costs Rs. 130 to place an order, and that the carrying costs of inventory is 10% per year. How frequently should orders for rivets be placed? Also determine the optimum size of each order.

Solution

Data: λ = 2,500 Kg. per year, C_3 = Rs. 130/-, i = 10%, p = Rs. 30/- per unit. ($C_1 = i \times p = 0.10 \times 30$ = Rs. 3/-)

$q_0 = \sqrt{(2 \times C_3 \times \lambda)} / ip$. $q_0 = \sqrt{(2 \times 130 \times 2500)} / 0.10 \times 30$ = App. 466 units

$t_0 = q_0 / \lambda = 466 / 2500 = 0.18$ year $= 0.18 \times 12 = 2.16$ month.

N = Number of orders = $\lambda / q_0 = 2500 / 466 = 5$ orders per year.

Problem 8.19

The data given below pertains to a component used by Engineering India (P) Ltd. in 20 different assemblies

Purchase price = p = Rs. 15 per 100 units,

λ Annul usage = 1,00,000 units,

Cost of buying office = Rs. 15,575 per annum, (fixed),

Variable cost = Rs. 12/- per order,

Rent of component = Rs. 3000/- per annum

Heating cost = Rs. 700/- per annum

Interest = Rs. 25/- per annum,

Insurance = 0.05% per annum based on total purchases,

Depreciation = 1% per annum of all items purchased.

(i) Calculate *EOQ* of the component.

(ii) The percentage changes in total annual variable costs relating to component if the annual usage happens to be (*a*) 125,000 and (*b*) 75,000.

Solution

λ = 100,000, C_3 = Rs. 12/-, C_1 = (15/100) × 0.25 + 0.0005 + 0.01) = 0.039075.

λ = 1,00,000, $q_0 = \sqrt{(2 \times 12 \times 10{,}000)}$ / 0.039075 = 7,873,

Ordering cost = 100000/ 7873 = Rs. 153.12

Carrying cost = (7873 / 2) × 0.039075 = Rs. 153.12

Total inventory cost = Rs. 153.12 + Rs. 153.12 = Rs. 306.25

Note that both ordering cost and carrying cost are same.

When λ = 125, 000

$q_0 = \sqrt{(2 \times 12 \times 1{,}25{,}000)}$ / 0.039075 = 8,762.

Ordering cost = (125,000/ 8762) = Rs. 171.12,

Carrying cost = (8762/2) × 039075 = RS. 171.12

Total inventory cost = RS. 171.12 + Rs. 171.12 = Rs. 342.31

When λ = 75,000

$q_0 = \sqrt{(2 \times 12 \times 75000)}$ / 0.039075 = 6,787 units.

Ordering cost = (75,000 / 6787) = Rs. 132.60

Carrying cost = (6787 / 2) × 0.39075 = Rs.132.60

Total cost = Rs. 132.60 + Rs. 132.60 = Rs. 264.20

Point to note: In all the three cases, ordering cost = Carrying cost, because they are at optimal order quantity. Also when the annual demand is 1,25,000, the total variable cost has increased by 12% (app) and when the demand is 75,000, it is decreased by 13%.

Economic lot size with different rates of demand in different periods

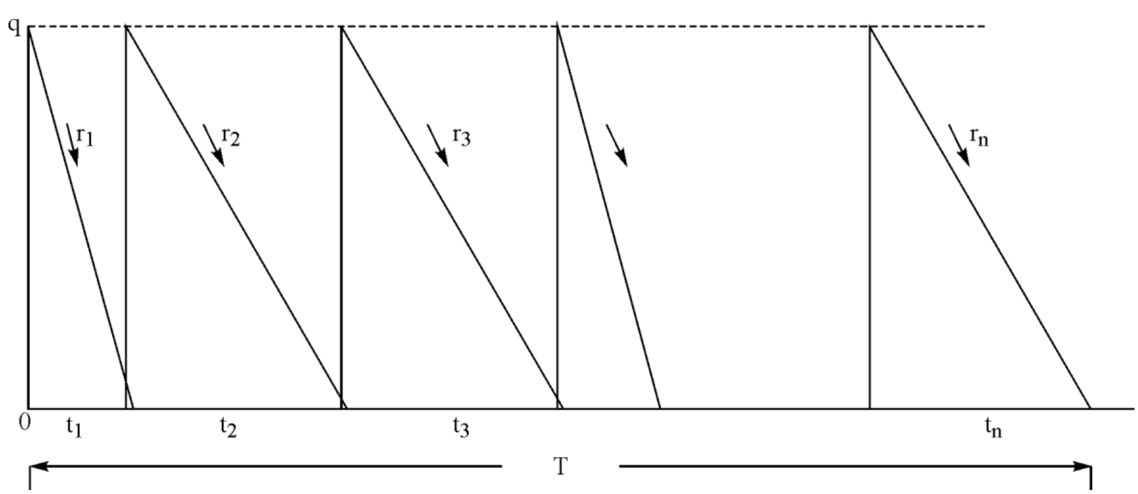


Figure 8.10

In the previous models, we have assumed that the demand is uniform and known and the time period is uniform. In the present model, the demand rate is different and period of the cycle is different. Suppose the time periods are $t_1, t_2, t_3 \ldots\ldots t_n$ and $\sum_{i=1}^{N} t_i = T$ and the demand be $r_1, r_2, r_3 \ldots r_n$ and $\sum_{i=1}^{n} r_i = R$, then,

The inventory carrying cost for the time period $T = (q/2)C_1t_1 + (q/2)C_1qt_2 + \ldots\ldots\ldots + (q/2)C_nt_n = (q/2)\ C_1 (t_1 + t_2 + t_3 + \ldots\ldots\ldots + t_n) = (q/2)\ C_1\ T$

Number of orders = Total demand for the period T / quantity ordered = R/q

Therefore, total setup cost or ordering cost for the period $T = C_3\ (R/q)$

Hence the total cost for the period T is given by: $C_q = (q/2)\ C_1\ T + C_3\ (R/q)$. This will be minimum when,

$dC_q/dq = (1/2)C_1T - C_3(R/q^2) = 0$, Simplifying, we get

$C_{0q} = \sqrt{(2C_3 R)} / C_1$. (Remember in all the models the ratio $\sqrt{(2C_3)} / C_1$ remains constant and depending on the demand pattern the value of r or R changes.

Similarly, $C_0q_0 = \sqrt{(2C_3 / C_1)}\ (R/T)$. Here R/T is the average rate of demand.

Problem 8.20

The demand for an item and the time period of consumption is given below. The carrying cost C_1 = Rs.2 / per unit and the ordering cost is Rs. 75/- per order. Calculate economic order quantity and the cost of inventory.

Demand in units. (r):	*25*	*40*	*30*	*20*	*70*
Period in months. (*t*)	1	2	2	1	6

Solution

$\Sigma\ t = 12$ months, $\Sigma\ r = 185$ units. C_1 = Rs.2/- and C_3 = Rs. 75/-

$q_0 = \sqrt{\{2 \times 75 \times (185/12)\}} / 2 = \sqrt{\{2 \times 75 \times 15.42\}} / 2 = \sqrt{2313} / 2 = \sqrt{1156.5}$ = App. 34 units.

$C_0 = \sqrt{2 \times C_3C_1}\ \ R/T = \sqrt{2 \times 75 \times 2 \times 185} / 12 = \sqrt{300 \times 15.42} = \sqrt{4626}$ = App. Rs. 68/-

Quantity Discount Model

Sometimes, the seller may offer discount to the purchaser, if he purchases larger amount of items. Say for example, if the unit price is Rs. 10/-, when customer purchase 10 or more than 10 items, he may be given 1% discount on unit price of the item. That means the purchaser, may get the item at the rate of Rs. 9/- per item. This may save the material cost. But, as he purchases more than the required quantity his inventory carrying charges will increase, and as he purchases more items per order, his ordering cost will reduce. When he wants to work out the optimal order quantity, he has to take above factors into consideration. The savings part of discount model is: (*a*) lower unit price, (*b*) lower ordering cost. The losing part of the model is (*a*) inventory carrying charges. The discount will be accepted when the savings part is greater than the increase in the carrying cost.

There are two types of discounts. They are: (*a*) **All units discount**: Here the customer is offered discount on all the items he purchase irrespective of quantity.

(b) **Incremental discount:** Here, the discount is offered to the customer on every extra item he purchases beyond some fixed quantity, say '*q*'. Up to '*q*' units the customer pays usual unit price and over and above '*q*' he is offered discount on the unit price.

Problem 8.21.

A shopkeeper has a uniform demand of an item at the rate of 50 items per month. He buys from a supplier at a cost of Rs.6/- per item and the cost of ordering is Rs. 10/- per order. If the stock holding costs are 20% of stock value, how frequently should he replenish his stock? Suppose the supplier offers 5% discount on orders between 200 and 999 items and a 10% discount on orders exceeding or equal to 1000 units. Can the shopkeeper reduce his costs by taking advantage of either of these discounts?

Solution

Data: C_1 = 20% per year of stock value, C_3 = Rs. 10/-, r = 50 items per month, $\lambda = 12 \times 50 = 600$ units per year, p = Rs. 6/- per item. Discounted price a) Rs.6 – 0.05 × 6 = Rs. 5.70, from 200 to 999 items,

(c) Rs.6 – 0.10 × 6 = Rs. 5.40, for 1000 units and above and $i = 0.20$

$q_0 = \sqrt{2C_3\lambda} / ip = \sqrt{(2 \times 10 \times 600)} / (0.20 \times 6) = 100$ units.

$t_0 = q_0 / \lambda = 100 / 600 = 1/6$ of a year. = 2 months.

$C_0 = \lambda \times p + \sqrt{2C_3\, ip\lambda} = 600 \times 6 + \sqrt{2 \times 100.20 \times 6 \times 600}$ = Rs. 3720.

This may be worked out as below: Material cost + carrying cost + ordering cost = 600 × 6 + (100/2) × 0.20 × 6 + 600 / 6 × 10 = 3600 + 60 + 60 = Rs. 3720/-.

(a) To get a discount of 5% the minimum quantity to be purchased is 200. Hence, let us take $q_0 = 200$

Savings: Savings in cost of material. Now the unit price is Rs. 5.70. Hence the savings is 600 × Rs. 6 – 600 × Rs.5.70 = Rs. 3600 – Rs. 3420 = Rs.180/-

Savings in ordering cost. Number of orders = $\lambda / q_0 = 600 / 200 = 3$ orders. Hence ordering cost = 3 × Rs. 10/- = Rs. 30. Hence the savings = Ordering cost of *EOQ* – present ordering cost = Rs.60 – Rs. 30 = Rs. 30.

Hence Total savings = Rs. 180 + 30 = Rs. 210/-

Additional cost due to increased inventory = present carrying cost – Carrying cost of *EOQ* =

(200 / 2) × 0.20 × Rs. 5.70 – (100/2) × 0.20 × Rs.6/- = 100 × 1.14 – 50 × Rs.1.2 = 114 – 60 = Rs. 54/-

Therefore, **by accepting 5% discount, the company can save Rs. 210 – Rs. 54 = Rs. 156/- per year.**

(b) 10% discount on $q_0 \geq 1000$.

Savings: Ordering cost:

Since 1000 items will be useful for 1000 / 600 = 5/3 years, the number of orders = 1 / (5/3) = 3 / 5 times in a year. Hence number of orders = 6 – 3/5 = 5.4 orders. Hence ordering cost = 5.4 × 10 = Rs. 54/-.

Savings in material cost: (10/100) × 6 × 600 = Rs. 360/-

Hence total savings = Rs. 360 + Rs. 54 = Rs. 414/-

Increase in the holding cost : (1000/2) × 0.20 × 0.90 × Rs. 6/- = Rs. 480/- As the savings is less than the increase in the total cost the discount offer of 10% can not be accepted.

Problem 8.22.

A company uses annually 24,000 units of raw material, which costs Rs.1.25 per unit. Placing each order costs Rs. 22.50 and the carrying cost is 5.4% per year of the average inventory. Find the economic lot size and the total inventory cost including material cost. Suppose, the company is offered a discount of 5% by the supplier on the cost price of single order of 24,000 units, should the company accept?

Solution

λ = 24,000, C_3 = Rs. 22.5, p = Rs.1.25, i = 5.4%.

$q_0 = \sqrt{2C_3\lambda}$ / ip = $\sqrt{(2 \times 22.5 \times 24{,}000)}$ / (0.054 × 1.25 = 4000 units.

Total cost per year = 24,000 × 1.25 + $\sqrt{2 \times 22.5 \times 24{,}000 \times 0.054 \times 1.25}$ = Rs. 30,000 + Rs. 270 = Rs. 30270/-.

To get the benefit of discount the lot size is 24,000 units.

Savings in the ordering cost: For *EOQ* 24,000 / 4000 = 6 orders. Hence ordering cost is 6 × 22.5 = Rs. 135/-

For 24,000 units per order, number of orders is one hence the ordering cost is Rs. 22.50, Hence savings is

Rs. 135 – Rs. 22.50 = Rs. 112.50.

Savings in material cost: 0.95 × Rs. 1.25 × 24,000 = Rs. 28,500. Savings in material cost =

Rs. 30,000 – Rs. 28,500 = Rs. 1, 500/-

Total savings = Rs. 1,500 + Rs. 112.50 = Rs. 1,612.50.

Additional burden in inventory carrying cost = Inventory cost for 24,00 units – Inventory carrying cost for *EOQ* =(24,000/2) × 0.054 × 0.95 × 1.25 – (4000 / 2) × 0.054 × 1.25 = Rs. 769.50 – Rs. 135/- = Rs.634.50.

Savings is Rs. 1,612.50 and the extra burden is Rs, 634.50. As the savings is more than the extra burden, the discount offer is accepted.

Economic Lot Size with finite rate of replenishment or production and uniform demand rate with no shortages: (Manufacturing model with no shortages). Assumption: Manufacturing rate is greater than the demand rate

In previous discussed models we have assumed that the replenishment time is zero and the items are procured in one lot. But in real practice, particularly in manufacturing model, items are produced on a machine at a finite rate per unit of time; hence we cannot say the replenishment time as zero. Here we assume that the replenishment rate is finite say at the rate of k units per unit of time. The economic lot size is q_0, carrying cost is C_1 and ordering cost is C_3. The model is given in the figure 8.11.

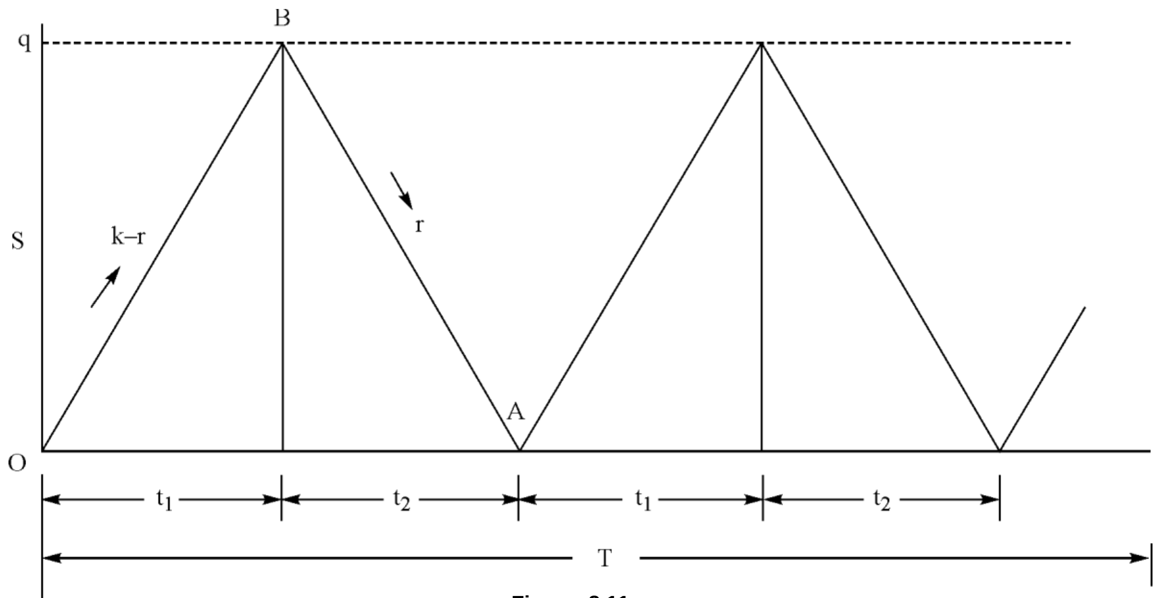


Figure. 8.11.

In the figure, we can see that in the first time period t_1 inventory build up, as the demand rate is less than the production rate ($r < k$), *i.e.* the constant rate of replenishment is ($k - r$). In the second period t_2 items are consumed at the demand rate 'r'. If we workout the total cost of inventory per unit of time as usual, we get:

$C_q = (q / 2) \{ (k - r) / k \} C_1 + C_3 (r/q)$ By equating the first derivative to zero, we get,

$dC_q / dq = (C_1 / 2) (1 - r/k) - (C_3 r / q^2) = 0$ which will give

$q_0 = \sqrt{(2C_3 / C_1)} \times \{r / 1 - (r/k)\}$ OR $q_0 = \sqrt{(k / k / r)} \times \sqrt{2C_3 r} / C_1$

$t_0 = q_0 / r = \sqrt{2C_3} / \{ rC_1 (1 - r/k)\}$ OR $t_0 = \sqrt{(k / k - r)} \times \sqrt{2C_3} / C_1 r$

$C_0 = \sqrt{2C_3 C_1} \; (r / 1 - r/k)$. OR $C_0 = \sqrt{(k - r)} / k \times \sqrt{2C_1 C_3}\, r$

Points to Remember

(a) **The carrying cost per unit of time is reduced from cost of first model by a ratio of [1 – (*r* / *k*).] But set up cost remains same.**

(b) **If we substitute a value of infinity to *k* in the model shown above, we will get the results of the first model.**

(c) **If the production rate is very low, then the lot size should be taken large, because much of the production will be consumed during the production period and hence the inventory in the second part of the graph will be built at a very low rate.**

(d) **If *r* > *k* then there will be no inventory.**

Problem 8.23

An item is produced at the rate of 50 items per day. The demand occurs at the rate of 25 units per day. If the set up cost is Rs. 100/- per run and holding cost is Rs.0.01 per unit of item per day, find the economic lot size for one run, assuming that the shortages are not permitted.

Solution

Data: r = 25 units per day, k = 50 items per day, C_3 = Rs. 100/- per run, C_1= Rs. 0.01 per item per day.

$q_0 = \sqrt{2C_3/C_1)} \times \{r / 1 - (r/k)\} = \sqrt{2 \times 100 \times 25}$ / 0.01 × (1 / 25 /50) = 1000 items.

$t_0 = q_0 / r$ = 1000 / 25 = 40 days.

Minimum daily cost $= \sqrt{2C_3C_1 r} \times \sqrt{(k / k - r)} = \sqrt{2 \times 100 \times 0.01 \times 25 \times (25/50)}$ = Rs. 5/

Minimum total cost per run = Rs.5/ – × 40 = Rs. 200/-

Problem 8.24.

A company has a demand of 12,000 units per year for an item and it can produce 2000 items per month. The cost of one setup is Rs. 400/- and the holding cost per unit per month is Rs. 0.15. Find the optimum lot size and the total cost per year, assuming the cost of one unit as Rs.4/-. Also find the maximum inventory, manufacturing time and total time.

Solution

Data: r = 12,000 units per year, k = 2000 units per month, C_3 = Rs. 400/- per set up, C_1 = Rs. 0.15 per unit per month, p = Rs. 4/- per item.

Now k = 2000 × 12 = 24,000 units per year and C_1 = Rs. 0.05 × 12 = Rs. 1.80 per unit per year.

$q_0 = \sqrt{2C_3 rk} / C_1 (k - r) = \sqrt{(2 \times 400 \times 12{,}000 \times 24{,}000)}$ / (1.8 × 12,000) = 3,264 units.

$C_0 = 12{,}000 \times 4 + \sqrt{2C_1C_3\, r(k - r/k)}$ = Rs. 48,000 + Rs. 2,940 = Rs. 50,940/-

Maximum Inventory = $q_{max} = \{(k - r) / k\} \times q_0$ = (24,000 – 12,000) 3,264 / 24,000 = 1,632 units.

Manufacturing time = $t_1 = (q_0 / k - r)$ = 1,632 / 12,000 = 0.136 year. = App 50 days. $t_0 = q_0 / r$ = 3,264 / 12,000 = 0.272 year. = App. 99 days.

Problem 8.25.

A contractor has to supply 10,000 bearings per day to an automobile manufacturer. He finds that, when he starts a production run, he can produce 25,000 bearings per day. The cost of holding a bearing in stock for one year is 20 paise, and setup cost of a production run is Rs. 180/-. How frequently should production run be made?

Solution

Data: r = 10,000 units, k = 25,000 units, C_1 = Rs. 0.20 / 365 = 0.00055 per bearing per day, C_3= Rs. 180/- per run.

$q_0 = \sqrt{2C_3 rk} / C_1 (k - r) = \sqrt{(2 \times 180 \times 10{,}000 \times 25{,}000)}$ / 0.00055 × (25,000 – 10,000) = $\sqrt{1.09 \times 10^{10}}$ = 1,05,000 bearings.

$t_0 = \sqrt{2C_3 k} / rC_1 (k - r) = \sqrt{(2 \times 180 \times 25000)}$ / (10,000 × 0.00055 × 15,000) = 0.3 day = 2.4 hours of 8 hour shift.

Problem 8.26.

In a paints manufacturing unit, each type of paint is to be ground to a specified degree of fineness. The manufacturer uses the same ball mill for a variety of paints an after completion of each batch, the mill has to be cleaned and the ball charge properly made up. The change over from one type of paint to another is estimated to cost Rs. 80/- per batch. The annual sales of a particular grade of paint are 30,000 liters and the inventory carrying cost is Re.1/- per liter. Given that the rate of production is 3 times the sales rate, determine the economic batch size.

Solution

Data: r = 30,000 liters, C_3 = Rs. 80/-, C_1 = Re. 1/-, k = 90,000 liters.

$q_0 = \sqrt{2 \times C_3\, rk} / C_1\,(k - r) = \sqrt{(2 \times 30{,}000 \times 80)} / 1.00 \times \{ 1 - (30{,}000 / 90{,}000) = 2683.28$ liters.

Number of batches per year = r / q_0 = 30,000 / 2683.28 = 11.18 batches.

Problem 8.27.

Amit manufactures 50,000 bottles of tomato ketch - up in a year. The factory cost per bottle is Rs.5/-, the setup cost per production run is estimated to be Rs.90/-, and the carrying cost on finished goods inventory amounts to 20% of the cost per annum. The production rate is 600 bottles per day, and sales amount to 150 bottles per day. What is the optimal production size and number of production runs?

Solution

Data: r = 150 bottles per day, k = 600 bottles per day, C_3 = Rs. 90 per run, $C_1 = 0.20 \times$ Rs.5/- = Rs. 1/-

$q_0 = \sqrt{2C_3\, rk} / C_1\,(k - r) = \sqrt{(2 \times 90 \times 150 \times 600)} / 1 \times (600 - 150) = \sqrt{16200000} / 450 = \sqrt{36000} = 189.73 = 190$ bottles.

Number of production runs = r / q_0 = 150 / 190 = 0.9 run or app. 1 batch.

Economic Order Quantity Model for Integrality of Items

In the previous models demand is considered to be a continuous variable and a straight line represents withdrawals. When the items are integral, the demand cannot be represented by straight line but appears to be stepped rectangles as shown in the figure 8.12.

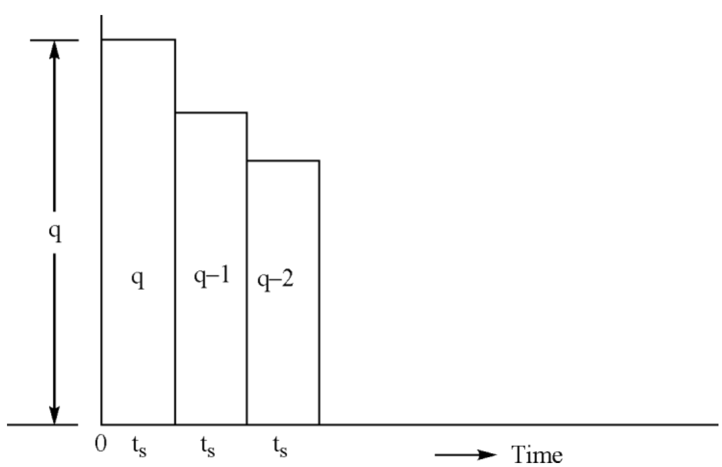


Figure. 8.12.

λ = Yearly demand, i = Inventory carrying rate, p = unit price in Rs. q = lot size, C_3 = ordering cost.

For this model, total cost of carrying inventory for one year = $C_3 (\lambda / q) + (q-1)/2 \times ip = C_q$

Optimal value of $q = q_0$ is obtained by: $C_{(q+1)} - C_q > 0$ and $C_{(q-1)} - C_q > 0$ by substituting the values for C_q and C_{q+1} and C_{q-1} we will get:

$$\mathbf{(q-1)\, q < (2C_3 \lambda / ip) < q\, (q+1)}$$

Hence optimal order quantity will occur when $(2C_3 \lambda / ip)$ is less than $(q-1)\, q$ and is greater than $q\,(q+1)$.

Problem 8.28.

A large automobile repair shop has a very low demand for certain component *i.e.* the demand is 8 items per year. The demand is assumed to be deterministic.The cost of placing an order for this part is Re. 1/-. The unit cost is Rs. 30/-. The inventory carrying cost is 20% of average inventory. Find the optimal order quantity, by considering the integrality of items. What would be the optimal time to place orders? What is the value of optimal quantity by using square root formula?

Solution

Data: C_3 = Re.1/-, λ = 8 items per year, i = 0.20, p = Rs. 30/-

Now $2C_3 \lambda / ip = (2 \times 1 \times 8) / (0.20 \times 30) = 2.66$

q =	1	2	3	4	5	6	7	8
$q\,(q-1)$ =	0	2	6	12	20	30	42	56
$q\,(q+1)$ =	2	6	12	20	30	42	56	72

Now, $q\,(q-1) < 2C_3 \lambda / ip < q\,(q+1)$ as per given data $2C_3 \lambda / ip = 2.66$ lies between 2 and 6. Taking the higher value $q_0 = 2$.

If we take $\sqrt{2C_3\lambda} / ip = (\sqrt{2 \times 1 \times 8}) / (0.20 \times 30) = 1.63$ taking the nearest whole number $q_0 = 2$ units.

Problem 8.29.

The annual demand of an item is 10 units and the ordering cost is Rs.2/- per order and the management has worked out the inventory carrying cost as 25 % of the average inventory. Assuming the integrality of items find the economic order quantity for the item. Given that the unit cost is Rs.40/

Solution

Data: $\lambda = 10$, C_3 = Rs.2/-, $i = 0.25$, p = Rs. 40/-

$2C_3 \lambda / ip = (2 \times 2 \times 10) / (0.25 \times 40) = 40 / 10 = 4$ units.

q =	1	2	3	4	5	6	7	8	9	10
$q\,(q-1)$ =	0	2	6	12	20	30	42	56	72	90
$q\,(q+1)$ =	2	6	12	20	30	42	56	72	90	110

$2C_3 \lambda / ip = 4$ lies between $q\,(q-1) = 2$ and $q\,(q+1) = 6$. Hence level of inventory = 2 units. If we take the square root of 4 which is equal to 2. Hence *EOQ* by square root formula is also 2.

Deterministic Models with Shortages

Shortages means when the demand for item is exists, the item is not available in the stores. This situation leads to the problem that the organization cannot keep up the delivery promises. In such case if the customer accepts, the organization can fulfill his order soon after the inventory is received. If the customer does not accept, the organization has to loose the order. The first situation is known as **back logged or back order situation** and the second one is known as **shortages or lost sales situation**. In back logged situation, the company has to loose the customer as well as the profit. In the first case, if the stock out position occurs frequently, the customer may get dissatisfied with the services provided by the organization and finally do not turnout to the organization.

(a) Instantaneous Production with back orders permitted

The figure 8.13 shows the model of instantaneous production, deterministic demand and the back orders permitted. Here the carrying cost is C_1 and the ordering cost is C_3. As the shortages are allowed (backlogged), the shortage cost C_2 is also taken into consideration. As usual, the lot size is 'q' and the inventory replenishment time is 't'. In addition the level of inventory in the beginning is shown as 'S' and the maximum level of short items is shown as 'z'. Hence lot size $q = S + z$.

From the figure, inventory carrying cost = $(S/2) \times t_1 \times C_1$ = Area of triangle $OBD \times C_1$

Shortage cost = Area of triangle $DAC \times C_2 = \{(q - S) / 2\} \times t_2 \times C_2$

Ordering cost = C_3

Hence the total cost for one run = $(S/2) \times C_1 \times t_1 + \{(q - S) / 2)\}\ t_2 \times C_2 + C_3$

Total cost per unit of time = $C_q = \{(SC_1t_1)/ 2\} + \{(q - S)\ C_2t_2 / 2)\} + C_3\ / t$

With mathematical treatment, we get:

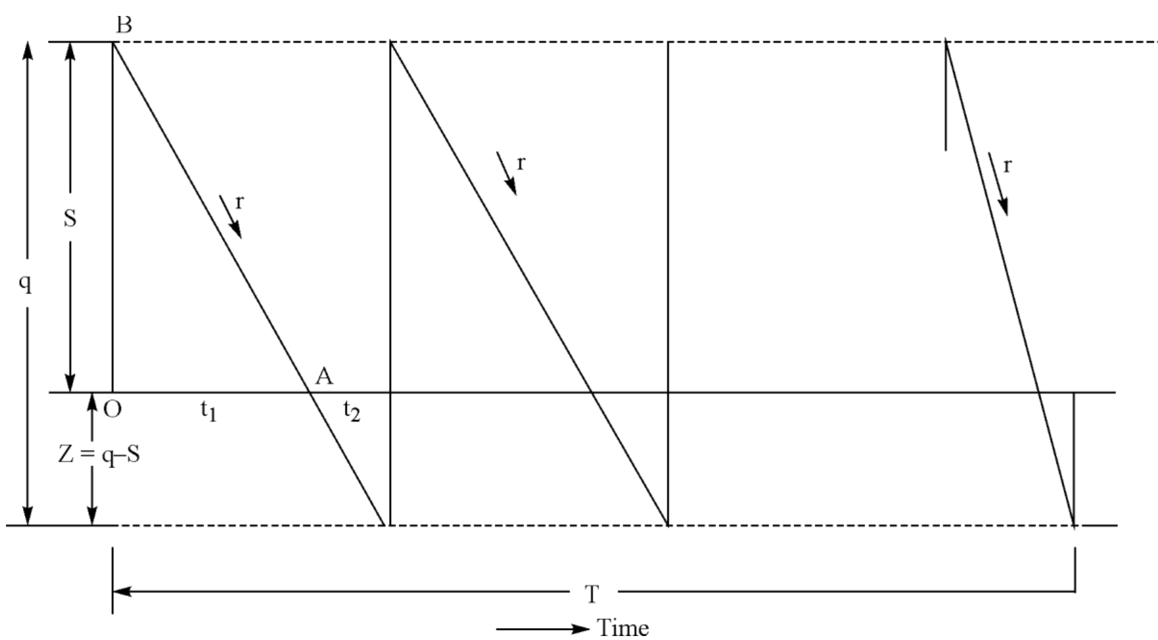


Figure 8.13.

$q_0 = r\,t_0 = \sqrt{\{(2C_3\,r\,(C_1 + C_2)/C_1C_2)\}}$ (Attention is to be given to see that the *EOQ* model is multiplied by a factor $(C_1 + C_2) / C_2$

OR $q_0 = \sqrt{(C_1 + C_2)} / C_2 \times \sqrt{(2C_3\lambda} / C_1)$

$t_0 = q_0 / r = \sqrt{\{2C_3\,(C_1 + C_2)\}} / C_1\,r\,C_2$ (Here also the optimal time formula is multiplied by $(C_1 + C_2) / C_2$

OR $t_0 = q_0 / \lambda = \sqrt{C_1 + C_2)} / C_2 \times \sqrt{(2C_3 / C_1\lambda)}$

$C_0 = \sqrt{(C_2 / (C_1 + C_2)} \times \sqrt{(2C_1C_3\lambda)}$

$\boldsymbol{I_{max}}$ **= Maximum inventory** $= S_0 = [C_2 / (C_1 + C_2)] \times q_0 = \sqrt{[C_2/(C_1 + C_2)]} \times \sqrt{(2C_3\lambda / C_1)}$

$S_0 = \sqrt{\{(2C_3\,r\,C_2)/C_1(C_1 + C_2)\}}$ this is also known as Order level model.

$z_0 = q_0 - S_0 = \sqrt{\{(2C_3\,r \times C_1)/C_2\,(C_1 + C_2)}$

(**Note:** By keeping $C_2 = \infty$, the above model reduces the deterministic demand *EOQ* model).

Problem 8.30.

The demand for an item is uniform at the rate of 25 units per month. The set up cost is Rs. 15/- per run. The production cost is Re.1/- per item and the inventory-carrying cost is Rs. 0. 30 per item per month. If the shortage cost is Rs. 1.50 per item per month, determine how often to make a production run and what size it should be?

Solution

Data: r = 25 units per month, C_3 = Rs.15/- per run, b = Re. 1/- C_1 Rs. 0.30 per item per month, C_2 = Rs. 1.50 per item per month. Where b = production cost, hence Set up cost is to be taken as C_3 + bq. In this example it will become Rs. 15./- + Re. 1/- = Rs.16/-. This will be considered when working the total cost of inventory and not the economic order quantity, as the any increase in C_3 will not have effect on q_0.

Remember when any thing is added to the setup cost, the optimal order quantity will not change.

$q_0 = \sqrt{\{2C_3\,r\,(C_1 + C_2)\}} / (C_1\,C_2) = \sqrt{\{(2 \times 15 \times 25 \times 1.80)} / (\,0.30 \text{ x } 1.50)\} = 10 \sqrt{30} = 54$ items. And optimal time $= q_0 / r = 54 / 25 = 2.16$ months.

Optimal cost $= C_{(S,\,t)} = (1/t) \times [\,(C_1S^2 / 2r) + C_2\,(tr - S^2) / 2r] + [(C_3/t) + br]$ because $(q / t) = r$

Problem 8.31.

A particular item has a demand of 9,000 units per year. The cost of one procurement is Rs. 100/- and the holding cost per unit is Rs. 2.40 per year. The shortages are allowed are the shortage cost Rs. 5/- per unit per year. (*a*) Find Economic lot size, (*b*) Number of orders per year, (*c*) The time between two orders, and

(*d*) Total cost per year including material cost, taking unit price as Re.1/- per unit.

Solution

Data: λ = 9,000 units per year, C_1 = Rs. 2.40 per unit per year, C_2 = Rs. 5/- per unit per year, C_3 Rs. 100/- per procurement.

$q_0 = \sqrt{(2C_3\lambda / C_1)} \times \sqrt{(C_1 + C_2)} / C_2 = \sqrt{[(2 \times 100 \times 9000)/2.40]} \times \sqrt{(2.40 + 5)} / 5 = \sqrt{11{,}10{,}000}$
= 1,053 units per run.

Total cost including material cost = C_0 = 9000 × 1 + $\sqrt{(C_2 / C_1 + C_2)} \times \sqrt{2C_1C_3\lambda}$ =

Rs. 9000 + $\sqrt{(2 \times 2.40 \times 100 \times 9000)}$ = Rs. 9,000 + Rs. 1,710 = Rs. 10, 710/- per year.

Number of orders per year = $N = \lambda / q_0$ = 9,000 / 1,053 = 8.55 orders = App. 9 orders

Time between orders = $t_0 = 1 / N$ = 1 / 8.55 = 0.117 year = 64.6 days = App. 65 days.

Problem 8.32.

A manufacturing firm has to supply 3,000 units annually to a customer, who does not have enough storage capacity. The contract between the supplier and the customer is if the supplier fails to supply the material in time a penalty of Rs. 40/- per unit per month will be levied. The inventory holding cost amounts to Rs. 20/- per unit per month. The set up cost is Rs. 400/- per run. Find the expected number of shortages at the end of each scheduling period.

Solution

Data: C_1 = Rs. 20/- per unit per month, C_2 = Rs. 40/- per unit per month, C_3 = Rs. 400/- per run, λ = 3000 units per year = 3000 / 12 = 250 units per month = r.

$I_{max} = S = \sqrt{[C_2/(C_1 + C_2)]} \times \sqrt{2C_3 r / C_1} = \sqrt{[40/(20+40)/40]} \times \sqrt{(2 \times 400 \times 250)} / 20 = 82$ units.

$q_0 = \sqrt{(C_1 + C_2)} / C_2 \times \sqrt{(2C_3 r} / C_1) = \sqrt{[(20+40)/40]} \times \sqrt{(2 \times 400 \times 250)} / 20 = 123$ Units.

Number of shortages per period = $q_0 - S_0$ = 123 – 82 = 41 units per period.

Problem 8.33.

The demand of a chemical is constant and at the rate of 1,00,000 Kg per year. The cost of ordering is Rs. 500/- per order. The cost per Kg of chemical is Rs. 2/-. The shortage cost is Rs.5/- per Kg per year if the chemical is not available for use. Find the optimal order quantity and the optimal number of back orders. The inventory carrying cost is 30 % of average inventory.

Solution

Data: λ = 1,00,000 Kg per year, p = Rs. 2/ per Kg., C_2 = Rs. 5/- per Kg per year, C_3 = Rs. 500 per order, C_1 Rs. 2 × 0.30 = Rs. 0.60 per Kg. per year.

$q_0 = \sqrt{[(C_1 + C_2)/C_2]} \times \sqrt{(2C_3\lambda / C_1)} = \sqrt{[(0.60+5)/5]} \times \sqrt{(2 \times 500 \times 1{,}00{,}000)} / 0.60 = 13{,}663$ Kg.

$I_{max} = S_0 = [C_2 / (C_1 + C_2)] \times q_0$ = [5 / (0.60 + 5)] × 13,663 = 12, 199 Kg.

Optimum back order quantity = $q_0 - S_0$ = 13,663 – 12, 199 = 1,464 Kg.

Problem 8.34.

The demand for an item is 18,000 units annually. The holding cost is Rs. 1.20 per unit time and the cost of shortage is Rs. 5.00. The production cost is Rs. 400/- Assuming that the replenishment rate is instantaneous determine optimum order quantity.

Solution

Data: λ = 18, 000 units per year, C_1 = Rs. 1.20 per unit, C_2 = Rs. 5/- and C_3 = Rs. 400/-

$q_0 = \sqrt{(2C_3\lambda / C_1)} \times \sqrt{(C_1 + C_2)} / C_2 = \sqrt{(2 \times 400 \times 18{,}000)} / 1.20 \times \sqrt{(1.20 + 5)} / 5$ = 3857 units.

$t_0 = q_0 / \lambda$ = 3857 / 18,000 = 0. 214 year = App. 78 days.

Number of orders = $N = \lambda / q0$ = 18000 / 3857 = 4.67 orders = App. 5 orders.

Problem 8.35.

The demand for an item is deterministic and constant over time and it is equal to 600 units per year. The per unit cost of the item is Rs. 50/- while the cost of placing an order is Rs. 5/-. The inventory carrying cost is 20% of the cost of inventory per year and the cost of shortage is Re.1/- per unit per month. Find the optimal order quantity when stock outs are permitted. If stock outs are not permitted what would be the loss to the company.

Solution

Data: λ = 600 units, i = 0.20, p = Rs. 50, $C_1 = ip$ = 0.20 × 50 = Rs. 10/-, C_3 = Rs. 5/-, C_2 = Re. 1/- per month = Rs. 12/ per unit per year.

$q_0 = \sqrt{2C_3\lambda / C_1)} \times \sqrt{(C_1 + C_2)/C_2} = \sqrt{(2 \times 5 \times 600)/10} \times \sqrt{(10{+}12)/12}$ = 77.46 × 1.35 =104.6 units.

Maximum number of back orders = $q_0 \times C_2/C_1 + C_2 = S_0$ = 12 / (10 + 12) × 104.6 = 0.55 × 105.6 = 57.05 units. = App. 57 units.

Expected yearly cost $C_0 = \sqrt{(2C_3C_1\lambda)} \times C_2 / (C_1 + C_2) = \sqrt{(2 \times 10 \times 5 \times 600)}$ × (12 / 10 + 12) = 245 × 0.55 = 134.75 = App. Rs. 135/-

If back orders are not allowed, $q_0 = \sqrt{(2 \times C_3 \times \lambda) / C_1} = \sqrt{(2 \times 5 \times 600)/10}$ = 24. 5 units.

Total cost $C_0 = \sqrt{(2 \times C_3 \times C_1 \times \lambda)} = \sqrt{(2 \times 5 \times 10 \times 600)} = \sqrt{60000}$ = Rs. 245/-

Hence the additional cost when backordering is not allowed is Rs. 245 - Rs.135 = Rs. 110/-

Problem 8.36.

The demand for an item is 12,000 units per year and shortages are allowed. If the unit cost is Rs. 15/- and the holding cost is Rs. 20/- per unit per year. Determine the optimum yearly cost. The cost of placing one order is Rs. 6000/- and the cost of one shortage is Rs.100/- per year.

Solution

Data: λ = 12,000 units, C_1 = Rs. 20/- per unit per year, C_2 = Rs. 100/- per year, C_3 = Rs. 6000/- per order. P = Rs. 15/-

$q_0 = \sqrt{(2C_3\lambda)/C_1} \times \sqrt{(C_1 + C_2) / C_2} = \sqrt{(2 \times 6000 \times 12{,}000)/20} \times \sqrt{(20 + 100)/200}$ = 2939 units.

Number of orders per year = λ / q_0 = 12,000 / 2939 = 4.08 = App. 4 orders.

Number of shortages = $z_0 = q_0 \times [C_1/(C_1 + C_2)]$ = 2939 × [20/(20 + 100)] = 489 units.

Total yearly cost = $p \times \lambda + \sqrt{(2C_3C_1\lambda)} + \sqrt{[C_2/(C_1 + C_2)]} =$

$15 \times 12{,}000 + \sqrt{(2 \times 6000 \times 20 \times 12{,}000)} \times \sqrt{(100/120)}$ = Rs. 1, 08, 989.79 = App. Rs.1, 08, 990

Problem 8.37.

A commodity is to be supplied at the constant rate of 200 units per day. Supplies of any amount can be had at any required time but each ordering costs Rs. 50/-. Cost of holding the commodity in inventory is Rs. 2/- per unit per day while the delay in the supply of the item induces a penalty of Rs.10/- per unit per delay of one day. Find the optimal policy, *q* and *t*, where t is the reorder cycle period and *q* is the inventory level after reorder. What would be the best policy if the penalty cost becomes infinity?

Solution

Data: C_1 = Rs. 2/- per unit per day, C_2 = Rs. 10/ per unit per day, C_3 = Rs. 50/- per order, $r = 200$ units per day.

$q_0 = \sqrt{(2C_3 r/C_1)} \times \sqrt{(C_1 + C_2)/C_2} = \sqrt{(2 \times 50 \times 200)/2} \times \sqrt{(2+10)/2}$ = 110 units.

$t_0 = q_0 / r = 110 / 200 = 0.55$ day.

The optimal order policy is $q_0 = 110$ units and the ordering time is 0.55 day.

In case the penalty cost becomes ∞ , then q_0 and t_0 are:

$q_0 = \sqrt{2C_3 r / C_1} = \sqrt{(2 \times 50 \times 200)/2}$ = 100 units.

$t_0 = q_0 / r = 100 / 200 = 0.5$ day.

Problem 8. 38.

A Contractor supplies diesel engines to a truck manufacturer at the rate of 20 per day. He has to pay penalty of Rs. 10/- per engine per day for missing the schedule delivery rate. Holding cost of a complete engine is Rs. 12/- per month. The manufacturing of engines starts with the beginning of the month and is completed at the end of the month. What should be the inventory level at the beginning of each month?

Solution

Data: $r = 20$ engines per day, C_1 = Rs. 12 per month = Rs. 12/30 = Rs. 0.40 per engine per day,

C_2 = Rs. 10/- per engine per day, t = 1 month = 30 days.

S_0 = Max. Inventory = $[(C_2)/(C_1 + C_2)]\, q_0 = [(C_2) / (C_1 + C_2)] \times r \times$ Max inventory =

$[(10) / (10 + 0.40)] \times 20 \times 30 = 577$ engines per month.

(b) Lost – Salesshortages

In the above case, due to shortages, back orders are allowed, *i.e.* the demand will be satisfied after the receipt of the material. But in the present case the assumption is the sales will be lost if there is a shortage. This is shown in figure 8.14. In this case, the unit shortage cost is proportional to quantity only and is independent of time as the shortages of any item is the shortage forever not for a finite interval of time. The shortage cost C_2 includes the loss of profit.

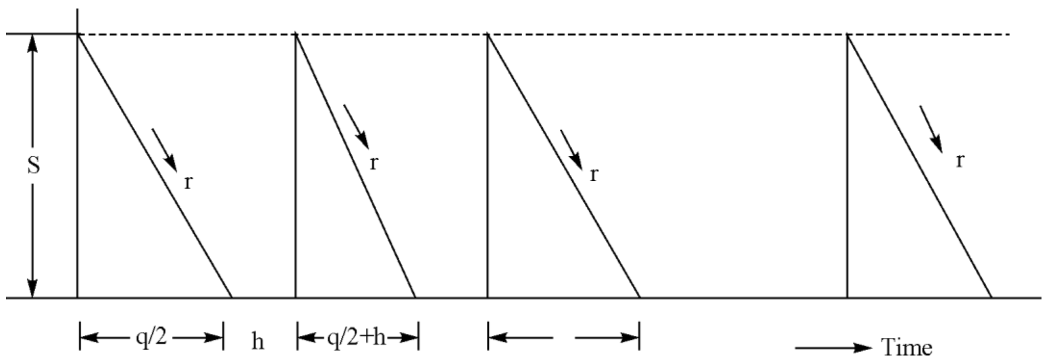


Figure 8. 14.

h = Stick out period. And $0 \le h \le \infty$ hence $t = (q/r) + h = (q + rh) / r$, where r = uniform rate of demand.

Carrying cost = $(q/2)\, q \times (1/r)\, C_1 = (q^2 / 2) \times (1/r)\, C_1 = (C_1 q^2) / 2r$

Shortage cost = $C_2 \times rh$ and set up cost = C_3

Total cost per cycle = $(C_1 q^2) / 2r + C_2 rh + C_3$

Average total cost per unit of time = $[(C_1 q^2) / 2\,(q + rh)] + [(C_2 r^2 h) / (q + rh)] + [(C_3 r) / (q + rh)$ $= C(q,k)$

Equating $dC_{(q,k)} / dq = 0$ and simplifying we get,

$$q_0 = \{C_2 r \pm [(\sqrt{(C_2 r)^2} - (2C_1 C_3 r)]\} / C_1$$

$$h = \{-C_2 r \pm \sqrt{[(C_2 r)^2} - (2C_1 C_3 r)]\} / C_1 r$$

Problem 8.39.

The demand for an item is continuous and deterministic at 200 units per month. The holding cost is Rs. 2/- per unit per month and ordering cost is Rs. 5/- per order. In case of shortage, the loss of sales causes a loss of profit to an extent of Rs. 200/ per month. Find the optimal order quantity.

Solution

Data: r = 200 units, C_2 = Rs. 20 / month, C_1 = Rs.2/- per unit per month. C_3 = Rs. 5 /- per order.

$$q_0 = \{C_2 r \pm [(\sqrt{(C_2 r^2} - (2C_1 C_3 r)]\} / C_1 = \{(200 \times 200 \pm [\sqrt{(200 \times 200)^2} - (2 \times 2 \times 5 \times 200)] / 2\}$$

$$q_0 = \{(40000) \pm [\sqrt{40000)^2} - (4000 / 2)]\} = 40000 \pm \sqrt{(1600000000\,)} - 2000)]\}$$

$$= 40000 \pm 39999.9$$

Economic Order quanity for finite rate of replenishment of inventory with back orders permitted

As in the previous finite rate of replenishment model, the rate of replenishment is at the rate of 'k' units per unit of time and shortages are allowed. The figure 8.14 shows the graphical representation of the model.

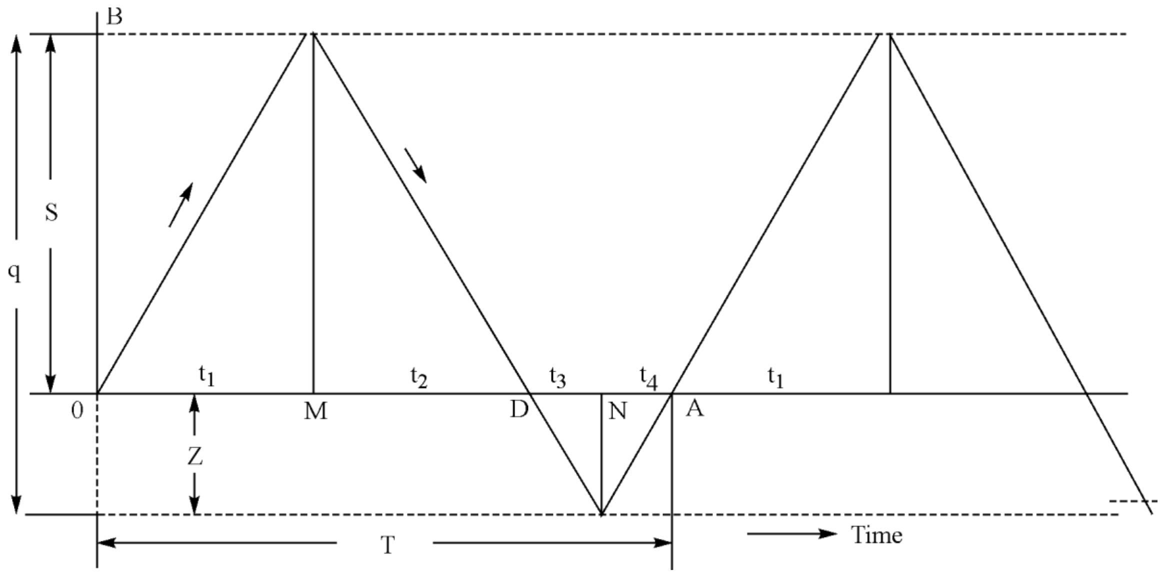


Figure 8.15.

From the figure, total cost per cycle = $C = (S/2) \times (t_1 + t_2) \times C_1 + (z/2) \times (t_3 + t_4) \times C_2 + C_3$

Now, $S = t_1 (k + r)$ or $t_1 = S / (k - r)$, similarly, $t_2 = (S / r)$, $t_3 = (z / r)$, and $t_4 = (z / k - r)$,
Substituting the values and simplifying, The cost function becomes,

$C = C_3 + (C_1S^2 + C_2 z^2) / \{2 r [1 - (r / k)]\}$

Hence cost per unit of time (by substituting $t = (q / r) = C_0 (s,t) = C_3 r / q + (C_1S^2 + C_2 z^2) / \{[2q (1 - (r / k)]$

After mathematical treatment, the optimal value of $C_0 (q_0,z_0)$ =

$$C_0\ (q_0,z_0) = \sqrt{\{2C_3C_1\, r[1-(r/k)\}} \times (C_2 / C_1 + C_2)\ \text{OR}$$

$$C_0\ (q_0,x_0) = \sqrt{C_2/(C_1 + C_2)} \times \sqrt{(k-r)/k} \times \sqrt{2C_1C_3\, r}$$

The other models are:

$$q_0 = \sqrt{(2C_3\, r / C_1)} \times [(C_1 + C_2) / C_2 \{1 - (r/k)\}]\ \text{OR}$$

$$q_0 = \sqrt{(C_1 + C_2) / C_2} \times \sqrt{k/(k-r)} \times \sqrt{(2C_3\, r) / C_1}$$

$$z_0 = \sqrt{[2C_3\, r\{1-(r/k)\}C_{1/}C_2\, (C_1 + C_2)]}\ \text{OR}\ z_0 = C_1 / (C_1 + C_2) \times (k - r) / k \times q_0$$

$$t_0 = (q_0 / r) = \sqrt{2C_3}(C_1 + C_2) / C_1C_2\, r\, [1 - (r/k)]\ \text{OR}$$

$t_0 = q_0 / r = \sqrt{(C_1 + C_2)/C_2} \times \sqrt{k/(k-r)} \times \sqrt{2C_3 / C_1 r}$

As we know that $S = [q\,(1 - r/k) - z]$ we can get,

$S_0 = \sqrt{2C_3\, r[1-(r/k)]} \times [C_2 / C_1\ (C_1 + C_2)]$ OR

Max Inv = $S_0 = \sqrt{C_2/(C_1 + C_2)} \times \sqrt{(k-r)/k} \times \sqrt{(2C_3\, r)/C_1}$

(**Note**: By keeping $k = \infty$, $C_2 = \infty$ and $k = \infty$, the above models reduces to the models without shortages.).

Problem 8.40.

The demand for an item in a company is 18,000 units per year, and the company can produce the item at a rate of 3,000 items per month. The cost of one setup is Rs. 500/- and holding cost of one unit per month is 15 paise. The shortage cost of one unit is Rs. 20/- per year. Determine the optimum manufacturing quantity and the number of shortages. Also determine the manufacturing time and the time between setups.

Solution

Data: λ = 18,000 units per year, or r = 1,500 units per month, k = 3000 units per month, C_1 = Rs. 0.15 per unit per month, C_2 = Rs. 20 /- per unit per year = Rs. 1.67 per unit per month, C_3 = Rs. 500/ - per set up.

$q_0 = \sqrt{(C_1 + C_2)/C_2} \times \sqrt{k/(k-r)} \times \sqrt{2C_3\, r / C_1} =$

$\sqrt{(0.15 + 1.67)/1.67} \times \sqrt{3000/(3000-1500)} \times \sqrt{(2 \times 500 \times 1500)/0.15}$ = 4,669 units.

Max inventory = $I_{max} = \sqrt{C_2/(C_1 + C_2)} \times \sqrt{(k-r)/k} \times \sqrt{2C_3\, r / C_1} =$

$\sqrt{1.67/(0.15 + 1.67)} \times \sqrt{(3000 - 1{,}500)/3000} \times \sqrt{(2 \times 500 \times 1500)/0.15}$ = 2,142 units.

Therefore, number of shortages = $q_0 - I_{max}$ = 4669 – 2142 = 2,527 units.

Manufacturing time = q_0 / k = 4667 / 3000 = 1.56 months.

Time between setups = $t_0 = q_0 / r$ = 4669 / 1500 = 3.12 months. = App. 3 months.

Problem 8.41.

The demand for an item in a company is Rs. 12,000 per year and the company can produce the item at a rate of 2000 units per month. The cost of one setup is Rs. 400/- and the holding cost is 15 paise per unit per month. The shortage cost of one unit is Rs. 20/- per year. Unit cost of material is Rs. 4/- Determine q_0, $C_0\,(q,s)$, Maximum inventory, Manufacturing time interval, Total time interval.

Solution

Data: λ = 12,000 units per year, k = 2000 units per month or 24000 units per year, C_1 = Rs. 0.15 x `12 = Rs. 1.80 per unit per year, C_2 = Rs. 20/- per year, C_3 = Rs. 400/- per set up. P = Rs. 4/- per unit.

$q_0 = \sqrt{(2C_3\lambda)/C_1} \times \sqrt{(C_1 + C_2)/C_2} \times \sqrt{k/(k-r)}$

$= \sqrt{(2 \times 400 \times 12000)} \times \sqrt{(1.8 + 20)/20} \times \sqrt{24000/(24000 - 12000)}$ = 3, 410 units.

$C_0\ (q,s)$ = Material cost + inventory cost = 12,000 × 4 + $\sqrt{(2 \times C_1 \times C_3 \times \lambda)}$ × $\sqrt{C_2 / C_1 + C_2)}$ × $\sqrt{(k - r) / k}$

= 48,000 + $\sqrt{2 \times 1.8 \times 400 \times 12{,}000)}$ + $\sqrt{20/(20+1.8)}$ × $\sqrt{(24000 - 12000)/12000}$

= Rs. 50,815 per year.

I_{max} = Maximum inventory = $\sqrt{(2C_3\lambda)/C_1}$ × $\sqrt{C_2\,(C_1 + C_2)}$ × $\sqrt{(k - r) / k}$

= $\sqrt{(2 \times 400 \times 12000)/1.80}$ × $\sqrt{20/(1.80 + 20)}$ × $\sqrt{(24{,}000 - 12{,}000)}$ /24000

= 1564 units/ per setup.

Manufacturing time interval = $t_1 + t_4 = q_0 / k$ = 3410 / 24000 = 0.1421 year = 51.86 days = App. 52 days.

Total time interval = $t_0 = q_0 / \lambda$ = 3410/12000 = 0.2842 year = 103.73 days = App. 104 days.

Fixed Time Model

In this case, the production is instantaneous and the shortages are allowed and the inventory is to be replaced at a fixed interval, say at time 't'. This appears to be similar with the model, where production is instantaneous and back orders are allowed (8.7.9.1). The difference between the two models is that in this model the cycle time for one period is fixed. The graphical representation of the model is given in figure number 8.16.

As the period is fixed, quantity 'q' is known exactly and is equals to 'rt'. The decision variable is level of inventory 'S' and the level of shortage.

Carrying cost = $(S/2) \times t_1 \times C_1$
Shortage cost = $[(q - S)/2] \times t_2 \times C_2$

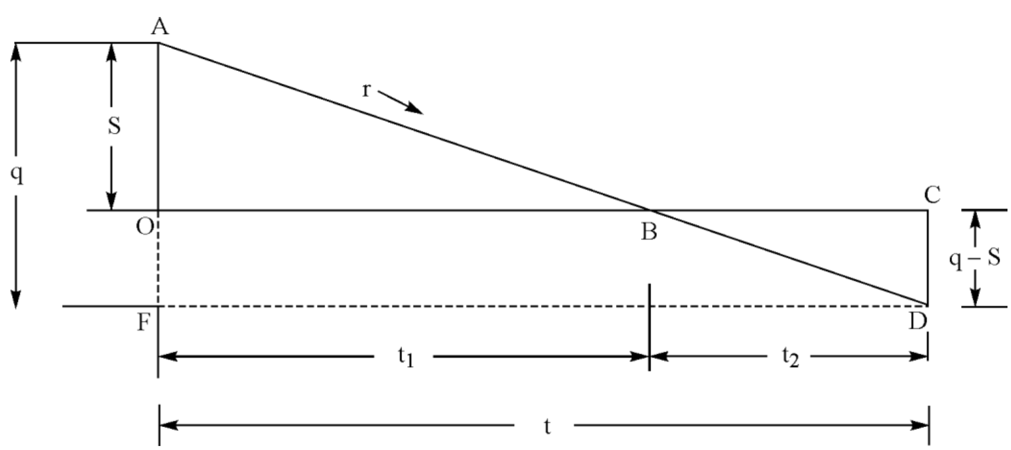


Figure 8.16.

From the triangles, *OAB* and *FBD*, the relations are:

$(t_1/t) = (S/q)$ or $t_1 = (S/q)\, t$ and $t_2 = (q - S)\, t/q$

Hence we can write the total cost equation as:

$C_{(S)} = (C_1 S^2)\, 2r + [C_2 (rt - S)^2] / 2r$ by differentiating and equating to zero, we get,

$\boldsymbol{S_0 = rt \times [(C_2) / (C_1 + C_2)]}$

Problem 8.42.

A contractor has to supply Diesel engines to a Truck manufacturing company at a rate of 20 per day. The penalty in the contract is Rs. 10/- per engine per day late for missing the scheduled delivery date. The cost of holding an engine in stock for one month is Rs. 15/-. His production process is such that each month (30 days) he starts a batch of engines through the agencies and all are available for supply after the end of month. What should inventory level be in the beginning of each month?

Solution

Data: C_1 = Rs. 15/- / 30 days (Rs. 15/30 per day), C_2 = Rs. 10/- per day, r = 20 engines, t = 30 days.

$S_0 = [rt\, (C_2)] / (C_1 + C_2) = 20 \times 30 \times 10 / 10 + (15/30) = 571.4$ = App. 571 engines.

MODELS WITH RESTRICTIONS

Multi- Item, Deterministic Models with one Linear Constraint

Sometimes business may face problems of purchasing many items and storing them when there are some restrictions regarding the capital to be invested, or storage space etc., Here the materials manager has to workout the optimal quantity for each material which minimizes the total inventory cost under given limitations. Due to limitations, may be space or may be capital to be invested, there exists a relation among items, hence they cannot be considered separately. To simplify the procedure, we use **Lagrange's multiplier** technique as explained below:

Procedure: First neglect the constraint and solve the problem. Then consider the effect of constraint on solution.

Let number of items is 'n'. The assumed condition is instantaneous production and no lead-time and the demand is deterministic and uniform at the rate of '*ri*' items per unit of time for '*i th*' item. Let C_1 be the inventory carrying cost per unit of quantity per unit of time for '*i th*' item and C_3 is the set up cost per run for the '*i th*' item. As the no shortages are allowed $C_2 = 0$. The cost for '*i th*' item per unit of time is:

$C_{0i} = (q_i / 2) / C_{1i} + (r_i / q_i)\, C_{3i}$ here subscript 'i' indicates the costs and quantity of '*i th*' item stocked at the beginning of the cycle.

Hence total cost per unit of time: $C = C_{(q1,q2,\ldots..qn)} = \sum_{i=1}^{n} [(q_i / 2) C_{1i} + (r_i / q_i) C_{3i}$

$\partial C / \partial q_i = (C_{1i} / 2) - C_{3i} (r_i / q_i^{\,2})$, where $i = 1,2,3,\ldots.n$.

By equating $\partial C / \partial q_i$ to zero, we get, $q_{i0} = \sqrt{(2C_{3i} \times r_i) / C_{1i}}$ **which gives the optima value of $\boldsymbol{q_1}$ where $\boldsymbol{i}$ = 1, 2, 3, …..$\boldsymbol{n}$.**

Restriction on the Number of Stocked Units

(Consider a limitation that the average number (of any item is equals to [q_i/2] for any item at any time) of all stocked units should not exceed the number '*I*', i.e.

$$(1/2)\sum_{i=1}^{n} q_i \leq 1.$$

Now we have to minimize *C*, subject to if

$(1/2)\sum_{i=1}^{n} q_{i0} \leq |$, then the optimal value q_{i0} given above are the required values without any problem.

if, $(1/2)\sum_{i=1}^{n} q_{i0} \leq |$, is not satisfied, we use the Lagrange's multiplier technique. The multiplier

function is $L = \sum_{i=1}^{n}[(q_i/2)C_{1i} + (C_{3i}\,r_i)/q_i] + \lambda[\sum_{i=1}^{n} q_i - 2]$, where λ is a Lagrange's multiplier.

By finding $\partial L/\partial q_i$ and $\partial L/\partial\lambda$ and equating them to zero, we get $q_0 = \sqrt{[(2C_{3i}r_i)/(C_{1i}+2\lambda)]}$

gives the optimal value when they satisfy the condition $\sum_{i=1}^{n} q_i - 2 |= 0$ OR $\sum_{i=1}^{n} q_i = 2 |$ This constraint is used to find the value of *i* by trial and error by
interpolation.

(*a*) Limitation on Investment

Let us assume that the upper limit of investment to be invested on inventory in Rs. is *M*. Let p_i be the unit price of '*i*' th item. Then:

$$\sum_{i=1}^{n} p_i\, q_i \leq M$$

Now our problem is to minimize the total cost given in the cost equation in 8.8.1 subject to an additional cost constraint given above. By analyzing carefully, we can get two cases.

Case1

When $\sum_{i=1}^{n} p_i\, q_{i0} \leq M$ where q_{i0} is the optimal quantity given by the equation shown in 8.8.2. This case does not give any trouble as the optimal order quantity can be found by the equation $q_0 = \sqrt{(2C_3\, r)/C_1}$.

Case 2

When $\sum_{i=1}^{n} p_i\, q_{i0} > M$ where $q_0 = \sqrt{(2C_3\, r)/C_1}$. Here suppose q_{i0} for $i = 1,2,3 \ldots n$ are not the required optimal values of '*q*', we have to use Lagrange's multiplier technique as shown below:

$$L = \sum_{i=1}^{n} [(q_i / 2) \times C_{1i} + (r_i / q_i) \times C_{3i}] + \lambda[\sum_{i=1}^{n} p_i\, q_i - M],$$ **where λ is Lagrange's multiplier. By finding $\partial L / \partial\lambda$ and equating it to zero we get,**

$$q_{i0} = \sqrt{[(2C_{3i}\, r_i) / (C_{1i} + 2\lambda \times p_i]}$$

and $\sum_{i=1}^{n} p_i \times q_{i0} = M$, which says that investment constraint must be satisfied.

Problem 8.43.

A company producing three items has a limited storage space of averagely 750 items of all types. Determine the optimal production quantities for each item separately, when the following information is given.

	Products		
Cost.	*1*	*2*	*3*
Carrying cost C_1 Rs.	0.05	0.02	0.04
Setup cost C_3 Rs.	50	40	60
Demand = r units.	100	120	75

Solution

Let us first ignore the space constraint imposed and find the optimal order quantities of each item.

Data: Product 1: C_1 = 0.05, C_3 = Rs. 50, Demand r = 100 units.

Product 2. C_1 = 0.02, C_3 = Rs. 40/-, r = 120 units.

Product 3 C_1 = 0.04, C_3 = Rs. 60/-, r = 75 units.

$$q_{01} = \sqrt{(2C_{31}\, r_1) / C_{11}} = \sqrt{(2 \times 50 \times 100) / 0.05} = 100 \times \sqrt{20} = 447 \text{ units.}$$

$$q_{02} = \sqrt{(2C_{32}\, r_2) / C_{12}} = \sqrt{(2 \times 40 \times 120) / 0.02} = 100 \sqrt{48} = 693 \text{ units.}$$

$$q_{03} = \sqrt{(2C_{33} \times r_3) / C_{13}} = \sqrt{(2 \times 60 \times 75) / 0.04} = 100 \sqrt{21.5} = 474 \text{ units.}$$

Total average inventory at any time = (447/2 + 693 /2 + 474 /2) = 802 units.

This exceeds 750 units the given constraint. We have to find the Lagrange's Multiplier by trial and error.

$$q_{01} = \sqrt{(2C_3\, r) / C_1 + 2\lambda)]} = \sqrt{[2 \times 50 \times 100) / (0.05 + 2 \times 0.005)]} = 100 \sqrt{16.67} = 409 \text{ units.}$$

Where value of λ is taken as 0.005. Similarly,

$$q_{02} = \sqrt{[(2 \times 40 \times 120) / (0.02 + 2 \times 0.005)]} = 100 \sqrt{32} = 566 \text{ units.}$$

$$q_{03} = \sqrt{[(2 \times 60 \times 75) / 90.04 + 2 \times 0.005)]} = 100 \sqrt{18} = 424 \text{ Units.}$$

Average inventory level = q_{01} /2 + q_{02} /2 + q_{03} /2 = (409 /2 + 566 /2 + 424 / 2) = (204.5 + 283 + 212) = 699.5 = App. 700 units.

This value is less than the give constraint. We can test the above with the value of λ = 0.004, 0.003, 0.002 and 0.001 etc. We can construct a graph for value of λ against the average inventory level. From this graph we will be in a position to find the exact value of λ . When we get an arc in the graph, we can connect the two ends of the arc and draw straight line, which will help us to find the value of λ . Figure 8.17 is the graph showing the value of λ on X – axis and the value of average inventory level on Y - axis.

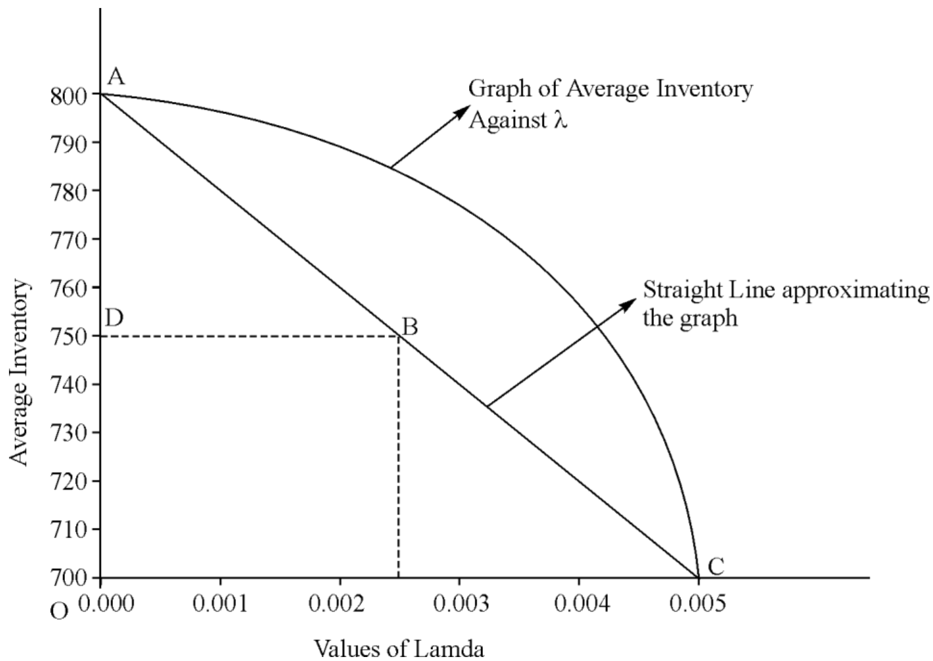


Figure 8.17. λ Vs Average inventory.

From the figure

(DB/OC) = (DA/OA) or (DB/0.005) = 52 / 100) or $\lambda = DB$ = (52 / 100) × 0.005 = 0.00256. By applying this value of λ we get the inventory levels as:

$q_{01} = \sqrt{(2 \times 50 \times 100)/(0.05 + 2 \times 0.00256)} = \sqrt{10000/0.05512} = 426$ units

$q_{02} = \sqrt{(2 \times 40 \times 120)/(0.02 + 2 \times 0.00256)} = \sqrt{9600/0.02256} = 652$ units

$q_{03} = \sqrt{(2 \times 60 \times 75)/(0.04 + 2 \times 0.00256)} = \sqrt{9000/0.04256} = 460$ units

Average inventory level = (426 /2) +(652 / 2) + (460 / 2) = 213 + 326 + 230 = 769

$q_{01} = \sqrt{[(2 \times 50 \times 100)/(0.05 + 2 \times 0.002)]} = 428$ units.

$q_{02} = \sqrt{[(2 \times 40 \times 120)/(0.02 + 2 \times 0.002)]} = 628$ units.

$q_{03} = \sqrt{[(2 \times 60 \times 75)/(0.04 + 2 \times 0.002)]} = 444$ units.

Average level of inventory = (428 / 2) + 628 / 2) + 444 / 2) = 214 + 314 + 222 = 750 units.

Hence optimal inventory of three items is: q_{01} = 428 units, q_{02} = 628 units, q_{03} = 444 units

Problem 8.44.

For the following data, determine approximately the economic order quantities, when the total value of average inventory level of the products is Rs. 1000/-

Costs.	*Product 1.*	*Product 2.*	*Product 3.*
Holding Cost C_1 (%)	20	20	20
Set up cost C_3 in Rs.	50	40	60
Cost per unit = p in Rs.	6	7	5
Yearly demand = r in units.	10000	12000	7500

Solution

Data: C_1 = Rs. 20/-, C_3 = Rs. 50/-, p = Rs.6/- per unit, r = 10000 units per year for product 1.
C_1 = Rs.20/-, C_3 = Rs. 40/- p = Rs. 7/- per unit, r = 12,000 units per year for product 2.
C_1 = RS.20/-, C_3 = Rs. 60/-, p = Rs. 5/- per unit, r = 7500 units per year for product 3.
Investment limit is Rs. 1000/-.

Ignoring constraint, if we find economic order quantity, we have:

$q_{01} = \sqrt{[(2\times 50\times 10000)/(20)]} = 100\sqrt{5}$ = App.223 units.

$q_{02} = \sqrt{[(2\times 40\times 12000)/(20)]} = 40\sqrt{30}$ = App. 216 units.

$q_{03} = \sqrt{[(2\times 60\times 7500)/(20)]} = 150\sqrt{2}$ =App. 210 units.

Value of inventory = $(q_{0i}/2) \times p_i$

Corresponding value of average inventory at any time is:

[(223 / 2) × 6 + (216 / 2) × 7 + 210 / 2) × 5] = Rs. 1950/-. This value is greater than the given financial limit of Rs. 1000/-. Now let us take the value of λ = 5 and find the value of inventory levels.

$q_{01} = \sqrt{[(2\times 50\times 10000)/(20+2\times 5\times 6)]}$ = App. 111 units.

$q_{02} = \sqrt{[(2\times 40\times 12000)/(20+2\times 5\times 7)]}$ = App. 102 units.

$q_{03} = \sqrt{[(2\times 60\times 7500)/(20+2\times 5\times 5)]}$ = App. 113 units.

Corresponding cost of Average inventory level = (111/2) × 6 + 102/2) × 7 + 113/2) × 5 = Rs. 972.50

This amount is slightly less than the given limit. Now let us try the value of λ as 4.

$q_{01} = \sqrt{[(2\times 50\times 10000)/(20+2\times 4\times 6)]}$ = App. 121 units.

$q_{02} = \sqrt{[(2\times 40\times 12000)/(20+2\times 4\times 7)]}$ = App. 112 units.

$q_{03} = \sqrt{[(2\times 60\times 7500)/(20+2\times 4\times 5)]}$ = App. 123 units.

Corresponding cost of Average inventory level = (121/2) × 6 + 112/2) × 7 + 123/2) × 5 = Rs. 1112.50. This is slightly greater than the given limit. Hence the value of λ must lie between 4 and 5. A graph is drawn for values of average inventory cost and λ and a straight line is drawn for the average inventory cost at λ = 4 and 5 and then a horizontal line from Rs. 1000/- is drawn to find the value of λ . This is shown in the figure 8.18. From the figure the value of λ is 4.7. Using this value let us find the value of optimal inventory.

$$q_{01} = \sqrt{[(2 \times 50 \times 10000)/(20 + 2 \times 4.7 \times 6)]} = \text{App. } 114 \text{ units.}$$

$$q_{02} = \sqrt{[(2 \times 40 \times 12000)/(20 + 2 \times 4.7 \times 7)]} = \text{App. } 105 \text{ units.}$$

$$q_{03} = \sqrt{[(2 \times 60 \times 7500)/(20 + 2 \times 4.7 \times 5)]} = \text{App. } 116 \text{ units.}$$

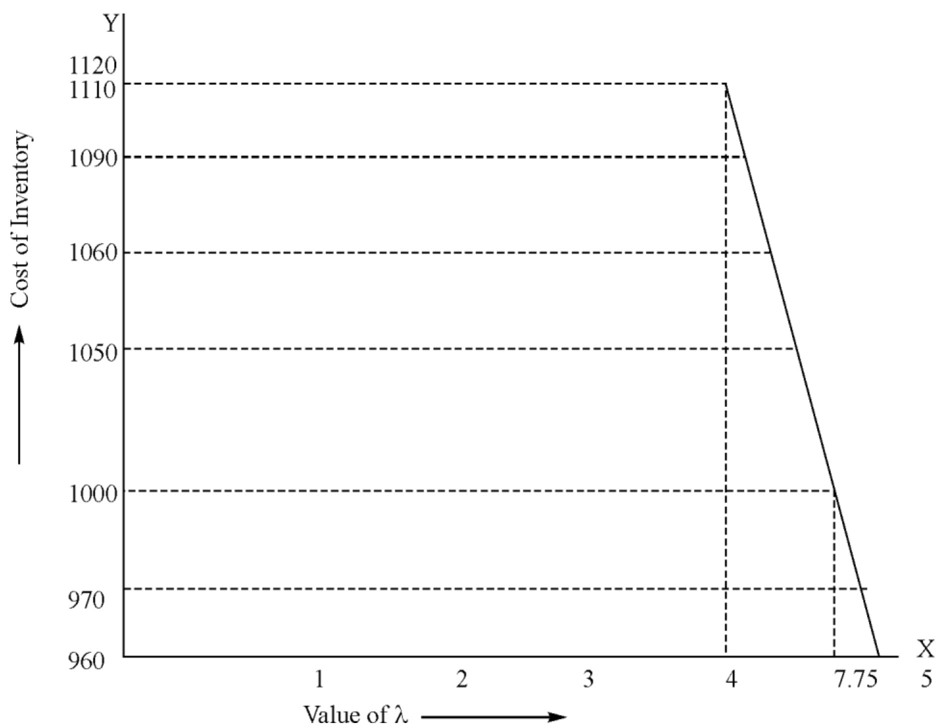


Figure 8.18.

Corresponding cost of Average inventory level = (114/2) × 6 + 105/2) × 7 + 116/2) × 5 = Rs. 999.50. This amount is very close to the given limit of financial commitment and hence this is accepted.
(Note: In the problems of the type shown above, we are concerned with total value of average inventories of three products. The constraint in the example is:
½ $\sum p_i q_i \leq M$ OR $\sum p_i q_i \leq 2M$. However, the values of q_i is worked out by same formula because by taking this constraint $\partial L / \partial q_i$ does not change).

(b) Restrictions on the area available for storage (storage space)

Now let us see when a restriction on storage space in square meters (or square feet) is made how to solve the problem. Let us assume that 'a' is the limit of floor space available in square meters (or square feet). Let a_i square meters (Square feet) of floor space is required for one of the material, say '*i th*' item, then the required constraint is:

$$\sum_{i=1}^{n} a_i \, q_i \le a$$

This is formally equivalent to the investment constraint *i.e.* $\sum^{n} p_i \, q_i \le M$, for which we have already obtained optimal order quantity. Hence in place of 'p_i', if we substitute, 'a_i' we get:

$$q_0 = \sqrt{[(2C_{3i}\, r_i)\,/(C_{1i} + 2\lambda\, a_i\,)]}$$

Problem 8.45.

A small shop produces three machine parts 1,2,and 3 in lots. The shop has only 650 square feet of storage space. The appropriate data for three items are represented fin the following table:

Item	*Product 1*	*Product 2*	*Product 3*
Demand rate in units per year	5000	2000	10000
Procurement cost in Rs.	100	200	75
Cost per unit in Rs.	10	15	5
Floor space required in square feet.	0.70	0.80	0.40

The carrying cost on each item is 20% of average inventory valuation per year. If no stock out are allowed, determine the optimal lot size for each item.

Solution

$$Q_{01} = \sqrt{[(2C_{31}\, r_1)\,/(i\, p_i)]} = \sqrt{[(2 \times 5000 \times 100)\,/(0.2 \times 10)]} = \text{App. 700 units.}$$

$$Q_{02} = \sqrt{[(2C_{32}\, r_2)\,/(i\, p_2)]} = \sqrt{[(2 \times 2000 \times 200)\,/(0.2 \times 15)]} = \text{App. 516 units.}$$

$$Q_{03} = \sqrt{[(2C_{33}\, r_3)\,/(i\, p_3)]} = \sqrt{[(2 \times 10000 \times 75)\,/(0.2 \times 5)]} = \text{App./ 1225 units.}$$

Corresponding floor space required = $\sum a_i \, q_{0i}$ = (0.07 × 707) + 0.8 × 516 + 0.4 × 1225 = 1397.7 square feet. But the given limit is only 650 square feet. Hence the space we got is more than the required. We can try with Lagrange's multiples λ to get the right answer. First let us try with the value of $\lambda = 4$.

$$q_{0i} = \sqrt{[(2C_{3i}\, r_i)\,/(i\, p_i + 2\lambda\, a_i\,)]}$$

$$q_{01} = \sqrt{[(2 \times 5000 \times 100)\,/(0.20 \times 10 + 2 \times 4 \times 0.70)]} = \text{App. 363 units.}$$

$$q_{02} = \sqrt{[(2 \times 2000 \times 200)\,/(0.20 \times 15 + 2 \times 4 \times 0.80)]} = \text{App 292 units.}$$

$$q_{03} = \sqrt{[(2 \times 10000 \times 75)\,/(0.20 \times 5 + 2 \times 4 \times 0.40)]} = \text{App 598 units.}$$

Corresponding floor space = (363 × 0.7) + (292 × 0.8) + (598 × 0.4) = 726.9 square feet. As this area is also more than the given limit, let us try with a value of $\lambda = 5$.

$q_{01} = \sqrt{[(2 \times 5000 \times 100)/(0.20 \times 10 + 2 \times 5 \times 0.70)]}$ = App. 333 units.

$q_{02} = \sqrt{[(2 \times 2000 \times 200)/(0.20 \times 15 + 2 \times 5 \times 0.80)]}$ = App 270 units.

$q_{03} = \sqrt{[(2 \times 10000 \times 75)/(0.20 \times 5 + 2 \times 5 \times 0.40)]}$ = App 578 units.

The required floor space = (333 × 0.7) + (270 × 0.8) + (578 × 0.4) = 668.3 Square feet. This value is slightly higher than the given limit. Hence by interpolation we can select a slightly higher value say λ = 5.4. Then the optimal quantities are:

$q_{01} = \sqrt{[(2 \times 5000 \times 100)/(0.20 \times 10 + 2 \times 5.4 \times 0.70)]}$ = App. 324 units.

$q_{02} = \sqrt{[(2 \times 2000 \times 200)/(0.20 \times 15 + 2 \times 5.4 \times 0.80)]}$ = App 263 units.

$q_{03} = \sqrt{[(2 \times 10000 \times 75)/(0.20 \times 5 + 2 \times 5.4 \times 0.40)]}$ = App 531 units.

The required floor space = (324 × 0.7) + (263 × 0.8) + (531 × 0.4) = 649.6 Square feet. This is very close to the given floor space. Hence the optimal quantities of products are:

q_{01} = 324 units, q_{02} = 263 units, and q_{03} = 531 units.

Problem 8.46.

Three items are produced in a company and they are to be stored in the available space, which is limited to 25 square meters. The other particulars are given in the table below. Find the optimal quantities of the products.

Item	*Demand in units.*	*C_3 Procurement cost in Rs,*	*Carrying cost in Rs.*	*Area required in meter square.*
1	20	100	30	1
2	40	50	10	1
3	30	150	20	1

Solution

By neglecting the constraint let us find optimal quantities, given by $q_{0i} = \sqrt{[(2C_{3i}\, r_i)/(C_{1i})]}$.

$q_{01} = \sqrt{[(2 \times 100 \times 20)/30]}$ = 11.5 units.

$q_{02} = \sqrt{[(2 \times 50 \times 40)/10]}$ = 20 units.

$q_{03} = \sqrt{[(2 \times 150 \times 30)/20]}$ = 21.2 units.

Corresponding space required = 11.5 × 1 + 20 × 1 + 21.2 × 1 = 52.7 Sq.mt. This is more than the required. Hence let us try the value of λ = 5,15,20 and 30.

λ = 5. for which $q_{0i} = \sqrt{[(2C_{3i}\, r_i) / C_i + 2\lambda\, a_i}$

$q_{01} = \sqrt{[(2 \times 100 \times 20)/(30 + 2 \times 5 \times 1)]}$ = 10 units.

$q_{02} = \sqrt{[(2 \times 50 \times 40)/(10 + 2 \times 5 \times 1)]}$ = 14.1 units.

$q_{03} = \sqrt{[(2 \times 150 \times 30)/(20 + 2 \times 5 \times 1)]}$ = 17.3 units.

Corresponding floor area = 10 × 1 + 14.1 × 1 + 17.3 × 1= 41.4 Sq. Mt. This is also more than the given limit.

Let take the value of λ = 15

$q_{01} = \sqrt{[(2 \times 100 \times 20)/(30 + 2 \times 15 \times 1)]}$ = 8.2 units.

$q_{02} = \sqrt{[(2 \times 50 \times 40)/(10 + 2 \times 15 \times 1)]}$ = 10.2 units.

$q_{03} = \sqrt{[(2 \times 150 \times 30)/(20 + 2 \times 15 \times 1)]}$ = 13.4 units.

Corresponding floor area = 8.2 × 1 + 10.2 × 1 + 13.4 × 1= 31.8 Sq. Mt. This is also more than the given limit.

Now let try with the value of λ = 20.

$q_{01} = \sqrt{[(2 \times 100 \times 20)/(30 + 2 \times 20 \times 1)]}$ = 7.6 units.

$q_{02} = \sqrt{[(2 \times 50 \times 40)/(10 + 2 \times 20 \times 1)]}$ = 8.9 units.

$q_{03} = \sqrt{[(2 \times 150 \times 30)/(20 + 2 \times 20 \times 1)]}$ = 12.2 units.

Corresponding floor area = 7.6 × 1 + 8.9 × 1 + 12.2 × 1= 28.7 Sq. Mt. This is also more than the given limit.

Now let take the value of λ = 30.

$q_{01} = \sqrt{[(2 \times 100 \times 20)/(30 + 2 \times 30 \times 1)]}$ = 6.7 units.

$q_{02} = \sqrt{[(2 \times 50 \times 40)/(10 + 2 \times 30 \times 1)]}$ = 7.6 units.

$q_{03} = \sqrt{[(2 \times 150 \times 30)/(20 + 2 \times 30 \times 1)]}$ = 10.6 units.

Corresponding floor area = 6.7 × 1 + 7.6 × 1 + 10.6 × 1= 24.9 Sq. Mt. This is very close to the given limit. Hence the optimal quantities of three items are: q_{01} = 6.7 units, q_{02} = 7.6 units and q_{03} = 10.6 units.

Problem 8.47.

A machine shop produces three products 1,2 and 3 in lots. The shop has a warehouse whose total floor area is 4000 square meters. The relevant data for three products is given below:

Item	*Product 1*	*Product 2*	*Product 3.*
Annual demand in units per year (r)	500	400	600
Cost per unit (p) in Rs.	30	20	70
Set up cost per lot C_3 in Rs.	800	600	1000
Floor area required in Sq. mt.	5	4	10

The inventory carrying chargers for the shop are 20% of the average inventory valuation per annum for each item. If no stock outs are allowed and at no time can the warehouse capacity be exceeded, determine the optimal lot size of each item.

Solution

Optimal quantities of each item is given by (ignoring the limitation on floor area) $\sqrt{[(2C_{3i} \times r_i)/(ip)]}$.

$q_{01} = \sqrt{[(2 \times 800 \times 500)/(0.20 \times 30)]}$ = App. 365 units.

$q_{02} = \sqrt{[(2 \times 600 \times 400)/(0.20 \times 20)]}$ = App 346 units.

$q_{03} = \sqrt{[(2 \times 1000 \times 600)/(0.20 \times 70)]}$ = App. 292 units.

Floor space required = $\Sigma\, q_{0i}\, a_i$ = 365 × 5 + 346 × 4 + 292 × 10 = 1825 + 1384 + 2920 = 6129 Sq. mt. This is greater than the given limit of 4000 Sq.mt. Let us use Lagrange's multiplier technique to find the required quantities. Let us try with values of λ = 1.0, 0.8.

$\lambda = 1.00$, $q_{0i} = \sqrt{[(2 \times C_{3i} \times r_i)/(i \times p_i \times 2 \times \lambda \times a_i)]}$

$q_{01} = \sqrt{[(2 \times 800 \times 500)/(0.20 \times 30 + 2 \times 1 \times 5)]}$ = 223 units.

$q_{02} = \sqrt{[(2 \times 600 \times 400)/(0.20 \times 20 + 2 \times 1 \times 4)]}$ = 200 units.

$q_{03} = \sqrt{[(2 \times 1000 \times 600)/(0.20 \times 70 + 2 \times 1 \times 10)]}$ = 187 units.

Required floor area = 223 × 5 + 200 × 4 + 187 × 10 = 3785 Sq. mt. This is also less than the given limit.

$\lambda = 0.8$, $q_{0i} = \sqrt{[(2 \times C_{3i} \times r_i)/(i \times p_i \times 2 \times \lambda \times 1_i)]}$

$q_{01} = \sqrt{[(2 \times 800 \times 500)/(0.20 \times 30 + 2 \times 0.8 \times 5)]}$ = 239 units.

$q_{02} = \sqrt{[(2 \times 600 \times 400)/(0.20 \times 20 + 2 \times 0.8 \times 4)]}$ = 214 units.

$q_{03} = \sqrt{[(2 \times 1000 \times 600)/(0.20 \times 70 + 2 \times 0.8 \times 10)]}$ = 200 units.

Required floor area = 239 × 5 + 215 × 4 + 200 × 10 = 4051 Sq.mt. This is slightly higher than the given limit. Now let us take the value of λ = 0.835 and find the optimal values of quantities.

$q_{01} = \sqrt{[(2 \times 800 \times 500)/(0.20 \times 30 + 2 \times 0.835 \times 5)]}$ = 236 units.

$q_{02} = \sqrt{[(2 \times 600 \times 400)/(0.20 \times 20 + 2 \times 0.835 \times 4)]}$ = 211 units.

$q_{03} = \sqrt{[(2 \times 1000 \times 600)/(0.20 \times 70 + 2 \times 0.835 \times 10)]}$ = 197 units.

Required floor area = 236 × 5 + 211 × 4 + 197 × 10 = 3994 Sq mt. This is very nearer to given value, hence is accepted. Hence q_{01} = 236 units, q_{02} = 211 units and q_{03} = 197 units. **(Remember always see that the obtained area must be slightly less than or equal to the given limit and it should never exceed the given value.)**

PROBABILISTIC OR STOCHASTIC MODELS

So far we have discussed the problems, where the demand for an item is known and deterministic in nature and it will not vary during the planning period. If the demand is not known exactly to us or it cannot be pre determined or in case it goes on changing / fluctuate with time in either way, the situation is known as ***Models with unknown demand or models with probabilistic demand***. This means that demand can be known with certain probability. When the probability of demand '*r*' is expected, then we cannot minimize the actual cost. But the optimal quantity of inventory is determined on the basis of **minimizing the total *expected* cost represented by (TEC)** instead of minimizing the actual cost. In many practical situations or in real world problems, it is observed that neither the consumption rate of material or commodity or the lead time is constant throughout the year. To face these uncertainties in consumption rate and lead time, an extra stock is maintained to meet the demand, in case any shortage is there. The extra stock is termed as **BUFFER STOCK OR SAFETY STOCK**.

Single period model with uniform demand (No set up cost model)

In this model the following assumptions are made:

(a) Reorder time is fixed and known say '*t*' units of time. Therefore the set up cost C_3 is not included in the total cost.

(b) Demand is uniformly distributed over period. Here the term period refers for the time of one cycle.

(c) Production is instantaneous, *i.e.* lead-time is zero.

(d) Shortages are allowed and they are backlogged. The costs included in this model is C_1 carrying cost per unit of quantity per unit of time and C_2 the shortage cost per unit of quantity per unit of time.

(e) Units are discrete and $p(r)$ is the probability of requiring '*r*' units per period.

If '*S*' is the level of inventory in the beginning of each period, and we have to find the optimum value of '*S*'. Hence the decision variable is *S*.

In this problem two situations will arise:

(a) Demand $r \leq S$, (*b*) demand $r > S$. The two situations are illustrated by means of graph in figure 8.19.

Inventory in one cycle = $\frac{1}{2}(S + S - r)\, t = \frac{1}{2}(2S - r)\, t = (S - r/2)\, t$. units.

Hence inventory Carrying cost = $C_1 \times (S - r/2) \times t$, this is true when $r \leq S$. But the demand is equal to '*r*' is with a probability of $p\,(r)$. Hence the expected carrying cost = $C_1 \times (S - r/2) \times t \times p\,(r)$. As '*r*' may have any values (because $r \leq S$), the total expected carrying cost when $r \leq S$ is given by:

$$\sum_{r=0}^{S} C_1\, ct \times (S - r/2) \times p\,(r)$$

In case $r > S$, then carrying cost and shortage cost are to be considered.

Carrying cost = $\frac{1}{2} \times S \times t \times C_1$ and shortage cost = $\frac{1}{2}(r - S) \times C_2\, t_2$

By mathematical treatment (Students are advised to refer for derivation the Operations Research book where mathematical approach is given), it can be shown that

$$L\,(S_0 - 1) < [(C_2) / (C_1 + C_2)] < L\,(S_0), \text{ where, } L\,(S) = \sum_{r=0}^{S} p\,(r) + (S + \tfrac{1}{2}) \times \sum_{r=S+1}^{\propto} [\,p\,(r)/r]$$

Total expected cost is given by the formula:

$$C_{(s)} = C_1 \sum_{r=0}^{S} [S - (r/2)]\, P(r) + C_1 \times \sum_{r=S+1}^{\infty} (S^2/2r) \times p(r) + C_2 \sum_{r=S+1}^{\infty} [(r - S)^2 / 2r] \times p(r)$$

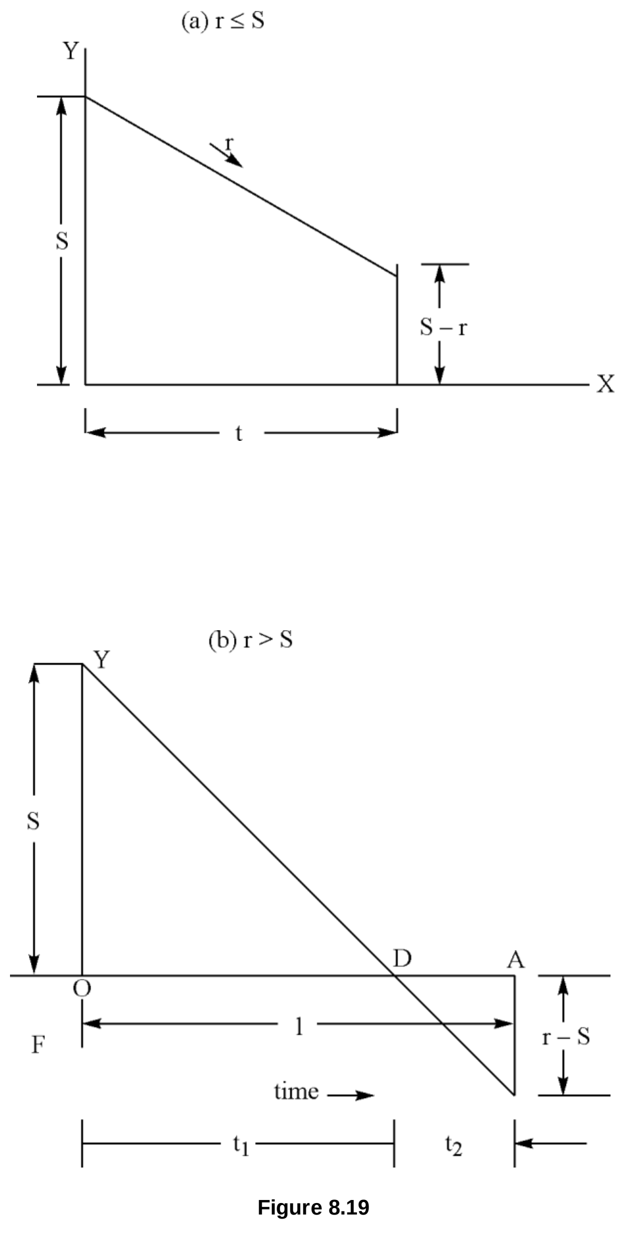


Figure 8.19

Problem 8.48.

A contractor of second hand motor trucks uses to maintain a stock of trucks every month. The demand of the trucks occurs at a constant rate but not in constant size. The probability distribution of the demand is as shown below:

Demand (r):	*0*	*1*	*2*	*3*	*4*	*5*	*6 or more.*
Probability $p(r)$:	0.40	0.24	0.20	0.10	0.05	0.01	0.00

The holding cost of an old truck in stock for one month is /Rs.100/- and penalty for a truck if not delivered to the demand, is Rs. 1000/-. Determine the optimal size of the stock for the contractor.

Solution

S	R	$P(r)$	$[p(r)/r]$	$\sum_{S+1}^{\infty}[p(r)/r$	$(S+1/2) \times \sum_{S+1}^{\infty}[p(r)/r]$	$\sum_{0}^{S} p(r)$	$L(S)$
0	0	0.40	∞	0.3875	0.19375	0.40	0.59375
1	1	0.24	0.2400	0.1475	0.22125	0.64	0.86125
2	2	0.20	0.1000	0.0475	0.11875	0.84	0.95874
3	3	0.10	0.0330	0.0145	0.05075	0.94	0.99075
4	4	0.05	0.0125	0.0020	0.00900	0.99	0.99900
5	5	0.01	0.0020	0.0000	0.00000	1.00	1.00000
≥6	≥6	0.00	0.0000	0.0000	0.00000	1.00	1.00000

Here the ratio $[C_2 / (C_1 + C_2)]$ = [1000 / (1000 + 100) = 1000 / 1100 = 0.9090. This figure lies between $L(2)$ and $L(1)$. Hence the optimal stock = 2 trucks.

Problem 8.49.

A manufacturer wants to determine the optimum stock level of a certain part. The part is used in filling orders, which come in at a constant rate. The delivery of these parts to him is almost instantaneous. He places his orders for these parts at the start of every month. The requirements per month are associated with probabilities shown in table below. Holding cost is Re.1/- per part per month and shortage cost is Rs. 19/- per part per month. Also find the expected cost associated with the optimum stock.

Demand in number							
of parts required per month:	0	1	2	3	4	5	6 or more.
Probability:	0.10	0.15	0.25	0.30	0.15	0.05	

Solution

S	R	$P(r)$	$[p(r)/r]$	$\sum_{S+1}^{\infty}[p(r)/r$	$(S+1/2) \times \sum_{S+1}^{\infty}[p(r)/r]$	$\sum_{0}^{S} p(r)$	$L(S)$
0	0	0.10	∞	0.4225	0.21125	0.10	0.31125
1	1	0.15	0.1500	0.2725	0.40875	0.25	0.65875
2	2	0.25	0.1250	0.1475	0.36875	0.50	0.86875
3	3	0.30	0.1000	0.0475	0.16625	0.80	0.96625
4	4	0.15	0.0375	0.0100	0.04500	0.95	0.99500
5	5	0.05	0.0100	0.0000	0.00000	1.00	1.00000
6 or > 6	6 or > 6	0.00	0.0000	0.0000	0.00000	1.00	1.00000

Here the ratio $C_2 / (C_1 + C_2) = 19 / (19 + 1) = 19 / 20 = 0.95$. This lies between (L_2) and (L_3). Hence we can take $S = 3$ units.

The optimal cost is given by:

$$C_{(s)} = C_1 \sum_{r=0}^{S} [S - (r/2)]\, P(r) + C_1 \times \sum_{r=S+1}^{\infty} (S^2/2r) \times p(r) + C_2 \sum_{r=S+1}^{\infty} [(r - S)^2 /2r] \times p(r)$$

$= \text{Rs.}[\, \sum (3 - r/2) \times p(r) + 1 \times \sum [(3^2 / 2) \times p(r) / r] + 19\ 3\ [\,(r-3)^2 / 2r\, p(r)]$

$= \{\, [(3 - 0)(0.10) + (3 - ½)(0.15)] + [(3 - 1)(0.25) + (3 - 3/2)(0.30)] + (9/2)\,[(0.15/4) + (0.05/4) + 0]$

$+ 19\, [\,(4 - 3)^2 / (2 \times 4) \times (0.15) + (5 - 3)^2 / (2 \times 4) \times (0.15) + 0]\}$

$= \text{Rs.}\ [\,(0.30 + 0.375 + 0.50 + 0.45) + 0.21375 + 0.73625] = \text{Rs. } 2.58.$

Problem 8.50.

The demand for a particular product is continuous and shows the following probability distribution:

Demand:	*0*	*1*	*2*	*3*	*4*	*5 or more*
Probability:	0.16	0.10	0.30	0.24	0.20	0.00

Find out the optimum stock level if the cost of shortage is Rs. 40/- per unit and the cost of holding is Rs. 10/- per unit. The shortage cost is proportional to both time and quantity short.

Solution

S	R	$P(r)$	$[p(r)/r]$	$\sum_{S+1}^{\infty}[p(r)/r]$	$(S+1/2) \times \sum_{S+1}^{\infty}[p(r)/r]$	$\sum_{0}^{S} p(r)$	$L(S)$
0	0	0.16	∞	0.38	0.190	0.16	0.350
1	1	0.10	0.10	0.28	0.420	0.26	0.680
2	2	0.30	0.15	0.13	0.325	0.56	0.885
3	3	0.24	0.08	0.05	0.175	0.80	0.975
4	4	0.20	0.05	0.00	0.000	1.00	1.000
≥ 5	≥ 5	0.00	0.00	0.00	0.000	1.00	1.000

Now the ratio $C_2 / (C_1 + C_2) = 40 / (40 + 10) = 0.8$.

$0.685 < 0.8 < 0.885$, in this case $S = 2$ satisfies the condition. Hence optimum stock level = 2 units.

Problem 8.51.

The probability distribution of monthly sales of a certain item is as follows:

Monthly sale in units:	*0*	*1*	*2*	*3*	*4*	*5*	*6*
Probability:	0.02	0.05	0.30	0.27	0.20	0.10	0.06

The cost of carrying inventory is Rs. 10/- per unit per month. The current policy is to maintain a stock of four items at the beginning of each month. Assuming that the cost of shortage is proportional to both time and quantity short, obtain the imputed cost of a shortage of one item for one unit of time.

Solution

As the data given is in discrete values, the imputed value will have a range.

Data: C_1 = Rs. 10/- per unit per month. S = Stock level = 4 units.

As the demand is uniformly distributed over the month,

$L(S_0 - 1) < C_2 / (C_1 + C_2) < L(S_0)$

$$L(S_0 - 1) = L(4 - 1) = \sum_{r=0}^{4} p(r) + (4 + ½) \times \sum_{r=4}^{6} p(r)/r$$

$= 0.84 + (7/2) [(0.20/4) + (0.10 / 5) + (0.06/ 6)] = 0.92$

Thus the least value of C_2 is given by $C_2 / (C_1 + C_2) = 0.92$ or $(C_2 /10) + C_2 = 0.92$. Which gives that the value of C_2 = Rs. 115/-.

Similarly, the highest value of C_2 is given by considering the right-hand side of $C_2 / (C_2 + C_2)$, *i.e.*

$$C_2 / (C_1 + C_2) = \sum_{r=0}^{4} p(r) + (4 + ½) \times \sum_{r=5}^{6} p(r)/r = 0.84 + (9 / 2) [(0.10 / 5) + (0.06 / 6)] = 0.975.$$

Hence $C_2 = 0.975 (10 + C_2)$, because C_1 = Rs. 10/-. This gives C_2 = Rs. 390/-.

Therefore imputed cost of Shortage is given by Rs. 115/- < C_2 < Rs. 390/-

Problem 8.52.

The probability distribution of monthly sales of certain item is as follows:

Monthly sales:	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
Probability:	0.01	0.04	0.25	0.30	0.23	0.08	0.05	0.03	0.01

The cost of holding inventory is Rs.8/- per unit per month. A stock of 5 items is maintained at the start of each month. If the shortage cost is proportional to both time and quantity short, find the imputed cost of shortage of unit item for unit time.

Solution

As the given data has discrete units the imputed cost will have a range.

Given that $S = 5$, C_1 = Rs.8/- per unit per month, Range of monthly sales = 0 to 8 and the probability of sales are as given below:

p_0	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8
0.01	0.04	0.25	0.30	0.23	0.08	0.05	0.03	0.01

The range of S is given by: $L(S_0 - 1) < C_2 / (C_1 + C_2) < L(S_0)$ *i.e.* The least value of C_2 is given by:

$$L(S_0 - 1) = L(S_0 - 1) = \sum_{r=0}^{4} p(r) + (S_0 + ½) \times \sum_{r=5}^{8} p(r)/r$$

$$L(5 - 1) = L(5 - 1) = \sum_{r=0}^{4} p(r) + (5 + ½) \times \sum_{r=5}^{8} p(r)/r = C_2/(8 + C_2)$$

$(p_0 + p_1 + p_2 + p_3 + p_4) \times (9/2)$ [$(p_5 / 5) + (p_6/6) + (p_7 / 7) + (p_8 / 8)$] $= C_2 / (C_1 + C_2)$

= (0.01 + 0.04 + 0.25 + 0.30 + 0.23) + (9/2) [(0.08/5) + 0.05 / 6) + (0.03 / 7) + (0.01 / 8)

$= C_2 / (8 + C_2)$

= 0.83 + 4.5 (0.016 + 0.0083 + 0.0043 + 0.00125) = 0.9643 = $C_2/(8 + C_2)$,

C_2 = (.9643 × 8) / 0.0357 = Rs. 216/-

Similarly upper limit of C_2 can be obtained by $C_2/(C_1 + C_2) < L(S_0) = C_2 / (C_1 + C_2) =$

$$\sum_{r=0}^{5} p(r) + (5 + ½) \times \sum_{r=6}^{8} p(r)/r$$

$(p_0 + p_1 + p_2 + p_3 + p_4 + p_5) + (11 / 2) \times [(p_6 / 6) + (p_7 / 7) + (p_8 / 8)] = C_2 / (8 + C_2)$

(0.01 + 0.04 + 0.25 + 0.30 + 0.23 + 0.08) + (11 / 2) [(0.05 / 6) + (0.03 / 7) + (0.01 / 8)]

$= C_2/(8 + C_2)$

0.91 + (11/ 2) × (0.01385) = 0.91 + 0.076165 = $C_2 / (8 + C_2)$

$(8 + C_2) \times (0.91 + 0.076165) = C_2$, Which gives C_2 = Rs. 570.25

Hence the range of C_2, *i.e.* imputed value = Rs. 216 < C_2 < Rs. 570.25

8.9.2. Single period problem with instantaneous demand (or discontinuous demand and time independent costs - no set up cost model)

This Model is very much similar to the previous one but here the withdrawal of items form the inventory is not uniformly distributed over the period and the cost C_1 and C_2 are independent of time. There are two cases here.

Case (*a*)- Demand '*r*' is $\leq S$: Here the cost is $(S - r)\, C_1$.

Case (*b*)- Here the demand '*r*' is $> S$: In this case we will not consider the holding cost of an item if it is used or interpret that the demand is fulfilled in the beginning of the period. The cost is $(r - S)\, C_2$. Both the cases are illustrated by means of graphs in the figure 8.20.

Here the total expected cost $[TEC(s)] = \sum_{r=0}^{S} (S - r)\, p\,(r) = C_2 \sum_{r=S+1}^{\infty} (r - S\,)\, p\,(r)$

The value of optimal value of $S = S_0$ is given when,

$TEC\,(S_0 + 1) > TEC\,(S_0)$ and $TEC\,(S_0 - 1) > TEC\,(S_0)$

Now TEC $(S+1)$ $= C_1 \sum_{r=0}^{S+1} (S + 1 - r)\, p\,(r) + C_2 \sum_{r=S+2}^{\infty} (r - S - 1)\, p\,(r)$

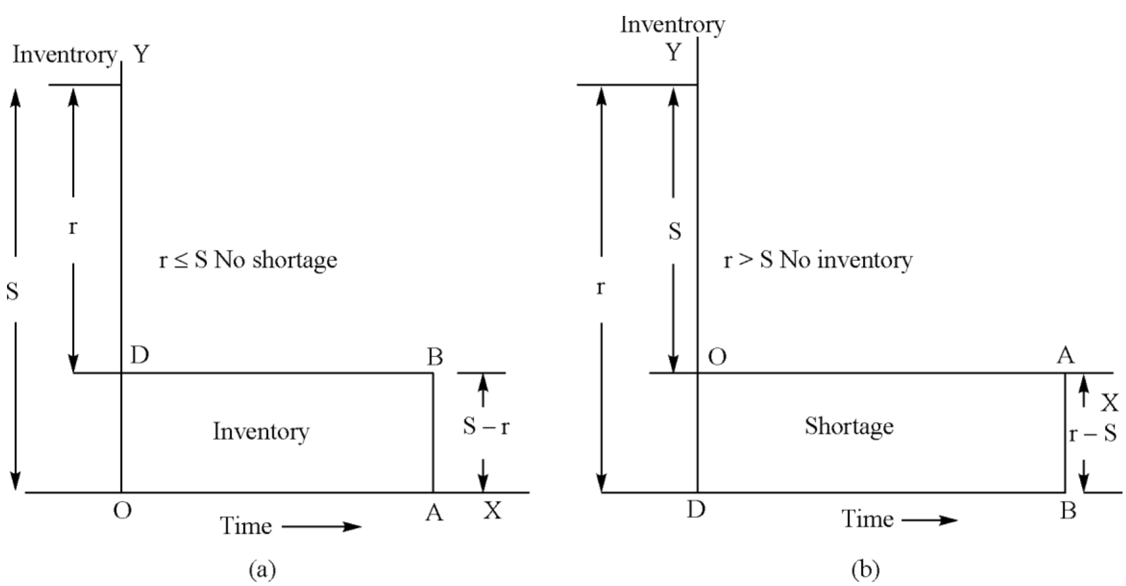


Figure 8.20

Similarly, $TEC\,(S - 1) - TEC\,(S) = C_2 - (C_1 + C_2) \sum_{r=0}^{S-1} p\,(r).$

By mathematical treatment (students are advised to refer to a book on O.R. with mathematical approach for derivation) we can show that S_0 is optimum when,

$$\sum_{r=0}^{S_0-1} p\,(r) < C_2/(C_1 + C_2) < \sum_{r=0}^{S_0} p\,(r)$$

If the units are not discrete or '*r*' is capable of being considered as continuous variable, then the optimal value of *S i.e.* S_0 is given by:

$$\int_0^S f\,(r)\,dr = C_2/(C_1 + C_2)$$

Problem 8.53.

A newspaper boy buys papers for 0.05 paise each and sells them for 0.06 paise each. He cannot return unsold newspapers. Daily demand '*r*' for newspapers follows the distribution:

Demand 'r':	*10*	*11*	*12*	*13*	*14*	*15*	*16*
Probability p (r):	0.05	0.15	0.40	0.20	0.10	0.05	0.05

If each day's demand is independent of the previous day's demand, how many papers should be ordered each day?

Solution

Let S be the number of newspapers ordered per day and 'r' be the demand for it (number of papers sold each day). Given that C_1 = Rs. 0.05 and C_2 = Rs. 0.06 – 0.05 = Rs. 0.01. Now let us work out the cumulative demand for the newspaper (because the demand for units is discrete).

Demand = r	*10*	*11*	*12*	*13*	*14*	*15*	*16*
Probability p (r)	0.05	0.15	0.40	0.20	0.10	0.05	0.05
Cumulative probability = $\sum_{r=0}^{S} p\,(r)$	0.05	0.20	0.60	0.80	0.90	0.95	1.00

Now, $\sum_{r=0}^{S_0-1} P\,(r) < C_2/(C_1 + C_2) < \sum_{r=0}^{S_0} p\,(r)$ and $C_2/(C_1 + C_2) = 0.01/(0.05 + 0.01) = (1/6) = 0.167$.

This value lies between demand 10 and 11. Hence the newspaper boy has to purchase 11 papers. (0.05 < 0.167 < 0.20).

Problem 8.54

The demand for certain product has a rectangular distribution between 4000 and 5000 units, find the optimal order quantity if storage cost is 'Re. 1.00 per unit and shortage cost is Rs. 7/- per unit.

Solution

Data: C_1 = Re.1.00 per unit and C_2 = Rs. 7/- per unit and the demand is rectangular between 4000 and 5000 units.

Since the demand is rectangular between 4000 and 5000, assuming it a continuous variate, the density function is given by: $f(r) = (1 / 1000)$ Therefore, **(Note: $f(x) = 1 / (b - a)$ where $a \leq x \leq b$)**

$$\sum_{4000}^{S} (1/1000)\, dr = 7/(1+7) = (7/8) \text{ or } (1/1000)\,(S - 4000) = (7/8) \text{ OR } S = 4875 \text{ Units.}$$

Problem 8.55.

Some of the spare parts of a ship cost Rs. 50,000 each. These spare parts can only be ordered together with the ship. If not ordered at the time the ship is constructed, these parts cannot be available on need. Suppose that a loss of Rs. 4,500,000 is suffered for each spare that is needed when none is available in the stock. Further suppose that the probabilities that the spares will be needed as replacement during the life term of the class of ship discussed are:

Demand (r) =	*0*	*1*	*2*	*3*	*4*	*5*	*6 or more.*	
Probability $p(r)$	0.9000	0.040	0.025	0.020	0.010	0.005	0.000	Total = 1.000

How many spare parts are to be procured with the ship?

Solution

Data: C_1 = Rs. 50,000/-, C_2 = Rs. 4,500,000.

Now the ratio $C_2 / (C_1 + C_2) = 4,500,000 / 4,550,000 = 0.989$.

Now cumulative probability is to be worked out because units are discrete.

Demand = r	*0*	*1*	*2*	*3*	*4*	*5*	*6 or more*
Probability $p(r)$	0.900	0.040	0.025	0.020	0.010	0.005	0.000
Cumulative Probability: $\sum_{r=0}^{S} p(r)$	0.900	0.940	0.965	0.985	0.995	1.000	1.000

Now the ratio 0.989 lies between demand 3 and 4, hence the optimal quantity to be purchased along with ship is 4 units.

Problem 8.56.

A company uses to order a new machine after a certain fixed time. It is observed that one of the parts of the machine is very expensive if it is ordered without machine and is Rs. 500/-. The cost of down time of machine and the cost of arranging the part is Rs. 10000/-. From the previous records it is observed that spare part is required with the probabilities as shown below:

Demand = r =	*0*	*1*	*2*	*3*	*4*	*5*	*6 or > 6*
Probability p (r) =	0.90	0.05	0.02	0.01	0.01	0.01	0.00

Find the optimum number of spare parts, which should be ordered with the order of machine.

Solution

Data: C_1 = Rs. 500 per part, C_2 = Rs. 10000 per part. The ratio $C_1 / (C_1 + C_2)$ = 10000 / 10500 = 0.952. As the demand for units is discrete, the cumulative probability is to be found.

Demand = r	*0*	*1*	*2*	*3*	*4*	*5*	*6 or > 6*
Probability $p(r)$	0.90	0.05	0.02	0.01	0.01	0.01	0.00
Cumulative Probability: $\sum_{r=0}^{S} p(r)$	0.90	0.95	0.97	0.98	0.99	1.00	1.00

Since the ratio (= 0.952) lies between 0.95 and 0.97, *i.e.* demand points 1 and 2, the optimal order quantity to be placed with machine is 2.

Problem 8.57.

A firm is to order a new lathe. Is power units is an expensive part and can be ordered only with the lathe. Each of these units is uniquely built for a particular lathe and cannot be used on other lathe. The firm wants to know how many spare units should be incorporated in the order for each lathe. Cost of the unit when ordered with the lathe is Rs. 700/- per units. If a spare unit is needed (because of failure during the service) and is not available, the whole lathe becomes useless. The cost of the unit made to order and the down time cost of lathe is Rs. 9,300/-. The analysis of 100 similar lathes yields the following information given below.

Number of Spared Required.	0	1	2	3	4	5	6	7 or more.
Number of Lathes Requiring Number of Spare parts.	87	5	3	2	1	1	1	0
Estimated Probability of occurrence of Indicated number of failures:	0.87	0.05	0.03	0.02	0.01	0.01	0.01	0.00

(b) If in the above problem, the shortage cost of the part is unknown and the firm wants to maintain stock level of 4 parts, find the shortage cost.

Solution

Data: C_1 = Rs. 700/- per unit, C_2 = Rs. 9300 per unit.

The ratio $C_2 / (C_1 + C_2)$ = 9300 / (700 + 9300) = 9300 / 10000 = 0.93.

S	r	$P(r)$	$\sum_{r=0}^{S} p(r)$
0	0	0.87	0.87
1	1	0.05	0.92
2	2	0.03	0.95
3	3	0.02	0.97
4	4	0.01	0.98
5	5	0.01	0.99
6	6	0.01	1.00
7	7	0.00	1.00

As 0.92 < 0.93 < 0.95 which falls between demand points 1 and 2. Hence optimal order quantity is 2 units.

(*b*) Here $S = 4$ that the level of inventory.

Now $p(r \leq 3) < C_2 / (700 + C_2) < p(r<4)$ **OR** $0.97 < C_2 / (700 + C_2) < 0.98$. Therefore the least value of C_2 is given by:

$C_2 / (700 + C_2) = 0.97$ **OR** $C_2 = (700 \times 0.97) / 0/03$ = Rs. 22, 633. 33 or app: Rs.22,633/-

The maximum value of C_2 is given by: $C_2 / (700 + C_2) = 0.98$ **OR** $C_2 = (700 \times 0.98) / 0.02$ = Rs. 34,300/-.

Rs. 22,633 < C_2 < Rs. 34,300.

Problem 8.58.

The cost of holding an item in stock is Rs.2/- per unit and the shortage cost is Rs. 8/- per unit. If Rs.2/- is the purchasing cost per unit, determine the optimal order level of inventory, given the following probability distribution of demand.

R = Demand =	*0*	*1*	*2*	*3*	*4*	*5*
Probability p (r) =	0.05	0.25	0.20	0.15	0.20	0.15

Solution

Data: C_1 = Rs. 2/- per unit, C_2 = Rs.8/- per units, p = Rs. 2/- per unit.

In this problem as the purchase price is given we have to work out the ratio $(C_2 - p) / (C_1 + C_2)$ *i.e.* (8 – 2) / (2 + 8) = 6 / 10 = 0.60. The cumulative probability is:

r = demand =	*0*	*1*	*2*	*3*	*4*	*5*
$p(r)$ =	0.05	0.25	0.20	0.15	0.20	0.15
$\sum_{r=0}^{S} p(r)$ =	0.05	0.30	0.50	0.65	0.85	1.00

Now 0.05 < 0.60 < 0.65 Hence the optimal order quantity lies between 2 and 3. The order quantity is 3 units.

NEWSPAPER BOY PROBLEM: (GENERAL SINGLE PERIOD MODEL OF PROFIT MAXIMIZATION WITH TIME INDEPENDENT COST)

In newspaper boy problem, he wants to know how many papers he has to purchase and sell to maximize his daily profit. The model can be generalized so as to apply the technique to other type of problem, where the person wants to maximize his profit.

Let us consider an item, which is purchased and sold. The condition here is once he purchases, he cannot return it and if he does not sell it, he sells it after the period for lesser price, i.e. the item is to be discarded. Here we have to find out the expected number of items to be purchased at the beginning of the period, so that the businessman can maximize his expected profit. Let

a = Unit price of an item at which it is procured (independent of number of items procured).

b = Unit selling price of the item during the period and $b>a$.

c = Unit selling price of the item, after the end of the period, in the beginning of which, items were procured. And $c<a$.

d = Unit cost per item if there is a shortage.

$P(x)$ Probability that the demand is of 'x' items during the period under consideration.

n = Items procured at the beginning of the period.

Here we can consider two cases: Case (i) = $x \le n$ *i.e.*, no shortages, and (ii) $x > n$ with shortages.

Expected return from sales when $x \le n$ is:

$$\sum_{x=0}^{n} b \times p(x) + \sum_{x=0}^{n} c\,(n-x)\,p(x)$$

Expected return from sales when $x > n$

$$\sum_{x=n+1}^{\infty} bn\,p(x) - \sum_{x=n+1}^{\infty} d\,(x-n)\,p(x)$$

By mathematical treatment we can arrive that 'n' is the optimal quantity if (demand is discrete)

$$\sum_{x=n+1}^{\infty} p(x) < (a-c)/(b-c+d) < \sum_{x=n}^{\infty} p(x)$$

If demand is continuous, we get the optimal value of 'n' by:

$$\int_{n}^{\infty} f(x)\,dx = (a-c)/(b=d-c)$$

Problem 8.59.

A newspaper boy buys papers for 30 paise each and sells them for 70 paise each. He cannot return unsold newspapers. Daily demand has the following distribution:

Number of Customers.:	*23*	*24*	*25*	*26*	*27*	*28*	*29*	*30*	*31*	*32*
Probability p (x):	0.01	0.03	0.06	0.10	0.20	0.25	0.15	0.10	0.05	0.05

If each day's demand is independent of previous day's demand, how many papers should be order each day?

Solution

Data: a = Rs. 0.30, b = Rs. 0.70, c = Rs. 0.00 and d = Rs. 0.00

Optimal value of 'n' is given by:

$$\sum_{x=n+1}^{32} p(x) < (30/70) < \sum_{x=n}^{32} p(x) \qquad (30 / 70) = 0.428$$

x =	23	24	25	26	27	28	29	30	31	32
n =	23	24	25	26	27	28	29	30	31	32
$p(x)$ =	0.01	0.03	0.06	0.10	0.20	0.25	0.15	0.10	0.05	0.05
$\sum_{x=n}^{32} p(x)$ =	1.00	0.99	0.96	0.90	0.80	0.60	0.35	0.20	0.10	0.05

As 0.428 lies between 0.60 and 0.35, that is demand of 28 and 29. Hence the newspaper boy has to purchase 28 papers.

Problem 8.60.

A baking company sells cake by the pounds. It makes a profit of 50 paise per pound on every pound sold on the day it is baked. It disposes of all cake not sold on the day it is baked at a loss of 12 paise per pound. If its demand is known to be rectangular between 2000 to 3000 pounds, determine the optimal daily amount to be baked.

Solution

Data: $b - a$ = Rs. 0.50, $a - c$ = Rs. 0.12, $d = 0$.

$(a - c) / (b + d - c) = 12 / (50 + 12) = 12 / 62 = 0.193$.

The demand is a continuous variate and it is rectangular distribution between 2000 and 3000. Hence the density function is $f(x) = (1 / 1000)$. **(Note: $f(x) = 1 / (b - a)$, where $a \leq x > b$**

Hence the daily amount to be baked is given by: 3000

$$\int_{n} (1/1000)\, dx = 0.193$$

OR $(3000 - n) = 193$ or $n = 2807$ pounds.

INVENTORY PROBLEMS WITH UNCERTAIN DEMAND (MODELS WITH BUFFER STOCK)

Many a time inventory manager comes across a situation where demand cannot be completely pre-determined. The demand fluctuates in either way. In fact in many practical situations, we see that both demand for an item or lead-time, *i.e.*, the time between placing order and procurement of material will remain constant. In many situations, both demand and lead-time are fluctuating due to uncontrollable reasons. They are highly uncertain in nature. To face these uncertainties in consumption rate and lead time, an extra stock is maintained to meet out the demands, if any. This extra stock is generally known as ***Safety stock***' or ***'Buffer stock'***.

To Determine the Buffer Stock and Re-order Level (ROL)

We must know the maximum lead-time and normal lead-time and the demand during these periods to estimate the **buffer stock or safety stock** required. The buffer stock is calculated by multiplying the consumption rate during the lead – time by the difference between maximum lead-time and normal lead-time. Let

B = Buffer stock,

L = Lead time,

L_d = Difference between maximum lead-time and minimum lead- time.

r = Demand rate.

Total inventory consumption during lead-time, if buffer stock is not maintained = $L \times r = Lr$.

Thus as soon as stock level reaches 'Lr', quantity 'q' should be ordered. This point where we order is known as ***reorder level or ROL.*** However due to uncertainty in supply, this policy of ordering when stock level reaches 'Lr' will create shortages and leads to back orders or lost sales. In order to avoid the shortages, a buffer stock is maintained. Hence,

$ROL = Lr$ + Buffer stock $= Lr + B. = Lr + L_d\, r = (L + L_d) \times r$

Now maximum inventory = $q + B$,

Minimum inventory = S

Average inventory = $[(q + B) + B] / 2 = (q / 2) + B$.

To illustrate the above, let us consider a simple example.

Suppose the demand for an item is 200 units per month, the normal lead-time is 15 days and maximum lead time is 2 months, then the buffer stock B = (2 - ½) × 200 = 300 units.

If L is the lead -time and 'r' is the demand, then the inventory during the lead-time = Lr, which is nothing but the *ROL* as discussed above. If we maintain buffer stock, then placed an order when stock level reaches the level = $B + Lr$. Say for example, the monthly consumption rate for an item is 100 units, the normal lead time is 5 days and the buffer stock is 150 units, then ROL = 150 + (1/2 + 100) = 200 units.

Optimum Buffer stock: When buffer stock maintained is very low, the inventory holding cost would be low but the shortages will occur very frequently and the cost of shortages would be very high. As against this if the buffer stock maintained is rather large, storages would be rather rare, resulting into low shortage costs but inventory holding costs would be high. Hence it becomes necessary to strike balance between the cost of shortages and cost of inventory holding to arrive at an Optimum Buffer Stock.

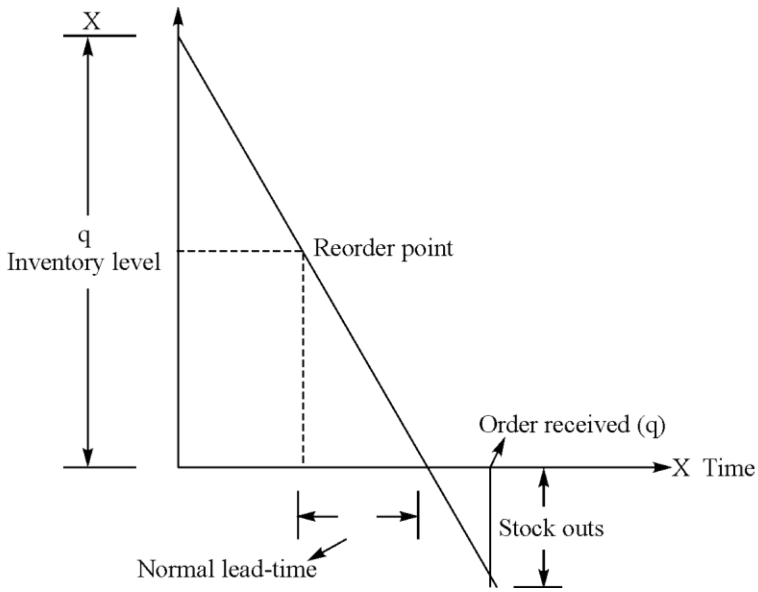


Figure 8.21

Problem 8.61.

The average monthly consumption for an item is 300 units and the normal lead-time is one month. If the maximum consumption has been up to 370 units per month and maximum lead-time is 1 ½ months, what should be the buffer stock for the item.

Solution

Maximum lead – time demand = Maximum lead-time × maximum demand rate = (3/2) × 370 = 555 units.

Normal lead-time demand = 1 × 300 = 300 units.

Buffer stock = Maximum lead-time demand – Normal lead- time demand = 555 – 300 = 255 units.

Problem 8.62.

For a fixed order quantity system find the various parameters for an item with the following data:

Annual demand = λ = 10000 units, Unit price = p = Rs. 1.00, i = Carrying cost = Rs. 0.24 per unit,

C_3 = Set up cost = Rs. 12/- per production run, Past lead times in days are = 15,25,13,14,30, 17 days.

Solution

(a) $E.O.Q = q_0 = \sqrt{(2C_3\lambda) / ip} = \sqrt{(2 \times 10000 \times 12) / 0.24 \times 1}$ = 1000 units.

(b) Optimum buffer = (Maximum lead- time – Normal lead time) × monthly consumption = = [(30 – 125) / 30] × 10000 / 12 = 416.66 = App 417 units. (Here the optimum lead-time = 15 days = 15/30 months).

Note: for more safety some times it is advisable to round off the buffer stock to 450 units.

Another way of getting the same is:

(Normal lead-time consumption = Norma lead time × monthly consumption = (15/30) × (10000/12) = 416.66 or approximately = 417 units).

Hence Re-order level or *ROL* = Safety stock + normal lead-time consumption = 450 + 417 = 867 or App. 870 units.

The inventory would fluctuate from a maximum, of 1450 to a minimum of 450 units. Hence the average inventory = (1450 + 450)/2 = 950 units.

Problem 8.63.

A company uses annually 50,000 units of an item each costing Rs. 1.20. Each order costs Rs. 45/- and inventory carrying costs are 15% of the annual average inventory value. Find *EOQ*.

If the company operates 250 days a year, the procurement time is 10 days and safety stock is 500 units, find re -order level, maximum, minimum and average inventory.

Solution

Data = λ = 50,000 units, p = Rs. 1.20, i = 15%, C_3 = Rs. 45/- L = 10 days, B = 500 units.

$q_0 = \sqrt{(2C_3\lambda)/ip} = \sqrt{[(2 \times 45 \times 50000)/(0.15 \times 1.20)]}$ = 5000 units.

The company operates 50 days a year. Hence requirement per day = 50000 / 250 = 200 units per day.

Lead-time demand = 10 × 200 = 2000 units.

Safety stock = 500 units.

Hence *ROL* = 2000 + 500 = 2500 units.

Maximum inventory = 5000 + 500 = 5500 units.

Minimum inventory = 5000 units.

Average inventory = (5000 / 2) + 500 = 3000 units.

Problem 8.64.

A firm uses every year 12000 units of a raw material costing Rs. 1.25 per units. Ordering cost is Rs. 15/- per order and the holding cost is 5 % per year of average inventory. (*i*) Find Economic Order Quantity,

(*ii*) The firm follows EOQ purchasing policy. It operates for 300 days per year. Procurement time is 14 days and safety stock is 400 units. Find the re-order point, the maximum inventory and the average inventory.

Solution

Data: λ = 12,000 units, p = Rs. 1.25 per units, C_3 Rs. 15/-, i = 0.05, Number of working days = 300, L = 14 days, B = 400 units.

$E.\ O.\ Q = \sqrt{(2C_3\lambda)/ip} = \sqrt{[(2 \times 15 \times 12{,}000)/(0.05 \times 1.25)]}$ = 2,400 units.

Re order level = Buffer stock + Consumption during the lead-time = 400 + (12,000 / 300) × 14 = 960 units.

Maximum inventory = $q_0 + B$ = 2400 + 400 = 2800 units.
Minimum inventory = B = 400 units.
Average inventory = $(q_0 / 2) / B$ = (2800 / 2) + 400 = 1600 units.

Problem 8.65

Calculate the various parameters when the following data is available for an item, which is maintained on EOQ system.

Annual consumption = λ = 12000 units, Unit price = Rs. 7.50, Set up cost = Rs. 6.00 per run, Inventory carrying cost = Rs. 0.12 per unit, Normal lead-time = L_n = 15 days and maximum lead-time = L_m = 20 days.

Solution

$E.O.Q = \sqrt{(2C_3\lambda) / C_1} = \sqrt{[(2 \times 7.50 \times 12000) / 0.12]}$ = 1096 units.

Optimum buffer stock = $B = (L_m - L_n) \times$ consumption = [(20 – 15) / 30] / (12000 / 12)] = 167 units.

Re-order level = $ROL = B$ + Normal lead-time consumption = 167 + [(15 / 30 × 12) × 12000] = 167 + 500 = 667 units.

Problem 8.66

In an inventory model, suppose that the shortages are not allowed and the production rate is infinite and the following data is available:

Yearly demand λ = 600 units, Carrying chargers = i = 0.20, C_3 = Rs. 80/- per order, p = Rs. 3.00 per unit, Lead-time = L = 1 year.

Solution

$q_0 = \sqrt{(2C_3\lambda) / ip} = \sqrt{[(2 \times 80 \times 600) / (0.20 \times 3)]}$ = 400 units.
The time of the cycle = $t_0 = (q_0 / \lambda)$ = 400 / 600 = (2/3) year
$ROL = B$ + Normal lead time consumption
Buffer stock = (Maximum lead time – Normal lead-time) × consumption = (1 – 2/3) × 600 = (1/3) × 600 = 200 units.
Hence ROL = 200 + 1 × 600 = 800 units.

The minimum average yearly cost of ordering and holding = $\sqrt{(2 \times C_3 \times \lambda \times ip}$
= $\sqrt{(2 \times 80 \times 600 \times 0.20 \times 3)}$ = Rs. 240/-.

Problem 8.67.

The following is the distribution of lead-time and daily demand during lead-time:

Lead- time in days:	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
Frequency:	0	0	1	2	3	4	4	3	2	2	1
Demand per day in units:	0	1	2	3	4	5	6	7			
Frequency:	3	5	4	5	2	3	2	1			

What is the buffer stock?

Solution

First let us find the average lead-time.

Lead - time = L	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*Total*
Frequency = f	0	1	2	3	4	4	3	2	2	1	22
L × f	0	2	6	12	20	24	21	16	18	10	129

Average lead-time = 129 / 22 = App. 5.86 days.

Average demand rate is:

Demand = r	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*Total*
Frequency = *f*	3	5	4	5	2	3	2	1	25
F × *r* =	0	5	8	15	8	15	12	7	70

Average demand rate = (70 / 25) = 2.8 units.

Average lead-time demand = 5.86 × 2.8 = 16.4 units.

Maximum lead-time demand = Maximum lead-time × maximum demand = 10 × 7 = 70 units.

Therefore Buffer stock = B = (70 – 16.4) = 53.6 units = App. 54 units.

Problem 8.68.

A company uses annually 24000 units of a raw material, which costs Rs. 1.25 per units. Placing each order costs Rs. 22.50 and the carrying cost is 5.4 percent per year of the average inventory. Find the economic order quantity and the total inventory costs including cost of material. Should the company accept the offer made by the supplier of a discount of 5% on the cost price on a single order of 24000 units? Suppose the company works for 300 days a year. If the procurement time is 12 days and safety stock is 400 units, find the re-order point, the minimum, maximum, and average inventory.

Solution

Data: λ = 24000 units, P = Rs. 1.25 per unit, C_3 = Rs. 22.50, i = 5.4%, discount = 5% for 24000 units. Number of working days = 300 days, L = 12 days, B = 400 units.

$q_0 = \sqrt{(2C_3\lambda)/ip} = \sqrt{[(2 \times 22.50 \times 24000)}$ / (0.054 × 1.25) = 4000 units.

$t_0 = q_0 / \lambda$ = 4000 / 24000 = 1/6 of a year = 2 months.

Total inventory cost = $\sqrt{(2 \times C_3 \times ip \times \lambda) + \lambda \times p} = \sqrt{(2 \times 22.5 \times 0.054 \times 1.24 \times 24000)}$ + 1.25 × 24000 = Rs. 270 + Rs. 30000 = Rs. 30270/-

If we want to use the discount facility, we have to purchase 24000 units, then each units cost 0.95 × 1.25 = Rs. 1.1875 say app. = Rs. 1.19.

Hence annual material cost = Rs. (0.95 × 1.25) × 24000 = Rs. 28500/-

As the company orders only once in a year, the ordering cost = Rs. 22.50

Annual carrying cost = (1.25 × 0.95) × 0.054 × (24000 / 2) = Rs. 769.50

Hence total cost = 769.50 + Rs. 22.50 + Rs. 28500 = Rs. 29292, this is less than Rs. 30270. Hence the company can accept the offer.

As the company works for 300 days in a year, the daily demand = 24000 / 300 = 80 units per day.

For this optimal time $t_0 = q_0 / r$ = (4000 / 80) = 50 days.

As t_0 is greater than the lead-time, and the safety stocks 400 units, the re-order level will be = Safety stock + Normal lead-time consumption = 400 + 12 × 80 = 1360 units.

Average inventory = $B + (q_0 / 2) = 400 + 4000 / 2 = 2400$ units.

Maximum inventory = $B + q_0 = 400 + 4000 = 4400$ units.

Minimum Inventory = B = 400 units.

Problem 8.69.

Consider the inventory system with the following data in usual notations: λ = 1000 units pr year, i = 0.30, p = Rs. 0.50 per unit, C_3 = Rs. 10 per order, L = 2 years, Determine (*a*) *E.O.Q*, (*b*) Re order point, (*c*) Minimum average cost.

Solution

$q_0 = \sqrt{(2C_3\lambda)/ip} = \sqrt{[(2 \times 10 \times 1000)/(0.30 \times 0.50)]}$ = 365 units., $t_0 = q_0 / \lambda$ = 365/1000 = 0.365 years. = 0.365 × 12 = 4.38 months.

Lead-time is given as 2 years. But optimal time = 4.38 months. Hence re-ordering occurs when the level of inventory is sufficient to satisfy the demand for $(L - t_0)$ = 2 – 0.365 = 1.635 years. Thus optimum quantity q_0 = 365 units is ordered when the re -order of inventory reaches 1.635 × 1000 = 1635 units.

Hence *R.O.P* = 1635 units.

Minimum average cost = $\sqrt{2C_3\,ip\lambda} = \sqrt{(2 \times 10 \times 0.3 \times 0.50 \times 1000)}$ = Rs. 54.77.

INVENTORY MODELS WITH VARIABLE PURCHASE PRICE OR PURCHASE INVENTORY MODELS WITH PRICE BREAKS

Previously in article 8.7.6 we have discussed quantity discount models, where, the seller will offer a discount on the quantity purchased between certain quantities. The extension of this model is the ***price break models***. In price break model the seller will offer discounts for the material purchased in a stepwise manner. This means to say that at every stage *i.e.,* quantity levels, he offers different prices. For example, let us say the price of material when the quantity purchased is from 1 to 100 Rs. 10 per unit, from 101 to 300 the price is Rs. 9/- per unit and for quantity above 301 the price will be Rs. 9/- per unit. This type of purchasing is known as price break models. Mathematically the model is represented as under:

Quantity purchased 'q'	*Unit purchasing price in Rs.*
$b_0 \le q < b_1$	p_1
$b_1 \le q < b_2$	p_2
…………….	………………
…………….	…………….....
$b_i - 1 \le q < b_i$	p_i
$b_n - 1 \le q < b_n$	p_n

(Note: Physically b_0 is meaningless as if $b = q = 0$, then there is no problem of inventory. Hence we consider lower bound as $b_0 = 1$.)

In general, $b_0 = 0$ and $b_n = \infty$ and $p_1 > p_2 > p_3 > \ldots\ldots\ldots p_{n-1}\ldots\ldots\ldots\ldots p_n$. The points b_1, b_2, b_3 $\ldots\ldots\ldots b_{n-1}$ are known as ***price breaks*** (in units) as price falls at these points. **The problem here is we have to find out the economic order quantity 'q_0' which minimizes the total cost including the material cost**. Here material cost is considered because the price varies at break points.

The notations used are:

p_j = Unit purchasing price in Rs.

i = Annual cost of carrying one rupee in the inventory value as percentage of average inventory value in Rs.

λ = Yearly demand in units.

C_3 = Ordering cost in Rs. per order.

q = Lot size.

Associated annual costs are:

Ordering cost = Number of orders × $C_3 = (\lambda / q) \times C_3$

Inventory carrying cost = Average inventory × Carrying cost = $(q / 2) \times ip_j$

Material cost = $\lambda \times p_j$

Total annual cost = $C_{j\,(q)}$ = material cost + ordering cost + carrying cost = $p_j \times \lambda + (\lambda / q) \times C_3 + (q / 2) \times ip_j$

If this cost is minimum for $q = q^i_j$, then $q^i{}_0$ is given by:

$(dC^i / dq) = -(C_3 \lambda / q^2) + ip_j (1/2) = 0$ Hence $q^j{}_0 = \sqrt{(2C_3\lambda) / ip_j}$

Procedure

1. For all the price breaks find the optimal order quantity. (Note: Better start from the last price break and move towards the first price break.)
2. Verify whether obtained optimal order quantity falls between the inventory range given in the problem for that particular break.
3. Once the obtained optimal order quantity falls between the given ranges, select that range for further treatment.
4. For example let us say our selected range is $100 \leq q < 200$. And the obtained optimal order quantity is 175. 175 lies between 100 and 200, hence this range is selected for further treatment.
5. Some times the price break may be as shown $100 \leq q < 200$ = Rs. 12, and $200 \leq q < 300$ = Rs. 10/- In such cases, calculate total cost for 175 units and also calculate the total cost for 200 units taking unit price as Rs. 10/-. For selecting required optimal order quantity, select the lowest one. Let us understand this by working some problems.

This model is represented graphically as under:

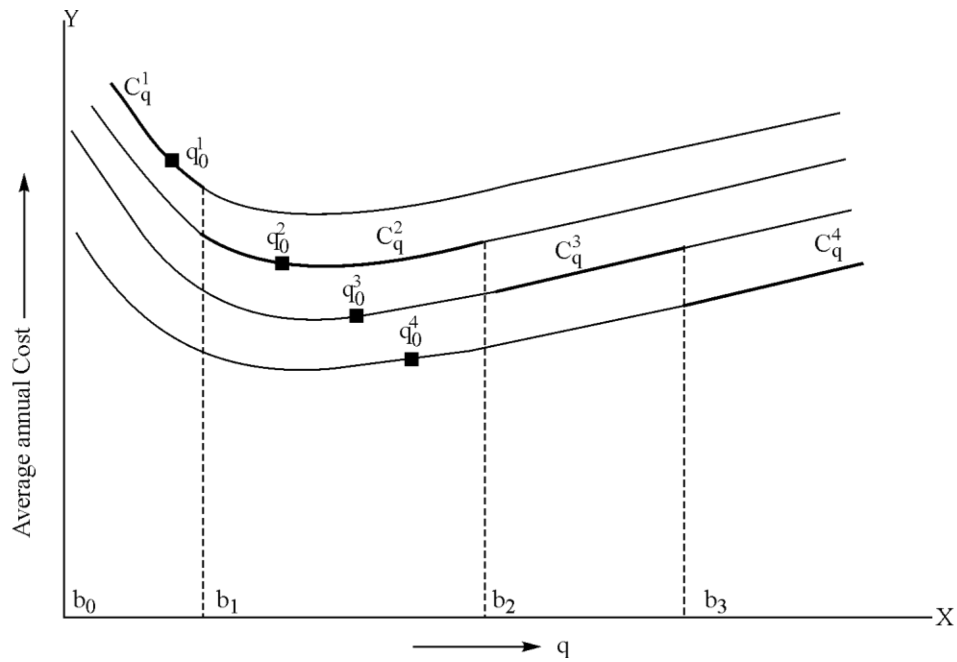


Figure 8.22

Problem 8.70 (Single price break)

Find the optimal order quantity for which the price breaks are as follows:

Quantity	*Unit price in Rs.*
$0 \le q < 500$	Rs. 10.00
$500 \le q < \infty$	Rs. 9.25

The monthly demand for the product is 200 units, the cost of storage is 2% of unit cost and the ordering cost is Rs. 350 per order.

Solution

Data: $b_0 = 0$, $\lambda = 200$, $I = 0.02$, $C_3 =$ Rs. 350/- $p_1 =$ Rs. 10/-, $p_2 =$ Rs. 9.25

$q^1{}_0 = \sqrt{(2C_3\lambda)/ip_1} = \sqrt{(2 \times 350 \times 200)/0.02 \times 10} = \sqrt{(140000/0.20} = \sqrt{700000} = 836.6 = 837$ units.

This does not fall in the range 0 to 500. Hence we have to take second range.

$q^2{}_0 = \sqrt{[(2 \times 350 \times 200)/(0.02 \times 9.25)]} = 870$ units. This is in the range 500 to ∞. Let us calculate the total cost for this quantity.

$C^2{}_q = 9.25 \times 200 + 350 \times (200 / 870) + 0.02 \times 9.25 \times (870 / 2) =$ Rs. 1850 + Rs. 80.45 + Rs.804.75 = Rs. 2735.20

Problem 8.71

Find the optimal order quantity for a product for which the price breaks are as follows:

Quantity	*Price in Rs. per unit.*
$0 \leq q < 100$	20
$100 \leq q < 200$	18
$200 \leq \infty$	16

The monthly demand for the product is 400 units. The storage cost is 20% of the unit cost and the ordering cost is Rs. 25 per order.

Solution

$I = 0.20$, $C_3 =$ Rs. 25/-, $\lambda = 400$,

$q^3{}_0 = \sqrt{[(2 \times 25 \times 400) / 0.20 \times 20)]}$ = 82.5 units = 83 units.

$q^2{}_0 = \sqrt{[(2 \times 25 \times 400) / 0.20 \times 18)]}$ = 74.3 units = 74 units.

$q^1{}_0 = \sqrt{[(2 \times 25 \times 400) / 0.20 \times 16)]}$ = 70 units.

From the above $q^1{}_0 = 75$ falls in the given range 0 to 100. Hence we have to find the total cost for 75 units at the rate of Rs. 20 per unit and cost of 100 units at the rate of 18 units. Which ever is less that is taken as the optimal order quantity.

$C^{75}{}_0 = 20 \times 400 + 25 \times (400 / 70) + 0.20 \times 20 \times (70 / 2) =$ Rs. 8282.80

$C^{100}{}_0 = 18 \times 400 + 25 \times (400 / 100) + 0.20 \times 18 \times (100 / 2) =$ Rs. 7480. Let us also find the cost 200 units at Rs. 16/- per unit.

$C^{200}{}_0 = 16 \times 400 + (400 / 200) \times 25 + 0.02 \times 16 \times (200 / 2) =$ Rs. 6770/-

As $C^{200}{}_0$ is the minimum, the optimal order quantity is 200 units.

Problem 8.72

Find the optimal order quantity for a product for which the price breaks are as under:

Quantity	*Unit cost in Rs. per unit.*
$0 \leq q_1 < 500$	10.00
$500 \leq q_2 < 750$	9.25
$750 \leq q_3 < \infty$	8.75

The monthly demand for the product is 200 units. The cost of storage is 2% of the unit cost and the cost of ordering is Rs. 350/- per order.

Solution

Data; $C_3 =$ Rs. 350 per order, $i = 0.02$, $\lambda = 200$ units.

$q^1{}_0 = \sqrt{[(2 \times 350 \times 200) / (0.02 \times 10)]}$ = 836.6 units.

$q^2{}_0 = \sqrt{[(2 \times 350 \times 200) / (0.02 \times 9.25)]}$ = 869.9 units

$q^3{}_0 = \sqrt{[(2 \times 350 \times 200)/(0.02 \times 8.75)]}$ = 894 units.

From the above $q^3{}_0$ = 894 units is within the given range. Hence we have to calculate the total cost of $C^{894}{}_0$

$C^{894}{}_0$ = 200 × 8.75 + 350 × (200 / 984) + 0.02 × 8.75 × (894 / 2) = Rs.1750 + Rs. 78.30 + Rs. 78.22 = Rs. 1906.52 = App. 1907/-

Problem 8.73

Find the optimal order quantity for a product when the annual demand for the product is 500 units, the cost of storage per unit per year is 10% of the unit cost and ordering cost per order is Rs. 180/-, the units costs are given below:

Quantity	*Unit cost in Rs.*
$0 \le q_1 < 500$	25.00
$500 \le q_2 < 1500$	24.80
$1500 \le q_3 < 3000$	24.60
$3000 \le q_4 < \infty$	24.40

Solution

Data: λ = 500 units, I = 0.10, C_3 = Rs. 180/-

$q_0^4 = \sqrt{(2C_3\lambda)/ip_4} = \sqrt{[(2 \times 180 \times 500)/(0.10 \times 24.40)}$ = 271.60 units. This is not in the given range.

$q_0^3 = \sqrt{(2C_3\lambda)/ip_3} = \sqrt{[(2 \times 180 \times 500)/(0.10 \times 24.60)}$ = 270.5 units. This is not in the given range

$q_0^2 = \sqrt{(2C_3\lambda)/ip_2} = \sqrt{[(2 \times 180 \times 500)/(0.10 \times 24.80)}$ = 260.4 units. This is not in the given range.

$q_0^1 = \sqrt{(2C_3\lambda)/ip_1} = \sqrt{[(2 \times 180 \times 500)/(0.10 \times 25.00)}$ = 268.3 units. This is within the given range.

Now we calculate the total cost for $C^{268}{}_0$ at Rs. 25/- per unit and $C^{500}{}_0$ at Rs. 24.80 and select the minimum one as the optimal order quantity.

$C^{268}{}_0$ = 500 × 25 + (500 / 268.3) × 180 + (268.3 / 2) × 0.10 × 25 = Rs. 13,170.82.

$C^{500}{}_0$ = 500 × 24.80 + (500 / 500) × 180 + 0.10 × 24.80 × (500 / 2) = 13200/-.

As Rs. 13, 170.82 is less than Rs.13, 200/-. The optimal order quantity is 268.3 or App. 268 units.

EXERCISE PROBLEMS

1 In each of the following cases, stock is replenished instantaneously and no shortages are allowed. Find the economic lot size, the associated total costs and length of time between orders and give your comments.

(a) C_3 = Rs. 100/- per order, C_1 = Re. 0.05 per unit and λ = 30 units per year.

(b) C_3 = Rs. 50/0 /- per order, C_1 = Re. 0.05 per unit and λ = 30 units per year.

(c) C_3 = Rs. 100/- per order, C_1 = 0.01 per unit and λ = 40 units per year.

(d) C_3 = Rs. 100/- per order, C_1 = Rs. 0.04 per unit and λ = 20 units per year.

2. The *XYZ* manufacturing company has determined from an analysis of its accounting and production data for part number 625, that its cost to purchase is Rs.36 per order and Rs. 2/- per part. Its inventory carrying charge is 18% of the average inventory cost. The demand for this part is 10,000 units per annum. Find (*a*) What is the economic order quantity; (*b*) What is the optimal number of days supply per optimum order.
3. A manufacturer receives an order for 6890 items to be delivered over a period of a year as follows:

   At the end of the first week = 5 items.

   At the end of the second week = 10 items.

   At the end of the third week = 15 items. etc.

   The cost of carrying inventory is Rs. 2.60 per item per year and the cost of set up is Rs. 450/- per production run.

   Compute the costs of following policies:

   (a) Make all 6890 at start of the year.

   (b) Make 3445 now and 3445 in 6 months,

   (c) Make 1/12 th the order each month.

   (d) Make 1/52 th order every week.
4. A product is produced at the rate of 50 items per day. The demand occurs at the rate of 30 items per day. Given that setup cost per order is Rs. 100/-and holding cost per item per unit time is Rs. 0.05, and shortages being allowed, what is the shortage cost per unit under optimal conditions if the lot size is of 600 units.
5. The probability distribution of monthly sales of a cretin item is as follows:

Monthly sales:	*0*	*1*	*2*	*3*	*4*	*5*	*6*
Probability:	0.01	0.06	0.25	0.35	0.20	0.03	0.10

   The cost of carrying inventory is Rs. 30/- per unit per month and the cost of unit charges is Rs. 70/- per month. Determine the optimum stock level, which minimizes the total expected cost.
6. Determine the optimal order rule for the following case:

   r = Demand per month = 5000 units.

   C_3 = Ordering cost = Rs. 50/- per order,

   i = Carrying charges = Rs. 0.50 per unit per month.

Quantity.					*Price in Rs. per unit.*
0	$\leq$	q_1	$<$	100	Rs. 1.50
100	$\leq$	q_2	$<$	250	Rs. 1.40
250	$\leq$	q_3	$<$	500	Rs. 1.30
500	$\leq$	q_4	$<$	1000	Rs. 1.00
1000	$\leq$	q_5	$<$	5000	Rs. 0.90
5000	$\leq$	q_6	$<$	∞	Rs. 0.85

7. For a fixed order quantity system, find the Economic order quantity, Safety stock, Re-order level and average inventory for an item with the following data:

   Annual demand = 10000 units, Unit Price p = Re. 1/-, Ordering cost = Rs. 12 per order, Carrying charges = i = 24 % of average inventory, Lead - time = 15, 25, 13, 14, 30, and 17 days.

8. From the data given below draw a plan for ABC control:

Item	*Units*	*Unit cost in Rs.*
1	7000	5.00
2	24000	3.00
3	1500	10.00
4	600	22.00
5	38000	1.50
6	40000	0.50
7	60000	0.20
8	3000	3.50
9	300	8.00
10	29000	0.40
11	11500	7.10
12	4100	6.20

9. An oil engine manufacturer purchases lubricants at the rate of Rs. 42/- per piece from a vendor. The requirement of this lubricant is 1800 per year. What should be the order quantity per order, if the cost per placement of an order is Rs.16/- and inventory-carrying charge per rupee per year is only 20 paise.

10. A manufacturer has to supply his customer with 24000 units of his product per year. This demand is fixed and known. Since the customer in an assembly line operation uses the unit and the customer has no storage space for the unit, the manufacturer must supply a day's requirement each day. If the manufacturer fails to supply the required units, he will lose the amount and probably his business. Hence, the cost of a shortage is assumed to be infinite, and consequently, none will be tolerated. The inventory holding cost amounts to 0.10 per unit per month, and the set up cost per unit is Rs. 350/-. Find the optimum lot size, the, length of optimum production run.

11. The demand for an item in a company is 18000 units per year, and the company can produce the item at a rate of 3000 per month. The cost of one set up is Rs. 500/- and the holding cost of 1 unit per month is 15 paise. Determine the optimum manufacturing quantity and the total cost per year assuming the cost of 1 unit is Rs. 2/-.

12. The demand for a purchase item is 1000 units per month, and shortages are allowed. If the unit cost is Rs. 1.50 per unit and the cost of making one purchase is Rs.600/-and the holding cost for one unit is Rs. 2/- per year and the cost of one shortage is Rs. 10/- per year, determine: (*a*) Optimum purchase quantity, (*b*) The number of orders per year, (*c*) The optimum yearly cost. Represent the model graphically.

13. A company has a demand of 12000 units per year for an item and it can produce 2000 such items per month. The cost of one setup is Rs. 400/-and the holding cost per unit per month is Rs. 0.15. The shortage cost of one unit is Rs. 20/- per year. Find the optimum lot size and the total cost per year, assuming the cost of 1 unit is Rs.4/-. Also find the maximum inventory, manufacturing time and total time.
14. A company producing three items has limited storage space of average 750 items of all types. Determine the optimal production quantities for each item separately, when the following information is available:

Item	*1*	*2*	*3*
Demand rate units per year	1000	5000	2000
Set up cost per unit in Rs.	50	75	100
Cost per unit in Rs.	20	100	50
Holding cost in Rs. per unit.	20	20	20

15. The following relations to inventory costs have been established for a company:

    Orders must be placed in multiples of 100 units.

    Requirement for the year = 3,00,000 units.

    The unit price of the product is Rs.3/-

    Carrying cost is 25% of the purchase price of gods.

    Ordering cost per order is Rs. 20/-

    Desired safety stock is 10,000 units. This amount on hand initially.

    Three days are required for delivery of goods.

    Calculate:

    EOQ, Number of orders per year, and Re-order level of inventory.
16. A contractor of second hand motor trucks uses to maintain a stock of trucks every month. Demand of the truck occurs at a relatively constant rate not in a constant size. The demand follows the following probability distribution:

Demand:	*0*	*1*	*2*	*3*	*4*	*5*	*6 or more.*
Probability:	0.40	0.24	0.20	0.10	0.05	0.01	0.00

    The holding cost of an old truck in stock for one month is Rs.100/- and the penalty for a truck if not supplied on demand is Rs. 1000/-. Determine the optimal size of the stock for the contractor.
17. What is inventory? Describe the types of inventory you know.
18. Explain the various costs associated with inventory with examples.
19. Describe the characteristics of inventory system.
20. Explain what is ABC analysis and what is its significance.
21. What is VED analysis? How is it useful to an Inventory manager?
22. Explain *p*– System and *q* – system of ordering material, which one you prefer? Give your reasons.
23. Derive EOQ formula with usual notations.

24. Distinguish between deterministic and stochastic models of inventory.
25. What function does inventory perform? State the two basic inventory decisions management must make as they attempt to accomplish the functions of inventory just described by you.
26. Describe six important components that constitute the stock holding costs.
27. Explain the significance of lead - time and safety stock in inventory control.
28. Explain the following terms with suitable examples:
    (a) Set up cost
    (b) Holding cost
    (c) Shortage cost,
    (d) Lead - time,
    (e) Re order point
    (f) Fixed order quantity
    (g) Fixed order interval
29. What is selective inventory control?
30. Explain the basic steps taken in conducting ABC analysis.

MULTIPLE CHOICE QUESTIONS

Inventory Models

1. One of the important basic objective of Inventory management is:
    (a) To calculate EOQ for all materials in the organization.
    (b) To go in person to the market and purchase the materials,
    (c) To employ the available capital efficiently so as to yield maximum results,
    (d) Once materials are issued to the departments, personally check how they are used.( )
2. The best way of improving the productivity of capital is:
    (a) Purchase automatic machines,
    (b) Effective labour control,
    (c) To use good financial management,
    (d) Productivity of capital is to be increased through effective materials management.( )
3. Materials management is a body of knowledge, which helps manager to:
    (a) Study the properties of materials,
    (b) Search for needed material,
    (c) Increase the productivity of capital by reducing the cost of material,
    (d) None of the above. ( )
4. The stock of materials kept in the stores in anticipation of future demand is known as:
    (a) Storage of materials,
    (b) Stock of materials,
    (c) Inventory,
    (d) Raw materials. ( )

5. The stock of animals reared in anticipation of future demand is known as:
   (a) Live stock Inventory,
   (b) Animal inventory,
   (c) Flesh inventory,
   (d) None of the above. ( )
6. The working class of human beings is a class of inventor known as:
   (a) Live stock,
   (b) Human inventory
   (c) Population,
   (d) Human resource inventory. ( )
7. In general the percentage of materials cost in product is approximately equals to:
   (*a*) 40 to 50 % (*b*) 5 to 10 % (*c*) 2 to 3 % (*d*) 90 to 95% ( )
8. Materials management bring about increased productivity of capital by:
   (a) Very strict control over use of materials,
   (b) Increasing the efficiency of workers,
   (c) Preventing large amounts of capital locked up for long periods in the form of inventory.
   (d) To apply the principles of capital management, ( )
9. We can reduce the materials cost by:
   (a) Using systematic inventory control techniques,
   (b) Using the cheap material,
   (c) Reducing the use of materials,
   (d) Making hand to mouth purchase. ( )
10. The basis for ABC analysis is
    (a) Interests of Materials manager,
    (b) Interests of the top management,
    (c) Pareto's 80-20 rule,
    (d) None of the above. ( )
11. ABC analysis depends on the:
    (a) Quality of materials,
    (b) Cost of materials,
    (c) Quantity of materials used,
    (d) Annual consumption value of materials. ( )
12. '*A*' class materials consumes:
    (a) 10 % of total annual inventory cost,
    (b) 30% of total annual inventory cost,
    (c) 70 to 75% of total inventory cost,
    (d) 90 % of total annual inventory cost. ( )

13. 'B' class of materials consumes --------- % of annual inventory cost.
    (*a*) 60 to 70% (*b*) 20 to 25%
    (*c*) 90 to 95% (*d*) 5 to 8% ( )
14. 'C' class materials consume--------% of annual inventory cost.
    (*a*) 5 to 10 % (*b*) 20 to 30%
    (*c*) 40 to 50% (*d*) 70 to 80% ( )
15. The rent for the stores where materials are stored falls under:
    (a) Inventory carrying cost, (*b*) Ordering cost,
    (*c*) Procurement cost, (*d*) Stocking cost. ( )
16. Insurance charges of materials cost falls under:
    (b) Ordering cost, (*b*) Inventory carrying cost,
    (c) Stock out cost (*d*) Procurement cost. ( )
17. As the volume of inventory increases, the following cost will increase:
    (a) Stock out cost, (*b*) Ordering cost,
    (*c*) Procuring cost, (*d*) Inventory carrying cost. ( )
18. As the order quantity increases, this cost will reduce:
    (a) Ordering cost, (*b*) Insurance cost
    (*c*) Inventory carrying cost, (*d*) Stock out cost. ( )
19. Procurement cost may be clubbed with:
    (a) Inventory carrying charges, (*b*) Stock out cost,
    (*c*) Loss due to deterioration, (*d*) Ordering cost. ( )
20. The penalty for not having materials when needed is:
    (a) Loss of materials cost, (*b*) Loss of order cost,
    (*c*) Stock out cost, (*d*) General losses. ( )
21. Losses due to deterioration, theft and pilferage comes under,
    (a) Inventory Carrying charges, (*b*) Losses due to theft,
    (*c*) Does not come under any cost, (*d*) Consumption cost. ( )
22. Economic Batch Quantity is given by: (where, C_1 = Inventory carrying cost, C_3 = Ordering cost, r = Demand for the product)
    (*a*) $(2C_1 / C_3)^{1/2}$, (*b*) $(2 C_3 / C_1 r)^{1/2}$,
    (*c*) $2C_3 r / C_1$, (*d*) $(2C_3 r / C_1)^{1/2}$. ( )
23. If λ is the annual demand, C_1 = Inventory carrying cost, i = rate of inventory carrying charges, p = unit cost of material in Rs., then EOQ =
    (*a*) $(2C_3 \lambda / ip)^{1/2}$, (*b*) $2C_3 \lambda / ip$,
    (*c*) $(2 C_3 / ip \lambda)^{1/2}$, (*d*) $(2 \lambda / C_3 ip)^{1/2}$, ( )
24. If C_1 = Carrying cost, C_3 is the ordering cost, r = demand for the product, then the optimal period for placing an order is given by:
    (*a*) $(2 C_3/C_1 r)^{1/2}$ (*b*) $(2C_1 C_3/r)^{1/2}$
    (*c*) $(2C_3 r/C_1)^{1/2}$ (*d*) $(2C_1 C_3 r)^{1/2}$ ( )

25. When C_1 = Inventory carrying cost, C_3 = ordering cost, r = demand for the product, the total cost of inventory is given by:
    (*a*) $(2C_1C_3r)$ (*b*) $(2C_1C_3)^{1/2}$
    (*c*) $(2C_3r/C_1)^{1/2}$ (*d*) $(2C_1C_3r)^{1/2}$ ( )
26. When λ is the annual demand for the material, p = unit price of the material in Rs., C_3 is the ordering cost, q = order quantity, then the total cost including the material cost is given by:
    (*a*) $(q/2)\, ip + \lambda /q\, C_3 + \lambda\, p$ (*b*) $2C_3\, \lambda\, ip + \lambda\, p$
    (*c*) $(q/2)\, ip + \lambda\, p$ (*d*) $(2C_3q\, \lambda\, ip)^{1/2}$ ( )
27. In VED analyses, the letter V stands for:
    (*a*) Very important material, (*b*) Viscous material
    (*c*) Weighty materials, (*d*) Vital materials. ( )
28. In VED analysis, the letter D strands for:
    (*a*) Dead stock, (*b*) Delayed material
    (*c*) Deserved materials, (*d*) Diluted materials. ( )
29. The VED analysis depends on:
    (*a*) Annual consumption cost of materials, (*b*) Unit price of materials,
    (*c*) Time of arrival of materials, (*d*) Criticality of materials. ( )
30. In FSN analysis the letter S stands for:
    (*a*) Slack materials, (*b*) Stocked materials,
    (*c*) Slow moving materials, (*d*) Standard materials. ( )
31. In FSN analyses, the letter N stands for:
    (*a*) Non moving materials, (*b*) Next issuing materials,
    (*c*) No materials, (*d*) None of the above. ( )
32. FSN analysis depends on:
    (*a*) Weight of the material, (*b*) Volume of material,
    (*c*) Consumption pattern, (*d*) Method of moving materials. ( )
33. MRP stands for:
    (*a*) Material Requirement Planning, (*b*) Material Reordering Planning,
    (*c*) Material Requisition Procedure, (*d*) Material Recording Procedure. ( )
34. A system where the period of placing the order is fixed is known as:
    (*a*) q - system, (*b*) Fixed order system
    (*c*) p - system (*d*) Fixed quantity system. ( )
35. A system in which quantity for which order is placed is constant is known as:
    (*a*) q - System, (*b*) p - system,
    (*c*) Period system, (*d*) Bin system. ( )
36. LOB stands for:
    (*a*) Lot of Bills, (*b*) Line of Batches
    (*c*) Lot of Batches, (*d*) Line of Balance. ( )
37. High reliability spare parts in inventory are known as:
    (a) Reliable spares, (*b*) Insurance spares,
    (*c*) Capital spares, (*d*) Highly reliable spares. ( )

38. The property of capital spares is:
    (a) They have very low reliability;
    (b) These can be purchased in large quantities, as the price is low,
    (c) These spares have relatively higher purchase cost than the maintenance spares.
    (d) They are very much similar to breakdown spares.
39. Re-usable spares are known as:
    (a) Multi use spares, (*b*) Repeated useable stores,
    (*c*) Scrap materials, (*d*) Rotable spares. ( )
40. JIT stands for:
    (a) Just in time Purchase, (*b*) Just in time production,
    (*c*) Just in time use of materials (*d*) Just in time order the material. ( )
41. The cycle time, selected in balancing a line must be:
    (a) Must be greater than the smallest time element given in the problem,
    (b) Must be less than the highest time element given in the problem,
    (c) Must be slightly greater than the highest time element given in the problem,
    (d) Left to the choice of the problem solver. ( )
42. The lead-time is the time:
    (a) To place orders for materials,
    (b) Time of receiving materials,
    (c) Time between receipt of material and using materials
    (d) Time between placing the order and receiving the materials. ( )
43. The PQR classification of inventory depends on:
    (a) Unit price of the material, (*b*) Annual consumption value of material,
    (*c*) Criticality of material, (*d*) Shelf life of the materials. ( )
44. The classification made on the weight of the materials is known as:
    (a) PQR analysis, (*b*) VED analysis,
    (*c*) XYZ analysis, (*d*) FSN analysis. ( )
45. At *EOQ*
    (a) Annual purchase cost = Annual ordering cost
    (b) Annual ordering cost = Annual carrying cost
    (c) Annual carrying cost = annual shortage cost
    (d) Annual shortage cost = Annual purchase cost. ( )
46. If shortage cost is infinity,
    (a) No shortages are allowed; (*b*) No inventory carrying cost is allowed,
    (*c*) Ordering cost is zero, (*d*) Purchase cost = Carrying cost. ( )
47. The most suitable system for a retail shop is
    (a) *FSN* Analysis, (*b*) *ABC* analysis,
    (*c*) *VED* analysis, (*d*) *GOLF* analysis. ( )
48. The inventory maintained to meet unknown demand changes is known as
    (a) Pipeline inventory, (*b*) Anticipatory inventory
    (*c*) De coupling inventory, (*d*) Fluctuatory inventory. ( )

49. The most suitable inventory system for a Petrol bunk is
   (a) *P*- System, (b) 2 Bin system,
   (c) *Q*- System, (d) Probabilistic model ( )

50. The water consumption from a water tank follows
   (a) *P* - system, (b) *PQ* -system,
   (c) *Q* - System, (d) *EOQ* System ( )

51. Which of the following inventory is maintained to meet expected demand fluctuations:
   (a) Fluctuatory Inventory, (b) Buffer stock
   (c) De- coupling inventory, (d) Anticipatory inventory. ( )

52. Which of the following increases with quantity ordered per order:
   (a) Carrying cost, (b) Ordering cost,
   (c) Purchase cost, (d) Demand, ( )

53. The ordering cost per order and average unit carrying cost are constant, and demand suddenly falls by 75 % then EOQ will:
   (a) Decreases by 50 %, (b) Does not change
   (c) Increases by 50 %, (d) Decreases by 40% ( )

54. In JIT system, the following is assumed to be zero.
   (a) Ordering cost, (b) Transportation cost
   (c) Carrying cost, (d) Purchase cost. ( )

55. Which of the following analysis neither considers cost nor value:
   (a) *ABC* (b) *XYZ*
   (c) *HML* (d) *VED* ( )

ANSWERS

1. (*c*)	2. (*d*)	3. (*c*)	4. (*c*)
5. (*a*)	6. (*d*)	7. (*a*)	8. (*c*)
9. (*a*)	10. (*c*)	11. (*d*)	12. (*c*)
13. (*b*)	14. (*a*)	15. (*a*)	16. (*b*)
17. (*d*)	18. (*a*)	19. (*d*)	20. (*c*)
21. (*a*)	22. (*b*)	23. (*c*)	24. (*a*)
25. (*d*)	26. (*a*)	27. (*d*)	28. (*c*)
29. (*d*)	30. (*c*)	31. (*a*)	32. (*c*)
33. (*a*)	34. (*c*)	35. (*a*)	36. (*d*)
37. (*b*)	38. (*c*)	39. (*d*)	40. (*b*)
41. (*c*)	42. (*d*)	43. (*c*)	44. (*d*)
45. (*b*)	46. (*b*)	47. (*a*)	48. (*d*)
49. (*c*)	50. (*a*)	51. (*d*)	52. (*a*)
53. (*c*)	54. (*c*)	55. (*d*)	

CHAPTER – 9

Waiting Line Theory or Queuing Model

INTRODUCTION

Before going to ***waiting line theory or queuing theory***, one has to understand two things in clear. They are ***service and customer or element.*** Here customer or element represents a person or machine or any other thing, which is in need of some service from servicing point. Service represents any type of attention to the customer to satisfy his need. For example,

1. Person going to hospital to get medical advice from the doctor is an element or a customer,
2. A person going to railway station or a bus station to purchase a ticket for the journey is a customer or an element,
3. A person at ticket counter of a cinema hall is an element or a customer,
4. A person at a grocery shop to purchase consumables is an element or a customer,
5. A bank pass book tendered to a bank clerk for withdrawal of money is an element or a customer,
6. A machine break down and waiting for the attention of a maintenance crew is an element or a customer.
7. Vehicles waiting at traffic signal are elements or customers,
8. A train waiting at outer signal for green signal is an element or a customer

Like this we can give thousands of examples.

In the above cases, the service means,

1. Doctor is a service facility and medical care is a service,
2. Ticket counter is a service facility and issue of ticket is service.
3. Ticket counter is a service facility and issue of ticket is service.
4. Shop owner is a service facility and issue of items is service.
5. Bank clerk is a service facility and passing the cheque is service.
6. Maintenance crew is service facility and repairing the machine is service.
7. Traffic signals are service facility and control of traffic is service.
8. Signal post is a service facility and green signaling is service.

Above we have seen elements or customer and service facility and service. We can see here that all the customer or elements (hereafter called as customer only) will arrive and waits to avail the service at service station. When the service station has no desired capacity to serve them all at a time the customer has to wait for his/its chance resulting the formulation of a waiting line of customers which is generally known as a queue. In general we can say that ***a flow of customers***

from infinite or finite population towards the service facility forms a queue or waiting line on account of lack of capability to serve them all at a time. The above discussion clarifies that the term customer we mean to the arriving unit that requires some service to be performed at the service station. **Queues** or **waiting lines** stands for a number of customers waiting to be serviced. Queue does not include the customer being serviced. The process or system that performs the services to the customer is termed as **service channel** or **service facility.**

Thus from the above we see that waiting lines or not only the lines formed by human beings but also the other things like railway coaches, vehicles, material etc.

A.K.Erlang, a Danish telephone engineer, did original work on queuing theory. Erlang started his work in 1905 in an attempt to determine the effects of fluctuating service demand (arrivals) on the utilization of automatic dialing equipment. It has been only since the end of World War II that work on waiting line models has been extended to other kinds of problems. In today's scenario a wide variety of seemingly diverse problems situations are recognized as being described by the general waiting line model. In any queuing system, we have an input that arrives at some facility for service or processing and the time between the arrivals of individual inputs at the service facility is commonly random in nature. Similarly, the time for service or processing is commonly a random variable.

Table 9.1 shows waiting line model elements for some commonly known situations. Servers may be in parallel or in service. When it is parallel, the arriving customers may form a single queue as in the case of post offices, ticket windows in railway station and bus station or a cinema theatre etc. shown in figure 9.1. If the serves are in series, then number of queues is formed in front of service facilities, for example we can take repair of break down machines. This is illustrated in figure number 9.2.

S.No	Population	Arrivals	Queue (Channel)	Service Facility (Phase)	Out going element	Name of the system	Remarks.
1	0000000 → 0000000 → 0000000 →	0	00000	0 □	0 →	Single channel single phase.	
2	0000000 → 0000000 → 0000000_ →	0	00000 00000 00000	0 □ 0 □ 0 □	0 → 0 → 0 →	Multi Channel Single Phase	
3	0000000 → 0000000 → 0000000 →	0	00000	0 0 0 □ □ □	0 →	Single Channel Multi Phase	
4	0000000 → 0000000 → 0000000 →	0	00000 00000 00000	0 0 0 □ □ □ 0 0 0 □ □ □ 0 0 0 □ □ □	0 → 0 → 0 →	Multi channel Multi Phase.	

Figure 9.1. Four basic structures of waiting line situations.

Table 9.1. Waiting line model elements for some commonly known situations.

S.No.	*Situation.*	*Arriving element.*	*Service facility*	*Service or process*	*Remarks.*
1	Ship entering a port.	Ships	Docks	Unloading and loading.	
2.	Maintenance and repair of machines	Machine break downs	Repair crew.	Repairing of machines.	
3.	Non automatic assembly line	Parts to be assembled.	Individual assembly operations or entire line.	Assembly.	
4.	Purchase of groceries at super market.	Customer with loaded grocery carts.	Checkout Counter.	Tabulation of bill, receipt of payment and bagging of groceries.	
5.	Automobile and other vehicles at an intersection of roads.	Automobiles and vehicles.	Traffic signal lights.	Control of traffic.	
6.	Inventory of items in stores or warehouse.	Order for withdrawal.	Store or warehouse.	Replenishment of inventory.	
7.	Patients arriving at an hospital	Patients	Medical craw	Health care of the patient.	

In figure number 9.2 arrows between service centers indicates possible routes for jobs processed in the shop. In this particular system, we see that the service center moves to the customer rather than the customer coming to service center for service. So, it may be understood here that there is no rule that always the customers has to move to service centers to get the service. Depending on the situation, the service center may also move to the customer to provide service. In this system the departure from one-service center may become input to the other service center.

In our everyday activity, we see that there is a flow of customer to avail some service from service facility. The rate of flow depends on the nature of service and the serving capacity of the station. In many situations there is a congestion of items arriving from service because an item cannot be serviced immediately on arrival and each new arrival has to wait for some time before it is attended. This situation occurs where the total number of customers requiring service exceeds the number of facilities. So we can define a queue as **“A group of customers / items waiting at some place to receive attention / service including those receiving the service.”**

In this situation, if queue length exceeds a limit, the customer get frustrated and leave the queue to get the service at some other service station.In this case the organization looses the customer goodwill.

Similarly some service facility waits for arrival of customers when the total capacity of system is more than the number of customers requiring service. In this case service facility remains idle for a considerable time causing a burden of exchequer.

So, in absence of a perfect balance between the service facility and the customers, waiting is required either by the customer or by the service facility. The imbalance between the customer and service facility, known as congestion, cannot be eliminated completely but efforts / techniques can be evolved and applied to reduce the magnitude of congestion or waiting time of a new arrival in the system or the service station. The method of reducing congestion by the expansion of servicing counter may result in an increase in idle time of the service station and may become uneconomical for the organization. Thus both the situation namely of unreasonable long queue or expansion of servicing counters are uneconomical to individual or managers of the system.

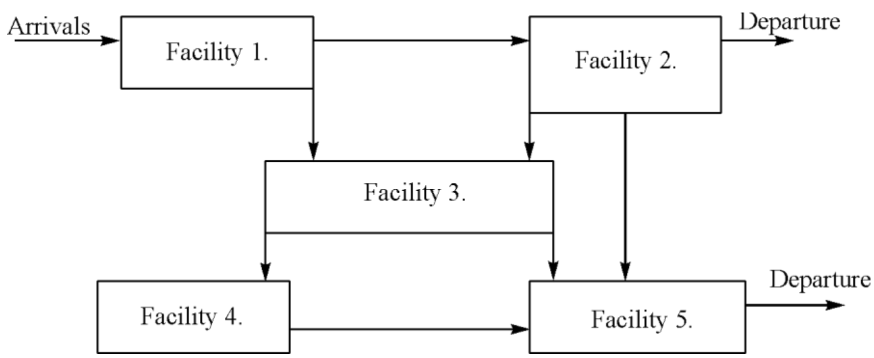


Figure 9.2. Complex queue for a maintenance shop.

As discussed above, if the length of the queue is longer, the waiting time of the customer will increase causing dissatisfaction of customer and to avoid the longer waiting time of customer, if the management increases the service facilities, then many a time we see that the service facilities will remain idle causing burden on the organization. To avoid this situation, the theory of waiting line will help us to reduce the waiting time of the customer and suggest the organization to install optimal number of service facilities, so that customer will be happy and the organization can run the business economically.

The arrival pattern of the customer and the service time of the facility depend on many factors and they are not under the control of the management. Both cannot be estimated or assessed in advance and moreover their arrival pattern and service time are random in nature. The waiting line phenomenon is the direct result of randomness in the operation of service facility and random arrival pattern of the customer. The customer arrival time cannot be known in advance to schedule the service time and the time required to serve each customer depends on the magnitude of the service required by the customer. For example, let us consider two customers who come to the ticket counter to purchase the counter. One-person tenders exact amount and purchase one ticket and leaves the queue. Another person purchases 10 tickets and gives a Rs. 500/- currency note. For him after giving the ticket, the counter clerk has to give the remaining amount back. So the time required for both customers will vary. The randomness of arrival pattern and service time makes the waiting line theory more complicated and needs careful study. The theory tries to strike a balance between the costs associated with waiting and costs of preventing waiting and help us to determine the optimal number of service facilities required and optimal arrival rate of the customers of the system.

HISTORICAL DEVELOPMENT OF THE THEORY

During 1903 Mr. A.K. Erlang, a Swedish engineer has started theoretical analysis of waiting line problem in telephone calls. In 1927, Mr. Millins developed the theory further and then by Mr. Thornton D Fry. But Mr. D.G.Kendall has given a systematic and mathematical approach to waiting line problem in 1951. After 1951 significant work has been done in waiting line theory, so as to enable it to apply to varieties of problems come across in industries and society. One best example of this may be quoted as the control of waiting time of a customer in queue complex of Tirupathi Temple. The present system of tying a belt with time to the hands of a customer is the results of application of queuing theory. Another example is computerized reservation of rail journey.

QUEUING SYSTEM OR PROCESS

One thing we have to remember is that when we speak of queue, we have to deal with two elements, *i.e.* Arrivals and Service facility. Entire queuing system can be completely described by:

(a) The input (Arrival pattern)
(b) The service mechanism or service pattern,
(c) The queue discipline and
(d) Customer behavior.

Components of the queuing system are arrivals, the element waiting in the queue, the unit being served, the service facility and the unit leaving the queue after service. This is shown in figure 9.3.

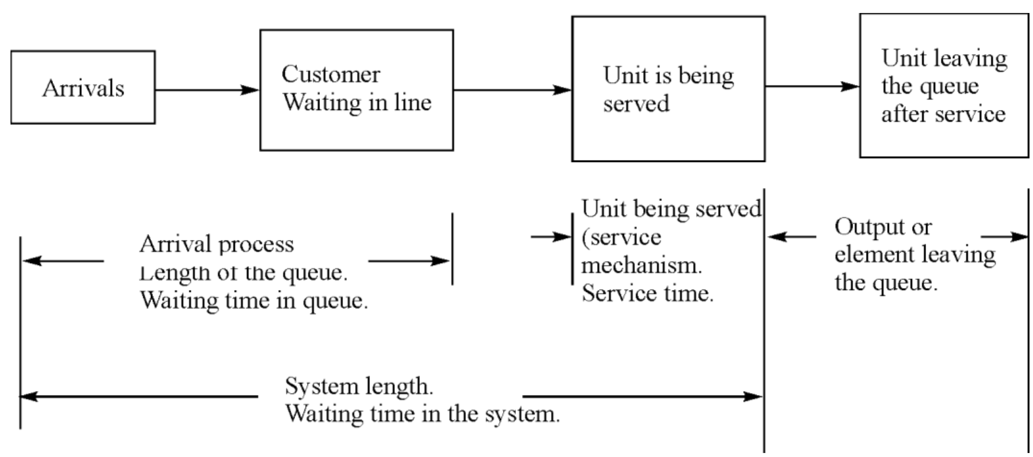


Figure 9.3. Components of queuing system.

Input Process

The input describes the way in which the customers arrive and join the system. In general customer arrival will be in random fashion, which cannot be predicted, because the customer is an independent individual and the service organization has no control over the customer. The characteristics of arrival are shown in figure 9.4.

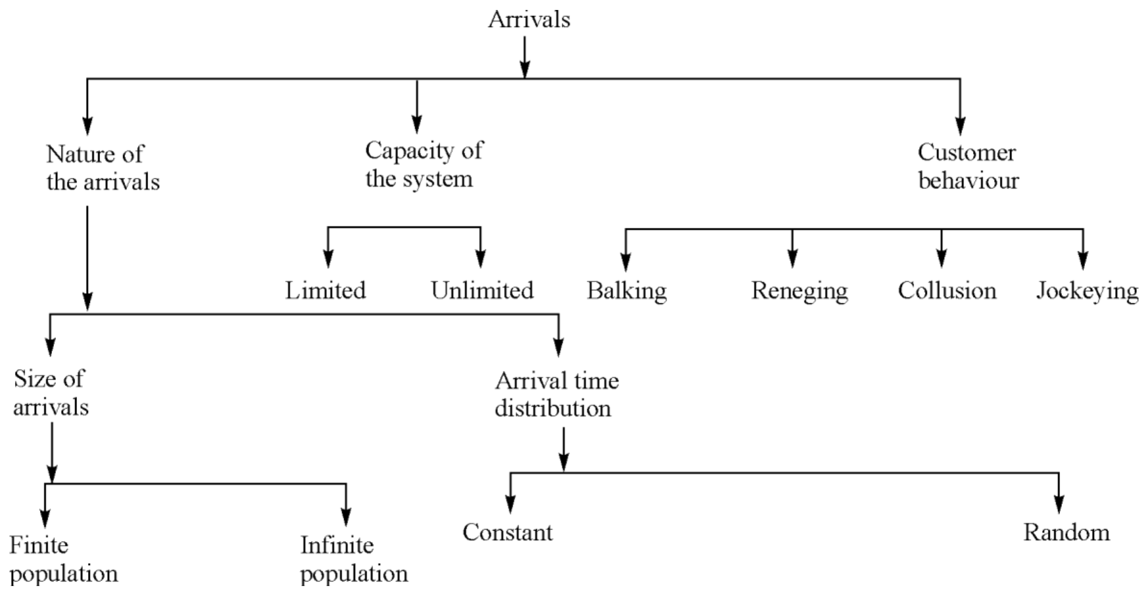


Figure 9.4. Characteristics of Arrivals or input.

Input to the queuing system refers to the pattern of arrival of customers at the service facility. We can see at ticket counters or near petrol bunks or any such service facility that the customer arrives randomly individually or in batches. The input process is described by the following characteristics (as shown in the figure 9.4) nature of arrivals, capacity of the system and behavior of the customers.

(a) *Size of arrivals:* The size of arrivals to the service system is greatly depends on the nature of size of the population, which may be infinite or finite. The arrival pattern can be more clearly described in terms of probabilities and consequently the probability distribution for inter- arrival times i.e. the time between two successive arrivals or the distribution of number of customers arriving in unit time must be defined. In our discussion in this chapter, it is dealt with those queuing system in which the customers arrive in ***Poisson*** or ***Completely random*** fashion. In fact there are many more arrival patterns are available but for simplicity, only Poisson arrivals are considered.

(b) *Inter-arrival time:* The period between the arrival of individual customers may be constant or may be scattered in some distribution fashion. Most queuing models assume that the some inter-arrival time distraction applies for all customers throughout the period of study. It is true that in most situations that service time is a random variable with the same distribution for all arrivals, but cases occur where there are clearly two or more classes of customers such as a machine waiting for repair with a different service time distribution. Service time may be constant or random variable. In this chapter mostly distribution of service time, which are important, are considered and they are ***Negative exponential distribution*** and ***Erlang or Gamma distribution.*** The most convenient way is to designate some random variables corresponding to the time between arrivals. In general the arrivals follow Poisson distribution when the total number of arrivals during any given time interval of the number of arrivals that have already occurred prior to the beginning of time interval. Figures 9.5 and 9.6 shows the Poisson distribution and negative exponential distribution curves.

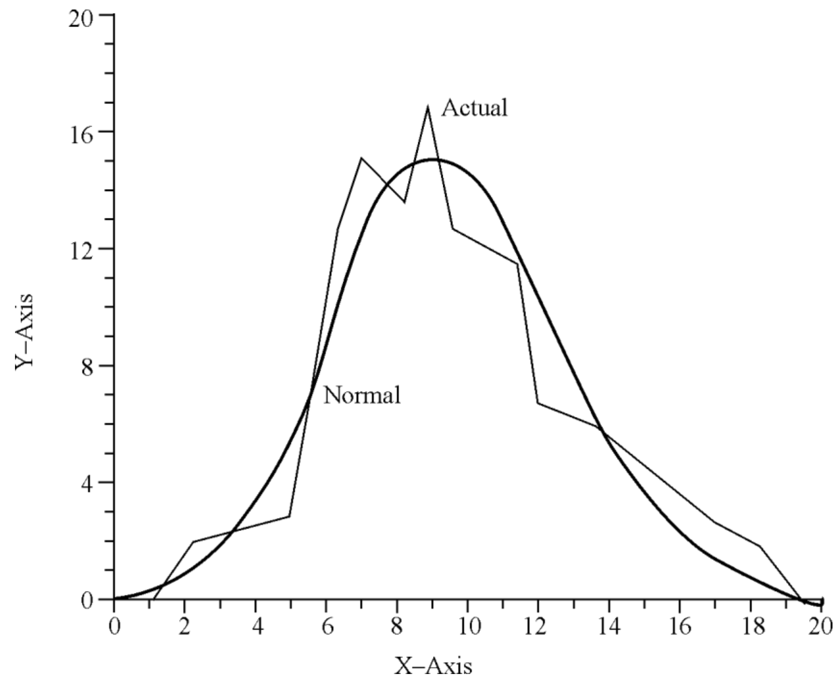


Figure 9.5. Poisson Distribution

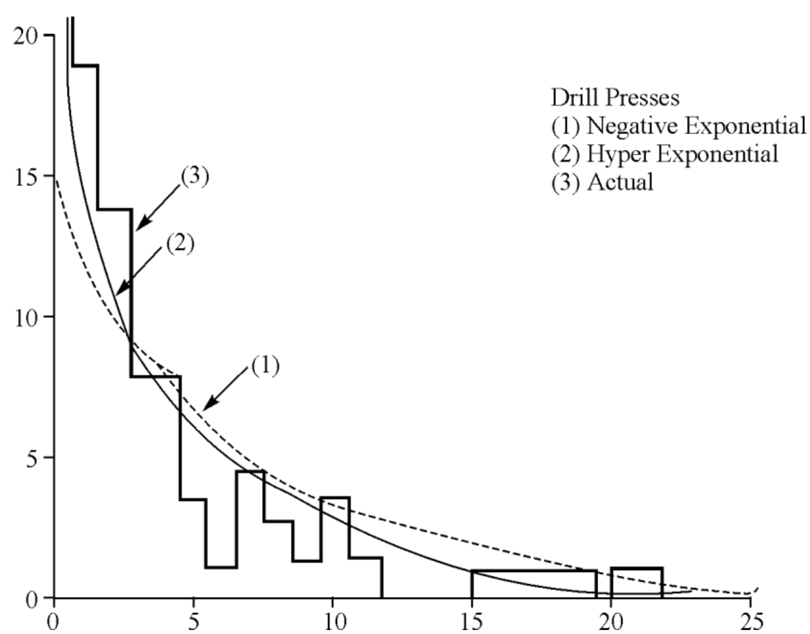


Figure 9.6. Negative exponential dist

(c) *Capacity of the service system*

In queuing context the capacity refers to the space available for the arrivals to wait before taken to service. The space available may be limited or unlimited. When the space is limited, length of waiting line crosses a certain limit; no further units or arrivals are permitted to enter the system till some waiting space becomes vacant. This type of system is known as *system with finite capacity* and it has its effect on the arrival pattern of the system, for example a doctor giving tokens for some customers to arrive at certain time and the present system of allowing the devotees for darshan at Tirupathi by using the token belt system.

(e) *Customer behaviour*

The length of the queue or the waiting time of a customer or the idle time of the service facility mostly depends on the behaviour of the customer. Here the behaviour refers to the impatience of a customer during the stay in the line. Customer behaviour can be classified as:

(i) **Balking:** This behaviour signifies that the customer does not like to join the queue seeing the long length of it. This behaviour may effect in loosing a customer by the organization. Always a lengthy queue indicates insufficient service facility and customer may not turn out next time. For example, a customer who wants to go by train to his destination goes to railway station and after seeing the long queue in front of the ticket counter, may not like to join the queue and seek other type of transport to reach his destination.

(ii) **Reneging:** In this case the customer joins the queue and after waiting for certain time looses his patience and leaves the queue. This behaviour of the customer may also cause loss of customer to the organization.

(iii) **Collusion:** In this case several customers may collaborate and only one of them may stand in the queue. One customer represents a group of customer. Here the queue length may be small but service time for an individual will be more. This may break the patience of the other customers in the waiting line and situation may lead to any type of worst episode.

(iv) **Jockeying:** If there are number of waiting lines depending on the number of service stations, for example Petrol bunks, Cinema theaters, etc. A customer in one of the queue after seeing the other queue length, which is shorter, with a hope of getting the service, may leave the present queue and join the shorter queue. Perhaps the situation may be that other queue which is shorter may be having more number of Collaborated customers. In such case the probability of getting service to the customer who has changed the queue may be very less. Because of this character of the customer, the queue lengths may goes on changing from time to time.

Service Mechanism or Service Facility

Service facilities are arranged to serve the arriving customer or a customer in the waiting line is known as *service mechanism*. The time required to serve the customer cannot be estimated until we know the need of the customer. Many a time it is statistical variable and cannot be determined by any means such as number of customers served in a given time or time required to serve the customer, until a customer is served completely. Service facility design and service discipline and the channels of service as shown in figure 9.7 may generally determine the service mechanism.

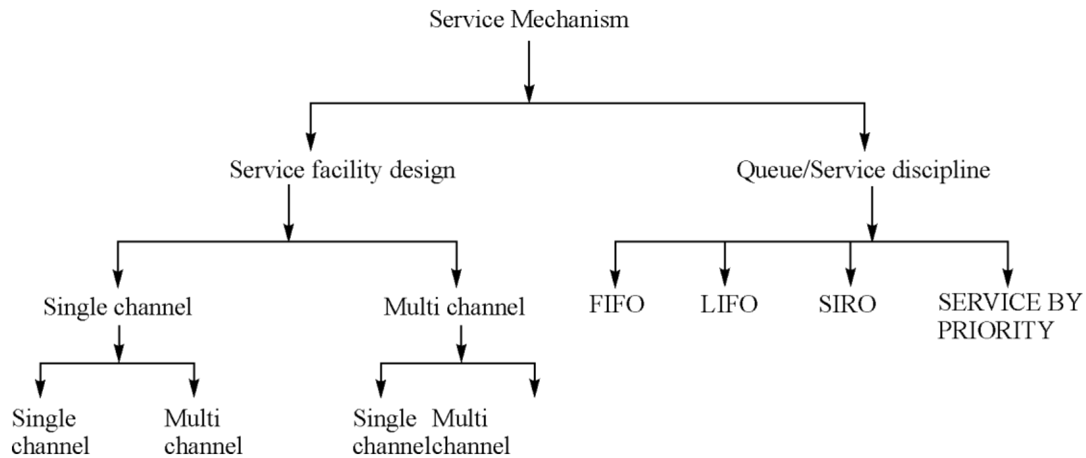


Figure 9.7 Service Mechanisms.

(a) ***Service facility design:*** Arriving customers maybe asked to form a single line (Single queue) or multi line (multi queue) depending on the service need. When they stand in single line it is known as ***Single channel facility*** when they stand in multi lines it is known as ***multi channel facility.***

(i) *Single channel queues:* If the organization has provided single facility to serve the customers, only one unit can be served at a time, hence arriving customers form a queue near the facility. The next element is drawn into service only when the service of the previous customer is over. Here also depending on the type of service the system is divided into *Single phase* and *Multi phase service facility*. In Single channel Single Phase queue, the customer enters the service zone and the facility will provide the service needed. Once the service is over the customer leaves the system. For example, Petrol bunks, the vehicle enters the petrol station. If there is only one petrol pump is there, it joins the queue near the pump and when the term comes, get the fuel filled and soon after leaves the queue. Or let us say there is a single ticket counter, where the arrivals will form a queue and one by one purchases the ticket and leaves the queue.

In single channel multi phase service design, the service needed by the customer is provided in different stages, say for example, at petrol station, the customer will first get the tank filled with fuel, then goes to pollution check point get the exhaust gas checked for carbon dioxide content and then goes to Air compressor and get the air check and leaves the petrol station. Here each service facility is known as a phase. Hence the system is known as multi phase system. Another good example is a patient enters the queue near the doctor's room, get examined by doctor and take prescription goes to compounder takes medicine and then goes to nurse have the injection and leaves the hospital. Here doctor, compounder and nurse all are facilities and serve the customer one by one. This is shown in figure 9.1.

(ii) *Multi Channel queues*

When the input rates increases, and the demand for the service increases, the management will provide additional service facilities to reduce the rush of customers or waiting time of customers. In such cases, different queues will be formed in front of different service facilities. If the service is provided to customers at one particular service center, then it is known as *Multi channel Single-phase* system. In case service is provided to customer in different stages or phases, which are in parallel, then it is known as *multi channel multi phase* queuing system. This is shown in figure 9.1.

(b) Queue discipline or Service discipline

When the customers are standing in a queue, they are called to serve depending on the nature of the customer. The order in which they are called is known as *Service discipline*. There are various ways in which the customer called to serve. They are:

(i) First In First Out (FIFO) or First Come First Served (FCFS)

We are quite aware that when we are in a queue, we wish that the element which comes should be served first, so that every element has a fair chance of getting service. Moreover it is understood that it gives a good morale and discipline in the queue. When the condition of FIFO is violated, there arises the trouble and the management is answerable for the situation.

(ii) Last in first out (LIFO) or Last Come First Served (LCFS)

In this system, the element arrived last will have a chance of getting service first. In general, this does not happen in a system where human beings are involved. But this is quite common in Inventory system. Let us assume a bin containing some inventory. The present stock is being consumed and suppose the material ordered will arrive that is loaded into the bin. Now the old material is at the bottom of the stock where as fresh arrived material at the top. While consuming the top material (which is arrived late) is being consumed. This is what we call Last come first served). This can also be written as First In Last Out (FILO).

(iii) Service In Random Order (SIRO)

In this case the items are called for service in a random order. The element might have come first or last does not bother; the servicing facility calls the element in random order without considering the order of arrival. This may happen in some religious organizations but generally it does not followed in an industrial / business system. In religious organizations, when devotees are waiting for the darshan of the god man / god woman, the devotees are picked up in random order for blessings. Some times we see that in government offices, the representations or applications for various favors are picked up randomly for processing. It is also seen to allocate an item whose demand is high and supply is low, also seen in the allocation of shares to the applicants to the company.

(iv) Service By Priority

Priority disciplines are those where any arrival is chosen for service ahead of some other customers already in queue. In the case of Pre-emptive priority the preference to any arriving unit is so high that the unit is already in service is removed / displaced to take it into service. A non- pre-emptive rule of priority is one where an arrival with low priority is given preference for service than a high priority item. As an example, we can quote that in a doctors shop, when the doctor is treating a patient with stomach pain, suddenly a patient with heart stroke enters the doctors shop, the doctor asks the patient with stomach pain to wait for some time and give attention to heart patient. This is the rule of priority.

QUEUING PROBLEMS

The most important information required to solve a waiting line problem is the nature and probability distribution of arrivals and service pattern. The answer to any waiting line problem depending on finding:

(a) *Queue length*: The probability distribution of queue length or the number of persons in the system at any point of time. Further we can estimate the probability that there is no queue.

(b) *Waiting time*: This is probability distribution of waiting time of customers in the queue. That is we have to find the time spent by a customer in the queue before the commencement of his service, which is called *his waiting time in the queue*. The total time spent in the system is the waiting time in the queue plus the service time. The waiting time depends on various factors, such as:

(i) The number of units already waiting in the system,

(ii) The number of service stations in the system,

(iii) The schedule in which units are selected for service,

(iv) The nature and magnitude of service being given to the element being served.

(c) *Service time:* It is the time taken for serving a particular arrival.

(d) *Average idle time or Busy time distribution:* The average time for which the system remains idle. We can estimate the probability distribution of busy periods. If we suppose that the server is idle initially and the customer arrives, he will be provided service immediately. During his service time some more customers will arrive and will be served in their turn according to the system discipline. This process will continue in this way until no customer is left unserved and the server becomes free again after serving all the customers. At this stage we can conclude, that the busy period is over. On the other hand, during the idle periods no customer is present in the system. A busy period and the idle period following it together constitute a *busy cycle*. The study of busy period is of great interest in cases where technical features of the server and its capacity for continuous operation must be taken into account.

STEADY, TRANSIENT AND EXPLOSIVE STATES IN A QUEUE SYSTEM

The distribution of customer's arrival time and service time are the two constituents, which constitutes of study of waiting line. Under a fixed condition of customer arrivals and service facility a queue length is a function of time. As such a queue system can be considered as some sort of random experiment and the various events of the experiment can be taken to be various changes occurring in the system at any time. We can identify three states of nature in case of arrivals in a queue system. They are named as ***steady state, transient state, and the explosive state.***

(a) *Steady State:* The system will settle down as steady state when the rate of arrivals of customers is less than the rate of service and both are constant. The system not only becomes steady state but also becomes independent of the initial state of the queue. Then the probability of finding a particular length of the queue at any time will be same. Though the size of the queue fluctuates in steady state the statistical behaviour of the queue remains steady. Hence we can say that *a steady state condition is said to prevail when the behaviour of the system becomes independent of time.*

A necessary condition for the steady state to be reached is that elapsed time since the start of the operation becomes sufficiently large *i.e*. ($t \to \infty$), but this condition is not sufficient as the existence of steady state also depend upon the behaviour of the system *i.e.* if the rate of arrival is greater than the rate of service then a steady state cannot be reached. Hence we assume here that the system acquires a steady state as $t \to \infty$ *i.e.* the number of arrivals during a certain interval becomes independent of time. *i.e.*

$$\lim_{t \to \infty} P_n(t) \to P_n$$

Hence in the *steady state system, the probability distribution of arrivals, waiting time, and service time does not depend on time.*

(*b*) *Transient State*

Queuing theory analysis involves the study of a system's behaviour over time. A system is said to be in 'transient state' when its operating characteristics or behaviour are dependent on time. This happens usually at initial stages of operation of the system, where its behaviour is still dependent on the initial conditions. So when the probability distribution of arrivals, waiting time and servicing time are dependent on time the system is said to be in transient state.

(*b*) *Explosive State*

In a situation, where arrival rate of the system is larger than its service rate, a steady state cannot be reached regardless of the length of the elapsed time. Here queue length will increase with time and theoretically it could build up to infinity. Such case is called the *explosive state*.

In our further discussion, all the problems and situations are dealt with steady state only.

DESIGNATION OF QUEUE AND SYMBOLS USED IN QUEUING MODELS

A queue is designated or described as shown below: A model is expressed as
A/B/S : (*d* / *f*) where,

A: Arrival pattern of the units, given by the probability distribution of inter - arrival time of units. For example, Poisson distribution, Erlang distribution, and inter arrival time is 1 minute or 10 units arrive in 30 minutes etc.

B: The probability distribution of service time of individual being actually served. For example the service time follows negative exponential distribution and 10 units are served in 10 minutes or the service time is 3 minutes, etc.

S: The number of service channels in the system. For example the item is served at one service facility or the person will receive service at 3 facilities etc.

d: Capacity of the system. That is the maximum number of units the system can accommodate at any time. For example, the system has limited capacity of 40 units or the system has infinite capacity etc.

f: The manner or order in which the arriving units are taken into service *i.e*. FIFO / LIFO / SIRO /Priority.

NOTATIONS

X: Inter arrival time between two successive customers (arrivals).

Y: The service time required by any customer.

w: The waiting time for any customer before it is taken into service.

v: Time spent by the customer in the system.

n: Number of customers in the system, that is customers in the waiting line at any time, including the number of customers being served.

$\mathbf{P_n (t)}$**:** Probability that '*n*' customers arrive in the system in time '*t*'.

Φ_n **(t):** Probability that '*n*' units are served in time '*t*'.

U (T): Probability distribution of inter arrival time $P(t \leq T)$.

V (T): Probability distribution of servicing time $P(t \leq T)$.

F (N): Probability distribution of queue length at any time $P(N \leq n)$.

$\mathbf{E_n}$**:** Some state of the system at a time when there are '*n*' units in the system.

λ_n : Average number of customers arriving per unit of time, when there are already '*n*' units in the system.

λ : Average number of customers arriving per unit of time.

μ_n : Average number of customers being served per unit of time when there are already '*n*' units in the system.

μ **:** Average number of customers being served per unit of time.

1 / λ **:** Inter arrival time between two arrivals.

1 / μ : Service time between two units or customers.

$\rho = (\lambda / \mu)$:System utility or traffic intensity which tells us how much time the system was utilized in a given time. For example given time is 8 hours and if $\rho = 3 / 8$, it means to say that out of 8 hours the system is used for 3 hours and (8 – 3 = 5) 5 hours the is idle.

DISTRIBUTION OF ARRIVAL AND SERVICE TIME

Distribution of Arrivals

The common basic waiting line models have been developed on the assumption that arrival rate follows the **Poisson** distribution and that *service times* follow the **negative exponential distribution**. This situation is commonly referred to as the Poisson arrival and Exponential holding time case. These assumptions are often quite valid in operating situations. Unless it is mentioned that arrival and service follow different distribution, it is understood always that arrival follows Poisson distribution and service time follows negative exponential distribution.

Research scholars working on queuing models have conducted careful study about various operating conditions like **-** arrivals of customers at grocery shops, Arrival pattern of customers at ticket windows, Arrival of breakdown machines to maintenance etc. and confirmed almost all arrival pattern follows nearly Poisson distribution. One such curve is shown in figure 9.5. Although we cannot say with finality that distribution of arrival rates are always described adequately by the Poisson, there is much evidence to indicate that this is often the case. We can reason this by saying that always Poisson distribution corresponds to completely random arrivals and it is assumed that arrivals are completely independent of other arrivals as well as any condition of the waiting line. The commonly used symbol for average arrival rate in waiting line models is the Greek letter Lamda (λ), arrivals per time unit. It can be shown that when the arrival rates follow a Poisson processes with mean arrival rate λ, the time between arrivals follow a negative exponential distribution with mean time between arrivals of $1/\lambda$. This relationship between mean arrival rate and mean time between arrivals does not necessarily hold for other distributions. The negative exponential distribution then, is also representative of Poisson process, but describes the time between arrivals and specifies that these time intervals are completely random. Negative exponential curve is shown in figure 9.6.

Let us try to understand the probability distribution for time between successive arrivals, which is known as exponential distribution as described above. The distribution of arrivals in a queuing system can be considered as a **pure birth process**. The term birth refers to the arrival of new calling units in the system the objective is to study the number of customers that enter the system, *i.e*. only arrivals are counted and no departures takes place. Such process is known as pure birth process. An example may be taken that the service station operator waits until a minimum-desired customers arrives before he starts the service.

Exponential Service Times

The commonly used symbol for average service rate in waiting line models is the Greek letter 'mu' 'μ', the number of services completed per time unit. As with arrivals it can be shown that when service rates follow a Poison process with mean service rate μ, the distribution of serviced times follow the negative exponential distribution with mean service time $1 / \mu$. The reason for the common reference to rates in the discussion of arrivals and to times in the discussion of service is simply a matter of practice. One should hold it clearly in mind, however, for both arrivals and services, that in the general Poison models, rates follow the Poisson distribution and times follow the negative exponential distribution. One must raise a doubt at this point why the interest in establishing the validity of the Poisson and Negative exponential distributions. The answer is that where the assumptions hold, the resulting waiting line formulas are quite simple. The Poison and Negative exponential distributions are single parameters distributions; that is, they are completely described by one parameter, **the mean**. For the Poisson distribution the standard deviation is the square root of the mean, and for the negative exponential distribution the standard deviation is equal to the mean. The result is that the mathematical derivations and resulting formulas are not complex. Where the assumptions do not hold, the mathematical development may be rather complex or we may resort to other techniques for solution, such as simulation.

QUEUE MODELS

Most elementary queuing models assume that the *inputs / arrivals* and *outputs / departures* follow a **birth** and **death** process. Any queuing model is characterized by situations where both arrivals and departures take place simultaneously. Depending upon the nature of inputs and service faculties, there can be a number of queuing models as shown below:

(i) **Probabilistic queuing model:** Both arrival and service rates are some unknown random variables.

(ii) **Deterministic queuing model:** Both arrival and service rates are known and fixed.

(iii) **Mixed queuing model:** Either of the arrival and service rates is unknown random variable and other known and fixed.

Earlier we have seen how to designate a queue. Arrival pattern / Service pattern / Number of channels / (Capacity / Order of servicing). (A /B/ S / (d / f).

In general ***M*** is used to denote Poisson distribution (**Markovian**) of arrivals and departures.

D is used to constant or Deterministic distribution.

$\boldsymbol{E_k}$ is used to represent Erlangian probability distribution.

G is used to show some general probability distribution.

In general queuing models are used to explain the descriptive behavior of a queuing system. These quantify the effect of decision variables on the expected waiting times and waiting lengths as well as generate waiting cost and service cost information. The various systems can be evaluated through these aspects and the system, which offers the minimum total cost is selected.

Procedure for Solution

(a) List the alternative queuing system
(b) Evaluate the system in terms of various times, length and costs.
(c) Select the best queuing system.

(Note: Students / readers are advised to refer to the books on Operations Research written with mathematical orientation for the derivation of formulas for various queuing models. In this book, the application of formula is made.)

Poisson Arrival / Poisson output / Number of channels / Infinite capacity / FIFO Model

M / M / 1 / (∞ / FIFO):

Formulae used

1. Average number of arrivals per unit of time = λ
2. Average number of units served per unit of time = μ
3. Traffic intensity or utility ratio = $\rho = \frac{\lambda}{\mu}$ the condition is : $(\mu > \lambda)$
4. Probability that the system is empty = $P_0 = (1 - \rho)$
5. Probability that there are '*n*' units in the system = $P_n = \rho^n P_0$
6. Average number of units in the system = $E(n) = \frac{\rho}{(1-\rho)}$ or $= \frac{\lambda}{(\mu-\lambda)} = L_q + \frac{\lambda}{\mu}$
7. Average number of units in the waiting line = $E_L = \frac{\rho^2}{(1-\rho)} = \frac{\lambda^2}{\mu(\mu-\lambda)}$
8. Average waiting length (mean time in the system) = $E(L / L > 0)$

$$= \frac{1}{(\mu-\lambda)} = \frac{1}{(1-\rho)}$$
$$= E(w) + \frac{1}{\mu} = \frac{L}{\lambda}$$

9. Average length of waiting line with the condition that it is always greater than zero

$$= V(n) = \frac{\rho}{(1-\rho)^2} = \frac{\lambda}{(\mu-\lambda)^2}$$
$$= \frac{L_q}{\lambda} = \frac{\lambda}{\mu(\mu-\lambda)}$$

11. Average time an arrival spends in the system = $E(v) = \dfrac{1}{\mu(1-\rho)} = \dfrac{1}{(\mu-\lambda)} = E(w / w > 0)$

12. P (w > 0) = System is busy = ρ

13. Idle time = $(1 - \rho)$

14. Probability distibution of waiting time = $P(w)\, dw =$
 $\mu\rho(1 - \rho)\, e^{-\mu w(1-\rho)}$

15. Probability that a consumer has to wait on arrival = (P (w > 0) = ρ

16. Probability that a new arrival stays in the system =
 $P(v)\, dv = \mu(1 - \rho)\, e^{-\mu v(1-\rho)}\, dv$,

Problem 9.1.

A T.V. Repairman finds that the time spent on his jobs have an exponential distribution with mean of 30 minutes. If he repairs sets in the order in which they come in, and if the arrival of sets is approximately Poisson with an average rate of 10 per 8 hour day, what is repairman's expected idle time each day? How many jobs are ahead of the average set just brought in?

Solution

This problem is Poisson arrival/Negative exponential service / single channel /infinite capacity/ FIFO type problem.

Data: λ = 10 sets per 8 hour day = 10 / 8 = 5/4 sets per hour.

Given $1/\mu$ = 30 minutes, hence $\mu = (1/30) \times 60 = 2$ sets per hour.

Hence, Utility ratio = ρ = (λ / μ) = (5/4) / 2 = = 5 / 8. = 0.625. This means out of 8 hours 5 hours the system is busy *i.e.* repairman is busy.

Probability that there is no queue = The system is idle = $(1 - \rho)$ = 1 – (5 / 8) = 3 / 8 = That is out of 8 hours the repairman will be idle for 3 hours.

Number of sets ahead of the set just entered = Average number of sets in system = $\lambda / (\mu - \lambda)$ = = $\rho / (1 - \rho)$ = 0.625 / (1 – 0.625) = 5 / 3 ahead of jobs just came in.

Problem 9.2.

The arrivals at a telephone booth are considered to be following Poisson law of distribution with an average time of 10 minutes between one arrival and the next. Length of the phone call is assumed to be distributed exponentially with a mean of 3 minutes.

(a) What is the probability that a person arriving at the booth will have to wait?

(b) What is the average length of queue that forms from time to time?

(c) The telephone department will install a second booth when convinced that an arrival would expect to wait at least thee minutes for the phone. By how much must the flow of arrivals be increased in order to justify a second booth?

Solution

Data: Time interval between two arrivals = 10 min. = $1/\lambda$, Length of phone call = 3 min. = $1/\mu$.

Hence $\lambda = 1/10 = 0.1$– per min and $\mu = 1/3 = 0.33$ per min., and $\rho = \lambda / \mu = 0.10 / 0.33 = 0.3$

(a) Any person who is coming to booth has to wait when there is somebody in the queue. He need not wait when there is nobody in the queue *i.e.* the queue is empty. Hence the probability of that an arrival does not wait = $P_0 = (1 - \rho)$.
Hence the probability that an arrival has to wait = 1 – The probability that an arrival does not wait = $(1 - P_0) = 1 - (1 - \rho) = \rho = 0.3$. That means 30% of the time the fresh arrival has to wait. That means that 70 % of the time the system is idle.

(b) Average length of non- empty queue from time to time = **(Average length of the waiting line with the condition that it is always greater than zero =** $1 (1 - \rho)$ ***i.e.*** $E\,(L/L > 0)$
$= 1 / (1 - 0.3) = 1.43$ persons.

(c) The installation of the second booth is justified if the waiting time is greater than or equal to three. If the new arrival rate is λ', then for $\mu = 0.33$ we can work out the length of the waiting line. In this case $\rho = \lambda' / \mu$.

Length of the waiting line for λ' and $\mu = 0.33 = E\,(w) = \{\lambda' / \mu(\mu - \lambda')\} \geq 3$ or $\lambda' = (3\mu^2 - 3\rho\mu\lambda')$ or $\lambda' = (3\mu^2)/(1 + 3\mu) = (3 \times 0.33^2) / (1 + 3 \times 0.33)$ *i.e.* $\lambda' \geq 0.16$. That is the arrival rate must be at least 0.16 persons per minute or one arrival in every 6 minutes. This can be written as 10 arrivals per hour to justify the second booth.

Problem 9.3.

In a departmental store one cashier is there to serve the customers. And the customers pick up their needs by themselves. The arrival rate is 9 customers for every 5 minutes and the cashier can serve 10 customers in 5 minutes. Assuming Poisson arrival rate and exponential distribution for service rate, find:

(a) Average number of customers in the system.
(b) Average number of customers in the queue or average queue length.
(c) Average time a customer spends in the system.
(d) Average time a customer waits before being served.

Solution

Data: Arrival rate is $\lambda = (9 / 5) = 1.8$ customers per minute.

Service rate = $\mu = (10 / 5) = 2$ customers per minute. Hence $\rho = (\lambda / \mu) = (1.8 / 2) = 0.9$

(a) Average number of customers in the system = $E\,(n) = \rho / (1 - \rho) = 0.9 / (1 - 0.9) = 0.9 / 0.1 = 9$ customers.

(b) Average time a customer spends in the system = $E\,(v) = 1/\mu\,(1 - \rho) = 1/(\mu - \lambda) = 1 / (2 - 1.8) = 5$ minutes.

(c) Average number of customers in the queue = $E\,(L)$
$= \rho^2 / (1 - \rho) = \lambda^2 / \mu(\mu - \lambda) = (\rho) \times \lambda / (\mu - \lambda) = 0.9 \times 1.8 / (2 - 1.8) = 8.1$ customers.

(d) Average time a customer spends in the queue = $\rho / \mu (1 - \rho) = \lambda / \mu(\mu - \lambda) = 0.9 / 2\,(1 - 0.9) = 0.9 / 0.2 = 4.5$ minutes.

Problem 9.4.

A branch of a Nationalized bank has only one typist. Since typing work varies in length (number of pages to be typed), the typing rate is randomly distributed approximating a Poisson distribution with a mean service rate of 8 letters per hour. The letter arrives at a rate of 5 per hour during the entire 8-hour workday. If the typist is valued at Rs. 1.50 per hour, determine:

(*a*) Equipment utilization, (*b*) The percent time an arriving letter has to wait, (*c*) Average system time, and d) Average idle time cost of the typewriter per day.

Solution

Data = arrival rate = $\lambda = 5$, Service rate $\mu = 8$ per hour.

Hence $\rho = (\lambda / \mu) = 5 / 8 = 0.625$

(a) Equipment utilization = Utility ratio = $\rho = 0.625$, *i.e.* 62.5 percent of 8 hour day the equipment is engaged.

(b) Percent time that an arriving letter has to wait = As the machine is busy for 62.5 % of the day, the arriving letter has to wait for 62.5 % of the time.

(c) Average system time = Expected (average) a customer spends in the system = $1/(\mu - \lambda)$ = $[1 / (8 - 5)] = 1/3$ hour. = 20 minutes.

d) Average idle time cost of the typewriter per day = 8 hours × idle time × idle time cost =
= $8 \times (1 - 5 / 8) \times$ Rs. 1.50 = Rs. 4.50.

Problem 9.5.

A product manufacturing plant at a city distributes its products by trucks, loaded at the factory warehouse. It has its own fleet of trucks plus trucks of a private transport company. This transport company has complained that sometimes its trucks have to wait in line and thus the company loses money paid for a truck and driver of waiting truck. The company has asked the plant manager either to go in for a second warehouse or discount prices equivalent to the waiting time. The data available is:

Average arrival rate of all trucks = 3 per hour.

Average service rate is = 4 per hour.

The transport company has provided 40% of the total number of trucks. Assuming that these rates are random according to Poisson distribution, determine:

(a) The probability that a truck has to wait?

(b) The waiting time of a truck that has to wait,

(c) The expected waiting time of company trucks per day.

Solution

Data: $\lambda = 3$ trucks per hour, $\mu = 4$ trucks per hour. Hence ρ = utilization factor = $(\lambda / \mu) = 3 / 4 = 0.75$. This means that the system is utilized 75% of the time. Hence 75% the time the truck has to wait.

The waiting time of truck that waits = $E(v) = 1/(\mu - \lambda) = 1 / (4 - 3) = 1$ hour.

Total expected waiting time of company trucks per day = (Trucks per day) × (% company trucks) × Expected waiting time per truck. = $(3 \times 8) \times (0.40) \times [\lambda / \mu(\mu - \lambda)] = 24 \times 0.40 \times [3 / 4(4-3) = 24 \times 0.40 \times 0.75 = 7.2$ hours per day.

Problem 9.6.

A repairman is to be hired to repair machines, which break down at an average rate of 3 per hour. The breakdown follows Poisson distribution. Non - productive time of a machine is considered to cost Rs. 16/- per hour. Two repairmen have been interviewed. One is slow but cheap while the other is fast but expensive. The slow worker charges Rs. 8/- per hour and the services breakdown machines at the rate of 4 per hour. The fast repairman demands Rs. 10/- per hour and services at an average rate of 6 per hour. Which repairman is to be hired?

Solution

Data: λ = 3 machines per hour, Idle time cost of machine is Rs. 16/- per hour, Slow repair man charges Rs. 8/- per hour and repairs 4 machines per hour = μ_1 , Fast worker demands Rs. 10 per hour and repairs 6 machines per our = μ_2

S.No.	*Particulars.*	*Formula/Symbol*	*Slow worker*	*Fast worker*	*Remarks.*
1.	Arrival rate	λ	3 machines per hour	3 machines per hour	
2.	Service rate	μ	4 machines per hour	6 machines per hour	
3.	Idle time cost	*C*	16 per hour	16 per hour.	
4.	Labour charges.	*L*	Rs. 8/- per hour	Rs. 10 per hour	
5.	Average down time of the machine =	= Average time spent by the machine in the system = E (v) = $1/\mu - \lambda$	1 / (4 - 3) = 1 hour.	1 / (6 – 3) = 1/3 hour.	
6.	Per hour total cost of slow worker.	*C* = 3 machines × 1 hour × Rs. 16 = Rs. 48	L = Rs. 8	---	Rs. 48 + 8 = Rs. 56/-
7.	Per hour Total cost of fast worker =	*C* = 1/3 × 3 machines × Rs. 16/- = Rs. 16/-	---	L = Rs. 10/-	Rs. 16 + Rs. 10 = Rs. 26

As total cost of fast worker is less than that of slow worker, fast workman should be hired.

Problem 9.7.

There is a congestion of the platform of a railway station. The trains arrive at the rate of 30 trains per day. The waiting time for any train to hump is exponentially distributed with an average of 36 minutes. Calculate: (*a*) The mean queue size, (*b*) The probability that the queue size exceeds 9.

Solution

Data: Arrival rate 30 trains per day, service time = 36 minutes.

λ = 30 trains per day. Hence inter arrival time = $1/\lambda$ = (60 × 24) / 30 = 1 / 48 minutes. Given that the inter service time = $1/\mu$ = 36 minutes. Therefore $\rho = (\lambda / \mu)$ = 36 / 48 = 0.75.

(a) The mean queue size = $E(n) = \rho / (1 - \rho) = 0.75 / (1 - 0.75) = 0.75 / 0.25 = 3$ trains.

(b) Probability that queue size exceeds 9 = Probability of queue size ≥ 10 = 1 – Probability of queue size less than 10 = $1 - (p_0 + p_1 + p_2 \ldots.. + p_9) = p_0 (1 + \rho + \rho^2 + \ldots.\rho^9)$ =

$1 - \{(1 - \rho)[(1 - \rho^{10}) / (1 - \rho)]\} = 1 - (1 - \rho^{10}) = \rho^{10} = (0.75)^{10} = 0.06$ approximately).

Problem 9.8.

Let on the average 96 patients per 24-hour day require the service of an emergency clinic. Also on average, a patient requires 10 minutes of active attention. Assume that the facilities can handle only one emergency at a time. Suppose that it costs the clinic Rs. 100/- per patient treated to obtain an average servicing time of 10 minutes and that each minute of decrease in this average time would costs Rs. 10/- per patient treated. How much would have to be budgeted by the clinic to decrease the average size of the queue from one and one third patient to half a patient?

Solution

Data: $\lambda = 96 / 24 = 4$ patients per hour., $\mu = (1 / 10) \times 60 = 6$ patients per hour. Hence $\rho = (\lambda / \mu) = 4 / 6 = 2 / 3$.

Average number of patients in waiting line = $E(L) = \rho^2 / (1 - \rho) = (4 / 9) / [1 - (2 / 3)] = 4 / 3$ patients. = One and one third patients. Now this is to be reduced to $1/2 = E'(L)$.

$E'(L) = (\lambda / \mu') \times (\lambda / \mu' - \lambda)$ or

$1/2 = (4 / \mu') \times (4 / \mu' - 4)$ or $\mu'^2 - 4\mu' - 32 = 0$ or $(\mu' - 8)(\mu' + \mu) = 0$ or $\mu' = 8$ patients per hour.

(Note $\mu' = -4$ is not considered as it does not convey any meaning.)

Therefore, average time required by each patient = 1/8 hour = 15 / 2 minutes = 7 ½ minutes.

Decrease in time required by each patient 10 – (15 / 2) = 5 / 2 minutes or 2 ½ minutes.

The budget required for each patient = Rs. [100 + 10 × (5/2)] Rs. 125/-

Thus decrease the size of the queue; the budget per patient should be increased from Rs. 100/- to Rs. 125/-.

Problem 9.9.

Arrival rate of telephone calls at a telephone booth is according to Poisson distribution, with an average time of 9 minutes between consecutive arrivals. The length of telephone call is exponentially distributed with a man of 3 minutes. Find:

(a) Determine the probability that a person arriving at the booth will have to wait.
(b) Find the average queue length that forms from time to time.
(c) The telephone company will install a second booth when conveniences that an arrival would expect to have to wait at least four minutes for the phone. Find the increase in flow of arrivals, which will justify a second booth.
(d) What is the probability that an arrival will have to wait for more than 10 minutes before the phone is free?

(e) What is the probability that they will have to wait for more than 10 minutes before the phone is available and the call is also complete?

(f) Find the fraction of a day that the phone will be in use.

Solution

Data: Arrival rate $\lambda = 1 / 9$ per minute and service rate $\mu = 1 / 3$ per minute.

(a) Probability that a person has to wait (person will wait when the system is busy *i.e.* we have to find ρ) $\rho = \lambda / \mu = (1 / 9) / (1 / 3) = 3 / 9 = 0.33$ *i.e.* 33% of the time the customer has to wait. This means that 67% of the time the customer will get the phone soon after arrival.

(b) Average queue length that forms from time to time $= \mu / (\mu - \lambda) = (1 / 3) / [(1 / 3) - (1 / 9)] = (1 / 3) / (2 / 9) = (1 / 3) \times (9 / 2) = 1.5$ persons.

(c) Average waiting time in the queue $= E(w) = \lambda_1 / \mu(\mu - \lambda_1) = 4$

$4 = \lambda_1 / (1/3) \times [(1/3) - \lambda_1] = (1 / 9) - (\lambda_1 / 3) = (\lambda_1 / 4)$ or $\lambda_1 \times (7/2) = 1/9$

$\lambda_1 = 12 / (7 \times 9) = (4 / 21)$ arrivals per minute. Hence increase in the flow of arrivals $= (4 / 21) - (1 / 9) = 5/63$ per minute.

(d) Probability of waiting time $\geq 10 = \int_{10}^{\infty} (\lambda / \mu)(\mu - \lambda) \times e^{-(\mu-\lambda)t} \, dt.$

$$= (\lambda / \mu)(\mu - \lambda) \times [(e^{-(\mu-\lambda)t}) / - (\mu - \lambda)]_{10}^{\infty} \quad = (\lambda / \mu) \times [0 - e^{-(\mu-\lambda)10}] = (\lambda / \mu) \times e^{-(\mu-\lambda)10}$$

$$= (1/3) \times e^{-(1/3 - 1/9) \times 10} = (1/3) \times e^{-(20/9)} = 1/30$$

(e) Probability that an element spends in system $\geq 10 = \int_{10}^{\infty} (\mu - \lambda)\, e^{-(\mu-\lambda)t} \, dt.$

$$= (\mu / \lambda) \int_{10}^{\infty} (\lambda / \mu)(\mu - \lambda).e^{-(\mu-\lambda)t} \, dt = (\mu / \lambda)\,(1/30) = [(1/3)/(1/9)] \times (1/30) = 1/10 = 0.1$$

(*g*) The expected fraction of a day that the phone will be in use $= \rho = (\lambda / \mu) = 0.33$.

Problem 9. 10.

In large maintenance department fitters draw parts from the parts stores, which is at present staffed by one storekeeper. The maintenance foreman is concerned about the time spent by fitters in getting parts and wants to know if the employment of a stores helper would be worthwhile. On investigation it is found that:

(a) A simple queue situation exists,

(b) Fitters cost Rs. 2.50 per hour,

(c) The storekeeper costs Rs. 2/- per hour and can deal on an average with 10 fitters per hour.

(d) A labour can be employed at Rs. 1.75 per hour and would increase the capacity of the stores to 12 per hour.

(e) On an average 8 fitters visit the stores each hour.

Solution

Data: λ = 8 fitters per hour, μ = 10 per hour.

Number of fitters in the system = $E(n) = \lambda/(\mu - \lambda)$ or $\rho/(1-\rho) = 8/(10-8) = 4$ fitters.

With stores labour λ = 8 per hour, μ = 12 per hour.

Number of fitters in the system = $E(n) = \lambda/(\mu - \lambda) = 8/(12-8) = 2$ fitters.

Cost per hour = Cost of fitter per hour + cost of labour per hour = 2 × Rs. 2.50 + Rs. 1.75 = Rs. 6.75.

Since there a net savings of Rs. 3.25 per hour, it is recommended to employ the labourer.

Model II. Generalization of model (M /M / 1) : (FCFS/ ∞/∞) : (Birth - Death process)

In waiting line system each arrival can be considered to be a birth *i.e.* if the system is in the state E_n, *i.e.* there are n units in the system and there is an arrival then the state of the system changes to the state E_{n+1}. Similarly when there is a departure from the system the state of the system becomes E_{n-1}. Hence whole system is thus viewed as a birth and death process. When λ is the arrival rate of the system, will never be fixed and dependent on the queue length 'n', then it will mean that some person interested in joining the queue may not join due to long queue. Similarly, if μ is also dependent on the queue length it may affect the service rate. Hence in this case both λ and μ cannot be taken to be fixed. Three cases may occur, which are described below.

In this model, arrival rate and service rate *i.e.* λ and μ do not remain constant during the queuing phenomenon and vary to $\lambda_1, \lambda_2 \ldots \lambda_n$ and $\mu_1, \mu_2 \ldots \mu_n$ respectively. Then:

$p_1 = (\lambda_0/\mu_1)\, p_0$

$p_2 = (\lambda_0/\mu_1)(\lambda_1/\mu_2)\, p_0$

……………………………..

……………………………..

$p_n = (\lambda_0/\mu_1)(\lambda_1/\mu_2)\ldots(\lambda_{n-2}/\mu_{n-1}) \times (\lambda_{n-1}/\mu_n)\, p_0$

But there are some special cases when:

1. $\lambda_n = \lambda$ and $\mu_n = \mu$ then,

   $p_0 = 1 - (\lambda/\mu)$, $p_n = (\lambda/\mu)^n \times [1 - (\lambda/\mu)]$

2. When $\lambda_n = \lambda/(n+1)$ and $\mu_n = \mu$

   $p_0 = e^{-\rho}$

   $p_n = [\rho^n/(n!)] \times e^{-\rho}$, where $\rho = (\lambda/\mu)$

3. When $\lambda_n = \lambda$ and $\mu_n = n \times \mu$ then $p_0 = e^{-\rho}$ and $p_n = (\rho^n/n!)\, xe^{-\rho}$.

Problem 9.11.

A transport company has a single unloading berth with vehicles arriving in a Poisson fashion at an average rate of three per day. The unloading time distribution for a vehicle with 'n' unloading workers is found to be exponentially with an average unloading time (1/2) xn days. The company has a large labour supply without regular working hours, and to avoid long waiting lines, the company has a policy of using as many unloading group of workers in a vehicle as there are vehicles waiting in line or being unloaded. Under these conditions find (*a*) What will be the average number of unloading group of workers working at any time? (*b*) What is the probability that more than 4 groups of workers are needed?

Solution

Let us assume that there are '*n*' vehicles waiting in line at any time. Now service rate is dependent on waiting length hence $\mu_n = 2n$ vehicles per day (when there are '*n*' groups of workers in the system).

Now $\lambda = 3$ vehicles per day and $\mu = 2$ vehicles per day. (With one unloading labour group)

Hence, $p_n = (\rho^n / n!) \times e^{-\rho}$ for $n \geq 0$

Therefore, expected number of group of workers working any specified instant is

$$E(n) = \sum_{n=0}^{\infty} n \times p_n = \Sigma\, n \times (\rho^n e^{-\rho}) / n! = \rho \times e^{-\rho} \times \sum_{n=1}^{\infty} (\rho^{n-1}) / (n-1) = (\lambda / \mu)$$

$$= 1.5 \text{ labour group.}$$

The probability that the vehicle entering in service will require more than four groups of workers is

$$\sum_{n=5}^{\infty} p_n = 1 - \sum_{n=0}^{4} (\rho^n / n!)\, e^{-\rho} = 0.019.$$

Model III. Finite Queue Length Model: (M / M / 1) : FCFS / N / ∞

This model differs from the above model in the sense that the maximum number of customers in the system is limited to *N*. Therefore the equations of above model is valid for this model as long as $n < N$ and arrivals will not exceed *N* under any circumstances. The various equations of the model is:

1. $p_0 = (1 - \rho) / (1 - \rho^{N+1})$, where $\rho = \lambda / \mu$ and $\lambda / \mu > 1$ is allowed.
2. $p_n = (1 - \rho)\, \rho^n / (1 - \rho^{N+1})$ for all $n = 0, 1, 2, \ldots N$
3. Average queue length $E(n) = \rho[1 - (1 + N)\, \rho^N + N\rho^{N+1}] / (1 - \rho)(1 - \rho^{N+1})$.

$$= [(1 - \rho) / (1 - \rho^{N+1})] \times \sum_{n=0}^{n} n\, \rho^n = p_0 \times \sum_{n=0}^{N} n \times \rho^n$$

4. The average length of the waiting line $= E(L) = [1 - N\rho^{N+1} + (N-1)\rho^N]\rho^2 / (1-\rho)(1-\rho^{N+1})$
5. Waiting time in the system $= E(v) = E(n) / \lambda'$ where $\lambda' = \lambda(1 - \rho_N)$
6. Waiting time in the queue $= E(w) = E(L) / \lambda' = [(E(n) / \lambda' / (1/\mu)]$.

Problem 9.12.

In a railway marshalling yard, good train arrives at the rate of 30 trains per day. Assume that the inter arrival time follows an exponential distribution and the service time is also to be assumed as exponential with a mean of 36 minutes. Calculate: (*a*) The probability that the yard is empty, (*b*) The average length assuming that the line capacity of the yard is 9 trains.

Solution

Data: $\lambda = 30 / (60 \times 24) = 1 / 48$ trains per minute. And $\mu = 1 / 16$ trains per minute. Therefore $\rho = (\lambda / \mu) = 36 / 48 = 0.75$.

(a) The probability that the queue is empty is given by = $p_0 = (1-\rho)/(1-\rho^{N+1})$, where $N = 9$.
{1 - 0.75) / [1- $(0.75)^{9+1}$ = 0.25 / 0.90 = 0.28. *i.e.* 28 % of the time the line is empty.

Average queue length is = $[(1-\rho) / (1-\rho^{N+1})] \times \sum_{n=0}^{N} n\,\rho^n$

$[(1-0.75)/(1-0.75^{10})] \times \sum_{n=0}^{9} n\ (0.75)^n = 0.28 \times 9.58 = 3$ trains.

Problem 9.13.

A barbershop has space to accommodate only 10 customers. He can serve only one person at a time. If a customer comes to his shop and finds it is full he goes to the next shop.

Customers randomly arrive at an average rate $\lambda = 10$ per hour and the barber service time is negative exponential with an average of $1/\mu = 5$ minute. Find p_0 and p_n

Solution

Data: $N = 10$, $\lambda = 10 / 60$, $\mu = 1 / 5$. Hence $\rho = (\lambda / \mu) = 5 / 6$.

$p_0 = (1-\rho)/(1-\rho^{11}) = [\ 1-(5/6)] / [\ 1-(5/6)^{11} = 0.1667 / 0.8655 = 0.1926$

$p_n = (1-\rho)\rho^N /(1-\rho^{N+1}) = (0.1926) \times (5/6)^n$ where $n = 0, 1, 2, \ldots 10$.

Problem 9.14.

A Car park contains 5 cars. The arrival of cars is Poisson at a mean rate of 10 per hour. The length of time each car spends in the car park is exponential distribution with a mean of 5 hours. How many cars are in the car park on an average?

Solution

Data: $N = 5$, $\lambda = 10 / 60 = 1/6$, $\mu = 1 / 2 \times 60 = 1 / 120$. Hence $\rho = (\lambda / \mu) = [\ (1/6) / (1/120) = 20$.

$p_0 = (1-\rho) / 1-\rho^{N+1} = (1-20) / (1-20^6) = 2.9692 \times 10^{-3}$

Average cars in car park = length of the system = $E(n) = p_0 \times \sum_{n=0}^{N} n \times \rho^n$

$= (2.9692 \times 10^{-3}) \sum_{n=0}^{N} n\,(0.9692 \times 10^{-3})^n$ = Approximately = 4

Problem 9.15.

Trains arrive at the yard every 15 minutes and the service time is 33 minutes. If the line capacity of the yard is limited to 4 trains, find (*a*) the probability that the yard is empty and (*b*) The average number of trains in the system.

Solution

Data: $\lambda = 1/15$ per minute, $\mu = 1/33$ per minute, $N = 4$. Hence $\rho = \lambda / \mu = 33/15 = 2.2$.

$p_0 = (1 - \rho) / (1 - \rho^{N+1}) = (1 - 2.2) / 1 - 2.2^5) = -1.2 / -50.5 = 0.0237$.

(b) Average number of trains in the system = $E(n) = \sum_{n=0}^{4} np_n = 0 + p_1 + 2p_2 + 3p_3 + 4p_4 =$

$= p_0 (\rho + 2\rho^2 + 3\rho^3 + 4\rho^4) = 0.0237 (2 \times 2.2^2 + 3 \times 2.2^3 + 4 \times 2.2^4) = 3.26$ trains.

Problem 9.16.

A railway station only one train is handled at a time. The railway yard is sufficient for two trains to wait while other is given signal to leave the station. Trains arrive at a station at an average rate of 6 per hour and the railway station can handle them on an average rate of 12 per hour. Assuming Poisson arrivals and exponential service distribution, find the steady state probabilities of the various number of trains in the system. Also find the average number of trains in the system.

Solution

Data: $\lambda = 6$ trains per hour, $\mu = 12$ trains per hour. As the maximum queue length is 2, the maximum number of trains in the system is $N = 3$.

Now $p_0 = (1-\rho)/(1-\rho^{N+1}) = (1 - 0.5) / (1 - 0.5^4) = 0.53$.

$$p_n = \rho^n \cdot p_0$$

$$p_1 = \rho_2 \times p_0 = 0.5 \times 0.53 = 0.256$$

$$p_2 = \rho^2 \times p_0 = 0.5^2 \times 0.53 = 0.132$$

$$p_3 = \rho^3 \times p_0 = 0.5^3 \times 0.53 = 0.066.$$

$$E(n) = \sum^{3} np_n = 0 + p_1 + 2p_2 + 3p_3 = 0 + 0.265 + 2 \times 0.132 + 3 \times 0.066$$

$= 0.727$ trains. *i.e.* 1 train.

MODEL IV: (M / M / 1) : FCFS / N /N (Limited Popultion or Source Model)

In this model, we assume that customers are generated by limited pool of potential customers *i.e*. finite population. The total customer's population is ***M*** and ***n*** represents the number of customers already in the system (waiting line), any arrival must come from *M* - *n* number that is not yet in the system. The formulae for this model are:

$$p_0 = 1 / \sum_{n=0}^{M} [M!/(M-n)!](\lambda / \mu)^n$$

$$p_n = [M!/(M-n)!] \times (\lambda/\mu)^n \times p_0 = \{[M!/(M-n)!] \times (\lambda/\mu)^n\} / \left\{ \sum_{n=0}^{M} M!/(M-n)! \times (\lambda/\mu)^n \right\}$$

Average number of customers in the system = $E(n) = \sum_{n=0}^{M} n\, p_n = M - (\mu / \lambda)(1 - p_0)$

Average number in the queue = $E(L) = M - [(\mu + \lambda) / \lambda] \times (1 - p_0)$

Problem 9.17.

A mechanic repairs 4 machines. The mean time between service requirements is 5 hours for each machine and forms an exponential distribution. The mean repair time is 1 hour and also follows the same distribution pattern. Machine down time costs Rs. 25/- per hour and the mechanic costs Rs. 55/- per day. Find (*a*) Expected number of operating machines, (*b*) the expected down time cost per day, (c) Would it be economical to engage two mechanics, each repairing only two machines?

Solution

Data: Finite population, λ = Arrival rate = (1 / 5) = 0.2, μ = Service rate = μ = (1 / 1) = 1 Probability of the empty system = p_0 =

$$p_0 = 1 / \sum_{n=0}^{4} [4!/(4-n)!](0.2/1)^n =$$

$1 / 1 + (4 \times 0.2) + (4 \times 3 \times 0.2^2) + (4 \times 3 \times 2 \times 0.2^3) + (4 \times 3 \times 2 \times 1 \times 0.2^4) = 0.4$ *i.e.* 40 percent of the time the system is empty and 60 percent of the time the system is busy.

(a) Expected number of breakdown machines I the system = $E(n) = M - (\mu/\lambda)(1 - p_0)$ $= 4 - (1 / 0.2)(1 - 0.4) = 4 - 5 \times 0.6 = 4 - 3 = 1$. *i.e.* Expected number of operating machines in the system = 4 – 1 = 3.

(b) Expected down time cost per day of 8 hours = 8 × (expected number of breakdown machines × Rs. 25 per hour) = 8 × 1 × 25 = Rs. 200 / - day.

(c) When there are two mechanics each serving two machines, $M = 2$, p_0 =

$$p_0 = 1 / \sum_{n=0}^{2} [2!/(2-n)!](0.2/1)^n = 1 / 1 + (2 \times 0.2) + (2 \times 1 \times 0.2^2) = 1 / 1.48 = 0.68$$ *i.e.* 68

percent of the time the system is idle. It is assumed that each mechanic with his two machines constitutes a separate system with no interplay. Expected number of machines in the system =

$M - (\mu / \lambda) \times (1 / p_0) = 2 - (1 / 0.2) \times (1 - 0.68) = 0.4$.

Therefore expected down time per day = 8 × 0.4 × Number of mechanics or machine in system = 8 × 0.4 × 2 = 6.4 hours per day. Hence total cost involved =

Rs. 55 × 2 + 6.4 × Rs. 25/- = Rs. (110 + 160) = Rs. 270 per day.

But total cost with one mechanic is Rs. (55 + 200) = Rs. 255/- per day, which is cheaper compared to the above. Hence use of two mechanics is not advisable.

9.9. MULTI CHANNEL QUEUING MODEL: M / M / c: ($\propto$ / FCFS)

The above symbols indicate a system with Poisson input and Poisson output with number of channels = c, where c is > 1, the capacity of line is infinite and first come first served discipline. Here the length of waiting line depends on the number of channels engaged. In case the number of customers in the system is less than the number of channels *i.e.* $n < c$, then there will be no problem of waiting and the rate of servicing will be $n\mu$ as only n channels are busy, each servicing at the rate m. In case $n = c$, all the channels will be working and when $n \geq c$, then $n - c$ elements will be in the waiting line and the rate of service will be $c\mu$ as all the c channels are busy.

Various formulae we have to use in this type of models are:

$$p_0 = 1/\sum_{n=0}^{c-1}[(\lambda/\mu)^n/n!] + [(\lambda/\mu)^c/c!] \times [(c\mu/c\mu - \lambda)]$$

$$OR = 1/\left[\sum_{n=0}^{c-1}(c\rho)^n/n!] + [(c\rho)^c/c!(1-\rho)\right]$$

$$p_n = \{[(\lambda/\mu)^n/n!]/n!\} \times p_0, \text{ when } 1 \leq n \leq c$$

$$p_n = [1/(c^{n-c} \times c!)]\,(\lambda/\mu)^n \times p_0 \quad \text{when } n \geq c$$

Average number of units in waiting line of the system = $E(n) = [\rho\, p_c/(1-\rho)^2]$

$$= \{[\lambda.\mu.(\lambda/\mu)^c]/[(c-1)!(c\mu-\lambda)^2]\}\, p_0 + (\lambda/\mu)$$

Average number in the queue = $E(L) = [\,pc\rho/(1-\rho)^2] + c\mu =$

$$= \{[\lambda.\mu.(\lambda/\mu)^c]/[(c-1)!\,(c\mu-\lambda)^2]\}\, p_0$$

Average queue length = Average number of units in waiting line + number of units in service

Average waiting time of an arrival = $E(w)$ = (Average number of units in waiting line) / λ = $[(p_c\rho)/\lambda(1-\rho)^2 = [\rho/\lambda(1-\rho)^2 \times (1/c!) \times (\lambda/\mu)^c \times p_0$

$$= E(L)/\lambda = \{[\mu \times (\lambda/\mu)^c]/[(c-1)!(c\mu-\lambda)^2]\} \times p_0$$

Average time an arrival spends in system = $E(v)$ =

(Average number of items in the queue) / λ = $[(p_c\rho)/\lambda(1-\rho)] + (C\mu/\lambda)$

$$E(n)/\lambda = \{[\mu \times (\lambda/\mu)^c]/[(c-1)!(c\mu-\lambda)^2]\} \times p_0 + (1/\mu)$$

Probability that all the channels are occupied = $p(n \geq c) = [1/(1-\rho)]\, p_c$

$$= [\mu \times (\lambda/\mu)^c]\, p_0/[(c-1)!(c\mu-\lambda)]$$

Probability that some units has to wait = $p(n \geq c+1) = [\rho\, p_c/(1-\rho)]$

$$= 1 - p(n \geq c) = 1 - [\mu \times (\lambda/\mu)^c]\, p_0/[(c-i)!(c\mu-\lambda)]$$

The average number of units which actually wait in the system =

$$\left[\sum_{n=c+1}^{\infty}(n-c)\, p_n\right] \div \sum_{n=c+1}^{\infty} p_n = 1/(1-\rho)$$

Average waiting time in the queue for all arrivals = $(1/\lambda)\sum_{n=0}^{\infty}(n-c)\,p_n = p_c / c\mu\,(1-\rho)^2$

Average waiting time in queue for those who acutely wait = $1/(c\mu - \lambda)$

Average number of items served = $\sum_{n=0}^{c-1} n\,p_n + \sum_{n=c}^{\infty} p_n$

Average number of idle channels = c – Average number of items served

Efficiency of M/M/c model: = (Average number of items served) / (Total number of channels)

Utilization factor = $\rho = (\lambda / c\mu)$

Problem 9.18.

A super market has two girls ringing up sales at the counters. If the service time from each customer is exponential with a mean of 4 minutes, and if people arrive in a Poisson fashion at the rate of 10 an hour, find (*a*) what is the probability of having an arrival has to wait for service?

(*b*) What is the expected percentage of idle time for each girl?

Solution

Data: Model: M/M/c model, $c = 2$, $\mu = ¼$ services per minute, $\lambda = 1 / (60 / 10) = 1 / 6$ per minute. $\rho = (\lambda / c\mu) = (1 / 6) / 2 \times (1 / 4) = (1 / 3)$

$$p_0 = 1/\left[\sum_{n=0}^{c-1} (c\rho)^n / n!\right] + [(c\rho)^c / c!(1-\rho)]$$

$$= 1/\left[\sum_{n=0}^{2-1} (2\rho)^n / n!\right] + [(2\rho)^c / 2!(1-\rho)] = 1/(1+2\rho) + [4\rho^2 / 2\,(1-\rho)]$$

$= 1 / 1 + (2 / 3) + (1 / 2\,!)\,(2 / 3)^2 \times 1 / [1 - (1 / 3)]$ (because $\rho = 1/3$

$= 1/2$

$$p_1 = (\lambda / \mu) \times p_0 = (2 / 3) \times (1 / 2) = 1/3$$

The probability that a customer has to wait = The probability that number of customers in the system is greater than or equal to 2. $= p\,(n \geq 2) = 1 - p\,(n < 2) = 1 - p_0 - p_1 = 1\,(1/2) - (1 / 3) = 0.167$

Expected percent of idle time of girls or expected number of girls who are idle =

Let X denotes number of idle girls. $X = 2$ when the system is empty and both girls are free. X=1 when the system contains only one unit and one of the girls is free. Hence X can take two values 2 or 1.Probability p_0 and p_1 respectively.

$E\,(X) = X_1 p\,(X = X_1) + X_2\,p\,(X = X_2) = (2 \times p_0) + (1 \times p_1) = (2 \times 1 / 2) + (1 \times 1 / 3) = (4 / 3)$

Probability of any girl being idle = (Expected number of idle girls) / (Total number of girls) = $(4 / 3) / 2 = 0.67$.

Expected percentage of idle time of each girl is 67%.

Problem 9.19.

A tax-consulting firm has 3 counters in its office to receive people who have problems concerning their income, wealth and sales taxes. On the average 48 persons arrive in an 8- hour day. Each tax adviser spends 15 minutes on the average on an arrival. If the arrivals are Poisson distributed and service times are according to exponential distribution find:

(a) The average number of customers in the system,

(b) Average number of customers waiting to be serviced,

(c) Average time a customer spends in the system

(d) Average waiting time for a customer,

(e) The number of hours each week a tax adviser spends performing his job,

(*f*) The probability that a customer has to wait before he gets service,

(*g*) The expected number of idle tax advisers at any specified time.

Solution

Data: $c = 3$, $\lambda = 48 / 8 = 6$ customers per hour, $\mu = (1 / 15) \times 60 = 4$ customers per hour, $(\lambda / \mu) = (6 / 4) = (3 / 2)$.

$$p_0 = 1/\sum_{n=0}^{c-1}[(\lambda/\mu)^n/n!]+[(\lambda/\mu)^c/c!]\times[(c\mu/c\mu-\lambda)]$$

$$p_0 = 1/\sum_{n=0}^{2}[(\lambda/\mu)^n/n!]+[(\lambda/\mu)^3/3!]\times[(3\mu/3\mu-\lambda)]$$

$1/[1 + (\lambda / \mu) + (1/ 2)(\lambda / \mu)^2] + [(\lambda / \mu)^3 / 6] \times (3\mu) /(3\mu - \lambda)$.

$1 / [(1) + (3 / 2) + (9 / 6)] + (27 / 48) \times 12 / (12 - 6) = 1 / [(29 / 8(+ (9 / 8)] = 8 / 38 = 0.21 = 21\%$

(a) Average number of customers in the system $= E (n) =$

$= \{[\lambda.\mu.(\lambda / \mu)^c] /[(c -1)!(c\mu - \lambda)^2]\}p_0 + (\lambda + \mu)$

$[6 \times 4 \times (3 / 2)^3] / [2! (12 - 6)^2] \times 0.21 + (3 / 2) = 1.74$ customers *i.e.* approximately 2 customers.

(b) Average number of customers waiting to be served $= E (L) = E (n) + (\lambda / \mu)$

$= 1.74 + (3 / 2) = 0.24$ customers. *i.e.* one customer approximately.

(c) Average time a customer spends in the system $= E (n) = E (L) / \lambda = 1.74 / 6 = 0.29$ hours $= 17.4$ minutes.

(d) Average waiting time for a customer $= E (L) / \lambda = 0.24 / 6 = 0.04$ hours $= 2.4$ minutes.

(e) Utilization factor $= \rho = (\lambda / c\mu) = (6 / 3 \times 4) = 1 / 2 = 50$ % of the time.

Hence number of hours each day a tax adviser spends doing his job $= (1 / 2) \times 8 = 4$ hours.

(*f*) Probability that a customer has to wait $= p (n > c) =$

$= [\mu \times (\lambda / \mu)^c] p_0/(c -1)!(c\mu - \lambda)] = \{[4 \times (3 / 2)^3] / 2! \text{ X } (12 - 6)]\} \times 0.21 = 0.236.$

(*g*) When the probability of no customers waiting is p_0, all the tax advisers are idle. Now we have to find probability of one tax adviser and probability of two tax advisers are idle, which are represented as p_1 and p_2 respectively. Now we know that

$p_n = \{(\lambda / \mu)^n / n!\} \times p_0$ when $1 \le n \le c$. Hence

$p_1 = \{(3 / 2) / 1!\} \times 0.21 = 0.315$ and

$p_2 = \{(3 / 2) / 2!\} \times 0.21 = 0.236$.

Therefore, expected number of idle adviser at any specified time = $3p_0 + 2p_1 + 1p_2$ =
$3 \times 0.21 + 2 \times 0.315 + 1 \times 0.236 = 1.496$ *i.e.* approximately = 1.5

Problem 9.20.

A telephone exchange has two long distance operators. The telephone company finds that during the peak load, long distance calls arrive in a Poisson fashion at an average rate of 15 per hour. The length of service on these calls is approximately exponentially distributed with mean length 5 minutes. What is the probability that a subscriber will have to wait for his long distance call during the peak hours of the day? If subscribers wait and are serviced in turn, what is the expected waiting time.

Solution

Data: λ = 15 calls per hour, μ = 60 / 5 = 12 calls per hour. Therefore ρ = (15) / (2 × 12) = 5 / 8.p_0 =

$$1/\left[\sum_{n=0}^{c-1} (c\rho)^n / n!\right] + [(c\rho)^c / c!(1-\rho)] = 1 / 1 + (5 / 4) + (1 / 2) \times (25 / 16) \times [1 / (1 - 5 / 8)]$$

= (12 / 52)

(*a*) Probability that a subscriber has to wait = $p\,(n \ge 2) = 1 - p_0 - p_1$ =
[1 – (12 / 52) – (15 / 32)] = 25 / 52 – 0.48. *i.e.* 48 % of the time the subscriber has to wait.

Expected waiting time = $E\,(w) = [\rho / \lambda(1-\rho)^2] \times (1/ c!) \times (\lambda / \mu)^c \times p_c$
= $\{ (5 / 8) / 15 [1 - (5 / 8)]^2 \times (1 / 2!) \times (15 / 12)^2 \times (12 / 32)$ hours = 3.2 minutes.

Problem 9.21.

A bank has two tellers working on savings account. The first teller handles withdrawals only. The second teller handles deposits only. It has been found that the service time distribution for deposits and withdrawals both are exponentially with a mean service time of 3 minutes per customer. Depositors are found to arrive in a Poisson fashion throughout the day with a mean arrival rate of 16 per hour. Withdrawals also arrive in a Poisson fashion with a mean arrival rate of 14 per hour. What would be the effect on the average waiting time for depositors and withdrawers, if each teller could handle both withdrawals and deposits? What would be the effect of this could only be accomplished by increasing the services time to 3.5 minutes?

Solution

Data: Mean service rate for both tellers = μ = (1 / 3) customers per hour, Mean arrival rate of depositors = λ 1 = 16 customers per hour, Rate of arrival of withdrawals = λ_2 = 14 withdrawals per hour.

First let us consider that both depositors and withdrawers under $M / M / 1$ system with one teller attending depositors and the other attending withdrawals, we get

Expected waiting time for depositors = $E(w_1) = \lambda_1 / \mu(\mu - \lambda_1)$ = 16 / 20 (20 – 16) = (1 / 5) hours = 12 minutes.

Expected waiting time for withdrawals = $E(w_2) = \lambda_2 / \mu(\mu - \lambda_2)$ = 14 / 20 (20 – 14) = 0.117 hours = 7 minutes.

If both tellers do service for withdrawals and deposits, then the problem becomes that the two service stations with $\lambda' = \lambda_1 + \lambda_2$ = 16 + 14 = 30 customers per hour. Here as usual μ = 20 per hour, and c = 2.

$$p_0 = 1 / \sum_{n=0}^{c-1} [(\lambda/\mu)^n / n!] + [(\lambda/\mu)^c / c!] \times [(c\mu / c\mu - \lambda)]$$

$$\left\{ \sum_{n=0}^{1} (1/n!)(3/2)^n + (1/2!)(3/2)^2 \, (40/40-30) \right\}^{-1}$$

$$= [\ (1/0)\ (3/2)^0 + (1/1)\ (3/2\ )^1 + (1/2 \times 1)\ (9\ /\ 4) \times 40/30]^{-1}$$

$$[1 + (3\ /2) + (9\ /\ 2)]^{-1} = (1\ /\ 7)$$

$$\rho = 30\ /\ 2 \times 20 = (3\ /\ 4) = 0.75.$$

$$E(w) = E(L) / \lambda = [1/\ (c - 1)! \times (\lambda\ /\ \mu)^c \times [\mu\ /\ (c\mu - \lambda)^2 \times p_0$$

$$= [1\ /\ (2 - 1)! \times (3\ /\ 2)^2 \times [20\ /\ (40 - 30)^2 \times (1/7) = (9/4) \times (20/100) \times (1/7) = 9/140 \text{ hours}$$

= 3.86 minutes.

Combined waiting time with increased service time when λ' = 30 per hour and

$1/\mu'$ = 3.5 minutes or μ' = 60 / 3.5 = 120 / 7 hours and $\rho' = \lambda' / c\mu'$ = 30 / 2 (120 / 7) = 7 /8 which is less than 1, and λ' / μ' = 30 / (120 / 7) = (7 / 4) which is greater than 1.

$$p_0 = \{(1\ /\ n!) \times (7\ /\ 4)^n + (1\ /\ 2!) \times (7\ /\ 4)^2 \times [2 \times (120\ /\ 7)]\ /[2 \times (120\ /\ 7) - 30]\}^{-1}$$

$$= (1\ /\ 0!)\ (7\ /\ 9)^0 + (1\ /\ 1!)\ (7\ /\ 4)^1 + (1\ /\ 2 \times 1) \times (49\ /\ 16) \times [2 \times (120\ /\ 7)]\ /\ (30\ /\ 7)]\}^{-1}$$

$$= [1 + (7\ /\ 4) + (49\ /\ 4)]^{-1} = (1\ /\ 15)$$

Average waiting time of arrivals in the queue =

$$E(w) = [1\ /\ (c - 1)!] \times (\lambda'\ /\ \mu')^c \times [\mu'\ /\ (c\mu' - \lambda'\ )^2 \times p_0$$

$$1\ /\ (2 - 1)! \times (7\ /\ 4)^2\ [(120/7)]\ /\ [2 \times (120\ /\ 7) - 30]^2 \times (1/15) = (343)\ /\ (30 \times 60) =$$

11.433 Minutes.

Problem 9.22.

Ships arrive at a port at the rate of one in every 4 hours with exponential distribution of inter arrival times. The time a ship occupies a berth for unloading has exponential distribution with an average of 10 hours. If the average delay of ships waiting for berths is to be kept below 14 hours, how many berths should be provided at the port?

Solution

Data: λ = ¼ ships per hour, μ = 1 / 10 ships per hour, λ / μ = 5 / 2. For multi channel queuing system $(\lambda / c\mu) < 1$, to ensure that the queue does not explode. Therefore, (1 / 4) / (1 / 10) c < 1 or c = 5 / 2. Let us consider c = 3 and calculate waiting time.

$$p_0 = 1/\sum_{n-0}^{c-1}[(\lambda/\mu)^n/n!]+[(\lambda/\mu)^c/c!]\times[(c\mu/c\mu-\lambda)]$$

1 / [1 + (5 / 2) + (1 / 2) × (5 / 2)^2] + (125 / 6 × 8) × (3 / 10) × (20 / 1) = 1 / [6.625 + 15.625] = 0.045

Average waiting time for ship = $E(w)$ = $\{[\mu \times (\lambda / \mu)^c]/[(c-1)! \times (c\mu - \lambda)^2]\} \times p_0$

= {[(1 / 10) × (5 / 2)^3] / 2! × [(3 / 10) – (1 / 4)]^2} × 0.045 = 14.06 hours, this is greater than 14 hours. Therefore three berths are sufficient. Let us take c = 4, then

$$p_0 = 1/\sum_{n=0}^{3}[(5/2)^2/n!] + [(5/2)^4/4] \times \{(3/10)/[(3/10)-(1/4)]\}$$

1 / [1 + (5 / 2) + (1 / 2) × (5 / 2)^2 + (1 / 6) (5 / 2)^3] + 625 / (24 × 16) × (3 / 10) × (20 / 1)

= 1 / (9.23 + 9.765) = 0.0526 = 13.7 hours. This is less than the allowable time of 14 hours. Hence 4 berths must be provided at the port.

9.10. (M / E_k / 1): (First Come First Served) / ∝ / ∝ : ONE UNIT SERVED IN MULTI PHASES / FIRST COME FIRST SERVED / INFINITE CAPACITY: (System with Poisson input, Erlangian service time with k phases single channel, infinite capacity and first in first out discipline.)

We assume that Arrival of one unit means addition of 'k' phases in the system and Departure of one unit implies reduction of 'k' phases in the system.

1. $\lambda_n = \lambda$ and $\mu_n = \mu^k$
2. k = number of phases.
3. System length = $E(n) = [(k + 1) / 2k] \times (\lambda / \mu) \times [\lambda / (\mu - \lambda)] + (\lambda / \mu)$
4. Length of the queue = $E(w) = [(k + 1) / 2k] \times (\lambda / \mu) \times [\lambda / (\mu - \lambda)]$
5. Waiting time in the system = $E(v) = [(k = 1) / 2k] \times [\lambda / \mu(\mu - \lambda)] + (1/\mu)$
6. Waiting time in the queue = $E(w) = [(k + 1) / 2k] \times [\lambda / \mu(\mu - \lambda)]$

For constant service time equating 'k' to ∝ , we get:

$E(n)$ System length = (1 / 2) $(\lambda/\mu)[\lambda/(\mu-\lambda)]+(\lambda/\mu)$

$E(w)$ Length of the queue = (1 / 2) $(\lambda/\mu)[\lambda/(\mu - \lambda)]$

Waiting time in the system = $E(v)$ = (1 / 2) $[\lambda/\mu(\mu - \lambda)] + (1/\mu)$

Waiting time in the queue = $E(w)$ = (1 / 2) $[\lambda / \mu(\mu - \lambda)]$. When k = 1 Erlang service time distribution reduces to exponential distribution.

Problem 9.23.

Repairing a certain type of machine, which breaks down in a given factory, consists of 5 basic steps that must be performed sequentially. The time taken to perform each of the 5 steps is found to have an exponential distribution with a mean of 5 minute and is independent of the other steps. If these machines breakdown in Poisson fashion at an average rate of two per hour and if there is only one repairman, what is the average idle time for each machine that has broken down?

Solution

Data: Number of phases = k = 5, Service time per phase = 5 minutes, λ = 2 units per hour,
Service time per unit = 5 × 5 = 25 minutes, hence μ = 1 / 25 minutes per minute.
Average idle time of the machine = $E(v) = [(k + 1) / 2k] \times (\lambda / \mu) \times [1/ (\mu - \lambda) + (1/ \mu)$
= 9(5 + 1) / (2 × 5) × (2 × 5) / 12] × 1 / [(12 / 5) – 2] + (5 / 12)
= (1 / 2) × (5 / 2) + (5 / 12) = (20 / 12 = 5 / 3 hours = 100 minutes.

Problem 9.24.

A colliery working one shift per day uses a large number of locomotives which breakdown at random intervals, on the average one failing per 8 - hour shift. The fitter carries out a standard maintenance schedule on each faulty locomotive. Each of the five main parts of this schedule takes an average of 1 / 2 an hour but the time varies widely. How much time will the fitter have for other tasks and what is the average time a locomotive is out of service.

Solution

Data: k = 5, λ = 1/8 per hour, Service time per part = 1 / 2 an hour.
Service time per locomotive = 5 / 2 hours. Hence μ = 2 / 5 hours.
Fraction of time the fitter will have for other tasks = Fraction of time for which the fitter is idle =
1 / (λ / μ) = 1 – [(1 / 8) / (2 / 5)] = 1 – (5 / 16) = 11 / 16.
Therefore, time the fitter will have for other tasks in a day = (11 / 16) × 8 = 5.5 hours
Average time a locomotive is out of service = Average time spent by the locomotive in the system =
$[(k + 1) / 2k] \times (\lambda / \mu) \times [1/ (\mu - \lambda)] + (1/ \mu)$ =
[(5 + 1) / (2 × 5)] × [(1 / 8) / (2 / 5)] × {1 / [(2 / 5) – (1 / 8)] + (5 / 2) =
(6 / 10) × (5 / 16) × (40 / 11) + (5 / 2) = (15 / 22) + (5 / 2) = 70 / 22 = 3.18 hours.

QUESTIONS

1. Explain with suitable examples about the queue. Why do you consider the study of waiting line as an important aspect?
2. Explain with suitable examples about Poisson arrival pattern and exponential service pattern.
3. Explain the various types of queues by means of a sketch and also give the situations for which each is suitable.
4. Customers arrive at one window drive in a bank according to a Poisson distribution with a mean of 10 per hour. Service time per customer is exponential with a mean of 5 minutes. The space in front of the window, including that for the serviced car can accommodate a maximum, of three cars. Other cars can wait outside the space.
   (a) What is the probability that an arriving customer can drive directly to the space in front of the window?
   (b) What is the probability that an arriving customer will have to wait outside the indicated space?

(c) How long an arriving customer is expected to wait before starting service?

(d) How much space should be provided in front of the window so that all the arriving customers can wait in front of the window at least 90 percent of the time?

5. A barber with a one-man shop takes exactly 25 minutes to complete one hair cut. If customers arrive in a Poisson fashion at an average rate of every 40 minutes, how long on the average must a customer wait for service?

6. At a public telephone booth in a post office arrivals are considered to be Poisson with an average inter-arrival time of 12 minutes. The length of phone call may be assumed to be distributed exponentially with an average of 4 minutes. Calculate the following:

(a) What is the probability that a fresh arrival will not have to wait for phone?

(b) What is the probability that an arrival will have to wait more than 10 minutes before the phone is free?

(c) What is the average length of queues that form from time to time?

(d) What is the fraction of time is the phone busy?

(e) What is the probability that an arrival that goes to the post office to make a phone call will take less than 15 minutes to complete his job?

(f) The telephone company will install a second booth when convinced that an arrival would expect to have to wait at least 5 minutes for the phone?

7. At what average rate must a clerk at a super market work in order to ensure a probability of 0.90 that the customer will not wait longer than 12 minutes? It is assumed that there is only one counter at which customer arrive in a Poisson fashion at an average rate of 15 per hour. The length of service by the clerk has an exponential distribution.

9. Consider a self-service store with one cashier; assume Poisson arrivals and exponential service times. Suppose that nine customers arrive on the average every 5 minutes and the cashier can serve 10 in 5 minutes. Find:

(a) The average number of customers queuing for service, (*b*) The probability of having more than 10 customers in the system, (*c*) The probability that a customer has to queue for more than 2 minutes.

If the service can be speeded up to 12 in 5 minutes, by using a different cash register, what will be the effect on the quantities of (*a*), (*b*) and (*c*) above?

10. The mean rate of arrival of planes at an airport during the peak period is 20 per hour, but the actual number of arrivals in an hour follows the Poisson distribution. The airport can land 60 planes per hour on an average in good weather, or 30 per hour in bad weather, but the actual number landed in any hour follows a Poisson distribution with the respective averages. When there is congestion, the planes are forced to fly over the field in the stock awaiting the landing of other planes that arrived earlier.

(a) How many planes would be flying over the field in the stack on an average in good weather and in bad weather?

(b) How long a plane would be in the stack and the process of landing in good and bad weather?

(c) How much stack and landing time to allow so that priority to land out of order would have to be requested only one time in twenty.

11. Customers arrive at a booking office window, being manned by a single individual at a rate of 25 per hour. Time required to serve a customer has exponential distribution with a mean of 120 seconds. Find the average time of a customer.
12. A repair shop attended by a single machine has average of four customers an hour who bring small appliances for repair. The mechanic inspects them for defects and quite often can fix them right away or otherwise render a diagnosis. This takes him six minutes, on the average. Arrivals are Poisson and service time has the exponential distribution. You are required to:
    (a) Find the proportion of time during which the shop is empty.
    (b) Find the probability of finding at least one customer in the shop?
    (c) What is the average number of customers in the system?
    (d) Find the average time spent, including service.
13. The belt snapping for conveyors in an open cast mine occur at the rate of 2 per shift. There is only one hot plate available for vulcanizing; and it can vulcanize on an average 5 belts snap per shift.
    (a) What is the probability that when a belt snaps, the hot plate is readily available?
    (b) What is the average number in the system?
    (c) What is waiting time of an arrival?
    (d) What is the average waiting time plus vulcanizing time?
14. A repairman is to be hired to repair machines which breakdown at an average rate of 6 per hour. The breakdown follows Poisson distribution. The productive time of a machine considered costing Rs. 20/- per hour. Two repairmen, Mr. X and Mr. Y have been interviewed for this purpose. Mr. X charges Rs. 10/- per hour and he services breakdown machines at the rate of 8 per hour. Mr. Y demands Rs. 14/- per hour and he services on an average rate of 12 per hour. Which repairman should be hired? Assume 8- hour shift per day.
15. A super market has two girls ringing up sales at counters. If the service time for each customer is exponential with mean of 4 minutes, and if people arrive in a Poisson fashion at the rate of 10 per hour. Find
    (a) What is the probability of having to wait for service?
    (b) What is the expected percentage of idle time for each girl?
    (c) If a customer has to wait, what is the expected length of waiting time?
16. Given an arrival rate of 20 per hour, is it better for a customer to get service at a single channel with mean service rare of 22 customers or at one of two channels in parallel, with mean service rate of 11 customers for each of the two channels? Assume that both queues are of M/M/S type.
17. In machine maintenance, a mechanic repairs four machines. The mean time between service requirement is 5 hours for each machine and forms an exponential distribution. The men repair time is one hour and also follows the same distribution pattern. Machine down time cost Rs. 25/- per hour and the mechanic costs Rs 55/- per day of 8 hours.
    (a) Find the expected number of operating machines.
    (b) Determine expected down time cost per day
    (c) Would it be economical to engage two mechanics each repairing two machines?

18. Four counters are being run on the frontier of a country to check the passports and necessary papers of the tourists. The tourists choose a counter at random. If the arrivals at the frontier is Poisson at the rate λ and the service is exponential with parameter μ, what is the steady state average queue at each counter?.
19. In a huge workshop tools are store in a tool crib. Mechanics arrive at the tool crib for taking the tools and lend them back after they have used them. It is found that the average time between arrivals of mechanics at the crib is 35 seconds. A clerk at the crib has been found to take on an average 50 seconds to serve a mechanic (either hand him the tools if he requests them or receive tools if he is returning the tools). If the labour cost of a clerk is Re. 1/- per hour and that of a mechanic is Rs. 2.50 per hour, find out how many clerks should be appointed at the tool crib to minimize the total cost of mechanic's waiting time plus clerk's idletime.
20. A barber runs his own saloon. It takes him exactly 25 minutes to complete on haircut. Customers arrive in a Poisson fashion at an average rate of one every 35 minutes.
    (a) For what percent of time would the barber be idle?
    (b) What is the average time of a customer spent in the shop?

MULTIPLE CHOICE QUESTIONS

1. As per queue discipline the following is not a negative behaviour of a customer:
   (*a*) Balking (*b*) Reneging
   (*c*) Boarding (*d*) Collusion. ( )
2. The expediting or follow up function in production control is an example of
   (*a*) LIFO (*b*) FIFO
   (*c*) SIRO (*d*) Pre emptive. ( )
3. In M/M/S: N/FIFO the following does not apply
   (*a*) Poisson arrival (*b*) Limited service
   (*c*) Exponential service (*d*) Single server ( )
4. The dead bodies coming to a burial ground is an example of:
   (*a*) Pure Birth Process (*b*) Pure death Process
   (*c*) Birth and Death Process (*d*) Constant rate of arrival ( )
5. The system of loading and unloading of goods usually follows:
   (*a*) LIFO (*b*) FIFO
   (*c*) SIRO (*d*) SBP ( )
6. A steady state exist in a queue if:
   (*a*) $\lambda > \mu$ (*b*) $\lambda < \mu$
   (*c*) $\lambda \leq \mu$ (*d*) $\lambda \geq \mu$ ( )
7. If the operating characteristics of a queue are dependent on time, then is said to be:
   (*a*) Transient state, (*b*) Busy state
   (*c*) Steady state (*d*) Explosive state. ( )

8. A person who leaves the queue by losing his patience to wait is said to be: ( )
   (a) Reneging (*b*) Balking
   (*c*) Jockeying (*d*) Collusion. ( )
9. The characteristics of a queuing model is independent of:
   (a) Number of service stations (*b*) Limit of length of queue
   (*c*) Service Pattern (*d*) Queue discipline. ( )
10. The unit of traffic intensity is:
   (a) Poisson (*b*) Markow
   (*c*) Erlang (*d*) Kendall ( )
11. In (M /M /1) : (∞ / FCFS) model, the length of the system L_s is given by:
   (*a*) $\rho^2/1/\rho$ (*b*) $\rho/1-\rho$
   (*c*) $\lambda^2/(\mu-\lambda)$ (*d*) $\lambda^2/\mu(\mu-\lambda)$
12. In (M / M / 1): (∞ / FIFO) model, $1/(\mu-\lambda)$ represents:
   (a) L_s, Length of the system (*b*) L_q length of the queue
   (*c*) W_q Waiting time in queue (*d*) W_s Waiting time in system. ( )
13. The queue discipline in stack of plates is:
   (a) SIRO (*b*) Non-Pre-Emptive
   (*c*) FIFO (*d*) LIFO ( )
14. Office filing system follows:
   (a) LIFO (*b*) FIFO
   (*c*) SIRO (*d*) SBP ( )
15. SIRO discipline is generally found in:
   (a) Loading and unloading (*b*) Office filing
   (*c*) Lottery draw (*d*) Train arrivals at platform. ( )
16. The designation of Poisson arrival, Exponential service, single server and limited queue selected randomly is represented by:
   (*a*) (M / E / S) : (∞ / SIRO) (*b*) (M / M / 1) : (∞ / SIRO)
   (*c*) (M / M / S) : (N / SIRO) (*d*) (M / M / 1) : (N / SIRO) ( )
17. For a simple queue (M / M / 1), $\rho = \lambda / \mu$ is known as:
   (a) Poisson busy period, (*b*) Random factor,
   (*c*) Traffic intensity (*d*) Exponential service factor. ( )
18. With respect to simple queuing model which on of the given below is wrong:
   (a) $L_q = \lambda\ W_q$ (*b*) $\lambda = \mu\ \rho$
   (*c*) $W_s = W_q + \mu$ (*d*) $L_s = L_q + \rho$ ( )
19. When a doctor attends to an emergency case leaving his regular service is called:
   (a) Reneging (*b*) Balking
   (*c*) Pre-emptive queue discipline (*d*) Non-Pre-Emptive queue discipline ( )

20. A service system, where customer is stationary and server is moving is found with:
    (a) Buffet Meals, (*b*) Out patient at a clinic
    (c) Person attending the breakdowns of heavy machines
    (d) Vehicle at Petrol bunk. ( )
21. In a simple queuing model the waiting time in the system is given by:
    (*a*) $(L_q - \lambda) + (1/\mu)$ (*b*) $1/(\mu - \lambda)$
    (*c*) $\mu/(\mu - \lambda)$ (*d*) $W_q + \mu$ ( )
22. This department is responsible for the development of queuing theory:
    (a) Railway station, (*b*) Municipal office
    (*c*) Telephone department (*d*) Health department.
23. If the number of arrivals during a given time period is independent of the number of arrivals that have already occurred prior to the beginning of time interval, then the new arrivals follow -----------distribution.
    (a) Erlang (*b*) Poisson
    (*c*) Exponential (*d*) Normal ( )
24. Arrival $\rightarrow$ Service $\rightarrow$ Service $\rightarrow$ Service $\rightarrow$ Out $\rightarrow$
    The figure given represents:
    (a) Single Channel Single Phase system
    (b) Multi channel single-phase system
    (c) Single channel multi phase system
    (d) Multi channel multi phase system. ( )
25. In queue designation $A/B/S$: (d/f), what does S represents:
    (a) Arrival Pattern (*b*) Service Pattern
    (*c*) Number of service channels, (*d*) Capacity of the system ( )
26. When the operating characteristics of the queue system dependent on time, then it is said to be:
    (a) Steady state (*b*) Explosive state
    (*c*) Transient state (*d*) Any one of the above ( )
27. The distribution of arrivals in a queuing system can be considered as a:
    (a) Death Process (*b*) Pure Birth Process
    (*c*) Pure live process (*d*) Sick process ( )
28. Queuing models measure the effect of:
    (a) Random arrivals (*b*) Random service
    (c) Effect of uncertainty on the behaviour of the queuing system
    (d) Length of queue. ( )
29. Traffic intensity is given by:
    (a) Mean arrival rate/Mean service rate,
    (b) (*b*) $\lambda \times \mu$ (*c*) μ / λ
    (*d*) Number present in the queue / Number served ( )

30. Variance of queue length is:

(*a*) $\rho = \lambda / \mu$ (*b*) $\rho^2 / 1 - \rho$

(*c*) $\lambda / \mu - \lambda$ (*d*) $\rho/(1-\rho)^2$ ( )

ANSWERS

1. (*c*)	2. (*d*)	3. (*d*)	4. (*a*)	5. (*a*)
6. (*c*)	7. (*a*)	8. (*a*)	9. (*d*)	10. (*c*)
11. (*b*)	12. (*c*)	13. (*d*)	14. (*a*)	15. (*c*)
16. (*d*)	17. (*c*)	18. (*c*)	19. (*d*)	20. (*c*
21. (*a*)	22. (*c*)	23. (*b*)	24. (*c*)	25. (*a*)
26. (*c*)	27. (*b*)	28. (*c*)	29. (*a*)	30. (*d*)

CHAPTER – 10

Theory of Games or Competitive Stratagies

INTRODUCTION

In previous chapters like Linear Programming, Waiting line model, Sequencing problem and Replacement model etc., we have seen the problems related to individual industrial concern and problems are solved to find out the decision variables which satisfy the objective of the industrial unit. But there are certain problems where two or more industrial units are involved in decision making under *conflict situation*. This means that decision-making is done to maximize the benefits and minimize the losses. The decision-making much depends on the decision made or decision variables chosen by the opponent business organization. Such situations are known as **competitive strategies**. Competitive strategies are a type of **business games**. When we here the word game, we get to our mind like pleasure giving games like Foot ball, Badminton, Chess, etc., In these games we have two parties or groups playing the game with definite well defined rules and regulations. The out come of the game as decided decides winning of a group earlier. In our discussion in Theory of Games, we are not concerned with pleasure giving games but we are concerned with **business games**. What is a business game?

Every business manager is interested in capturing the larger share in the market. To do this they have to use different strategies (course of action) to motivate the consumers to prefer their product. For example you might have seen in newspapers certain company is advertising for its product by giving a number of (say 10) eyes and names of 10 cine stars and identify the eyes of the stars and match the name with the eyes. After doing this the reader has to write why he likes the product of the company. For right entry they get a prize. This way they motivate the readers to prefer the product of the company. When the opponent company sees this, they also use similar strategy to motivate the potential market to prefer the product of their company. Like this the companies advertise in series and measure the growth in their market share. This type of game is known as **business game**. Managers competing for share of the market, army chief planning or execution of war, union leaders and management involved in collective bargaining uses different strategies to fulfill their objective or to win over the opponent. All these are known as business games or **competitive situation**. In business, competitive situations arise in advertising and marketing campaigns by competing business firms.

Hence, **Game theory is a body of knowledge that deals with making decisions when two or more intelligent and rational opponents are involved under conditions of conflict or competition. The competitors in the game are called *players.***

The beginning of **theory of games** goes back to 20 th century. But **John Von Neumann and Morgenstern** have mathematically dealt the theory and published a well-known paper **"theory of Games and Economic Behavior"** in 1944. The mathematical approach of Von Neumann utilizes the

Minimax principle, which involves the fundamental idea of **minimization of the maximum losses**. Many of the competitive problems can be handled by the game theory but not all the competitive problems can be analyzed with the game theory. Before we go to game theory, it is better for us to discuss briefly about decision-making.

DECISION MAKING

Making decision is an integral and continuous aspect of human life. For child or adult, man or woman, government official or business executive, worker or supervisor, participation in the process of decision-making is a common feature of everyday life. What does this process of decision making involve? What is a decision? How can we analyze and systematize the solving of certain types of decision problems? Answers of all such question are the subject matter of **decision theory**. Decision-making involves listing the various alternatives and evaluating them economically and select best among them. Two important stages in decision-making is: (*i*) making the decision and (*ii*) Implementation of the decision.

Analytical approach to decision making classifies decisions according to the amount and nature of the available information, which is to be fed as input data for a particular decision problems. Since future implementations are integral part of decision-making, available information is classified according to the degree of certainty or uncertainty expected in a particular future situation. With this criterion in mind, three types of decisions can be identified. First one is that these decisions are made when **future can be predicted with certainty.** In this case the decision maker assumes that there is only one possible future in conjunction with a particular course of action. The second one is that decision making under **conditions of risk**. In this case, the future can bring more than one state of affairs in conjunction with a specific course of action. The third one is decision making under **uncertainty**. In this case a particular course of action may face different possible futures, but the probability of such occurrence cannot be estimated objectively.

The Game theory models differ from **decision-making under certainty (DMUC)** and **decision-making under risk (DMUR)** models in two respects. First the opponent the decision maker in a game theory model is an active and rational opponent in DMUC and DMUR models the opponent is the passive state of nature. Second point of importance is decision criterion in game model is the **maximin** or the **minimax** criterion. In DMUC and DMUR models the criterion is the maximization or minimization of some measure of effectiveness such as profit or cost.

DESCRIPTION OF A GAME

In our day-to-day life we see many games like Chess, Poker, Football, Baseball etc. All these games are pleasure-giving games, which have the character of a competition and are played according to well-structured rules and regulations and end in a **victory** of one or the other team or group or a player. But we refer to the word **game** in this chapter the competition between two business organizations, which has more earning competitive situations. In this chapter game is described as:

A competitive situation is called a game if it has the following characteristics (Assumption made to define a game):

1. There is finite number of competitors called **Players**. This is to say that the game is played by two or more number of business houses. The game may be for creating new market, or to increase the market share or to increase the competitiveness of the product.

2. A list of finite or infinite number of possible **courses of action is available** to each player. The list need not be the same for each player. Such a game is said to be in **normal form**. To explain this we can consider two business houses A and B. Suppose the player A has three strategies, as strategy I is to offer a car for the customer who is selected through advertising campaign. Strategy II may be a house at Ooty for the winning customer, and strategy III may a cash prize of Rs. 10,00,000 for the winning customer. This means to say that the competitor A has three strategies or courses of action. Similarly, the player B may have two strategies, for example strategy I is A pleasure trip to America for 10 days and strategy II may be offer to spend with a cricket star for two days. In this game A has three courses of action and B has two courses of actions. The game can be represented by mans of a matrix as shown below:

		B	
		I	II
A	I		
	II		
	III		

3. A play is played when each player chooses one of his courses o action. The choices are made simultaneously, so that no player knows his opponent's choice until he has decided his own course of action. But in real world, a player makes the choices after the opponent has announced his course of action.

Every play *i.e.* combination of courses of action is associated with an out come, known as ***pay off -*** (generally money or some other quantitative measure for the satisfaction) which determines a set of gains, one to each player. Here **a loss is considered to be negative gain**. Thus after each playoff the game, one player pays to other an amount determined by the courses of action chosen. For example consider the following matrix:

		B		
		I	II	III
A	I	2	4	–3
	II	–1	2	2

In the given matrix, we have two players. Among these the player who is named on the left side matrix is known as winner, *i.e.* here A is the winner and the matrix given is the matrix of the winner. The player named above is known as the loser. The loser's matrix is the negative version of the given matrix. In the above matrix, which is the matrix of A, a winner, we can describe as follows. If A selects first strategy, and *B* selects the second strategy, the out come is +4 *i.e. A* will get 4 units of money and *B* loses 4 units of money. *i.e. B* has to give 4 units of money to *A*. Suppose *A* selects second strategy and *B* selects first strategy A's out come is –1, *i.e. A* loses one unit of money and he has to give that to *B*, it means *B* wins one unit of money.

4. All players act rationally and intelligently.
5. Each player is interested in **maximizing his gains or minimizing his losses**. The winner, *i.e*. the player on the left side of the matrix always tries to maximize his gains and is known as **Maximin player**. He is interested in maximizing his minimum gains. Similarly, the player B, who is at the top of the matrix, a loser always tries to minimize his losses and is known as **Minimax player -** *i.e*. who tries to minimize his maximum losses.
6. Each player makes individual decisions without direct communication between the players. By principle we assume that the player play a strategy individually, without knowing opponent's strategy. But in real world situations, the player play strategy after knowing the opponent's choice to maximin or minimax his returns.
7. It is assumed that each player knows complete relevant information.

Game theory models can be classified in a number of ways, depending on such factors as the:

(i) Number of players,

(ii) Algebraic sum of gains and losses

(iii) Number of strategies of each player, which decides the size of matrix.

Number of players: If number of players is two it is known as **Two-person game**. If the number of players is is '*n*' (where $n \geq 3$) it is known as ***n*- person game**. In real world two person games are more popular. If the number of players is '*n*', it has to be reduced to two person game by two constant collations, and then we have to solve the game, this is because, the method of solving n- person games are not yet fully developed.

Algebraic sum of gains and losses: A game in which the gains of one player are the losses of other player or the algebraic sum of gains of both players is equal to zero, the game is known as **Zero sum game (ZSG)**. In a zero sum game the algebraic sum of the gains of all players after play is bound to be zero. i.e. If g_i as the pay of to a player in a n-person game, then the game will be a zero sum game if sum of all g_i is equal to zero.

In game theory, the resulting gains can easily be represented in the form of a matrix called **pay - off matrix** or **gain** matrix as discussed in S.No 3 above. A pay - off matrix is a table, which shows how payments should be made at end of a play or the game. Zero sum game is also known as **constant sum game**. Conversely, if the sum of gains and losses does not equal to zero, the game is a **nonzero -sum game.** A game where two persons are playing the game and the sum of gains and losses is equal to zero, the game is known as **Two-Person Zero-Sum Game (TPZSG).** A good example of two-person game is the game of chess. A good example of n- person game is the situation when several companies are engaged in an intensive advertising campaign to capture a larger share of the market.

BASIC ELEMENTS OF GAME THEROY

Let us consider a game by name **Two-finger morra,** where two players (persons) namely *A* and *B* play the game. *A* is the winner and *B* is the loser. The matrix shown below is the matrix of *A*, the winner. The elements of the matrix show the gains of *A*. Any positive element in the matrix shows the gain of *A* and the negative element in the matrix show the loss (negative gain) of *A*.

		(One finger) I	B	(Two finger) II
(One finger)	A I	2		–2
(Two fingers)	A II	–2		2

The game is as follows: Both the players *A* and *B* sit at a table and simultaneously raise their hand with **one** or **two** fingers open. In case the fingers shown by both the players is same, then *A* will gain Rs.2/-. In case the number of fingers shown is different (*i.e*. A shows one finger and *B* shows two fingers or vice versa) then *A* has to give *B* Rs. 2/- *i.e*. *A* is losing Rs.2/-. In the above matrix, strategy I refer to finger one and strategy II refers to two fingers. The above given matrix is the pay of matrix of A. **The negative entries in the matrix denote the payments from *A* to *B*.** The pay of matrix of *B* is the negative version of *A*'s pay of matrix; because in two person zeros sum game the gains of one player are the losses of the other player. Always we have to write the matrix of the winner, who is represented on the left side of the matrix. The winner is **the maximizing player, who wants to maximize his minimum gains. The loser is the minimizing player, who wants to minimize his maximum losses.**

Note the following and remember

1. **The numbers within the payoff matrix represent the *outcome* or the *payoffs* of the different *plays* or *strategies* of the game. The payoffs are stated in terms of a measure of effectiveness such as money, percent of market share or utility.**

   By convention, in a 2-person, zero-sum game, the positive numbers denote a gain to the row or maximizing player or winner, and loss to the *column* or *minimizing player or loser*. It is assumed that both players know the payoff matrix.
2. **A strategy is a course of action or a complete plan. It is assumed that a strategy cannot be upset by competitors or nature (chance). Each player may have any number of strategies. There is no pressure that both players must have same number of strategies.**
3. ***Rules of game* describe the framework within which player choose their strategies. An assumption made here that *player must choose their strategies simultaneously and that the game is repetitive*.**
4. **A strategy is said to be *dominant* if each payoff in the strategy is *superior* to each *corresponding* pay off of alternative strategy. For example, let us consider *A* (winner) has three strategies. The payoffs of first strategy are 2, 1, 6 and that of second strategy are -1, -2 and 3. The second strategy's outcomes are inferior to that of first strategy. Hence first strategy dominates or superior to that of second strategy. Similarly let us assume B (loser) has two strategies. The outcomes of first strategy 2, -1 and that of second strategy is 1 and -2. The payoffs of second strategy is better than that of first strategy, hence second strategy is superior and dominates the first strategy. The rule of dominance is used to reduce the size of the given matrix.**

5. **The *rule of game* refers to the expected outcome per play when both players follow their best or optimal strategies. A game is known as fair game if its value is zero, and unfair if its value is nonzero.**
6. **An *Optimal strategy* refers to the course of action, or complete plan, that leaves a player in the most preferred position regardless of the actions of his competitors. The meaning of the *most preferred position* is that any deviation from the optimal strategy, or plan, would result in decreased payoff.**
7. **The purpose of the game model is to identify the optimal strategy for each player. The conditions said in serial number 1 to 3 above, the practical value of game theory is rather limited. However the idea of decision-making under conditions of conflict (or cooperation) is at the core of managerial decision. Hence the concepts involved in game theory are very important for the following reasons.**
   - **It develops a framework for analyzing decision making in competitive (and sometimes in cooperative) situations. Such a framework is not available through any other analytical technique.**
   - **(i) It describes a systematic quantitative method (in two-person zero-sum games) that enables the competitors to select rational strategies for the attainment of their goals.**
   - **(ii) It describes and explains various phenomena in conflicting situations, such as bargaining and the formation of coalitions.**

THE TWO-PERSON, ZERO-SUM GAME: (Pure Strategy and Mixed Strategy games)

In our discussion, we discuss two types of Two-person, Zero-sum games. In one of the most preferred position for each player is achieved by adopting a **single strategy**. Hence this game is known as **pure-strategy game.** The second type requires the adoption by both players of **a mixture or a combination of** different strategies as opposed to a single strategy. Therefore this is termed as **mixed strategy game.**

In pure strategy game one knows, in advance of all plays that he will always choose only one particular course of action. **Thus pure strategy is a decision rule always to select the same course of action.** Every course of action is pure strategy.

A mixed strategy is that in which a player decides, in advance to choose on of his course of action in accordance with some fixed probability distribution. This in case of mixed strategy we associate probability to each course of action (each pure strategy). The pure strategies, which are used in mixed strategy game with non-zero probabilities, are termed as **supporting strategies**. Mathematically, a mixed strategy to any player is an ordered set of 'm' non-negative real numbers, which add to a sum unity (m is the number of pure strategies available to a player).

It is said above that in pure strategy game a player selects same strategy always, hence the opponent will know in advance the choice. But the superiority of mixed strategy game over pure strategy games is that the player is always kept guessing about the opponent's choice as innumerable combination of pure strategies one can adopt.

The purpose of the game theory is to determine the **best strategies** for each player on the basis of **maximin and minimax criterion of optimality. In this criterion a player lists his worst possible**

outcomes and then he chooses that strategy which corresponds to the best of those worst outcomes. The **value of the game** is the maxim guaranteed gain to player. The value is denoted by **'*v*'**. The game whose value $v = 0$ is known as zero sum **game** or **fair game**. Solving the game mean to find the best strategies for both the players and find the value of the game.

The game theory does not insist on how a game should he played, but only tells the procedure and principles by which the action should be selected. Hence, **the game theory is a decision theory useful in competitive situations.** *The fundamental theorem assures that there exists a solution and the value of a rectangular game in terms of mixed strategies*.

CHARACTERISTICS OR PROPERTIES OF A GAME

To classify the games, we must know the properties of the game. They are:

Number of persons or groups who are involved in playing the game

Number of strategies or courses of action each player or group have (they may be finite or infinite).

Type of course of action or strategy.

How much information about the past activities of other player is available to the players. It may be complete or partly or may be no information available.

The pay off may be such that the gains of some players may or may not be the direct losses of other players.

The players are independent in decision-making and they make the decision rationally.

THE MAXIMIN AND MINIMAX PRINCIPLES

To understand, the principles of **minimax** and **maximin** let us consider a pay of matrix of two players - Player *A*, the winner and Player *B*, the looser.

		B	
		I	II
	I	–3	–4
A	II	–2	3
	III	2	4

From the matrix above if *A* plays his first strategy, his worst outcome is –4, if he plays second strategy, his worst out come is -2 and if he plays his third strategy, his worst outcome (minimum gain) is +2. Out of all these strategy, for *A* the best strategy is third strategy. He can select the third strategy. But this outcome of +2 is possible when *B* selects his first strategy. But where is the guarantee that *B* will select first strategy. He may select his second strategy because where he has an out come of 4 (negative of A's outcome). Similarly, for *B* the worst outcome (minimum loss) if he selects his first strategy is 2 (i.e. negative version is –2, a loss of two units of money) and his worst out come if selects his second strategy is 4. Hence he selects the best among the two is first strategy. By doing so, *A* is sure of getting +2 units of money when he selects first strategy and *B* is sure of losing 2 units of money, which is minimum. Hence *A* will always selects his first strategy which will guarantee him

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minimum gain of +2 units of money and *B* will always selects his first strategy as he looses minimum amount i.e. 2 units of money. Here, *A* is ftnown **as Ma ximin player, as he is maximizing** his **minimum gains. And** *B* **is known as Minimax player** as **he is minimizing his maximum losses.**

Mathematically, if we denote the pay of matrix of a game by *[a,q*]p, then **minimax** for A and **maximin for** *B*

Max [min *a;*] and min [max a ;,] respectively.

Let Max [min *a ,q*] = *a pp* and Min [max ,a] = a *r*

Then a $q S the minimum element in the p th row, therefore, *a* is < *a pg* another element in the *p* th row. Similarly, *a ,s* is the maximum element in s th column. Hence, *a p$* < *a s*.

Combining these two we get, *a* p_s < *a* ,. or

Mix |min *a ;y* j < **Min |max a *,y***

Maximin of ,a is called the lower value of he game and is denoted by v and the minimax of,,a is called the upper value of the game and is denoted by v . The value of the game is always between v and *v* and

Satisfies the inequality: Max min for *A* < M in max for *B* OR *v* < *v* < v

lf row in in imum is equals to column maximum, then the element at the intersection of that row and column is known as the sadd le point and is the value of the game. In the matrix given above, thi: saddle point is

A fi and A's third strategy and fi 's first strategy are pure strategies. The element at the intersection *of A B* is the saddle point and the value of the game is 2. The answer for the problem is represented as shown below:

Optimal strategies: *A* (0,0, I) and *D* (1,0) and the value of the game *v* — 2.

Please remember that, the strategies used by both players are the pure strategies. This is the simplest type of galre. Here the solution is stable in the rninmax sense because in this case neither of the players can increase his gains after deviating from their optimal strategies. In case any player deviates from his optimal strategy,)ie will loose and his gains are reduced.

A game for which maxmin of winner is equals to ininmax of loser, is known as game with saddle point.

The game with optimal pure strategies is sometimes called as **strictly** determined. Here, *v_* — *v*

METHOD OF SOLVING THE GAME OR **STEPS** IN **SOLVING THE GAME**

Step 1. Find the saddle point. If the game has saddle point, the game is solved. Write the optimal strategies and the value of the game.

Step 2. If no saddle point, try to reduce the size of the matrix given ($m \times n$) to:

(a) 2×2 matrix, which has formula for optimal strategies and the value of the game. Use the formula to get the answer.

(b) 3×2 or 2×3 matrix and use Sub game method to get the answer. (The sub games are once again 2×2 games).

(c) To $m \times 2$ or $2 \times n$ matrix and use graphical method to get solution. Graphical solution will give us way to 2×2 matrix.

Step 3. Use algebraic method to get the solution.

Step 4. Use Linear-programming approach to get the solution. Use simplex method to get solution (Duality principle in Linear Programming is used).

Step 5. Use Iteration method or approximate method to get the solution.

All these methods are explained by using numerical examples in the following discussion.

Saddle Point Method

$\text{Maxi}_i \min_j a_{ij} = \text{mini}_j \max_i a_{ij}$ is called **a game with saddle point**. This makes us to understand that the players in the game always use pure strategies. The element at the intersection of their pure strategies is known as **saddle point**. The element at the saddle point is the value of the game. As the players uses the pure optimal strategies, the game is known as **strictly determined game**. A point to remember is that the saddle point is the **smallest element in the row and the greatest element in the column. Not all the rectangular games will have saddle point, but if the game has the saddle point, then the pure strategies corresponding to the saddle point are the best strategies and the number at the point of intersection of pure strategies is the value of the game.** Once the game has the saddle point the game is solved. The rules for finding the saddle point are:

1. *Select the minimums of each row and encircle them.*
2. *Select the maximums of each column and square them.*
3. *A point where both circle and square appears in the matrix at the same point is the saddle point.*

Another name given to saddle point is equilibrium point of the game and the corresponding strategies form the equilibrium pair of strategies.

Problem 10.1.

Solve the game given below:

		Player B		
		I	II	III
Player A	I	1	9	2
	II	8	5	4

Solution

		Player B			
		I	II	III	Row minimum
	I	1	9	2	1
Player A	II	8	5	$\boxed{4}$	**4**
Column Maximum		8	9	4	

In the matrix given, row minimums and column maximums are indicted. The element of A's second strategy and B's third strategy *i.e.* a $_{32}$ is both row minimum and column maximum. Hence **4** is the saddle point and pure strategy for A is second strategy and pure strategy for *B* is third strategy. Hence answer is:

A (0.1), B (0, 0, 1) and the value of the game is ***v* = +4**. This means A will gain 4 units of money *B* will loose 4 units of money and the sum of outcomes is zero.

Problem 10.2.

Solve the game whose pay of matrix is:

		B		
		I	II	III
	I	–3	–2	6
A	II	2	$\boxed{0}$	4
	III	5	–2	–4

Solution

		B			
		I	II	III	Row minimum
	I	–3	–2	6	–3
A	II	2	$\boxed{0}$	4	**0**
	III	5	–2	-4	–4
Column Maximum		5	**0**	6	

Element at *A*(II) and *B*(II) is both column maximum and row minimum. Hence the element **0** is the saddle point. The answer is: *A* (0, 1, 0) and *B* (0, 1, 0) and the value *v* = 0.

Problem 10.3.

The matrix given below illustrates a game, where competitors *A* and *B* are assumed to be equal in ability and intelligence. *A* has a choice of strategy 1 or strategy 2, while *B* can select strategy 3 or strategy 4. Find the value of the game.

		B	
		3	4
A	1	+4	+6
	2	+3	+5

Solution

		B		
		3	4	Row minimum
A	1	$\boxed{+4}$	**+6**	+4
	2	+3	+5	+3
Column Maximum:		**+4**	+6	

The element a_{11} is the row minimum and column maximum. Hence the element $a_{11} = 4$ is the saddle point and the answer is **A (1, 0) and B (1, 0) and value of the game = *v* = 4.**

Problem 10. 4.

In a certain game player has three possible courses of action *L*, *M* and *N*, while *B* has two possible choices *P* and *Q*. Payments to be made according to the choice made.

Choices	*Payments.*
L,P	A pays B Rs.3
L,Q	B pays A Rs. 3
M,P	A pays B Rs.2
M,Q	B pays A Rs.4
N,P	B pays A Rs.2
N,Q	B pays A Rs.3

What are the best strategies for players *A* and *B* in this game? What is the value of the game for *A* and *B*?

Solution

The pay of matrix for the given problem is:

		B: P	Q	Row minimum
	L	–3	+3	–3
A	M	–2	+4	–2
	N	$\boxed{+2}$	+3	**+2**
Column Maximum:		**+2**	+4	

Optimal strategies for *A* and *B* are: ***A* (0, 0, 1)** and ***B* (1, 0)** and the value of the game is ***v* = +2**

Problem 10.5.

Consider the game *G* with the following payoff.

		B: I	II
A	I	2	6
	II	–2	p

(a) Show that *G* is strictly determinable, whatever the value of *p* may be.

(b) Determine the value of *p*

Solution

(a) Ignoring whatever the value of *p* may be, the given payoff matrix represents:

		B: I	II	Row minimum
A	I	$\boxed{2}$	6	**2**
	II	–2	p	–2
Column Maximum:		**2**	6	

Maximin value = 2 and Minimax value = 2. Therefore, the game is strictly determinable as the saddle pointy is $a_{11} = 2$.

(b) The value of the game is $\boldsymbol{v = +2}$. And optimal strategies of players are ***A* (1, 0) and *B* (1, 0)**.

Problem 10.6.

For what value of *q*, the game with the following payoff matrix is strictly determinable?

		B		
		I	II	III
A	I	*q*	6	2
	II	–1	q	–7
	III	–2	4	q

Solution

Ignoring whatever the value of *q* may be, the given payoff matrix represents:

		B			Row minimum
		I	II	III	
A	I	*q*	6	2	**2**
	II	–1	q	–7	–7
	III	–2	4	q	–2
Column maximum:		**–1**	6	2	

Maximin value = 2 and Minimax value = –1. So the value of the game lies between –1 and 2. *i.e.* $-1 \le v \le 2$.

For strictly determinable game since maximin value = minimax value, we must have $-1 \le q \le 2$.

Problem 10.7.

Find the ranges of values of *p* and *q*, which will render the entry (2,2) a saddle point for the game.

		B		
		I	II	III
A	I	2	4	5
	II	10	7	q
	III	4	p	6

Solution

Let us ignore the values of p and q and find the row minimum and column maximum.

		B I	B II	B III	Row minimum
	I	2	4	5	2
A	II	10	**7**	q	**7**
	III	4	p	6	4
Column maximum:		10	**7**	6	

Maximin value = 7 = Minimax value. This means that $p \leq 7$ *i.e*. column maximum and $q \geq 7$ *i.e.* row minimum. Hence the range of p and q will be $p \leq 7$ and $q \geq 7$.

Problem 10. 8.

Find the solution of the game whose payoff matrix is given below:

		B I	B II	B III	B IV	B V
	I	–4	–2	–2	3	1
A	II	1	0	–1	0	0
	III	–6	–5	–2	–4	4
	IV	3	1	–6	0	–8

Solution

		B I	B II	B III	B IV	B V	Row Minimum
	I	–4	–2	–2	3	1	4
A	II	1	0	$\boxed{-1}$	0	0	**–1**
	III	–6	–5	–2	-4	4	–6
	IV	3	1	–6	0	–8	–8
Column Maximum:		3	1	**–1**	3	4	

Optimal strategies for A = ***A* (0, 1, 0, 0)** and for B = ***B* (0, 0, 1, 0, 0)** and the value of the game ***v* = –1.** This means that B always wins 1 unit of money.

Problem 10.9.

Find the range of values of p and q which will render the entry (2,2) a saddle point in the game with the following payoff matrix.

		B		
		1	2	3
A	1	1	q	3
	2	p	5	10
	3	6	2	3

Solution

		B			
		1	2	3	Row minimum
A	1	1	q	3	1
	2	p	$\boxed{5}$	10	**5**
	3	6	2	3	2
Column maximum		6	**5**	10	

In order to have element (2,2) as the saddle point *i.e.* 5 as the saddle point, q should be less than or equal to 5 and p should be greater than or equals to 5. Hence range for p and q are $p \geq 5$ and $\boldsymbol{q \leq 5}$ **or** $\boldsymbol{q \leq 5 \leq p}$**.**

Principle of Dominance in Games

In case there is no saddle point the given game matrix ($m \times n$) may be reduced to $m \times 2$ or $2 \times n$ or 2×2 matrix, which will help us to proceed further to solve the game. The ultimate way is we have to reduce the given matrix to 2×2 to solve mathematically.

To discuss the principle of dominance, let us consider the matrix given below:

		B				
		I	II	III	IV	Row minimum
A	I	2	–4	–3	4	–4
	II	4	–3	–4	2	–4
Column Maximum		4	–3	–3	4	

The row minimums and column maximums show that the problem is not having saddle point. Hence we have to use method of dominance to reduce the size of the matrix.

(i) Consider the first and second strategies of *B*. If *B* plays the first strategy, he looses 2 units of money when *A* plays first strategy and 4 units of money when *A* plays second strategy. Similarly, let us consider *B*'s second strategy, *B* gains 4 units of money when *A* plays his first strategy and gains 3 units of money when *A* plays second strategy. Irrespective of *A*'s choice, *B* will gain money. **Hence for *B* his second strategy is superior to his first**

strategy. In other words, *B*'s second strategy dominates *B*'s first strategy. Or *B*'first strategy is dominated by B's second strategy. Hence we can remove the first strategy of B from the game. The reduced matrix is:

$$\begin{array}{ccc} & & \text{B} \\ & & \begin{array}{ccc} \text{II} & \text{III} & \text{!V} \end{array} \\ \text{A} & \begin{array}{c} \text{I} \\ \text{II} \end{array} & \left| \begin{array}{ccc} -4 & -3 & 4 \\ -3 & -4 & 2 \end{array} \right. \end{array}$$

(ii) Consider *B*'s III and IV strategy. When *B* plays IV strategy, he loose 4 units of money when A plays his first strategy and 2 units of money when A plays his second strategy. Where as, when *B* plays his III strategy, he gains 3 units of money and 4 units of money, when A plays his I and II strategy respectively. Hence B's IV strategy (pure strategy) is dominating the third strategy. Hence we can remove the same from the game. The reduced matrix is:

$$\begin{array}{ccc} & & \text{B} \\ & & \begin{array}{cc} \text{II} & \text{III} \end{array} \\ \text{A} & \begin{array}{c} \text{I} \\ \text{II} \end{array} & \left| \begin{array}{cc} -4 & -3 \\ -3 & -4 \end{array} \right. \end{array}$$

In the above example, if we keenly observe, we see that the elements of second column are smaller or less than the elements of column 4, similarly elements of III column also smaller or less than the elements of I and IV column and I. Hence, we can write the dominance rule for columns as **When elements of a column, say *i*th are less than or equals to the corresponding elements of *j*th column, then *j*th column is dominated by *i*th column or *i*th column dominates *j*th column.**

Consider the matrix given below

$$\begin{array}{ccc} & & \text{B} \\ & & \begin{array}{cc} \text{I} & \text{II} \end{array} \\ \text{A} & \begin{array}{c} \text{I} \\ \text{II} \end{array} & \left| \begin{array}{cc} -2 & -4 \\ 1 & 2 \end{array} \right. \end{array}$$

Let *A* play his first strategy, then he looses 2 units of money and looses 4 units of money when *B* plays his second strategy. But when *A* plays his second strategy, he gains 1 unit of money for *B*'s first strategy and gains 2 units of money, for *B*'s second strategy. Hence, **A's second strategy (pure strategy) is superior to A's first strategy or A's second strategy dominates A's first strategy or**

***A's* first strategy is dominated by *A's* second strategy.** We can closely examine and find that elements of *A's* second strategy are greater than the elements of first strategy. Hence we can formulate general rule of dominance for rows. **When the elements of *r*th row are greater than or equals to elements of *s*th row, then *r*th row dominates *s*th row or *s*th row is dominated by *r*th row.**

The general rules of dominance can be formulated as below

1. **If all the elements of a column (say ith column) are greater than or equal to the corresponding elements of any other column (say *j*th column), then *i*th column is dominated by *j*th column.**
2. **If all the elements of *r*th row are less than or equal to the corresponding elements of any other row, say *s*th row, then *r*th row is dominated by *s*th row.**
3. **A pure strategy of a player may also be dominated if it is inferior to some convex combinations of two or more pure strategies, as a particular case, inferior to the averages of two or more pure strategies.**

Note: At every reduction of the matrix, check for the existence of saddle point. If saddle point found, the game is solved. Otherwise continue to reduce the matrix by method of dominance.

Solutions to 2 x 2 games without saddle point: (Mixed strategies)

In rectangular games, when we have saddle point, the best strategies were the pure strategies. Now let us consider the games, which do not have saddle points. In such cases, the best strategies are the **mixed strategies**. While dealing with mixed strategies, we have to determine the probabilities with which each action should be selected. Let us consider a 2 × 2 game and get the formulae for finding the probabilities with which each strategy to be selected and the value of the game.

Points to be remembered in mixed strategy games are

(a) If one of the players adheres to his optimal mixed strategy and the other player deviates from his optimal strategy, then the deviating player can only decrease his yield and cannot increase in any case (at most may be equal).

(b) If one of the players adheres to is optimal strategy, then the value of the game does not alter if the opponent uses his supporting strategies only either singly or in any combination.

(c) If we add (or subtract) a fixed number say 1,to (from) each elements of the payoff matrix, then the optimal strategies remain unchanged while the value of the game increases (or decreases) by 1.

Consider the 2 × 2 game given below:

			B	
			y_1 I	y_2 II
A	x_1	I	a_{11}	a_{12}
	x_2	II	a_{21}	a_{22}

Let x_1 and x_2 be the probability with which A plays his first and second strategies respectively. Similarly B plays his first and second strategies with probability of y_1 and y_2 respectively. Now $x_1 + x_2 = 1$, and $y_1 + y_2 = 1$. Let us work out expected gains of A and B when they play the game with probabilities of x_1, x_2 and $y_1$and y_2.

A's expected gains when:

B plays his first strategy = $a_{11}\, x_1 + a_{21}\, x_2$

When B plays his second strategy = $a_{12}\, x_1 + a_{22}\, x_2$

Similarly B's gains when:

A plays his first strategy = $a_{11}\, y_1 + a_{12}\, y_2$

When A plays his second strategy = $a_{21}\, y_1 + a_{22}\, y_2$

Now let us assume that the v is the value of the game. As A is the miximin player, he wants to see that his gains are $\geq v$. As B is the minimax player, he wants to see that his gains must be always $\leq v$.

Therefore, we have:

$$a_{11}\, x_1 + a_{21}\, x_2 \geq v$$
$$a_{12}\, x_1 + a_{22}\, x_2 \geq v \text{ and}$$
$$a_{11}\, y_1 + a_{12}\, y_2 \leq v$$
$$a_{21}\, y_1 + a_{22}\, y_2 \leq v$$

To find the value of x_1, x_2 and y_1, y_2 we have to solve the above given inequalities. For convenience, let us consider them to be equations to find the values of x_1, x_2 and y_1, y_2. Therefore, we have:

$$a_{11}\, x_1 + a_{21}\, x_2 = v$$
$$a_{12}\, x_1 + a_{22}\, x_2 = v \text{ and}$$
$$a_{11}\, y_1 + a_{12}\, y_2 = v$$
$$a_{21}\, y_1 + a_{22}\, y_2 = v$$

Always we workout a solution of a 2 × 2 game by considering the above inequalities as strict equalities. Now we can write above as:

$a_{11}\, x_1 + a_{21}\, x_2 = v = a_{12}\, x_1 + a_{22}\, x_2$ or this can be written as $x_1\,(a_{11} - a_{12}) = x_2\,(a_{22} - a_{21})$ or

$(x_1 / x_2) = (a_{22} - a_{21}) / (a_{11} - a_{12})$, Similarly we can write:

$(y_1 / y_2) = (a_{22} - a_{12}) / (a_{11} - a_{12})$, by simplifying, we get:

$x_1 = (a_{22} - a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})$ or $= 1 - x^2$

$x_2 = (a_{11} - a_{12}) / (a_{11} + a_{22}) - (a_{12} + a_{21})$ or $= 1 - x^1$

$y_1 = (a_{22} - a_{12}) / (a_{11} + a_{22}) - (a_{12} + a_{21})$ or $= 1 - y^2$

$y_2 = (a_{11} – a_{21}) / (a_{11} + a_{22}) – (a_{12} + a_{21})$ or $= 1 – y^1$, and the value of the game is $\boldsymbol{v = (a_{11} a_{22} - a_{12} a_{21}) / (a_{11} + a_{22}) – (a_{12} + a_{21})}$

Hints to remember formula:

The matrix is

			B	
			y_1	y_2
			I	II
	x_1	I	a_{11}	a_{12}
A				
	x_2	II	a_{21}	a_{22}

For x_1 Numerator = $a_{22} – a_{21}$ *i.e.* x_1 is in the first row, for numerator we have to take the difference of second row elements from right to left.

For x_2, which comes in second row, we have to take difference of the first row elements from left to right.

For y_1 which comes in the first column, we have to take the difference of second column elements from bottom to top.

For y_2, which comes in second column, we have to take the difference of the elements of first column from top to bottom.

As for the denominator is concerned, it is common for all formulae. It is given by sum of diagonal elements from right hand top corner to left-hand bottom corner minus the sum of the elements diagonally from left-hand top corner to right hand bottom corner.

For value of the game, the numerator is given by products of the elements in denominator in the first bracket minus the product of the elements in the second bracket.

When the game does not have saddle point, the two largest elements of its payoff matrix must constitute one of the diagonals.

Now, let us consider the 2 × 2 matrix we got by reducing the given matrix in the article 10.8.2 and get the answer by applying the formula.

The reduced matrix is:

		B		
		II	III	Row minimum.
	1	–4	–3	–4
A				
	II	–3	–4	–4
Column maximum:		–3	–3	

$\boldsymbol{x_1 = (a_{22} - a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$ **or** $\boldsymbol{= 1 - x^2}$

$x_1 = (-4 - [-3]) / (-4 + [-4]) - (-3 + [-3]) = (-4 + 3) / (-4 - 4) - (-3 -3) = -1 / (-8) - (-6) = -1 / -8 + 6 = -1 / -2 = 1/2 = 0.5.$

$x_2 = 1 - x_1 = 1 - 0.5 = 0.5.$

$\boldsymbol{y_1 = (a_{22} - a_{12}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$ **or** $\boldsymbol{= 1 - y_2} = [-4 + (-3)] / [-4 + (-3)] - [-3 + (-3)] = (-4 + 3) / (-4 - 3) - (-3 -3) = -1 / (-7 + 6) = 1$ (*i.e.* pure strategy).

Value of the game = $\boldsymbol{v = (a_{11}\, a_{22} - a_{12}\, a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$

[12 – 12) / [`–4 – 3] – [–3 – 3] = 0

Problem 10.10.

Solve the game whose payoff matrix is:

		B: I	II	III
A	I	1	7	2
	II	6	2	7
	III	5	1	6

Solution

		B: I	II	III	Row minimum.
A	I	1	7	2	1
	II	6	2	7	2
	III	5	1	6	1
Column Maximum.		6	7	7	

No saddle point. Hence reduce the matrix by method of dominance.

B's third strategy gives him 2,7,6 units of money when A plays his I, II, and III strategies. When we compare this with the *B*'s first strategy, it clearly shows that the payoffs of first strategy are superior or better to that of third strategy. Hence B's third strategy is dominated by the B's first strategy. Hence we remove the third of B strategy from the game.

The reduced matrix is

		B: II	III	Row minimum
A	I	1	7	1
	II	6	2	2
	III	5	1	1
Column maximum:		6	7	

No Saddle point. Reduce the matrix by method of dominance. Consider A's II strategy. The payoffs are 6 and 2 units of money when *B* plays his II and III strategy. When we compare this with A's III strategy, which fetches only 5 and 1 units of money, which is inferior to payoffs of II strategy. Hence we can remove A's third strategy form the game. The reduced matrix is:

		B II	B III	Row minimum
A	I	1	7	1
	II	6	2	2
Column maximum:		6	7	

No saddle point. Hence apply the formula.

$\boldsymbol{x_1 = (a_{22} - a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$ **or** $\boldsymbol{= 1 - x_2}$ $= (2 - 6) / (1 + 2) - (6 + 7) = -4 / -10 = (2/5)$ or 0.4

Hence x_2 $(1 - 2/5) = 3 / 5$ or 0.6.

$\boldsymbol{y_1 = (a_{22} - a_{12}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$ **or** $\boldsymbol{= 1 - y_2}$ $= (2 - 7) / (1 + 2) - (6 + 7) = (-5 / -10) = (1/2) = 0.5$

$y_2 = 1 - y_1 = 1 - (1/2) = 1 / 2 = 0.5$

Value of the game = $\boldsymbol{v = (a_{11}\, a_{22} - a_{12}\, a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$

$= (1 \times 2) - (6 \times 7) / (1 + 2) - 6 + 7) = -40 / -10 = 4$

Solution to the game is: *A* (2/5, 3/5, 0) and *B* (0, ½, ½) and value of the game is $v = 4$ *i.e.* A allays win 4 units of money.

Problem 10.11.

Use the concept of dominance to solve the game.

		B I	B II	B III	B IV	Row minimum
A	I	3	2	4	0	0
	II	3	4	2	4	2
	III	4	2	4	0	0
	IV	0	4	0	8	0
Column maximum		4	4	4	8	

No saddle point. Let us reduce the matrix by method of dominance.

Compare *A*'s I strategy and III strategy, we find that third strategy is superior to first strategy as the elements of III row are greater than or equal to that of elements of first row. Hence, A's III strategy dominates *A*'s I strategy. Hence *A*'s first strategy can be removed from the game. The reduced matrix is:

		B				
		I	II	III	IV	Row minimum
	II	3	4	2	4	2
A	III	4	2	4	0	0
	IV	0	4	0	8	0
Column maximum		4	4	4	8	

No saddle point, try to reduce the matrix by dominance method. Compare *B'*s first strategy and III strategy. As the elements of III strategy are less than or equal to that of first strategy, the III strategy dominates the first strategy. Hence, *B*'s first strategy is removed from the game. The reduced matrix is:

		B			
		II	III	IV	Row minimum
	II	4	2	4	2
A	III	2	4	0	0
	IV	4	0	8	0
Column Maximum		4	4	8	

No saddle point and there is no dominance among pure strategies. Hence let us take the averages of two or more pure strategies and compare with other strategies, to know whether there is dominance or not. Let take *B*'s III and IV strategy and take the average and compare with elements of first strategy.

Average of elements of *B*'s III and IV strategy are: (2 + 4 = 6/2 = 3), (4 + 0 = 4 /2 = 2) and (0 + 8 = 8/2 = 4).

Hence the reduced matrix is:

		B			
		II	Avg. of III & IV	Row minimum	
	II	4	3	3	
A	III	2	2	2	(do not consider saddle point)
	IV	4	4	4	
Column maximum		4	4		

As all the elements of *B'*s second strategy are greater than or equal to that of averages of III and IV strategies, *B*'s second strategy is inferior to that of III and IV strategies. Hence the matrix is:

		B III	B IV	Row minimum
	II	2	4	2
A	III	4	0	0
	IV	0	8	0
Column maximum		4	8	

No saddle point. Hence, let us try the dominance by comparing the averages of two *A*'s strategies with elements of other strategy. Averages of *A*'s II and IV pure strategies is:

(4 + 2 = 6 / 2 = 3) and (0 + 8 = 8 / 2 = 4). The matrix is:

		B III	B IV
A	II	2	4
	Avg. of III & IV	3	4

As the elements of *A*'s II strategy are inferior to averages of III and IV strategy, II strategy is removed from the matrix. The reduced matrix is:

		B III	B IV	Row minimum
A	III	4	0	0
	IV	0	8	0
Column maximum		4	8	

No saddle point. By applying the formulae:

$\boldsymbol{x_1 = (a_{22} - a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$ **or = 1 –** $\mathbf{x_2}$ = (8 – 0) / (4 + 8) – (0 + 0) = 8 / 12 = **2 / 3.** Hence x_2 = 1 – (2/3) = **1/3**.

$\boldsymbol{y_1 = (a_{22} - a_{12}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$ **or = 1 –** $\boldsymbol{y_2}$ = (8 – 0) / (4 + 8) – (0 + 0) = 8 / 12 = **2/ 3.** Hence y_2 = 1 – (2/3) = **1/3.**

Value of the game = $\boldsymbol{v = (a_{11}\, a_{22} - a_{12}\, a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$ **=**

(32 – 0) / (4 + 8) – (0 + 0) = 32 / 12 = **8/3.**

Hence the solution is ***A*** **(0, 0, 2/3, 1/3), B (0, 0, 2/3, 1/3) and** $\boldsymbol{v}$ **= 8/3**

A will always win 8/3 units of money.

Problem 10.12.

Two players *P* and *Q* play the game. Each of them has to choose one of the three colours: White (*W*), Black (*B*) and Red (*R*) independently of the other. Thereafter the colours are compared. If both *P* and *Q* has chosen white (*W*, *W*), neither wins anything If player *P* selects white and Player *Q* black (*W*, *B*), player *P* loses Rs.2/- or player *Q* wins the same amount and so on. The complete payoff table is shown below. Find the optimum strategies for *P* and *Q* and the value of the game.

		Q		
		W	B	R
	W	0	–2	7
P	B	2	5	6
	R	3	–3	8

Solution

The payoff matrix is:

		Q			
		W	B	R	Row minimum
	W	0	–2	7	–2
P	B	2	5	6	2
	R	3	–3	8	–3
Column maximum:		3	5	8	

No saddle point. Reduce the matrix by method of dominance. Comparing the elements of *B*'s strategy *R*, the elements of strategy *R* are greater than the elements of other strategies; hence it can be removed from the matrix as it is dominated by strategies *W* and *B*. Reduced matrix is:

		Q		
		W	B	Row minimum
	W	0	–2	–2
P	B	2	5	2
	R	3	–3	–3
Column maximum:		3	5	

There is no saddle point. Comparing *P*'s strategies, *W* and *B*, we see that the elements of *W* strategy are less than the elements of strategy *B*. Hence Strategy *B* dominates strategy *W* and is removed from the matrix. The reduced matrix is:

		Q		
		W	B	Row minimum
P	B	2	5	2
	R	3	–3	3
Column maximum:		3	5	

There is no saddle point. By applying the formulae:

$\boldsymbol{x_1 = (a_{22} - a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$ **or** $\boldsymbol{= 1 - x^2}$ = (–3 – 3) / [2 + (–3)] – [3 + 5] = –6 / [(-1) – (8) = –6 / –9 = 6 / 9 = 2 / 3. Hence x_2 = (1 – 2/3) = **1/3**

$\boldsymbol{y_1 = (a_{22} - a_{12}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$ **or** $\boldsymbol{= 1 - y^2}$ = (–3 – 5) / (–9) = –8 / –9 = **(8 / 9)**. Hence y_2 = 1 / (8 / 9) = **(1 / 9).**

Value of the game = $\boldsymbol{v = (a_{11}\, a_{22} - a_{12}\, a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$ = (–6 – 15) / – 9 = –21 / –9 = **(21 /9). The solution is: *P* (0, 2/3, 1/3), *Q* (8/9, 1/9, 0) and *v* = 21 /9.**

Problem 10.13.

Solve the game whose payoff matrix is:

				B		
		1	2	3	4	5
	1	1	3	2	7	4
A	2	3	4	1	5	6
	3	6	5	7	6	5
	4	2	0	6	3	1

Solution

				B			
		1	2	3	4	5	Row minimum
	1	1	3	2	7	4	1
A	2	3	4	1	5	6	1
	3	6	5	7	6	5	5
	4	2	0	6	3	1	0
Column maximum:		6	5	7	7	6	

The game has the saddle point (3, 2). Hence the value of the game is ***v* = 5** and the optimal strategies of *A* and *B* are: **A (0, 0, 1, 0) , B (0, 1, 0, 0, 0)**

Problem 10. 14.

Solve the following game whose payoff matrix is:

		B					
		I	II	III	IV	V	VI
	I	4	2	0	2	1	1
	II	4	3	1	3	2	2
A	III	4	3	7	–5	1	2
	IV	4	3	4	–1	2	2
	V	4	3	3	–2	2	2

Solution

		B						
		I	II	III	IV	V	VI	Row minimum
	I	4	2	0	2	1	1	0
	II	4	3	1	3	2	2	1
A	III	4	3	7	–5	1	2	–5
	IV	4	3	4	–1	2	2	–1
	V	4	3	3	–2	2	2	–2
Column maximum:		4	3	7	3	2	2	

The game has no saddle point. Let us reduce the size of the matrix by method of dominance.

Compare *A*'s I and II strategies, I strategy is dominated by II strategy. Similarly, compare *A*'s IV and V strategies, elements of IV strategy are greater than that of V strategy; hence V strategy is dominated by IV strategy. Hence *A*'s I ad V strategies can be eliminated and the reduced matrix is:\

		B						
		I	II	III	IV	V	VI	Row minimum
	II	4	3	1	3	2	2	1
A	III	4	3	7	-5	1	2	-5
	IV	4	3	4	-1	2	2	-1
Column maximum:		4	3	7	3	2	2	

The matrix has no dominance. Compare *B*'s I and II strategies are dominated by *B*'s V and VI strategy, as the elements of I and II columns are greater than that of V and VI columns. Hence, *B*'s I and II strategies can be eliminated. Similarly, elements of VI column are greater than that of V column, hence V strategy dominates VI strategy, and hence VI strategy is eliminated. Reduced matrix is:

		B			
		III	IV	V	Row minimum
	II	1	3	2	1
A	III	7	–5	1	–5
	IV	4	–1	2	–1
Column maximum:		7	3	2	

As the pure strategies do not have dominance, let us take the average of *B*'s III and IV strategies and compare with V strategy. The averages are: (1 + 3)/2 = 2, (7 – 5)/2 = 1, and (4 – 1)/2 = 3/2. These average when compared with the elements of V strategy, they are smaller, hence, III and IV strategies dominates V strategy of *B*. The reduced matrix is:

		B		
		III	IV	Row minimum.
	II	1	3	1
A	III	7	–5	–5
	IV	4	–1	–1
Column maximum.		7	3	

No saddle point. Let us take average of *A*'s II and III strategy and compare with IV strategy.

Average is: (1 + 7)/2 = 4, and (3 – 5)/2 = –1. II and III strategies of *A* dominate IV strategy. Hence is eliminated from the matrix. The reduced matrix is:

		B		
		III	IV	Row minimum
A	II	1	3	1
	III	7	–5	–5
Column maximum.		7	3	

No dominance. Hence applying the formulae:

$\boldsymbol{x_1 = (a_{22} - a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$ **or = 1 –** $\boldsymbol{x^2}$ **=** (–5 – 7) / ([1 + (– 5)] – (3 + 7) = –12 / (–4 – 10) = 12/14 = **6 / 7**, hence x_2 = 1 – (6 / 7) = **(1 / 7)**

$\boldsymbol{y_1 = (a_{22} - a_{12}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$ **or = 1 –** $\boldsymbol{y^2}$ **= (–5 – 3) / – 14 = 8 / 14 = (2 / 7)** hence, y_2 = 1 – 2 / 7 = **5 / 7.**

Value of the game = $\boldsymbol{v = (a_{11}\, a_{22} - a_{12}\, a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$ = (–5 – 21) / **–14 = 26 / 14 = 13 / 7**

The answer is: A (0, 6 / 7, 1 / 7, 0, 0), B (0, 0, 2 / 7, 7 / 7, 0, 0) and v = 13 / 7.

Problem 10.15.

A and *B* play a game in which each has three coins, *a* 5 paise, 10 paise and 20 paise coins. Each player selects a coin without the knowledge of the other's choice. If the sum of the coins is an odd amount, *A* wins *B*'s coins. If the sum is even, *B* wins *A*'s coins. Find the optimal strategies for the players and the value of the game.

Solution

The pay of matrix for the given game is: Assume 5 paise as the I strategy, **10 paise as the** II strategy and the 20 paise as the III strategy.

			B 5 I	10 II	20 III
	5	I	–10	15	25
A	10	II	15	–20	–30
	20	III	25	–30	–40

In the problem it is given when the sum is odd, *A* wins *B*'s coins and when the sum is even, *B* will win *A*'s coins. Hence the actual pay of matrix is:

			B 5 I	10 II	20 III	Row minimum
	5	I	–5	10	20	–5
A	10	II	5	–10	–10	–10
	20	III	5	–20	–20	–20
Column maximum.			5	10	20	

The problem has no saddle point. Column I and II are dominating the column III. Hence it is removed from the game. The reduced matrix is:

			B 5 I	10 II	Row minimum
	5	I	–5	10	–5
A	10	II	5	–10	–10
	20	III	5	–20	–20
Column maximum.			5	10	

The problem has no saddle point. Considering *A*, row III is dominated by row II, hence row III is eliminated from the matrix. The reduced matrix is:

		B: I	B: II	Row minimum
A	I	–5	10	–5
	II	5	–10	–10
Column maximum.		5	10	

No saddle point. By application of formulae:

$\boldsymbol{x_1 = (a_{22} - a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$ **or** $\boldsymbol{= 1 - x_2}$ = (–10 – 5) / [–5 + (–10)] – (10 – 5) = –15 / (–15 – 5) = (–15 / –20) = (15 / 20) = **3 / 4,** hence $x_2 = 1 - (3 / 4) =$ **1 / 4**

$\boldsymbol{y_1 = (a_{22} - a_{12}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$ **or** $\boldsymbol{= 1 - y_2}$ = (–10 – 10) / –20 = 20 / 20 = **1** and $y_2 = 0$

Value of the game = $\boldsymbol{v = (a_{11} a_{22} - a_{12} a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$ = (50 – 50) / – 20 = **0**

Answer is *A* (3/ 4, 1 / 4, 0), B (1, 0, 0) , *v* = 0.

Method of Oddments (for 2 × 2 games)

Once the game matrix is reduced to 2 × 2 the players has to resort to mixed strategies. We have already seen how using formulae can an algebraic method to find optimal strategies and value of the game. There is one more method available for the same that is the method of oddments. Steps involved in method of dominance are:

1. Subtract the two digits in column 1 and write them under column 2, ignoring sign.
2. Subtract the two digits in column 2 and write them under column 1 ignoring sign.
3. Similarly proceed for the two rows.

These values are called **oddments.** They are the frequencies with which the players must use their courses of action in their optimum strategies. Let us take a simple example and get the answer for a game.

Problem 10.16.

In a game of matching coins, player A wins Rs.2/-, if there are two heads, wins nothing if there are two tails and loses Re.1/- when there are one head and one tail. Determine the pay off matrix and best strategies and value of the game.

Solution: (by using method of oddments)

The payoff matrix is:

		B: I H	B: II T	Row minimum
	I H	2	–1	–1
A	II T	–1	0	–1
Column maximum.		2	0	

There is no saddle point. Let us apply method of oddments.

The given matrix is:

		B: H	B: T	Oddments of A	Probability
A	H	2	–1	(–1– 0) = 1	1 / (1 + 3) = 1/4
	T	–1	0	(2 – (–1) = 3	3 / (1 = 3) = 3/4
Oddments of *B*		(–1–0) = 1	(2 – (–1)) = 3		
Probability:		1/ (1 + 3) = 1 /4	3 / (3 + 1) = 3/4		

Hence optimal strategies for A and B are: A (1/4, 3/4), B (1/4, 3/4),

Value of the game can be written if the sum of oddments of both the players is equal. Other wise we have to apply the formula of 2 × 2 games and get the value. In this case the sum of oddments of both players is 4. Hence, we can find the value of the game.

Using *A*'s oddments:

When *B* lays his I strategy (*H*), $v = [1 \times 2 + (3 \times -1)] / (3 + 1) = (2 - 3) / 4 = -(1 / 4)$,

When *B* plays his II strategy, (*T*) = $v = (1 \times -1 + 3 \times 0) / (3 + 1) = -(1 / 4)$

Using *B*'s Oddments:

When *A* plays his I strategy (*H*): $v = (1 \times 2 + 3 \times -1) / (3 + 1) = -(1 / 4)$,

When *A* plays his II strategy (*T*) $v = (-1 \times 1 + 3 \times 0) / (3 + 1) = -(1 / 4)$.

Value of the game is – 1 / 4.

Problem 10.17.

By using the oddments of *A* and *B* solve the game.

		B: I	B: II
A	I	4	0
	II	0	8

There is no saddle point. The oddments are:

		B: I	B: II	Oddments of *A*	Probability
A	I	4	0	(0 – 8) = 8 = 2	(2/3)
	II	0	8	(4 – 0) = 4 = 1	(1/3)

Oddments of *B* (0 – 8)	= 8 (4 – 0) = 4	
	=2	= 1
Probability	= 2 / 3	1 / 3.

Value of the game:(The sums of two oddments is same) v = For *B* playing I strategy:

$v = 2 \times 4 + 1 \times 0 / (2 + 1) = (8 / 3)$

Answer is *A* (2 / 3, 1 / 3,), *B* (2 / 3, 1 / 3) and *v* = 8 / 3

Problem 10.18.

Two armies are at war. Army *A* has two air bases, one of which is thrice as valuable as the other. Army *B* can destroy an undefended air base, but it can destroy only one of them. Army *A* can also defend only one of them. Find the strategy for *A* to minimize the losses.

Solution

B (attacker)

	I (attack smaller)	II (attack larger)	Row min.
(Defend smaller) I	Both survive: 0	The larger destroyed: –3	–3
Defender: A			
(Defend larger) II	Smaller destroyed: –1	Both survive: 0	–1
Column Max:	0	0	

No saddle point. By method of oddments:

B

		I	II	Oddments of A	Probability
A	I	0	– 3	(–1 – 0) = 1	1 / 4
	II	– 1	0	(0 – (–3) = 3	3 / 4

Oddments of *B*	(0 – (–3) = 3	(–1–0) = 1
Probabilities:	3 / 4	1 / 4

Value of the game is: (Note the sum of oddments is same: Taking the oddments of *A*, When *B* plays his I strategy, *A*'s expected winning for army *A* =

$(3/4)\,[(0) \times (1 / 4) + (-1) \times (3 / 4)] + (1 / 4)\,[(-3) \times (1 / 4) + (0) \times (3 / 4)] = -(9 / 16) - (3 / 16) = - (12 / 16) = -(3 / 4)$

Solutions to 2 × *n* or *m* × 2 games

When we can reduce the given payoff matrix to 2 × 3 or 3 × 2 we can get the solution by method of **sub games**. If we can reduce the given matrix to 2 × *n* or *m* × 2 sizes, then we can get the solution by **graphical method**. A game in which one of the players has two strategies and other player has number of strategies is known as $2 \times n$ or $m \times 2$ games. If the game has saddle point it is solved. If no saddle point, if it can be reduced to 2×2 by method of dominance, it can be solved. When no more reduction by dominance is possible, we can go for Method of Sub games or Graphical method. We have to identify 2×2 sub games within $2 \times n$ or $m \times 2$ games and solve the game.

Problem 10.19.

Solve the game whose payoff matrix is:

		B		
		I	II	III
A	I	–4	3	–1
	II	6	–4	–2

Solution

Given pay of matrix is 2×3 matrix.

		B			
		I	II	III	Row minimum
A	I	–4	3	–1	–4
	II	6	–4	–1	–4
Column maximum.		6	3	–1	

No saddle point.
The sub games are:
Sub game I:

		B		
		I	II	Row minimum
A	I	–4	3	–4
	II	6	–4	–4
Column Maximum.		6	3	

No saddle point. First let us find the value of the sub games by applying the formula. Then compare the values of the sub games; which ever is favorable for the candidate, that sub game is to be selected. Now here as *A* has only two strategies and *B* has three strategies, the game, which is favorable to *B*, is to be selected.

Value of the game = $\boldsymbol{v_1 = (a_{11}\, a_{22} - a_{12}\, a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$
$= (-4 \times -4) - (3 \times 6) / [(-4 + -4] - (3 + 6) = 2 / 17$

Sub game II:

		B		
		I	III	Row minimum
A	I	–4	–1	–4
	II	6	–2	–2
Column maximum.		6	1	

No saddle point, hence value of the game = $\boldsymbol{v_2 = (a_{11}\, a_{22} - a_{12}\, a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$
= [(–4) × (–2)] – [(–1) × 6] / [(–4) + (–2) – (6 – 1) = – (14 / 11)

Sub game III:

		B		
		II	III	Row minimum
A	I	3	**–1**	**1**
	II	–4	–2	4
Column Maximum.		3	**1**	

The game has saddle point (1,3), the element is (–1). Hence the value of the game v_3 =**-**1.

Comparing the two values v_1 and v_2, v_2, and v_3, both v_2 and v_3 have negative values, which are favorable to player *B*. But v_2 is more preferred by *B* as it gives him good returns. Hence *B* prefers to play strategies I and III. Hence sub game II is selected. For this game we have to find the probabilities of strategies. For sub game II the probabilities of strategies are:

$\boldsymbol{x_1 = (a_{22} - a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$ **or = 1 –** $\boldsymbol{x_2}$ = [(–2) – 6] / (– 11) = (8 / 11), hence
x_2 = 1 – (8 / 11) = 3 / 11

$\boldsymbol{y_1 = (a_{22} - a_{12}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$ **or = 1 –** $\boldsymbol{y_2}$ = [(–2) – (–1)] / –11 = (1 / 11), Hence
y_2 = 1 / (1 / 11) = (10 / 11).

Hence optimal strategies for the players are:

A (8/11, 3 / 11), B (1 / 11, 0, 10 / 11) and the value of the game is – (14 / 11).

Problem 10.20.

Solve the following 2 × n sub game:

		B	
		I	II
	I	1	8
A	II	3	5
	III	11	2

Solution

The given game is $m \times 2$ game.

		B		
		I	II	Row minimum
	I	1	8	1
A	II	3	5	3
	III	11	2	2
Column maximum.		11	8	

No saddle point. Hence *A*'s Sub games are:
A's sub game No.1.

		B		
		I	II	Row minimum
	I	1	8	1
A				
	II	**3**	5	**3**
Column Maximum.		**3**	8	

The game has saddle point and hence value of the game is $v_1 = 3$
A's sub game No.2.

		B		
		I	II	Row minimum
	I	1	8	1
A				
	III	11	2	2
Column maximum.		11	8	

No saddle point. Hence the value of the game $v_2 = (a_{11}\, a_{22} - a_{12}\, a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})$
$= [(1) \times (2) - (8) \times (11)] / (3) - (19) = (3 - 88) / (-16) = (85 / 16)$
A's Sub game No. 3:

		B		
		I	II	Row minimum
	II	3	5	3
A	III	11	2	2
Column maximum.		11	5	

No saddle point. Hence the value of the game $v_3 = \mathbf{(a_{11}\, a_{22} - a_{12}\, a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$
$(3 \times 2) - (11 \times 5) / (3 + 2) - (5 + 11) = (6 - 55) / (5 - 16) = -(49 / 11)$

Now $v_1 = 3$, $v_2 = 85 / 16 = 5.31$, and $v_3 = 49 / 11 = 4.45$. Comparing the values, as far as *A* is concerned, v_2 gives him good returns. Hence *A* prefers to play the sub game No. 2. For this game we have to find out the probabilities of playing the strategies. For sub game No.2:

$\mathbf{x_1 = (a_{22} - a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$ **or = 1 –** $\mathbf{x_2}$ $= (2 - 11) / (-16) = (9 / 16)$,

$x_2 = 1 - (9 / 16) = (7/16)$

$\mathbf{y_1 = (a_{22} - a_{12}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$ **or = 1 –** $\mathbf{y_2}$ $= (2 - 8) / (-16) = (6 / 16)$

$y_2 = 1 - (6 / 16) = (10 / 16)$.

Therefore optimal strategies for *A* and *B* are:

A (9 / 16, 0, 7 / 16), B (6 / 16, 10 / 16) and value of the game *v* = (85 / 16) = 5.31.

Problem 10.21.

Solve the game by method of sub games whose payoff matrix is:

		B: I	B: II
A	I	6	5
	II	3	6
	III	8	4

Solution

The given payoff matrix is

		B: I	B: II	Row minimum
A	I	6	5	5
	II	3	6	3
	III	8	4	4
Column maximum:		8	6	

No saddle point. Let us form sub games of *A* and find the optimal strategies.

Sub game No. 1.

		B: I	B: II	Row minimum
A	I	6	5	5
	II	3	6	3
Column maximum.		6	6	

No saddle point. The value of the game = $\boldsymbol{v_1 = (a_{11}\, a_{22} - a_{12}\, a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21}) =}$
(5 × 6) – (5 × 3) / (6 + 6) – (5 + 3) = (36 – 15) / (12 – 8) = 21 / 4.

Sub game No. 2:

		B: I	B: II	Row minimum.
A	I	6	**5**	**5**
A	III	8	4	4
Column maximum.		8	**5**	

The game has saddle point (1,2) and the element is 5. Hence the value of the game is $v_2 = 5$.

Sub game No. 3.

		B: I	B: II	Row minimum.
A	II	3	6	3
A	III	8	4	4
Column maximum.		8	6	

No saddle point. Let us find the value of the game.

Value of the game = $\boldsymbol{v_3 = (a_{11}\, a_{22} - a_{12}\, a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21}) =}$
(12 – 48) / (7 – 14) = (36 / 7).

Now $v_1 = (21 / 4) = 5.25$, $v_2 = 5$, and $v_3 = (36 / 7) = 5.14$. Among all the three $v_1 = 5.25$ is good return to A. Hence he selects sub game No.1. Let us find the probabilities of strategies for this game.

$\boldsymbol{x_1 = (a_{22} - a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$ **or = 1 –** $\boldsymbol{x_2}$ = (6 – 3) / 4 = 3 / 4. Therefore $x_2 = 1/4$.

$\boldsymbol{y_1 = (a_{22} - a_{12}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$ **or = 1 –** $\boldsymbol{y_2}$ = (6 – 5) / 4 = 1/4, Therefore, $y_2 = 3 / 4$.

Answer: ***A*** **(3 / 4, 1 / 4, 0),** ***B*** **(1 / 4, 3 / 4) and the value of the game =** $\boldsymbol{v}$ **= 36 / 7 = 5.25.**

Problem 10. 22.

Solve the game whose payoff matrix is given below by method of sub games.

		B: I	B: II	B: III
A	I	–5	5	0
A	II	8	–4	–1

Solution

Given matrix is

		B I	B II	B III	Row minimum
A	I	–5	5	0	–5
A	II	8	–4	–1	–4
Column maximum.		8	5	0	

No saddle point.

Sub game No. 1 of B:

		B I	B II	Row minimum
A	I	–5	5	–5
A	II	8	–4	-4
Column maximum		8	5	

No saddle point. Hence, Value of the game = $\boldsymbol{v_1 = (a_{11}\, a_{22} - a_{12}\, a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$

[(–5 × –4) – (5 × 8) / [(–5) + (–4)] – [5 + 8] = (20 – 40) / (–9 – 13) = –20 / –21 = (20 / 21)

Sub game 2 of B:

		B I	B III	Row minimum
A	I	–5	0	–5
A	II	8	–1	–1
Column maximum.		8	0	

No saddle point. Hence, value of the game = $\boldsymbol{v_2 = (a_{11}\, a_{22} - a_{12}\, a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$

= [(–5 × –1) – (0)] / [(–5 + (–1)] – (0 + 8) = (5 – 0) / (–6 – 8) = – (5 / 14).

Sub game No.3 of B:

		B II	B II	Row minimum.
A	I	5	**0**	**0**
A	II	–4	–1	–4
Column maximum.		5	0	

The game has saddle point. $V_3 = 0$

Comparing all the three values, $v_1 = (20 / 21)$, $v_2 = -(5 / 14)$ and $v_3 = 0$. The sub game 2 will give good returns to *B*. Hence, *B* prefers to play the sub game 2. Now let us find the probabilities of the strategies.

$\boldsymbol{x_1 = (a_{22} - a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$ **or = 1 –** $\mathbf{x_2}$ = (–1 – 8) / (– 14) = –9 / –14 = 9/14

x_2 = 1 – (9 / 14) = 5 / 14,

$\boldsymbol{y_1 = (a_{22} - a_{12}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$ **or = 1 –** $\boldsymbol{y_2}$ = (–1 – 0) / (–14) =(– 1 / – 15) = 1/15

y_2 = 1 – (1 / 15) = 14 / 15.

Answer: *A* (9 / 14, 5 / 14), *B* (1 / 15, 0, 14 / 15) and the value of the game *v* = – (5 / 14).

Graphical Method

When a $m \times n$ pay of matrix can be reduced to $m \times 2$ or $n \times 2$ pay off matrix, we can apply the sub game method. But too many sub games will be there it is time consuming. Hence, it is better to go for **Graphical method** to solve the game when we have $m \times 2$ or $n \times 2$ matrixes.

Problem 10.23.

Solve the game whose pay of matrix is:

		B			
		I	II	III	IV
A	I	1	4	–2	–3
	II	2	1	4	5

Solution

Given payoff matrix is:

Solve the game whose pay of matrix is:

			B				
		I	II	III	IV	Row minimum
x	I	1	4	–2	–3	–3
A						
(1-*x*)	II	2	1	4	5	1
Column maximum		2	4	4	5	

No saddle point. If sub game method is to be followed, there will be many sub games. Hence, graphical method is used.

Let A play his first strategy with a probability of x, and then he has to play his second strategy with a probability of $(1 - x)$. Let us find the payoffs of A when B plays his various strategies.

Step 1

Find the payoffs of A when B plays his various strategies and A plays his first strategy with a probability x and second strategy with a probability $(1 - x)$. Let pay off be represented by P. Then A's payoffs, when

B plays his first strategy: $P_1 = 1\,x\,(x) + 2\,(1 - x) = 1x + 2 - 2x = 2 - x$.

B plays his second strategy: $P_2 = 4x + 1\,(1 - x) = 4x + 1 - x = 1 + 3x$.

B plays his third strategy: $P_3 = -2x + 4\,(1 - x) = -2x + 4 - 4x = 4 - 6x$.

B Plays his fourth strategy: $P_4 = -3x + 5\,(1 - x) = -3x + 5 - 5x = 5 - 8x$.

Step 2

All the above payoff equations are in the form of $y = mx + c$. Hence we can draw straight lines by giving various values to x. To do this let us write two **vertical lines,** keeping the distance between lines at least four centimeters. Then write a horizontal line to represent the probabilities. Let the left side vertical line represents, A's first strategy and the probability of $x = 1$ and right side vertical line represents A's second strategy and the probability of $1 - x$. Mark points 1, 2, 3 etc on vertical lines above the horizontal line and –1, –2, –3 etc, below the horizontal lines, to show the payoffs.

Step 3

By substituting $x = 0$ and $x = 1$ in payoff equations, mark the points on the lines drawn in step 2 above and joining the points to get the payoff lines.

Step 4

These lines intersect and form open polygon. These are known as upper bound above the horizontal line drawn and the open polygon below horizontal line is known as lower bound. The upper bound (open polygon above the horizontal line is used to find the decision of player B and the open polygon below the line is used to find the decision of player A. This we can illustrate by solving the numerical example given above.

Step 5

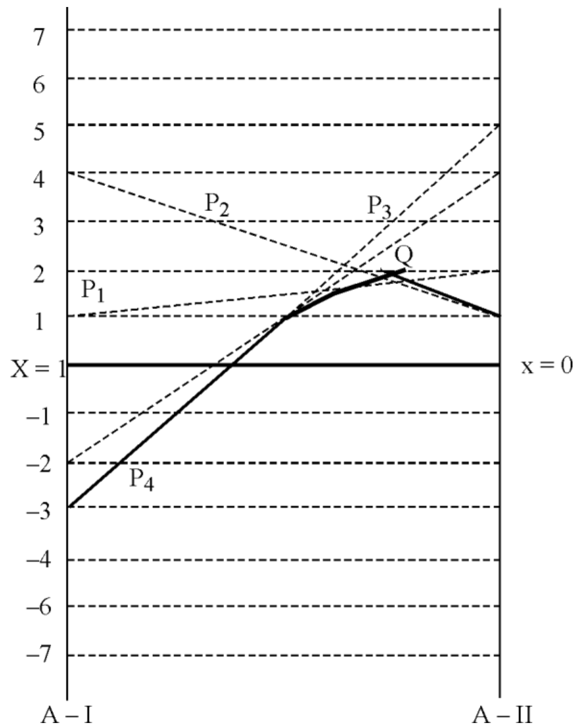
Remember that the objective of graphical method is also to reduce the given matrix to 2×2 matrix, so that we can apply the formula directly to get the optimal strategies of the players.

For $P_1 = 2 - x$, when $\times = 0$, $P_1 = 2$ and when $\times = 1$, $P_1 = 1$> Mark these points on the graph and join the points to get the line P_1. Similarly, we can write other profit lines.

$P_2 = 1 + 3x$, when $x = 0$, $P_2 = 1$, $x = 1$, $P_2 = 4$.

$P_3 = 4 - 6x$. When $x = 0$, $P_3 = 4$ and When $x = 1$, $P_3 = -2$.

$P_4 = 5 - 8x$, When $x = 0$, $P_4 = 5$ and When $x = 1$, $P_4 = -3$.



After drawing the graph, the lower bound is marked, and the highest point of the lower bound is point Q, lies on the lines P_1 and P_2. Hence B plays the strategies II, and I so that he can minimize his losses. Now the game is reduced to 2 × 2 matrix. For this payoff matrix, we have to find optimal strategies of A and B. The reduced game is:

		B: I	B: II	Row minimum
A	I	1	4	1
A	II	2	1	1
Column Maximum:		2	4	

No saddle point. Hence we have to apply formula to get optimal strategies.

$\boldsymbol{x_1 = (a_{22} - a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21}) \text{ or } = 1 - x_2 =}$

$x_1 = (1 - 2) / (1 + 1) - (4 + 2) = -1 / (2 - 6) = (-1 / -4) = 1 / 4.$ and $x_2 = 1 - (1 / 4) = 3 / 4$

$\boldsymbol{y_1 = (a_{22} - a_{12}) / (a_{11} + a_{22}) - (a_{12} + a_{21}) \text{ or } = 1 - y_2}$

$y_1 = (1 - 4) / (-4) = (3 / 4), y_2 = 1 - (3 / 4) = (1 / 4)$

Value of the game = $\boldsymbol{v = (a_{11}\, a_{22} - a_{12}\, a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21}) =}$
$(1 \times 1) - (4 \times 2) / -4 = (1 - 8) / -4 = -4 / -4 = (7 / 4)$
Answer: ***A* (1 / 4, 3 / 4), B (3 / 4, 1 / 4, 0, 0) , *v* = 7 /4. *A* always wins 7/4 units of money.**

Problem 10.24.

Solve the given payoff matrix by Graphical method and state optimal strategies of players *A* and *B*.

		B 1	B 2	B 3	B 4	B 5
A	1	–5	5	0	–1	8
A	2	8	–4	–1	6	–5

Solution

Given Payoff Matrix is

			B 1	B 2	B 3	B 4	B 5	Row minimum
x	A	1	–5	5	0	–1	8	–5
$(1-x)$	A	2	8	–4	–1	6	–5	–5
Column Max:			8	5	0	6	8	

No saddle point. Reduce the given matrix by using graphical method. Let us write the payoff equations of *B* when he plays different strategies. *A* has only two strategies to use. Let us assume that *A* plays his first strategy with a probability x and his second strategy with a probability $(1 - x)$. The *B*'s payoffs are:

P_1 for *B*'s first strategy $= -5x + 8\,(1 - x)$, *i.e.* $P_1 = -5x + 8 - 8x = 8 - 13x$. When, $x = 0$, $P_1 = 8$, $x = 1$,

$P_1 = -5$.

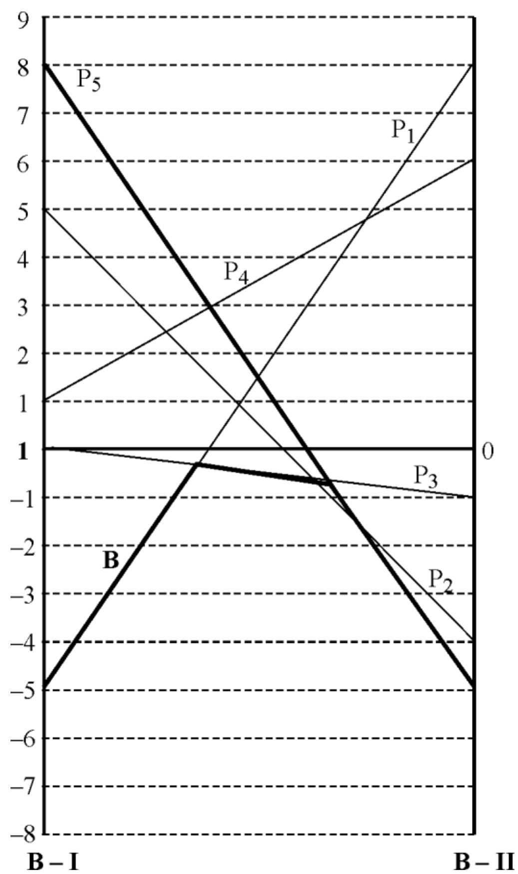
P_2 for *B*'s second Strategy $= 5x - 4\,(1 - x)$, *i.e.* $P_2 = 5x - 4 + 4x = 9x - 4$, When $x = 0$, $P_2 = -4$. When $x = 1$, $P_2 = 5$.

P_3 for *B*'s third strategy $= 0x - 1\,(1 - x) = x - 1$, When $x = 0$, $P_3 = -1$, and When $x = 1$, $P_3 = 0$.

P_4 for *B*'s fourth strategy $= -1x + 6\,(1 - x) = x + 6 - 6x = 6 - 5x$. When $x = 0$, $P_4 = 6$ and when $x = 1$, $P_4 = 1$

P_5 for *B*'s fifth strategy $= 8\,x - 5\,(1 - x) = 8x - 5 + 5x = 13x - 5$. When $x = 0$, $P_5 = -5$, when $x = 1$, $P_5 = 8$.

If we plot the above payoffs on the graph:



Now, player *B* has to select the strategies, as player *A* has only two strategies. To make A to get his minimum gains, *B* has to select the point *B* in the lower bound, which lies on both the strategies *B* –1 and *B* – 3. Hence now the 2 × 2 game is:

		B		
		1	3	Row minimum
A	1	–5	0	5
	2	8	–1	–1
Column maximum.		8	0	

No saddle point. Hence apply the formula to get the optimal strategies.

$\boldsymbol{x_1 = (a_{22} - a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$ **or** $\boldsymbol{= 1 - x_2 =}$

$x_1 = [-1 - 8]/(-5 - 1) - (0 + 8) = (-9)/(-6 - 8) = (-9/-14) = (9/14)$ and $x_2 = [1 - (9/14) = (5/14)$

$y_1 = (a_{22} - a_{12}) / (a_{11} + a_{22}) - (a_{12} + a_{21})$ or $= 1 - y_2$

$y_1 = [-1 - 0] / -(14) = -(-1/-14) = (1/14)$ and $y_2 = 1 - (1/14) = (13/14)$

Value of the game = $v = (a_{11} a_{22} - a_{12} a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21}) = [(-1 \times -5) - (0 \times 8)] / (-14) = (-5 / -14) = (5 / 14)$.

Answer: $v = (5/14)$, A (9 / 14, 5 / 14), B (1/14, 0, 13 / 14, 0, 0). A always wins a sum of 5/14.

Note: While calculating the profits to draw graph, it is shown that first to write the equation and then substituting the values of 0 and 1 to x we can get the profits for each strategy. Students as well can directly write the profit points, without writing the equation. For example, in the given problem, we know that A plays his first strategy with x and then the second strategy with (1– x) probability. When $x = 0$, the value is 8, *i.e.* the element a_{21} in the matrix. Similarly, when $x = 1$, the values is –5 *i.e.* the element a_{11}. We can write other values similarly. But it is advised it is not a healthy practice to write the values directly. At least show one equation and calculate the values and then write the other values directly. This is only a measure for emergency and not for regular practice.

Problem 10.25.

Solve the game graphically, whose pay off matrix is:

		B: I	B: II
A	1	–6	7
	2	4	–5
	3	–1	–2
	4	–2	5
	5	7	–6

Solution

The given pay off matrix is:

		B: I	B: II	Row minimum
A	1	–6	7	–6
	2	4	–5	–5
	3	–1	–2	–2
	4	–2	5	–2
	5	7	–6	–6
Column maximum:		7	7	

No saddle point. For graphical method, let us assume that *B* plays his first strategy with a probability of 'y' and the second strategy with a probability of $(1 - y)$. Then the equations of various pay offs when *A* plays his different strategies are:

A's pay off when he plays his strategy 1 is $P_1 = -6y + 7(1 - y) = -6y + 7 - 7y = 7 - 13y$, When $y = = 0$,

$P_1 = 7$ and when $y = 1$, $P_1 = -6$.

A's pay off when he plays his strategy 2 is $P_2 = 4y - 5(1 - y) = 4y - 5 + 5y = -5 + 9y$, When $y = 0$, $P_2 = -5$,

When $y = 1$, $P_2 = 4$.

A's pay off when he plays his strategy 3 is $P_3 = -1y - 2(1 - y) = -1y - 2 + 2y = y - 2$, When $y = 0$, $P_3 = -2$,

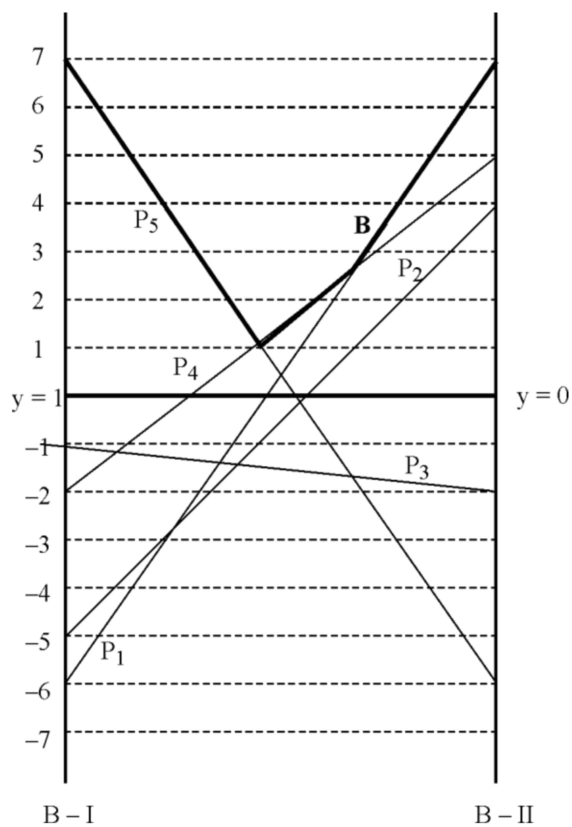
When $y = 1$, $P_3 = -1$.

A's pay off when he plays his strategy 4 is $P_4 = -2y + 5(1 - y) = -2y + 5 - 5y = 5 - 7y$, When $y = 0$, $P_4 = 5$

When $y = 1$, $P_4 = -2$

A's pay off when he plays his strategy 5 is $P_5 = 7y - 6(1 - y) = 7y - 6 + 6y = 13y - 6$, When $y = 0$, $P_5 = -6$

When $y = 1$, $P_5 = 7$.



In the figure, point *B* lies on the strategies 4 and 5 of player *A*. Now *B* has to select lowest point in the lower bound (Thick lines) *i.e.* point *B*. The reduced 2 × 2 game is

		B I	B II	Row minimum.
A	4	–2	5	–2
A	5	7	– 6	–6
Column maximum		7	5	

No saddle point. To find the optimal strategies of *A* and *B* and the value of the game the formula is used.

$\boldsymbol{x_1 = (a_{22} - a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$ **or = 1 –** $\boldsymbol{x_2}$ **=**

$x_1 = (-6-7) / (-2-6) - (5+7) = (-13) / (-8) - (12) = 13 / 20$, hence $x_2 = 7 / 20$.

$\boldsymbol{y_1 = (a_{22} - a_{12}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$ **or = 1 –** $\boldsymbol{y_2}$

$y_1 = (-6-5) / (-20) = 11 / 20$, $y_2 = 9 / 20$.

Value of the game = $\boldsymbol{v = (a_{11}\, a_{22} - a_{12}\, a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})} = (-2 \times -6 - 5 \times 7) / (-20)$ $= (12 - 35) / (-20) = 23 / 20. = 1.15$.

Answer: *A* (0, 0, 0, 13 / 20, 7 / 20), *B* (11 / 20, 9 / 20), and *v* = 1.15.

Problem 10.26.

Solve the game whose pay off matrix is:

		B I	B II	B III	B IV
A	I	3	4	10	12
A	II	8	4	3	2

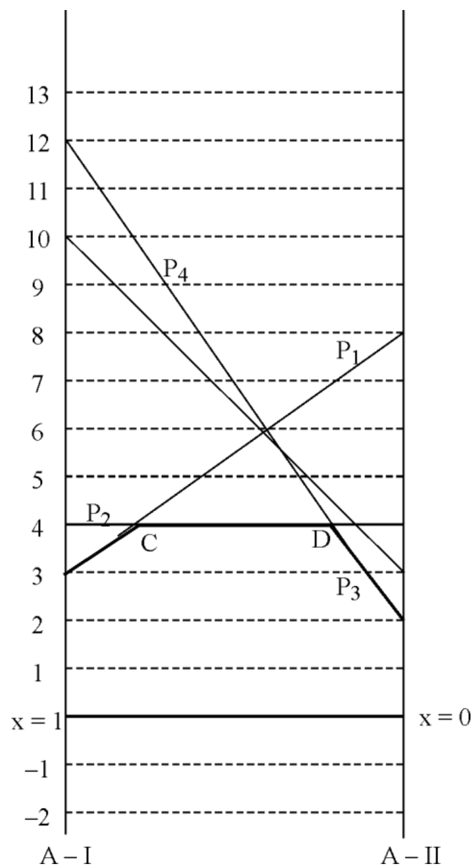
Solution

The pay of matrix is:

		B I	B II	B III	B IV	Row minimum
A	I	3	4	10	12	3
A	II	8	4	3	2	2
Column maximum.		8	4	10	12	

No Saddle point. Hence reduce the matrix by graphical method. Let *A* play his I strategy with a probability of *x* and the II strategy with a probability of $(1 - x)$. The pay offs of *B* when he plays different strategies is as follows:

B's Strategy	*Pay off Equation* (P_i)	*When* $x = 0$ $P_i =$	*When* $x = 1$ $P_i =$
I	$3x + 8(1 - x) = 3x + 8 - 8x = 8 - 5x.$	8	3
II	$4x + 4(1 - x) = 4x + 4 - 4x = 4$	4	4
III	$10x + 3(1 - x) = 10x + 3 - 3x = 3 + 7x$	3	10
1V	$12x + 2(1 - x) = 12x + 2 - 2x = 2 + 10x$	2	12



In the figure points *C* and *D* lies on a horizontal line and point *C* lies on P_2 and P_1, similarly, point *D* lies on P_2 and P_3. Hence we have two 2 × 2 games. If we solve by applying the formula, the pay offs or the value of the game is same for both games.

The games are:

		B		
		I	II	Row minimum
A	**I**	3	4	3
	II	8	$\boxed{4}$	**4**
Column maximum		8	**4**	

		B		
		II	IV	Row minimum
A	I	$\boxed{4}$	12	**4**
	II	4	2	2
Column maximum		**4**	12	

Both the games have saddle point and the value of the game in both cases is $v = 4$.

Optimal strategies are:

$\boldsymbol{x_1 = (a_{22} - a_{21}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$ **or** $\boldsymbol{= 1 - x_2}$ $= (4 - 8) / (7 - 11) = 1$, Then $x_2 = 0$.

$\boldsymbol{y_1 = (a_{22} - a_{12}) / (a_{11} + a_{22}) - (a_{12} + a_{21})}$ **or** $\boldsymbol{= 1 - y_2}$ $= (0) / -4 = 0$, then $y_2 = 1$. (This means they are playing pure strategy game.

Answer: *A* (1, 0), *B* (0, 1, 0, 0) and $\boldsymbol{v = 4}$.

Algebraic Method

When the given pay off matrix of a game cannot be reduced to 2 × 2, or 2 × 3 or $m \times 2$ or $2 \times n$, then, we can solve the game by using the algebraic method. This is a straightforward and lengthy and time-consuming method. Here we have to write a system of inequalities and consider them as equations and solve the simultaneous equations as usual. To illustrate the method, let us take a numerical example and solve by using Algebraic method.

Problem 10.27.

Solve the game whose pay off matrix is as given:

		B		
		I	II	III
	I	–1	2	1
A	II	1	–2	2
	III	3	4	–3

Solution

Given matrix is:

		B			
		I	II	III	Row minimum
	I	–1	2	1	–1
A	II	1	–2	2	–2
	III	3	4	–3	–3
Column maximum		3	4	2	

No saddle point. Let us solve the game by Algebraic method. Let A play his strategies with a probability of x_1, x_2 and x_3 and B play his strategies with a probability of y_1, y_2 and y_3. Now we know $x_1 + x_2 + x_3 = 1$ and $y_1 + y_2 + y_3 = 1$. Let us write the inequalities, which show the pay offs of both players. A is a maximizing (maximin) player and he expects his pay off should be $\geq v$ and B is the minimizing player (minimax) he expects his pay off must be $\leq v$. The inequalities are:

$$-1x_1 + 1x_2 + 3x_3 \geq v$$
$$2x_1 - 2x_2 + 4x_3 \geq v$$
$$1x_1 + 2x_2 - 3x_3 \geq v$$

$$-1y_1 + 2y_2 + 1y_3 \leq v$$
$$1y_1 - 2y_2 + 2y_3 \leq v$$
$$3y_1 + 4y_2 - 3y_3 \leq v \; x_1$$
$$+ x_2 + x_3 = 1$$
$$y_1 + y_2 + y_3 = 1 \text{ and } x_1, x_2, x_3, y_1, y_2, y_3 \text{ all} \geq 0$$

Now let us consider all the above inequalities into equations so that we can go ahead to solve the equations. The equations are:

$$-1x_1 + 1x_2 + 3x_3 = v \quad \ldots 1$$
$$2x_1 - 2x_2 + 4x_3 = v \quad \ldots 2$$
$$1x_1 + 2x_2 - 3x_3 = v \quad \ldots 3$$
$$-1y_1 + 2y_2 + 1y_3 = v \quad \ldots 4$$
$$1y_1 - 2y_2 + 2y_3 = v \quad \ldots 5$$
$$3y_1 + 4y_2 - 3y_3 = v \quad \ldots 6$$
$$x_1 + x_2 + x_3 = 1 \quad \ldots 7$$
$$y_1 + y_2 + y_3 = 1 \quad \ldots 8$$
$$\text{And } x_1, x_2, x_3, y_1, y_2, y_3 \text{ all } \geq 0 \quad \ldots 9$$

Add equations 1 and 3:

$$-1x_1 + 1x_2 + 3x_3 = v \quad \ldots 1$$
$$\underline{1x_1 + 2x_2 - 3x_3 = v} \quad \ldots 3$$
$$3x_2 = 2v \text{ or } x_2 = (2 / 3)\, v$$

Add two times of equation 1 to equation 2:

$$-2x_1 + 2x_2 + 6x_3 = 2v \quad \ldots 1$$
$$\underline{2x_1 - 2x_2 + 4x_3 = v} \quad \ldots 2$$
$$10x_3 = 3v \text{ or } x_3 = (3 / 10)\, v$$

Substituting the values of x_2 and x_3 in equations 1 to 3 we get $x_1 = (17 / 30)\, v$. Substituting the values of x_1, x_2 and x_3 in equation number 7 we get $\boldsymbol{v = (15 /23)}$. Substituting the value of v we get $\boldsymbol{x_1 = (17 / 46)}$**,**

$\boldsymbol{x_2 = (10 / 23)}$ **and** $\boldsymbol{x_3 = (9 / 46)}$**.**

As we know the value of v, we can substitute this value in equations 4, 5 and 6 and solving for y_1, y_2, y_3 we get the values as: $\boldsymbol{y_1 = (7 / 23)}$**,** $\boldsymbol{y_2 = (6 / 23)}$ **and** $\boldsymbol{y_3 = (10 / 23)}$**.**

Answer: $\boldsymbol{A}$ **(17 / 46, 10 / 23, 9 / 46),** $\boldsymbol{B}$ **(7 / 23, 6 / 23, 10 / 23) and** $\boldsymbol{v = 15 / 23}$**.**

Method of oddments for solving the games

Problem 10.29.

Solve the given game by method of oddments:

		B		
		I	II	III
	I	3	–1	–3
A	II	–3	3	–1
	III	–4	–3	3

Solution

The matrix given is:

		B			
		I	II	III	Row minimum
	I	3	–1	–3	–3
A	II	–3	3	–1	–3
	III	–4	–3	3	–4
Column maximum		3	3	3	

The game has no saddle point. Let us apply method of oddments.

*Step.*1: Subtract each row from the row above it. That is subtract second row from first row and third row from second row etc. Write the difference of these rows in the form of two-succssie rows below the rows of the matrix.

Step 2: Subtract each column from the column to its left i.e. subtract second column from the first and the third column from the second and so on and write the difference in the form of two successive columns to the right of the given matrix. This is shown below:

		B				
		I	II	III		
	I	3	–1	–3	4	2
A	II	–3	3	–1	–6	4
	III	–4	–3	3	–1	–6
		6	–4	–2		
		1	6	–4		

Step 3: Calculate the oddments for *A*'s I, II, and III strategies and *B*'s I, II, and III strategies.

Oddment for *A*'s first strategy = Determinant

$$= \begin{vmatrix} -6 & 4 \\ -1 & -6 \end{vmatrix} = (-6 \times -6) - (4 \times -1) = 40$$

Oddment for A’s Second strategy = Determinant

$$= \begin{vmatrix} 4 & 2 \\ -1 & -6 \end{vmatrix} = (4 \times -6) - (2 \times -1) = -24 + 2 = -22$$

Oddment for A’s third strategy = Determinant

$$= \begin{vmatrix} 4 & 2 \\ -6 & 4 \end{vmatrix} = (4 \times 4) - (2 \times -6) = 16 + 12 = 28$$

Oddment for B’s First strategy = Determinant

$$= \begin{vmatrix} -4 & -2 \\ 6 & -4 \end{vmatrix} = (-4 \times -4) - (-2 \times 6) = 16 + 12 = 28$$

Oddment for B’s second strategy = Determinant

$$= \begin{vmatrix} 6 & -2 \\ 1 & -4 \end{vmatrix} = (6 \times -4) - (-2 \times 1) = -24 + 2 = -22$$

Oddment for B's third strategy = Determinant

$$= \begin{vmatrix} 6 & -4 \\ 1 & 6 \end{vmatrix} = (6 \times 6) - (-4 \times 1) = 36 + 4 = 40.$$

Step 4: Write these oddments, neglecting the signs as shown below:

		B I	B II	B III	
A	I	3	–1	–3	40
	II	–3	3	–1	22
	III	–4	–3	3	28
		28	22	40	Col total = 90 Row total = 90

Step 5: Now verify the sums of oddments of *A* and *B*. They must be same to solve the game by matrix method. In this example both sums are equal to 90. This means that both players use their pure strategies and hence the game is conformable for matrix method. **The necessary condition for solving the game is the sums of two oddments must be same. In case the sums of oddments are different, then both the players do not use their all-pure strategies and hence matrix method fails.**

Step.6: Divide the oddments by the sum of the oddments to get the optimal strategies of players.

A (40 / 90, 22 / 90, 28. 90), B (28 / 90, 22 / 90, 40 / 90) OR A (20 / 45, 11 / 45, 14 / 45), B (14 / 45, 11 / 45, 20 / 45).

Value of the game is given by: $v = [40 \times 3 + 22 \times (-3) + 28 \times (-4)] / (40 + 22 + 28) = (-58 / 90) = (-29 / 45)$ OR

$v = [40 \times (-1) + 22 \times 3 + 28 \times (-3)] / 90 = -58 / 90$ OR

$v = [40 \times (-3) + 22 \times (-1) + 28 \times 3] / 90 = -58 / 90$ OR

$v = [28 \times 3 + 22 \times (-1) + 40 \times (-3)] / 90 = -58 / 90$ OR

$v = [28 \times (-3) + 22 \times 3 + 40 \times (-1)] / 90 = -58 / 90$ OR

$v = [28 \times (-4) + 22 \times (-3) + 40 \times 3] / 90 = -58 / 90$

The value can be found by any one of the above.

Problem 10.30.

Solve the given game by method of matrices:

		B I	B II	B III
A	I	1	–1	–1
	II	–1	–1	3
	III	–1	2	–1

Solution

The given pay of matrix is:

		B I	B II	B III	Row minimum
A	I	1	–1	–1	–1
	II	–1	–1	3	–1
	III	–1	2	–1	–1
Column Maximum:		1	2	3	

No Saddle point. To solve by method of oddments:

		B I	B II	B III		
A	I	1	–1	–1	2	0
	II	–1	–1	3	0	–4
	III	–1	2	–1	–3	3
		2	0	4		
		0	–3	4		

Oddment for AI = Determinant = $\begin{vmatrix} 0 & -4 \\ -3 & 3 \end{vmatrix}$ = 0 – 12 = – 12

Oddment for A II = Determinant = $\begin{vmatrix} 2 & 0 \\ -3 & 3 \end{vmatrix}$ =0 – 6 = – 6

Oddment for A III = Determinant = $\begin{vmatrix} 2 & 0 \\ 0 & -4 \end{vmatrix}$ = –8 – 0 = – 8

Oddment for B I = Determinant = $\begin{vmatrix} 0 & -4 \\ -3 & -4 \end{vmatrix}$ = 0 – 12 = – 12

Oddment for B II = Determinant = $\begin{vmatrix} 2 & -4 \\ 0 & -4 \end{vmatrix}$ = 0 – 8 = – 8

Oddment for B III = Determinant = $\begin{vmatrix} 2 & 0 \\ 0 & -3 \end{vmatrix}$ = –6 – 0 = – 6

Now neglecting the signs of oddments, we write as below:

		B					
		I	II	III			
	I	1	–1	–1	12	12 / 23	6 / 13
A	II	–1	–1	3	6	6 / 26	3 / 13
	III	–1	2	–1	8	8 / 26	4 / 13
12	8	6	26 = Total of oddments.				
12 / 26	8 / 26	6 / 26					
6 / 13	4 / 13	3 / 13					

Value of the game v = [12 × 1 – 6 × 1 – 8 × 1] / (12 + 6 + 8) = 2 / 26 = 1 / 13.

Answer = A (6 / 13, 3 / 13, 4 / 13), B (6 / 13, 4 / 13, 3 / 13) and v = 1 / 13.

Problem 10.31.

Solve the game by using oddments:

		B 1	B 2
A	1	5	1
A	2	3	4

Solution

Given matrix is

		B 1	B 2	Row minimum
A	1	5	1	1
A	2	3	4	3
Column maximum:		5	4	

No saddle point. To find the oddment it is easy as it is 2 × 2 matrix.

		B 1	B 2	
A	1	5	1	1
A	2	3	4	4
		3	2	5

Sum of both oddments is 5. Hence optimal strategies for A = (1 / 5, 4 / 5) and that for B = (3 / 5, 2 / 5)

And the value of the game is $v = (1 \times 1 + 4 \times 4) / 5 = 17 / 5$.

Answer: *A* (1/5, 4/5), *B* (3 / 5, 2 / 5) and *v* = 17 / 5.

Method of Linear Programming

When the given pay of matrix cannot be reduced into lesser degree, (in case it does not have pure strategy for players), the mixed strategy game can easily be solved by applying the principles of linear programming. If the problem of maximizing player is primal one, the problem of minimizing player will be the dual of the primal. Hence by solving either primal or dual, we can get the answer of the problem. As the linear programming problem insists on non-negativity constraint, we must take care to see that all the elements in the given matrix are positive elements. In case, there are negative elements in the given matrix, we can add a suitable, large and positive number to the matrix, so that all the elements in the matrix will become positive elements. Or by writing the row minimums and column maximums, we can know that the range of the value of ‘v’ and to keep the v as positive, a positive, sufficiently large number is added to the all elements of the matrix, so that we can satisfy the non - negativity constraint of the linear programming inequalities.

Problem 10.32.

Two oil companies; Indian oil Company and Caltex Company operating in a city are trying to increase their market at the expense of the other. The Indian Oil Company is considering possibilities of decreasing price; giving free soft drinks on Rs. 40/- purchases of oil or giving away a drinking glass with each 40-liter purchase. Obviously, Caltex cannot ignore this and comes out with its own programme to increase its share in the market. The payoff matrix from the viewpoints of increasing or decreasing market shares is given in the matrix below:

		Caltex Oil Company		
		Decrease Price.	Free soft drink on Rs. 40/- purchase.(II)	Free drinking glass On 40 liters or more.(III)
	Decrease Price. (I)	4%	1%	–3%
Indian Oil Co.,	Free soft Drink on Rs.40-/ Purchase. (II)	3	1	6
	Free drinking Glass on 40 Lts or more. (III)	–3	4	–2

Find the optimal strategies and the value of the game.

Solution

Given pay off matrix is:

		Caltex Company			
		I	II	III	Row minimum
	I	4	1	–3	–3
Indian Oil Company.	II	3	1	6	1
	III	–3	4	–2	–3
Column maximum		4	4	6	

The game has no saddle point. Moreover, the value of the game lies between 1 (best out of the worst or maximum of minimum gains for (A) and 4 *i.e.* minimum of maximum losses for B (minimax). As both are positive numbers, we can proceed further for linear programming method. If, in any way the value of the game lies between a negative element and a positive element, then we have to add suitable positive element to the matrix to see that the value lies between two positive elements. (Some times to make the things easy, we can make all the elements of matrix positive by adding a suitable positive element). Let us assume that A plays his strategies with probabilities of x_1, x_2, and x_3. Similarly *B* plays his strategies with probabilities of y_1, y_2 and y_3. The inequalities are:

For B:

$$1y_1 + 1y_2 - 1y_3 = 1$$
$$4y_1 + 1y_2 - 3y_3 \leq v$$
$$3y_1 + 1y_2 + 6y_3 \leq v$$
$$-3y_1 + 4y_2 - 2y_3 \leq v$$

For A:

$$1x_1 + 1x_2 + 1x_3 = 1$$
$$4x_1 + 3x_2 - 3x_3 \geq v$$
$$1x_1 + 1x_2 + 4x_3 \geq v$$
$$-3x_1 + 6x_2 - 2x_3 \geq v$$

Now divide all the inequalities and equations by 'v' and keep $(x_i / v) = X_i$ and $(y_j / v) = Y_j$ and write the inequalities and equations.

(Note: Dividing the above relations by 'v' is valid only if $v > 0$. If however, $V < 0$, the direction of inequality constraints must be reversed and if $v = 0$, division is meaning less). However, both these cases can be easily solved by adding a positive constant 'L' (Where $L \geq$ the negative game value) to all the elements of the matrix, thus ensuring that the game value for the *revised matrix* is greater than zero. After obtaining the optimal solution, the true value of the game can be obtained by subtracting L from the game value so obtained. In general, if the maxmin value of the game is non-negative, then the value of the game is greater than zero, provided the game does not have a saddle point.

If we keenly observe the above inequalities, and recollect the knowledge of linear programming, particularly the duality in linear programming, we understand that the *B*'s constraints are primal then the A's constraints are that of dual. After dividing by 'v' the inequalities and equations are:

$$(y_1 / v) + (y_2 / v) + (y_3 / v) = 1$$
$$(4y_1 / v) + (1y_2 / v) - (3y_3 / v) \leq 1$$
$$(3y_1 / v) + (1y_2 / v) + (6y_3 / v) \leq 1$$
$$(-3y_1 / v) + (4y_2 / v) - (2y_3 / v) \leq 1$$

Similarly we can write for A also. For A the inequalities are:

$$(x_1 / v) + (x_2 / v) + (x_3 / v) = 1$$
$$(4x_1 / v) + (3x_2 / v) - (3x_3 / v) \geq 1$$
$$(1x_1 / v) + (1x_2 / v) + (4x_3 / v) \geq 1$$
$$(-3x_1 / v) + (6x_2 / v) - (2x_3 / v) \geq 1$$

Now putting $x_i / v = X_i$ and $y_j / v = Y_j$ the above inequalities will become:

$$X_1 + X_2 + X_3 = 1/ v$$
$$4X_1 + 3X_2 - 3X_3 \geq 1$$

$$1X_1 + 1X_2 + 4X_3 \geq 1$$
$$-3X_1 + 6X_2 - 2X_3 \geq 1$$

For B

$$Y_1 + Y_2 + Y_3 = 1/v$$
$$4Y_1 + 1Y_2 - 3Y_3 \leq 1$$
$$3Y_1 + 1Y_2 + 6Y_3 \leq 1$$
$$-3Y_1 + 4Y_2 - 2Y_3 \leq 1$$

If B want to minimize 'v' he has to maximize $(1 / v)$, This means he has to maximize $Y_1 + Y_2 + Y_3$ which is the objective function for B.

Maximize $Z = Y_1 + Y_2 + Y_3 = 1/v$ S.T.

$$4Y_1 + 1Y_2 - 3Y_3 \leq 1$$
$$3Y_1 + 1Y_2 + 6Y_3 \leq 1$$
$$-3Y_1 + 4Y_2 - 2Y_3 \leq 1 \text{ and all } Y_1, Y_2 \text{ and } Y_3 \text{ are } \geq 0.$$

Writing this in Simplex format:

Maximize $Z = Y_1 + Y_2 + Y_3 + 0S_1 + 0S_2 + 0S_3$ S.T.

$$4Y_1 + 1Y_2 - 3Y_3 + 1S_1 + 0S_2 + 0S_3 = 1$$
$$3Y_1 + 1Y_2 + 6Y_3 + 0S_1 + 1S_2 + 0S_3 = 1$$
$$-3Y_1 + 4Y_2 - 2Y_3 + 0S_1 + 0S_2 + 1S_3 = 1 \text{ and all } Y_1, Y_2 \text{ and } Y_3 \text{ are } \geq 0.$$

Initial basic feasible solution is:

Table I $Y_1 = 0$, $Y_2 = 0$, $Y_3 = 0$, $S_1 = 1$, $S_2 = 1$, $S_3 = 1$ and $Z =$ Rs. 0.

Problem Variable	*Profit in* Rs.	*Capacity*	*1* Y_1	*1* Y_2	*1* Y_3	*0* S_1	*0* S_2	*0* S_3	*Replacement* Ratio.
S_1	0	1	**4**	1	–3	1	0	0	1 / 4 ←
S_2	0	1	3	1	6	0	1	0	1 / 3
S_3	0	1	–3	4	–2	0	0	1	1 / –3
		Net Ev.	1	1	1	0	0	0	
				↑					

Table II : $Y_1 = 1 / 4$, $Y_2 = 0$. $Y_3 = 0$, $S_1 = 0$, $S_2 = 1/4$, $S_3 = 7 / 4$ and $Z =$ Rs. $1 \times (1 / 4)$

Problem Variable	*Profit in Rs.*	*Capacity*	*1* Y_1	*1* Y_2	*1* Y_3	*0* S_1	*0* S_2	*0* S_3	*Replacement Ratio.*
Y_1	1	1 / 4	1	1 / 4	– 3 / 4	1 / 4	0	0	– 1 / 3
S_2	0	1 / 4	0	1 / 4	**33 / 4**	–3 / 4	1	0	1 / 33 ←
S_3	0	7 / 4	0	19 / 4	– 17 / 4	3 / 4	0	1	–7 / 17
		N.E	0	3 / 4	7 / 4	– 1 / 4	0	0	
					↑				

Table III: $Y_1 = 3 / 11$, $Y_2 = 0$. $Y_3 = 1 / 33$, $S_1 = 0$, $S_2 = 0$, $S_3 = 66 / 33$ and Z = Rs. 1 × (3 / 11) + 1 × (1 / 33)

Problem Variable	*Profit in Rs.*	*Capacity*	*1* Y_1	*1* Y_2	*1* Y_3	*0* S_1	*0* S_2	*0* S_3	*Replacement Ratio.*
Y_1	1	3 / 11	1	3 / 11	0	2 / 11	1 / 11	0	1
Y_3	1	1 / 33	0	1 / 33	1	–1 / 11	4 / 33	0	1
S_3	0	62 / 33	0	**161 / 33**	0	4 / 11	17 / 33	1	62 / 161←
		N.E.	0	23 / 33	0	– 1 / 11	– 7 / 33	0	

Table III: $Y_1 = 27 / 161$, $Y_2 = 62 / 161$. $Y_3 = 3 / 161$, $S_1 = 0$, $S_2 = 0$, $S_3 = 0$ and Z = Rs. 4 / 7

Problem Variable	*Profit in Rs.*	*Capacity*	*1* Y_1	*1* Y_2	*1* Y_3	*0* S_1	*0* S_2	*0* S_3	*Replacement Ratio.*
Y_1	1	27 / 161	1	0	0	26 / 161	10 / 161	–9 / 161	
Y_3	1	3 / 161	0	0	1	– 15 / 161	19 / 161	–1 / 161	
Y_2	1	62 / 161	0	1	0	12 / 161	17 / 161	33 / 161	
		N.E.	0	0	0	–23 / 161	–46 / 161	–23 / 161	

Now the value of $(1 / v) = (4 / 7)$

As, $y_j / v = Y_j$

$$y_1 = Y_1 \times v = (27 / 161) \times (7 / 4) = (27 / 92)$$
$$y_2 = Y_2 \times v = (62 / 161) \times ((7 / 4) = (62 / 92)$$
$$y_3 = Y_3 \times v = (3 / 161) \times (7 / 4) = (3 / 92).$$

A's best strategies we can get from net evaluation row and the elements under slack variables column. We have:

$$X_1 = (23 / 161),\ X_2 = (46 / 161), \text{ and } X_3 = 23 / 161.$$
$$x_1 = X_1 \times v = (23 / 161) \times (7 / 4) = (1 / 4)$$
$$x_2 = X_2 \times v = (46 / 161) \times (7 / 4) = (1 / 2)$$
$$x_3 = X_3 \times v = (23 / 161) \times (7 / 4) = (1 / 4)$$

Therefore the optimal strategies of A and B are:

Indian Oil Company = A (1/4, 1/2 , 1/4)

Caltex Company = B (27 / 92, 62 / 92, 3 / 92) and value of the game is (7 / 4) for A.

Problem 10.33.

Solve the game given in the pay off matrix below:

$$\begin{array}{ccc|ccc} & & & & B & \\ & & & 1 & 2 & 3 \\ \hline & & 1 & 3 & -4 & 2 \\ A & & 2 & 1 & -3 & -7 \\ & & 3 & -2 & 4 & 7 \end{array}$$

Solution

The given matrix is:

		B 1	2	3	Row minimum
	1	3	–4	2	–4
A	2	1	–3	–7	–7
	3	–2	4	7	**–2**
Column maximum:		3	4	7	

The game has no saddle point. The value lies between –2 and 3. Hence a constant $L = \geq 2$ (*i.e.* $L = 3$) is added to all the elements of the matrix, so that the value of the game will be positive. Let x_1, x_2 and x_3 be the probabilities with which A plays his strategies and y_1, y_2 and y_3 be the probabilities with which B plays his strategies. As the player B is the minimizing player, let us write his inequalities and solve by Linear Programming method. A's strategies can be found from the net evaluation row of the final table of B. Modified matrix is:

			B y_1 1	y_2 2	y_3 3
	x_1	1	6	–1	5
A	x_2	2	4	0	–4
	x_3	3	1	7	10

The inequalities of B when A plays his different strategies are:

$$6y_1 - 1y_2 + 5y_3 \leq v$$
$$4y_1 + 0y_2 - 4y_3 \leq v$$
$$1y_1 + 7y_2 + 10y_3 \leq v \text{ and } y_1 + y_2 + y_3 = 1.$$

Dividing by v and keeping $y_i / v = Y_i$ we get,

$$6Y_1 - 1Y_2 + 5Y_3 \leq 1$$
$$4Y_1 + 0Y_2 - 4Y_3 \leq 1$$
$$1Y_1 + 7Y_2 + 10Y_3 \leq 1 \text{ and } Y_1 + Y_2 + Y_3 = (1 / v) \text{ and all } Y_1, Y_2, Y_3 \text{ all} \geq 0.$$

Writing the same in Simplex format, we have:

As B has to minimize v, he has to maximize $(1 / v)$. Therefore,

Maximize $Z = 1 / v = Y_1 + Y_2 + Y_3 + 0S_1 + 0S_2 + 0S_3$ S.t.

$$6Y_1 - 1Y_2 + 5Y_3 + 1S_1 + 0S_2 + 0S_3 = 1$$
$$4Y_1 + 0Y_2 - 4Y_3 + 0S_1 + 1S_2 + 0S_3 = 1$$
$$1Y_1 + 7Y_2 + 10Y_3 + 0S_1 + 0S_2 + 1S_3 = 1 \text{ and } Y_1, Y_2, \text{ and } Y_3 \text{ all} \geq 0$$

Table I: $S_1 = 1$, $S_2 = 1$, $S_3 = 1$ and $Y_1 = 0$, $Y_2 = 0$, $Y_3 = 0$ and Z = Rs.0/-

Problem Variable	*Profit in* Rs.	*Capacity*	*1* Y_1	*1* Y_2	*1* Y_3	*0* S_1	*0* S_2	*0* S_3	*Replacement* Ratio.
S_1	0	1	**6**	–1	5	1	0	0	1 / 6 ←
S_2	0	1	4	0	–4	0	1	0	1 / 4
S_3	0	1	1	7	10	0	0	1	1
		N.E	1	1	1	0	0	0	

↑

Table II: $S_1 = 0$, $S_2 = 1 / 3$, $S_3 = 5 / 43$ and $Y_1 = 1 / 6$, $Y_2 = 0$, $Y_3 = 0$ and Z = Rs.1 × (1 / 6)

Problem Variable	*Profit in Rs.*	*Capacity*	*1* Y_1	*1* Y_2	*1* Y_3	*0* S_1	*0* S_2	*0* S_3	*Replacement Ratio.*
Y_1	1	1 / 6	1	–1 / 6	5 / 6	1 / 6	0	0	–1
S_2	0	1 / 3	0	2 / 3	– 22 / 3	– 2 / 3	1	0	1 / 2
S_3	0	5 / 43	0	**43 / 6**	55 / 6	– 1 / 6	0	1	5 / 43 ←
		N.E.	0	7 / 6	1 / 6	– 1 / 6	0	0	

↑

Table III: $S_1 = 0$, $S_2 = 11 / 43$, $S_3 = 0$,and $Y_1 = 8 / 43$, $Y_2 = 5 / 43$, $Y_3 = 0$ and Z = Rs.1 × (8 / 43) + 1 × (5 / 43) = 13 / 43.

Problem Variable	*Profit in Rs.*	*Capacity*	*1* Y_1	*1* Y_2	*1* Y_3	*0* S_1	*0* S_2	*0* S_3	*Replacement Ratio.*
Y_1	1	8 / 43	1	0	45 / 43	7 / 43	0	1 / 43	
S_2	0	11 / 43	0	0	– 352 /43	– 28 / 43	1	– 4 / 43	
Y_2	1	5 / 43	0	1	55 / 43	– 1 / 43	0	6 / 43	
		N.E.	0	0	– 57 / 43	– 6 / 43	0	– 7 / 43	

$Y_1 = (8 / 43)$, $Y_2 = (5 / 43)$, $Y_3 = 0$ and $v = (13 / 43)$.

Now we know that $Y_j = (y_j / v)$, therefore,

$$y_1 = Y_1 \times v = (8 / 43) \times 43 / 13) = (8 / 13)$$

$$y_2 = Y_2 \times v = (5 / 43) \times (43 / 13) = (5 / 13) \text{ and } y_3 = 0$$

$$X_1 = 6 / 43,\ X_2 = 0,\ X_3 = 7 / 43.$$

$$x_1 = (6 / 43) \times (43 / 13) = (6 / 13)$$

$$x_2 = 0 \text{ and } x_3 = (7 / 43) \times (43 / 13) = (7 / 13).$$

Optimal strategies for A = A (6 / 13, 0, 7 / 13),

For $B = (8 / 13, 5 / 13, 0)$ and value of the game = $v = (43 / 13) – 3 = 4 / 13$. (The element 3 was added to get the value of the v as positive).

Problem 10.34.

Solve the game by L.P.P. method whose pay off matrix is:

		B			
		I	II	III	IV
	I	3	2	4	0
A	II	3	4	2	4
	III	4	2	4	0
	IV	0	4	0	8

Solution: The given matrix is:

		B				
		I	II	III	IV	Row minimum
	I	3	2	4	0	0
A	II	3	4	2	4	2
	III	4	2	4	0	0
	IV	0	4	0	8	0
Column Maximum		4	4	4	8	

The game has no saddle point and the value of the game falls between 2 and 4. Hence we can write the inequalities directly, without adding any positive number to the matrix. As the player is minimizing player let us write his inequalities and apply Linear Programming approach.

Let y_j, where j = 1, 2, 3, and 4 be the probabilities with which player B plays his strategies and x_1, where, i = 1, 2, 3, and 4 with which player A plays his strategies. Then the inequalities of B are:

$$3y_1 + 2y_2 + 4y_3 + 0y_4 \le 1$$
$$3y_1 + 4y_2 + 2y_3 + 4y_4 \le 1$$
$$4y_1 + 2y_2 + 4y_3 + 0y_4 \le 1$$
$$0y_1 + 4y_2 + 0y_3 + 8y_4 \le 1 \text{ and}$$
$$y_1 + y_2 + y_3 + y_4 = 1$$

Dividing all the above inequalities by v and keeping $y_j / v = Y_j$ and writing entire thing in simplex model, we get:

Maximize $Z = (I / v) = Y_1 + Y_2 + Y_3 + 0S_1 + 0S_2 + 0S_3 + 0S_4$ s.t.

$3Y_1 + 2Y_2 + 4Y_3 + 0Y_4 + 1S_1 + 0S_2 + 0S_3 + 0S_4 = 1$

$3Y_1 + 4Y_2 + 2Y_3 + 4Y_4 + 0S_1 + 1S_2 + 0S_3 + 0S_4 = 1$

$4y_1 + 2y_2 + 4y_3 + 0y_4 + 0S_1 + 0S_2 + 1S_3 + 0S_4 = 1$

$0y_1 + 4y_2 + 0y_3 + 8y_4 + 0S_1 + 0S_2 + 0S_3 + 1S_4 = 1$

And all Y_j are ≥ 0 where j = 1, 2, 3 and 4.

Table I: $Y_1 = 0$, $Y_2 = 0$, $Y_3 = 0$, $Y_4 = 0$, $S_1 = 1$, $S_2 = 1$, $S_3 = 1$, $S_4 = 1$ and Z = Rs. 0

Problem Variable	*Profit in Rs.*	*Capacity*	*1* Y_1	*1* Y_2	*1* Y_3	*1* Y_4	*0* S_1	*0* S_2	*0* S_3	*0* S_4	*Replacement Ratio.*
S_1	0	1	3	2	4	0	1	0	0	0	∝
S_2	0	1	3	4	2	4	0	1	0	0	1 / 4
S_3	0	1	4	2	4	0	0	0	1	0	∝
S_4	0	1	0	4	0	**8**	0	0	0	1	1 / 8 ←
		N.E.	1	1	1	1	0	0	0	0	

Table II: $Y_1 = 0$, $Y_2 = 0$, $Y_3 = 0$, $Y_4 = 1 / 8$, $S_1 = 1$, $S_2 = 1 / 2$ $S_3 = 1$, $S_4 = 0$ and Z = Rs. 1 x (1 / 8)

Problem Variable	*Profit in Rs.*	*Capacity in Units*	*1* Y_1	*1* Y_2	*1* Y_3	*1* Y_4	*0* S_1	*0* S_2	*0* S_3	*0* S_4	*Replacement Ratio.*
S_1	0	1	3	2	4	0	1	0	0	0	1 / 4
S_2	0	1 / 2	3	2	2	0	0	1	0	–1 / 2	1 / 4
S_3	0	1	4	2	**4**	0	0	0	1	0	1 / 4 ←
Y_4	1	1 / 8	0	1/2	0	1	0	0	0	1 / 8	1
		N.E.	1	1/2	1	0	0	0	0	– 1/ 8	

Table III: $Y_1 = 0$, $Y_2 = 0$, $Y_3 = 1 / 4$, $Y_4 = 1 / 8$, $S_1 = 0$, $S_2 = 0$, $S_3 = 0$, $S_4 = 0$ and

Z = Rs. 1 × (1 / 8) + 1 × (1 / 4) = 3 / 8

Problem Variable	*Profit in Rs.*	*Capacity*	*1* Y_1	*1* Y_2	*1* Y_3	*1* Y_4	*0* S_1	*0* S_2	*0* S_3	*0* S_4	*Replacement Ratio.*
S_1	0	0	– 1	0	0	0	1	0	– 1	0	
S_2	0	0	1	1	0	0	0	1	–1 / 2	–1 / 2	
Y_3	1	1 / 4	1	1 / 2	1	0	0	0	1 / 4	0	
Y_4	1	1 / 8	0	1 / 2	0	1	0	0	0	1 / 8	
		N.E.	0	0	0	0	0	0	–1 / 4	–1 / 8	

$1 / v = (3 / 8)$, Hence the value of the game $= v = (8 / 3)$. As $y_j = Y_j \times v$

$y_1 = 0 \times (8 / 3) = 0$, $y_2 = 0 \times (8 / 3) = 0$, $y_3 = (1 / 4) \times (8 / 3) = (2 / 3)$, $y_4 = (1 / 8) \times (8 / 3) = (1 / 3)$. Therefore, B's optimal policy = (0, 0, 2 / 3, 1 / 3)

From simplex table, $X_1 = 0$, $X_2 = 0$, $X_3 = (1 / 4)$, $X_4 = (1 / 8)$ and $x_i = X_i \times v$

$X_1 = 0 \times (8 / 3) = 0$, $x_2 = 0 \times (8 / 3) = 0$, $x_3 = (1 / 4) \times (8 / 3) = (2 / 3)$, and $x_4 = (1 / 8) \times (8 / 3) = (1 / 3)$

Answer: *A* (0, 0, 2 / 3, 1 / 3), *B* (0, 0, 2 / 3, 1 / 3) and *v* = 8 / 3

Iterative Method for Approximate Solution

Many a times the operations research manager is satisfied with an approximate answer, nearer to the optimal answer for making quick decisions. It is well known fact that, to get an accurate solution for the problem on hand by using well programmed methods will take time and by the time one get the correct answer, the situation or variables of the may change due to change in the conditions of market or environment. Hence, the manager, who has to take quick decision to solve the problem on hand, will be more satisfied with an approximate answer rather than the correct answer. The same applies to game theory also. One of the methods of determining the approximate solution is the method of iteration. The principle of the approximate method is:

The two players are supposed to play the plays of the game iteratively and at each play the players choose a strategy which is best to himself or say worst to opponent, in view of what the opponent has done up to the iteration.

In the given game, one of the players starts the game by selecting one of his strategies. The second player looking to the out comes of the strategy selected by the first player, the second player selects his strategy, which is best to him or worst to the opponent. Then the pay offs are written as shown. The first player selects the strategy best to him in the same way and writes his outcomes. Every time the out comes are added and written as shown. More number of iterations they play, they get answer very close to the optimal answer. But it is time consuming. Hence, the students **can write up to 10 iterations while dealing with the problem.** Generally the approximate method or method iteration is selected, when the game has no saddle point, cannot be reduced due to domination, and when we need to get the solution quickly. But students are advised not to use this method directly, without trying - Saddle point, Domination, Sub games, Graphical method etc. Let us try to understand by taking a numerical example.

Problem 10.35.

Solve the following game by using method of iteration.

		B		
		I	II	III
A	I	–1	2	1
	II	1	–2	2
	III	3	4	–3

Solution

Given pay off matrix is:

		B													
		I	II	III	1	2	3	4	5	6	7	8	9	10	
	I	–1	2	1	2	$\boxed{3}$	$\boxed{4}$	$\boxed{5}$	$\boxed{6}$	$\boxed{5}$	4	5	6	7	(5)
A															
	II	1	–2	2	–2	–4	–2	0	2	4	5	$\boxed{7}$	$\boxed{9}$	$\boxed{11}$	(3)
	III	3	4	–3	4	1	–2	–5	–8	4	$\boxed{7}$	4	1	2	(2)
	1	1	$\boxed{-2}$	2											
	2	4	8	$\boxed{-1}$											
	3	3	10	$\boxed{0}$											
	4	2	12	$\boxed{1}$											
	5	4	14	$\boxed{2}$											
	6	$\boxed{3}$	16	3											
	7	$\boxed{2}$	18	4											
	8	5	22	$\boxed{1}$											
	9	6	20	$\boxed{3}$											
	10	7	18	$\boxed{5}$											
		(2)	(1)	(7)											

1. Let *A* select his second strategy. Write this below the last row of the matrix. Now, *B* looks at the out comes and selects the best strategy to him *i.e*. second strategy, because he gets 2 units of money. Write the outcomes *B*'s second strategy beside the third column of the matrix.
2. Now, *A* examines his out come for various strategies and selects the strategy which is most advantageous to him, that is third strategy, which gives him four units of money. The selected strategy is generally encircled or the element is written in thick letter.
3. Now, the outcomes of *A*'s third strategy is added to the elements written in the first row below the given matrix and the out come is written against S.No. 2, below the matrix. Once again in this *B* selects the best one for him, *i.e*. (–1) for third strategy. The elements of third strategy is added and added to the first column written right to the matrix and written against S.No. 2.
4. Continue the procedure until all the 10 plays are completed.
5. Look at the 6 th play of *B* (below the matrix), we have two elements having the same numerical value *i.e*. 3. As *B* is already played third strategy many times, we can give chance to his first strategy. *i.e*. Game is to be played judiciously.
6. To find the optimal strategies, let us see how many times each player has played each of his strategies.Now take *A*. *A* has played his first strategy 5 times, second strategy 3 times and his third strategy 2 times. Hence, the optimal strategy is ***A* (5 / 10, 3 / 10, 2 / 10)**. Similarly, for *B* the optimal strategy is ***B* (2 / 10, 1/ 10, 7 / 10).**

7. To find the values of the game, we can fix the higher and lower limits of value. Take the highest element in the last column i.e. 11 and lowest element, in the last row, *i.e.* 5. Divide these two by 10, *i.e*. the number of times the game is played. Then the **value of the game lies between**

$$v = 5 / 10 \le v \le 11 / 10.$$

Problem 10.36.

Solve the game given below by method of iteration.

		B			
		1	2	3	4
A	1	2	3	–1	0
	2	5	4	2	–2
	3	1	3	8	2

Solution

Given matrix is:

		B												
		1	2	3	4	1	2	3	4	5	6	7	8	9
	1	2	3	–1	0	–1	–1	1	3	5	5	5	5	5
A	2	5	4	2	–2	2	0	5	10	(15)	13	11	9	7
	3	1	3	8	2	(8)	(10)	(11)	(12)	13	(15)	(17)	(19	(21)
										Total				
	1	5	4	(2)	3			10						
	2	6	7	10	(5)			5	0					
	3	(7)	10	18	(7)			5	1					
	4	(8)	13	26	9			(23)	9					
	5	(9)	16	34	11									
	6	14	20	36	(9)									
	7	15	23	44	(11)									
	8	16	26	52	(13)									
	9	17	29	60	(15)									
	10	18	32	68	(17)									
Total		3	0	0	7									

Optimal strategies: ***A* (0, 1 / 10, 9 / 10), *B* (3 / 10, 0, 0, 7 / 10). Value of the game lies between: 0 $\le$ *v* $\le$ 23 / 10.**

Typical Problems

So far we have solved the problems, where the pay off matrix is given. Sometimes, we have to construct the payoff matrix, which is a very difficult job. Once the pay off matrix is written, solving can be done by any one of the suitable methods discussed so far. In the following problems, the pay off matrix is constructed and the students have to solve the game to get the optimal strategies of the players and the value of the game.

Problem 10.37.

A and *B* each take out one or two matches and guess how many matches the opponent has taken. If one of the players guesses correctly then the opponent has to pay him as many rupees as the sum of the numbers of matches had by both the players, otherwise the pay-out is zero. Write down the pay off matrix and obtain the optimal strategies for both the players.

Solution

Let *A* be the guessing fellow or the winner and the player *B* is the loser. They have two strategies; one to take one matches and the second is to take two matches. If both have taken one matches and winning player guesses correctly, then the opponent has to pay him the sum of rupees equal to the sum of matches. If he guesses wrongly the pay out is zero. The pay off matrix is as shown.

		B: 1 matches	B: 2 matches
A	1 matches	2	0
A	2 matches	0	4

By solving with the formulae of 2 × 2 game, we get, $v = 4 / 3$, *A* (2 / 3, 1 / 3), *B* (2 / 3, 1 / 3)

Problem 10.38.

Two players *A* and *B*, without showing each other put on a table a coin of Re.1/- with head or tail up. If the coin shows the same side (both head or both tail), the player *A* takes both the coins, otherwise *B* get them Construct the game and solve it.

Solution

If both are heads or tails, *A* will win the game and he gets Re.1/- + Re.1/- = Rs.2 /-. If the one is head and the other is tail, then *B* will get Rs.2/-. Therefore, the pay of matrix is:

	B: Head	B: Tail
A: Head	2	–2
A: Tail	–2	2

As the pay of matrix of *A* is written, *A*'s outcomes are positive and *B*'s outcomes are negative, because *B* is winning and *A* is losing.

Optimal strategies are: *A* (1 / 2, 1 / 2), *B* (1 / 2, 1 / 2) and value of the game is = 0

Problem 10.39.

In a game of matching coins with two players, suppose *A* wins one unit of vale when there are two heads; wins nothing when there are two tails and loses 1 / 2, unit of value when there is one head and one tail. Determine the pay off matrix, and the optimal strategies for the players.

Solution

A wins one unit of money when there are two heads (*H*, *H*), Wins nothing when there are two tails (*T*, *T*), loses ½ unit of money when there are two tails (*H*, *T*). The pay of matrix is:

		B	
		H	T
A	H	1	–1 / 2
	T	–1 / 2	0

The optimal strategies are: *A* (1 / 4, 3 / 4), *B* (1 / 4, 3 / 4), Value of the game is – (1 / 8). *B* gets always 1 / 8 units of money.

Problem 10.40.

Consider a modified form of ''matching coins'' game. The matching player is paid Rs. 8/- if the two coins are both heads, and Re.1/- if both are tails. The non-matching player is paid Rs.3 /- when the coins do not match. Given the choice of being the matching or non-matching player both, which would you choose and what would be your strategy.

Solution

Let *A* be the matching player. He gets Rs.8/- when both are heads (*H*, *H*), he gets Re. 1 /- when both are tails (*T*, *T*). The non-matching player gets Rs. 3/ - when there is (*H*, *T*) or (*T*, *H*). Hence the pay off matrix is:

		B	
		H	T
A	H	8	-3
	T	–3	1

The optimal strategies are: *A* (4 / 15, 11 / 15), *B* (4 / 15, 11 / 15), and value of the game is – (1 / 15). Because the non - matching player is getting the money, it is better to be a non-matching player.

Problem 10.41.

Consider the two person zero sum game in which each player selects independently an integer from the set of integers: 1, 2, and 3. The player with the smaller number wins one point unless his number is less than his opponents by one unit. When the numbers are equal, there is no score. Find the optimal strategies of the players.

Solution

		B		
		1	2	3
	1	0	1	1
A	2	–1	0	1
	3	1	–1	0

Value of the game is $v = 0$, Optimal strategies of *A* and *B* are *A* (1/3, 1/3, 1/3), *B* (1/3, 1/3, 1/3). This type of games is known as symmetric games.

Problem 10.42.

Two children play the following game, named as **'Scissors, Paper,** and **Stone' (*S, P,* St.).** Both players simultaneously call one of the three: Scissors, Paper or Stone. Scissors beat paper as paper can be cut by scissor, Paper beats stone as stone can be wrapped in paper, and stone beats scissors as stone can blunt the scissors. If both players name the same item, then there is a tie. If there is one point for win, zero for the tie and –1 for the loss. Form the pay of matrix and write the optimal strategies.

Solution

If both call same (*S*, *S*) or (*P*, *P*) or (*St*, *St*) the pay out is zero. There is one point for winning (*S*, *P*), (*P*, *St*) and (*St*, *S*) and –1 point for (*S*, *St*), (*P*, *S*) and (*St*., *P*). the pay off matrix is:

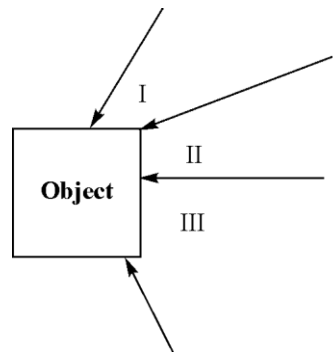
		B		
		S	P	St
	S	0	1	–1
A	P	–1	0	1
	St	1	–1	0

This is also symmetric game. Hence Value $= v = 0$. Strategies are: *A* (1/3, 1/3, 1/3), *B* (1 / 3, 1 / 3, 1/ 3).

Problem 10. 43.

Party *A* attacks an object party *B* defends it. *A* has two airplanes *B* has three anti-aircraft guns. To attack the object, it is enough that one airplane of *A* breaks through B's defenses. The planes of *A* can choose any one of the three regions I, II, and III of space for approaching the object as shown in figure. The party B can place his guns to defend in any of the regions of space. *A* gun can only defend one particular region and is incapable of engaging a plane approaching the object form a different region of approach. Each gun is capable of shooting down only one plane with a probability 1.

Party *A* does not know where the guns are placed, party *B* also does not know how the airplanes approach the object. The problem of *A* is to attack. The problem of party *B* is to defend this. Construct the game and determine best strategies for both the parties.



Solution

Strategies of party A are: A_1: Send airplanes in two different regions and A_2: Send both airplanes in one region.

Strategies of B are: B_1 To place three guns in three different regions.

B_2 To place one gun in one region, two guns in other region, and keep one region undefended.

B_3 To place all the three guns in only one region.

To analyze the situation: The pay of matrix is of the order 2 × 3 as A has only two strategies and B has three strategies. Now:

$A_1\ B_1 = a_{11} = 0$, because the three regions are defended by one gun and hence, probability of hitting the object = 0.

$A_2\ B_1 = a_{21}$ since one of the two airplanes, which attack the same region is shot down and the other hits the object, hence the probability of attacking the object = 1.

$A_1\ B_2 = a_{12}$ = Probability of attacking the object is probability of the airplanes selecting the undefended region is equals to (1 / 3) + (1 / 3) = (2 / 3), as in an undefended region, the airplane definitely hit the object.

$A_2\ B_2 = a_{22}$ = Probability of sending the airplanes to undefended region or the region defended by one gun, is (1 / 3) + (1 / 3) = (2 / 3).

$A_1\ B_3 = a_{13}$ = As all the guns are placed in one region and the two airplanes are sent in two different regions, at least one of the planes will definitely go to one of the undefended regions. Hence the probability of striking the object is 1.

$A_2\ B_3 = a_{23}$ = Probability of sending both the planes to any one of the two undefended regions is (1 / 3) + (1 / 3) = (2 / 3). Hence the required pay off matrix is given by:

			B	
		B_1	B_2	B_3
A	A_1	0	2/3	1
	A_2	1	2/3	2/3

B_2 is dominating B_3 hence the game is reduced to 2 × 2 game.

		B		
		B_1	B_2	Row minimum
A	A_1	0	2 / 3	0
	A_2	1	**2 / 3**	2 / 3
Column Maximum		1	**2 / 3**	

Optimal strategies are A (0, 1), B (0, 1, 0) and the value of the game is (2 / 3).

Problem 10.44.

A has two ammunition stores, one of which is twice as valuable as the other. *B* is an attacker, who can destroy an undefended store, but he can only attack one of them. *A* knows that *B* is about to attack one of the stores but does not know which one? What should he do? Note that A can successfully defend only one store at a time.

Solution

Let us assume the value of the smaller store is 1, and then the value of the bigger store is 2. By analysis, If both stores survive, A loses nothing *i.e*. 0 (a_{11}). If the smaller survives, *i.e*. larger is destroyed, then *A* loses 2 (a_{12}). If the larger store survives and the smaller is destroyed, *A* losses 1. Therefore the pay off matrix of *A* is

		B	
		B_1 attack smaller Store	B_2 Attack larger store
A	A_1 defend small Store.	0	–2
	A_2 defend larger Store.	–1	0

Solving the game by formulae, we get *A* (1 / 3, 2 / 3), v = – (2 / 3)

Problem 10.45.

The game known as **'The prisoner's dilemma'.** The district authority has two prisoners in different cells and knows that both are guilty. To provide the sufficient evidence to convict them, he plays a game. He offers them a chance of confessing and declares that if one confesses and the other refuses to confess, the penalty will be great particularly for the one who denies the charge say 10 years, whilst the one who confesses will go free for giving the testimony against the other. Both prisoners know that if neither confesses they will both receive at most a minor sentence say 1 year for a technical offence. Also if both confess, they will get 8 years. What the prisoners do?

Solution

Each prisoner's sentence (in years) may be represented by the matrix given below:

		II Prisoner	
		Denies	Confesses
	Denies	1	10
I Prisoner			
	Confesses	0 go free	8

The game is not a zero sum game as the II prisoner's matrix is not the negative of the above matrix as usual the case may be. The following matrix may also represent this sentenced of the two prisoners in order.

		II Prisoner	
		Denies	confesses
	Denies	(1, 1)	(10, 0)
I Prisoner			
	Confesses	(0, 10)	(8, 8)

We can analyze the game as follows: No doubt when both will study the situation both will decide to play the first strategy (I, I). But however, with some reflection first prisoner may give reasons as follows: If second prisoner plays his first strategy, then he should play second because he can go free. But when the first prisoner plays his second strategy, another prisoner also decided to play his second strategy. If both prisoners play their second strategy, both get a sentence of 8 years. The pair of strategies (II, II) forms an equilibrium point, because departing from this, neither, without the other doing so, can do better for himself. Therefore, both play the game (II, II)

Bidding Problems

Bidding problems are of two types. They are ***Open or Auction bids*** in which two or more bidders bid on an item of certain value until none is willing to increase the bid. The last bid is then the winner of the bid.

The second one is ***Closed bids*** in which each bidder submits his bid in a closed envelop and the envelopes are opened all at one time and the highest (or lowest) bid is accepted. In this case none knows his opponent's bid.

Problem 10.46.

Two items of worth Rs. 100/- and Rs. 150/- are to be auctioned at a public sale. There are only two bidders *A* and *B*. Bidder A has Rs. 125/- and the bidder *B* has Rs. 155/- with him. If each bidder wants to maximize his own return, what should be his strategy?

Solution

Let each bidder increase the bid successively by λ . At any bid, each player has the option to increase the bid or to leave the opponent's bid stand. If *B* bids Rs. x on the first item (Rs. 100/- value), then *A* has the following options.

If A lets B win the first item, for Rs. x/-, then B will be left with Rs. $(155 - x)$ only for bidding the second item i.e. he cannot make a bid more than Rs. $(155 - x)$ for the second item. Thus A will be positively able to win the second item for Rs. $(155 - x + \lambda)$. Therefore, A's gain by allowing B to win the first item for Rs. x/- will be Rs. $[150 - (155 - x + \lambda)] = (x - \lambda - 5)$.

On the other hand, if A bids Rs. $(x + \lambda)$, for the first item and B lets him to win the bid, then A's gain will be Rs. $([100 - (x + \lambda)] = (100 - x - \lambda)$.

Now since A wants to maximize his return, he should bid Rs. $(x + \lambda)$ for the first item provided $(100 - x - \lambda) \geq (x - \lambda - 5)$ or $x \leq$ Rs. 52.50

Thus A should bid for first item until $x \leq$ Rs. 52.50. In case $x >$ Rs. 5.50, he should allow B to win the first item.

Similarly, B's gains in the two alternatives are: Rs. $[150 - (125 - y) - \lambda]$ and Rs. $(100 - y - \lambda)$, where y denote A's bid for the firs item. Thus B should bid Rs. $(y + \lambda)$ for the first item provided:

$(100 - y - \lambda) \geq [150 - (125 - y) - \lambda]$ or $y \leq$ Rs. 37.50

Obviously, A will win the fist item for Rs. 37.50 because he can increase his bid without any loss up to Rs. 52.50, and B will get the second item for Rs. 125 – Rs. 37.50 = Rs. 87.50 because a, after winning the first item in Rs. 37.50 cannot increase his bid for the second item beyond Rs. 87.50. Thus B will get the second item for Rs. 87.50. Therefore A's gain is Rs. 100 – Rs.37.50 = Rs. 62.50 and B's gain is Rs. 250 – Rs. 87.50 = Rs. 62.50.

Problem 10.47.

Two items of values Rs.100/- and Rs. 120 respectively are to be bid simultaneously by two bidders A and B. Both players intend to devote a lot of sum of Rs. 130 to the two bids. If each uses a minimax criterion, find the resulting bids.

Solution

Here the bids are closed since they are to be made simultaneously. Let A_1 and A_2 are the A's optimum bids for the first and second items respectively. Obviously, A's optimum bids are the ones that fetch the same profit to A on both the items. If p denote the profit earned by the successful bid, then,

$2p = (100 - A_1) + (120 - A_2)$ or $2p = 220 - A_1 - A_2$

Since both A and B intend to devote only Rs. 130/- for both the bids, $A_1 + A_2 =$ Rs. 130/-

Therefore, $2p =$ Rs.(220 – 130) = Rs. 90/- or $p =$ Rs. 45/-

Now $p = 100 - A_1$ or $A_1 = 100 - p = 100 - 45 =$ Rs. 55/-

Also, $p = 120 - A_2$ or $A_2 = 120 - p = 120 - 45 =$ RS. 75/-

Thus optimum bids for A are Rs. 55/- and Rs. 75/- for the first and second items respectively. Likewise, optimum bids for B can be determined and will be Rs 55/- and Rs. 75/- respectively for the two items.

n- Person Zero sum games

Whenever more than two persons are involved in the game, they are treated as if two coalitions are formed by *n* - persons involved. The properties of such games are values of the various games between every possible pair of coalitions. For example, for a player *A*, *B*, *C*, and *D* the following coalitions can be formed:

A against *B*, *C*, *D*;

B against *A*, *C*, *D*;

C against *A*, *B*, *D*;

D against *A*, *B*, *C*;

A, *B* against *C*, *D*;

A, *C* against *B*, *D*;

A, *D* against *B*, *C*.

If the value of the games for *B*, *C*, *D* coalition is *V*, then the value of the game for *A* is – *V*, since it is zero sum game. Thus in a four person zero sum game, there will be seven values or characteristics for the game, which are obtained from the seven different coalitions.

Problem 10. 48.

Find the value of the three person zero sum game in which player *A* has two choices, X_1 and X_2; player *B* has two choices, Y_1, and Y_2 and player *C* has two choices, Z_1 and Z_2. They pay offs are as shown below:

Choices			*Pay offs.*		
A	*B*	*C*	*A*	*B*	*C*
X_1	Y_1	Z_1	3	2	–2
X_1	Y_1	Z_2	0	2	1
X_1	Y_2	Z_1	0	–1	4
X_1	Y_2	Z_2	1	3	–1
X_2	Y_1	Z_1	4	–1	0
X_2	Y_1	Z_2	–1	1	3
X_2	Y_2	Z_1	1	0	2
X_2	Y_2	Z_2	0	2	1

Solution

There are three possible coalitions:

(1). *A* against *B* and *C*; (2). *B* against *A* and *C* and (3). *C* against *A* and *B*.

Now we shall solve the resulting games.

1. *A* against *B* and *C*:

		B and C				
		$Y_1 Z_1$	$Y_1 Z_2$	$Y_2 Z_1$	$Y_2 Z_2$	Row minimum
A	X_1	3	$\boxed{0}$	0	1	**0**
	X_2	4	–1	1	0	–1
Column maximum:		4	**0**	1	1	

The game has saddle point. Hence *A*'s best strategy is X_1 and *B* and *C*'s best combination is $Y_1 Z_2$. Value of the game for $A = 0$, and that of *B* and *C* also equals to zero.

2. *B* against *A* and *C*:

		A and C				
		$X_1 Z_1$	$X_1 Z_2$	$X_2 Z_1$	$X_2 Z_2$	Row minimum
B	Y_1	2	2	–1	1	–1
	Y_2	–1	3	0	2	–1
Column maximum:		2	3	0	2	

Game has no saddle point. First and third columns dominate second and fourth columns respectively hence dominated columns are cancelled. The reduced matrix is:

		A and *C*			
		$X_1 Z_1$	$X_2 Z_1$	Oddments.	Probabilities.
B	Y_1	2	–1	1	1 / 4
	Y_2	–1	0	3	3 / 4
Oddments:		1	3		
Probability		1 / 4	3 / 4		

By solving the method of oddments, *B*'s best strategy is Y_1 with a probability of 1 / 4 and choice Y_2 with a probability of 3 / 4. Now *A* and *C* has to play X_1 and Z_1 with a probability of 1 / 4 and X_2 and Z_1 with a probability of 3 / 4.

Value of game for $B = [(2/4) - (3/4)] / [(1/4 + (3/4)] = -(1/4)$

Value of game for A and C is (1 / 4)

3. C against A and B. Pay of matrix is:

		A and *B*				
		$X_1 Y_1$	$X_1 Y_2$	$X_2 Y_1$	$X_2 Y_2$	Row minimum
C	Z_1	–2	4	0	2	–2
	Z_2	1	–1	3	1	–1
Column Maximum:		1	4	3	2	

The game has no saddle point. First column dominates third and fourth column. Hence the reduced matrix is:

		A and *B*			
		$X_1\ Y_1$	$X_1\ Y_2$	oddment	Probability
C	Z_1	–2	4	2	2 / 8
	Z_2	1	–1	6	6 / 8
Oddment		5	3		
Probability.		5 / 8	3 / 8		

C's best strategy is to play $(Z_1, Z_2) = [(2/8), (6/8)]$

For A and $B = (X_1\ Y_1, X_1\ Y_2) = [(5/8), (3/8)]$

Value of the game for $C = [-(10/8) + (12/8)] / [(5/8) + (3/8)] = (2/8) / 1 = (1/4)$

Value of the game for A and B is $-(1/4)$ as this is a zero sum game.

Therefore, the characteristics of the game are:

$V(A) = 0$, $V(B) = -(1/4)$, $V(C) = (1/4)$ and $V(B, C) = 0$, $V(A, C) = (1/4)$. $V(A, B) = -(1/4)$

QUESTIONS

1. What are competitive situations? Explain with the help of an example.
2. What is a business game? Enlist the properties of the game. What assumptions are made in game theory?
3. Explain Maximin and Minimax principle with respect to game theory.
4. By means of an example, explain what do you mean by Two Person Zero Sum game.
5. Solve the following game, whose pay of matrix is:

		B		
		I	II	III
	I	–2	15	–2
A	II	–5	–6	–4
	III	–5	20	–8

6. Solve the game whose pay of matrix is:

		B			
		I	II	III	IV
	I	3	2	4	0
A	II	2	4	2	4
	III	4	2	4	0
	IV	0	4	0	8

7. Explain the theory of Dominance in solving a given game.
8. Explain the graphical method of solving a game.
9. Solve the following game:

		B				
		1	2	3	4	5
A	1	2	–1	5	–2	6
	2	–2	4	–3	1	0

10. In a small town, there are two discount stores *ABC* and *XYZ*. They are the only stores that handle Sunday goods. The total number of customers is equally divided between the two; because the price and quality are equal. Both stores have good reputations in the community, and they render equally good customer service. Assume that a gain customer by *ABC* is a loss to *XYZ* and vice versa. Both stores plan to run annual pre Diwali sales during the first week of the month in which Diwali falls. Sales are advertised through local newspaper, radio and television media. With aid of an advertising firm *ABC* Store constructed the game matrix given below, which gives the gain and loss to each customer. Find the optimal strategies of the stores.

		XYZ		
		Newspaper	Radio	television.
	Newspaper	30	40	–80
ABC	Radio	0	15	–20
	Television.	90	20	50

11. Players *A* and *B* play the following game. *A* has a bag containing three coins, one worth 4 units, one 6 units and the rest 9 units of money. *A* takes one coin from the bag and before exposure *B* guesses. If *B* is right he takes the coin and if wrong he pays to *A* the same worth money to *A*. Find the optima strategies of *A* and *B* and the value of the game.

Game Theory: MULTIPLE CHOICE QUESTIONS

1. If the value of the game is zero, then the game is known as:
   (a) Fair strategy (*b*) Pure strategy
   (*c*) Pure game (*d*) Mixed strategy. ( )
2. The games with saddle points are:
   (a) Probabilistic in nature, (*b*) Normative in nature
   (*c*) Stochastic in nature, (*d*) Deterministic in nature. ( )
3. When Minimax and Maximin criteria matches, then
   (a) Fair game is exists. (*b*) Unfair game is exists,
   (*c*) Mixed strategy exists (*d*) Saddle point exists. ( )
4. When the game is played on a predetermined course of action, which does not change throughout game, then the game is said to be
   (a) Pure strategy game, (*b*) Fair strategy game
   (*c*) Mixed strategy game (*d*) Unsteady game. ( )
5. If the losses of player *A* are the gins of the player *B*, then the game is known as:
   (a) Fair game (*b*) Unfair game
   (*c*) Non- zero sum game (*d*) Zero sum game. ( )
6. Identify the wrong statement:
   (a) Game without saddle point is probabilistic
   (b) Game with saddle point will have pure strategies
   (c) Game with saddle point cannot be solved by dominance rule.
   (d) Game without saddle point uses mixed strategies, ( )
7. In a two person zero sum game, the following does not hold correct:
   (a) Row player is always a loser; (*b*) Column Player is always a winner.
   (c) Column player always minimizes losses
   (d) If one loses, the other gains. ( )
8. If a two person zero sum game is converted to a Linear Programming Problem,
   (a) Number of variables must be two only,
   (b) There will be no objective function,
   (c) If row player represents Primal problem, Column player represent Dual problem,
   (d) Number of constraints is two only. ( )

9. In case, there is no saddle point in a game then the game is
   (a) Deterministic game, (*b*) Fair game,
   (*c*) Mixed strategy game, (*d*) Multi player game. ( )
10. When there is dominance in a game then
   (a) Least of the row ≥ highest of another row
   (b) Least of the row ≤ highest of another row
   (c) Every element of a row ≥ corresponding element of another row.
   (d) Every element of the row ≤ corresponding element of another row. ( )
11. When the game is not having a saddle point, then the following method is used to solve the game:
   (a) Linear Programming method, (*b*) Minimax and maximin criteria
   (*c*) Algebraic method (*d*) Graphical method. ( )
12. Consider the matrix given, which is a pay off matrix of a game. Identify the dominance in it.

		B		
		X	Y	Z
	P	1	7	3
A	Q	5	6	4
	R	7	2	0

   (a) P dominates Q (*b*) Y dominates Z
   (*c*) Q dominates R (*d*) Z dominates Y ( )
13. Identify the unfair game:

(*a*)

	C	D
A	0	0
B	0	0

(*b*)

	C	D
A	1	–1
B	–1	1

(*c*)

	C	D
A	–5	+5
B	+ 10	– 10

(*d*)

	C	D
A	1	0
B	0	1

( )

14. 14If there are more than two persons in a game then the game is known as:
   (*a*) Non zero sum game (*b*) Open game
   (*c*) Multiplayer game (*d*) Big game ( )
15. For the pay of matrix the player *A* always uses:

		B	
		I	II
A	I	–5	–2
	II	10	5

   (*a*) First strategy (*b*) Mixed strategy of both II and I
   (*c*) Does not play game (*d*) Second strategy. ( )

16. For the pay off matrix the player prefers to play

$$\begin{array}{cc|cc} & & \multicolumn{2}{c}{B} \\ & & I & II \\ \hline & I & -7 & 6 \\ A & & & \\ & II & -10 & 8 \end{array}$$

(*a*) Second strategy (b) First strategy

(*c*) Keep quite (c) Mixed strategy. ( )

17. For the game given the value is:

$$\begin{array}{cc|cc} & & \multicolumn{2}{c}{B} \\ & & I & II \\ \hline & 1 & 2 & 3 \\ A & & & \\ & II & -5 & 5 \end{array}$$

(*a*) 3, (*b*) –5

(*c*) 5 (*d*) 2 ( )

19. In the game given the saddle point is:

$$\begin{array}{cc|ccc} & & & B & \\ & & I & II & III \\ \hline & I & 2 & -4 & 6 \\ A & II & 0 & -3 & -2 \\ & III & 3 & -5 & 4 \end{array}$$

(*a*) –2 (*b*) 0

(*c*) –3 (*d*) 2 ( )

20. A competitive situation is known as:

(*a*) Competition (*b*) Marketing

(*c*) Game (*d*) None of the above. ( )

21. One of the assumptions in the game theory is:

(a) All players act rationally and intelligently,

(b) Winner alone acts rationally

(c) Loser acts intelligently,

(d) Both the players believe luck ( )

22. A play is played when:
    (*a*) The manager gives green signal
    (*b*) Each player chooses one of his courses of action simultaneously
    (*c*) The player who comes to the place first says that he will start the game
    (*d*) When the latecomer says that he starts the game. ( )
23. The list of courses of action with each player ....
    (*a*) Is finite
    (*b*) Number of strategies with each player must be same
    (*c*) Number of strategies with each player need not be same
    (*d*) None of the above. ( )
24. A game involving '*n*' persons is known as:
    (*a*) Multi member game (*b*) Multi player game
    (*c*) *n* - person game (*d*) not a game. ( )
25. Theory of games and economic behavior is published by:
    (*a*) John Von Neumann and Morgenstern (*b*) John Flood
    (*c*) Bellman and Neumann (*d*) Mr. Erlang, ( )
26. In the matrix of a game given below the negative entries are:

$$\begin{array}{ccc|cc} & & & \multicolumn{2}{c}{B} \\ & & & \text{I} & \text{II} \\ \hline \multirow{2}{*}{A} & \text{I} & & 1 & -1 \\ & \text{II} & & -1 & 1 \end{array}$$

    (*a*) Payments from *A* to *B* (*b*) Payments from *B* to *A*
    (*c*) Payment from players to organizers (*d*) Payment to players from organizers. ( )

ANSWERS

1. (*c*)	2. (*d*)	3. (*d*)	4. (*a*)
5. (*d*)	6. (*c*)	7. (*a*)	8. (*c*)
9. (*c*)	10. (*d*)	11. (*b*)	12. (*d*)
13. (*d*)	14. (*c*)	15. (*d*)	16. (*b*)
17. (*d*)	18. (*c*)	19. (*c*)	20. (*c*)
21. (*a*)	22. (*b*)	23. (*c*)	24. (*c*)
25. (*a*)	26. (*a*)		

***CHAPTER* – 11**

Dynamic Programming

INTRODUCTION

In previous chapters, we have seen how to solve the problems, where decision is made in single stage, *i.e*. one time period. But we may come across situations, where we may have to make decision in multistage, *i.e*. optimization of multistage decision problems. Dynamic programming is a technique for getting solutions for multistage decision problems. A problem, in which the decision has to be made at **successive stages,** is called a **multistage decision problem**. In this case, the problem solver will take decision at every stage, so that the total effectiveness defined over all the stages is optimal. Here the original problem is broken down or decomposed into small problems, which are known as **sub problems** or **stages** which is much convenient to handle and to find the optimal stage. For example, consider the problem of a sales manager, who wants to start from his head office and tour various branches of the company and reach the last branch. He has to plan his tour in such a way that he has to visit more number of branches and cover less distance as far as possible. He has to divide the network of the route connecting all the branches into various stages and workout, which is the best route, which will help him to cover more branches and less distance. We can give plenty of business examples, which are multistage decision problems. The technique of Dynamic programming was developed by Richard Bellman in the early 1950.

The computational technique used is known as **Dynamic Programming** or **Recursive Optimization**. We do not have a standard mathematical formulation of the Dynamic Programming Problem (D.P.P). For each problem, depending on the variables given, and objective of the problem, one has to develop a particular equation to fit for situation. Though we have quite good number of dynamic programming problems, sometimes to take advantage of dynamic programming, we introduce multistage nature in the problem and solve it by dynamic programming technique. Nowadays, application of Dynamic Programming is done in almost all day to day managerial problems, such as, inventory problems, waiting line problems, resource allocation problems etc. Dynamic programming problem may be classified depending on the following conditions.

(i) Dynamic programming problems may be classified depending on the nature of data available as *Deterministic and Stochastic or Probabilistic models*. In deterministic models, the outcome at any decision stage is unique, determined and known. In Probabilistic models, there is a set of possible outcomes with some probability distribution.

(ii) The possible decisions at any stage, from which we are to choose one, are called '**states**'. These may be finite or infinite. States are the possible situations in which the system may be at any stage.

(iii) Total number of stages in the process may be finite or infinite and may be known or unknown.

Now let us try to understand certain terms, which we come across very often in this chapter.

Stage: A stage signifies a portion of the total problem for which a decision can be taken. At each stage there are a number of alternatives, and the best out of those is called **stage decision**, which may be optimal for that stage, but contributes to obtain the optimal decision policy.

State: The condition of the decision process at a stage is called its state. The variables, which specify the condition of the decision process, *i.e*. describes the **status** of the system at a particular stage are called **state variables**. The number of state variables should be as small as possible, since larger the number of the state variables, more complicated is the decision process.

Policy: A rule, which determines the decision at each stage, is known as Policy. A policy is optimal one, if the decision is made at each stage in a way that the result of the decision is optimal over all the stages and not only for the current stage.

Principle of Optimality: Bellman's Principle of optimality states that ''**An optimal policy (a sequence of decisions) has the property that whatever the initial state and decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.**''

This principle implies that a wrong decision taken at a stage does not prevent from taking optimal decision for the reaming stages. This principle is the firm base for dynamic programming technique. In the light of this, we can write a recurrence relation, which enables us to take the optimal decision at each stage.

Steps in getting the solution for dynamic programming problem:

- Mathematical formulation of the problem and to write the recursive equation (recursive relation connecting the optimal decision function for the 'n' stage problem with the optimal decision function for the $(n - 1)$ stage subproblems).
- To write the relation giving the optimal decision function for one stage subproblem and solve it.
- To solve the optimal decision function for 2-stage, 3-stage ....$(n - 1)$ stage and then n-stage problem.

COMPUTATIONAL PROCEDURE IN DYNAMIC PROGRAMMING

Discrete or Continuous systems: There are two ways of solving (computational procedure) recursive equations depending on the type of the system. If the system is continuous one the procedure is different and if the system is discrete, we use a different method of computation. If the system is discrete, a tabular computational scheme is followed at each stage. The number of rows in each table is equal to the number of corresponding feasible state values and the number of columns is equal to the number of possible decisions. In case of continuous system, the optimal decision at each stage is obtained by using the usual classical technique such as differentiation etc.

- **Forward and Backward Equations:** If there are 'n' stages, and recursive equations for each stage is f_1, f_2 ....f_n and if they are solved in the order f_1 to f_n and optimal return for f_1 is r_1 and that of f_2 is r_2 and so on, then the method of calculation is known as **forward computational procedure**.

- On the other hand, if they are solved in the order from f_n, f_{n-1}, ….. f_1, then the method is termed as **backward computational procedure**. (e.g. Solution to L.P.P. by dynamic programming).

The Algorithm

- Identify the decision variables and specify objective function to be optimized under certain limitations, if any.
- Decompose or divide the given problem into a number of smaller sub-problems or stages. Identify the state variables at each stage and write down the transformation function as a function of the state variable and decision variables at the next stage.
- Write down the general recursive relationship for computing the optimal policy. Decide whether forward or backward method is to follow to solve the problem.
- Construct appropriate stage to show the required values of the return function at each stage.
- Determine the overall optimal policy or decisions and its value at each stage. There may be more than one such optimal policy.

CHARACTERISTICS OF DYNAMIC PROGRAMMING

The basic features, which characterize the dynamic programming problem, are as follows:

(i) Problem can be sub-divided into stages with a policy decision required at each stage. A stage is a device to sequence the decisions. That is, it decomposes a problem into sub-problems such that an optimal solution to the problem can be obtained from the optimal solution to the sub-problem.

(ii) Every stage consists of a number of states associated with it. The states are the different possible conditions in which the system may find itself at that stage of the problem.

(iii) Decision at each stage converts the current stage into state associated with the next stage.

(iv) The state of the system at a stage is described by a set of variables, called **state variables**.

(v) When the current state is known, an optimal policy for the remaining stages is independent of the policy of the previous ones.

(vi) To identify the optimum policy for each state of the system, a recursive equation is formulated with 'n' stages remaining, given the optimal policy for each stage with (n – 1) stages left.

(vii) Using recursive equation approach each time the solution procedure moves backward, stage by stage for obtaining the optimum policy of each stage for that particular stage, still it attains the optimum policy beginning at the initial stage.

PROBLEMS

Problem 11.1. (Product allocation problem)

A company has 8 salesmen, who have to be allocated to four marketing zones. The return of profit from each zone depends upon the number of salesmen working that zone. The expected returns for different number of salesmen in different zones, as estimated from the past records, are given below. Determine the optimal allocation policy.

	SALES	*MARKETING IN*	*ZONES Rs. X 000*	
No. of . Salesmen	*Zone 1*	*Zone 2*	*Zone 3*	*Zone 4*
0	45	30	35	42
1	58	45	45	54
2	70	60	52	60
3	82	70	64	70
4	93	79	72	82
5	101	90	82	95
6	108	98	93	102
7	113	105	98	110
8	118	110	100	110

Solution

The problem here is how many salesmen are to be allocated to each zone to maximize the total return. In this problem each zone can be considered as a **stage**, number of salesmen in each stage as **decision** variables. Number of salesmen available for allocation at a stage is the **state variable** of the problem.

Here let us consider the first stage (zone 1) and add to it the second stage (zone 2) and see what will be the optimal return and optimal allocation. Remember, that allocation of salesmen for each zone may be 0, 1, 2, …and 8. See the table below to understand how we can allocate salesmen between zones 1 and 2.

In this problem, decision policy requires making four interrelated decisions. What should be the number of salesmen in each of the four marketing zones? If x_1, x_2, x_3 and x_4 are the number of salesmen allocated to the four zones and $f_1(x_1)$,… $f_4(x_4)$ are respectively the returns from the four zones, then the objective function is

Maximize $Z = f_1(x_1) + f_2(x_2) + f_3(x_3) + f_4(x_4)$

Subject to: $x_1 + x_2 + x_3 + x_4 \leq 8$ and x_1, x_2, x_3 and x_4 are non-negative integers.

Or can be written as: Maximize $Z = \sum_{i=1}^{4} f_i(x_i)$ s.t. $\sum_{i=1}^{4} x_i = 8$ where all x_i are nonnegative integers.

No. of Salesmen in zone 1.	0	1	2	3	4	5	6	7	8
No. of Salesmen in zone 2.	8	7	6	5	4	3	2	1	0

Construct a table to calculate the return from the above combination.

Zone 1 → Salesmen		0	1	2	3	4	5	6	7	8
Return		45	58	70	82	93	101	108	113	118
Zone 2 ↓ Salesmen	Return									
0	30	**75**	88	100	112	123	141	138	143	148
1	45	**90**	103	115	127	138	146	153	158	
2	60	**105**	**118**	**130**	**142**	**153**	161	168		
3	70	115	128	140	152	**163**	171			
4	79	124	137	149	161	**172**				
5	90	135	148	160	**172**					
6	98	143	156	168						
7	105	150	163							
8	110	155								

Procedure: If we want to allocate zero salesmen, then zero to zone 1 and zero to zone 2 and the total outcome is 30 + 45 = Rs. 75 × 1000. This is written in the table where lines from zero from zone 1 and zone 2 intersect. As this is the only entry in the diagonal line it is made bold.

When company wants to allocate 1 salesman to two zones, the allocation is zero to zone 1 and 1 to zone 2 or 1 to zone 1 and zero to zone 2. The outcomes are entered where the horizontals from zone 2 and verticals from zone 1 intersect. Higher number is written in bold numbers. In this example, the outcomes are 90 and 88, 90 is written in bold. Similarly we have to allocate 8 salesmen and write the outcomes and bold the highest outcome in the diagonal. Sometimes, it may happen that there may be two or more same numbers indicating highest outcome. All these are written in bold letter. (Note: Instead on writing highest in bold letter, we can encircle the element or enclose it in a square or superscribe with a star.)

Now let us write the outcomes below:

Number of salesmen.	0	1	2	3	4	5	6	7	8	
Zone 1	0	0	0	1	2	3	4	4	4	3
Zone 2	0	1	2	2	2	2	2`	3	4	5
Outcome in Rs. × 1000	75	90	105	118	130	142	153	162	172	172

Now in the second stage, let us combine zone 3 and zone 4 and get the total market returns.

Combination of zone 3 and zone 4:

Zone 3 → Salesmen		0	1	2	3	4	5	6	7	8
Return.		35	45	52	64	72	82	93	98	100
Zone 4 Salesmen	↓ Return.									
0	42	**77**	**97**	94	**106**	114	124	136	140	142
1	54	89	**99**	**106**	**118**	126	136	**147**	152	
2	60	95	105	112	124	132	142	153		
3	70	105	115	122	134	142	152			
4	82	117	127	134	146	154				
5	95	**130**	**140**	**147**	**159**					
6	102	137	**147**	154						
7	110	145	155							
8	110	145								

Now the table below shows the allocation and the outcomes for zone 3 and zone 4.'

Number of Salesmen	0	1	2	3	4	5	6	7		8
Zone 3	0	1	1	2	3	5	1	1	2	3
Zone 4	0	0	1	1	1	0	5	6	5	5
Return in Rs. × 1000	77	97	99	106	118	130	140	147	147	159

In third stage we combine both zones 1 & 2 outcomes and zones 3 and 4 outcomes.

Zones 1 and 2 and zones 3 and 4 combined.

Zones 1& 2 → Salesmen		(0, 0) 0	(0, 1) 1	(0, 2) 2	(1, 2) 3	(2, 2) 4	(3, 2) 5	(4, 2) 6	(4, 3) 7	(4, 4)(3, 5) 8
Return.		75	90	105	118	130	142	153	163	172
Zones 3 & 4 Salesmen	Return ↓									
0(0,0)	77	**152**	**187**	182	195	207	219	230	240	247
1(1,0)	97	172	**187**	**202**	**215**	**302**	**239**	**243**	**260**	
2(1,1)	99	174	189	204	217	229	241	252		
3(2,1)	106	181	196	211	224	236	248			
4 (3,1)	118	193	208	223	236	248				
5(5,0)	130	205	220	235	248					
6(1,5)	140	215	230	245						
7(1,6)										
(2,5)	147	222	237							
8(3,5)	159	234								

Optimal allocation is:

Salesmen	0	1	2	3	4	5	6	7	8
Zone 1	0	0	1	0	1	2	3	4	4
Zone 2	0	0	0	2	2	2	2	2	3
Zone 3	0	1	1	1	1	1	1	1	1
Zone 4	0	0	0	0	0	0	0	0	0
Total return in Rs. × 1000	152	187	187	202	215	302	239	243	260

The above table shows that how salesmen are allocated to various zones and the optimal outcome for the allocation. **Maximum outcome is Rs. 260 × 1000.**

Note: Students may try different combinations, *i.e*. first combining zone 1 and zone 3 and then zones 2 and 4 and then combining both. Then also the optimal outcome will be same. OR add 1 and 2 zones, then add zone 3 and then zone 4 to it. Then also the optimal outcome will be same.

Problem 11.2.

The owner of a chain of four grocery stores has purchased six crates of fresh strawberries. The estimated probability distribution of potential sales of the strawberries before spoilage differs among the four stores. The following table gives the estimated total expected profit at each store, when it is allocated various numbers of crates:

Stores.

Number of Crates	1	2	3	4
0	0	0	0	0
1	4	2	6	2
2	6	4	8	3
3	7	6	8	4
4	7	8	8	4
5	7	9	8	4
6	1	10	8	4

For administrative reasons, the owner does not wish to split crates between stores. However he is willing to distribute zero crates to any of his stores.

Solution

Let the four stores be considered as four stages in dynamic programming formulation. The decision variables x_i (i = 1, 2, 3 and 4) denote the number of crates allocated to the i th stage. Let f (x_i) be the expected profit from allocation of x_i crates to the store 'i', then the problem is:

Maximize $Z = f_1(x_1) + f_2(x_2) + f_3(x_3) + f_4(x_4)$ subject to

$x_1 + x_2 + x_3 + x_4 = 6$ and all $x_i \geq 0$

Store 3 →

Store 4 ↓

No. of crates		0	1	2	3	4	5	6
Profit. →		0	6	8	8	8	8	8
No. of crates	Profit. ↓							
0	0	0	**6**	**8**	8	8	8	8
1	2	2	3	**10**	10	10	10	
2	3	3	9	**11**	11	11		
3	4	4	10	**12**	**12**			
4	4	4	10	**12**				
5	4	4	10					
6	4	4						

No. of crates	0	1	2	3	4	5	6	
Store 3	0	1	2	2	2	2	3	2
Store 4	0	0	0	1	2	3	3	4
Profit	0	6	8	10	11	12	12	

Store 2 ↓

Store 1 →

No. of crates		0	1	2	3	4	5	6
Profit →		0	4	6	7	7	7	7
No. of crates	Profit ↓							
0	0	**0**	**4**	**6**	7	7	7	7
1	2	2	**6**	**8**	9	9	9	
2	4	4	**8**	**10**	11	11		
3	6	6	**10**	**12**	12			
4	8	8	**12**	**14**				
5	9	9	13					
6	10	10						

No. of crates	0	1	2		3		4		5		6
Store 1	0	1	2	1	2	1	2	1	2	1	2
Store 2	0	0	0	1	1	2	2	3	3	4	4
Profit.	0	4	6	6	8	8	10	10	12	12	14

Stores 3 & 4

Stores 1 & 2

No. of crates		0 0, 0	1 1, 0	2 2, 0	3 2, 1	4 2,2	5 2, 3	6 3 2
Profit →		0	6	8	10	11	12	12
No. of crates	↓ Profit							
0 (0, 0)	0	0	6	8	10	11	12	12
1 (1, 0)	4	4	10	12	14	15	16	
2 (2, 0), (1, 1)	6	6	12	14	16	17		
3 (2, 1), (1, 2)	8	8	14	16	18			
4 (2,2), (1, 3)	10	10	16	18				
5(2, 3), (1, 4)	12	12	18					
6 (2, 4)	14	14						

All the four stores combined at 3 rd stage.

No. of crates	0	1	2	3			4					5						6					
Store 1				2	1	1	2	1	2	1	1	2	1	2	1	2	1	2	1	2	1	2	1
Store 2				0	1	0	1	2	0	1	0	2	3	1	2	0	1	3	4	2	3	1	2
Store 3				1	1	2	1	1	2	2	2	1	1	2	2	2	2	1	1	2	2	2	2
Store 4				0	0	0	0	0	0	0	1	0	0	0	0	1	1	0	0	0	0	1	1
Profit.	0	6	10	12			14					16						18					

Maximum profit is Rs. 18/-

Problem 11.3. (Cargo load problem)

A vessel is to be loaded with stocks of 3 items. Each item '*i*' has a weight of w_i and a value of v_i. The maximum cargo weight the vessel can take is 5 and the details of the three items arte as follows:

j	w_j	v_j
1	1	30
2	3	80
3	2	65

Develop the recursive equation for the above case and find the most valuable cargo load without exceeding the maximum cargo weight by using dynamic programming.

Solution

Let us represent the three items as x_j (j = 1, 2, 3) and we have to take decision how much of each item is to be loaded into the vessel to fulfil the objective. Let $f_j\,(x_j)$ is the value of optimal allocation for the three items, and if $f_j(s, x_j)$ is the value associated with the optimum solution $f_j^*(s)$ for (j = 1, 2 and 3) then the objective function is:

$$f_j^*(\text{s}) = \underset{0 \le x_j \le s}{\text{Max}}\, f_1\ (s, x_j) \text{ and}$$

$f_j^*(\text{s}) = \text{Max}\,[p_j(x_j) + f_{j-1}^*(s - x_j)$, for j = 1, 2, 3 and $p_j(x_j)$ is the expected value obtained form allocation of x_j units of weight to the item 'j'.

As there are three items this is a three-stage problem. First let us allocate the item number 1 and see what is the outcome. For the first stage, *i.e.* loading one item in the cargo we have: $f_1^*(s) = \underset{x_1}{\text{Max}}\,[30x_1]$

Now as the weight value of item number 1 is 1 (= w_1) only and the maximum load (W) that can be loaded is 5 the largest value of item number one that can be loaded is = W/w_1 = 5/1 = 5. The tabular computation for stage 1 is:

Optimum Soln.

$x_1 \rightarrow$ / $\downarrow$	*0*	*1*	*2*	*3*	*4*	*5*	$f_1^*(s)$	x_1^*
s	-	-	-	-	-	-	-	-
0	0						0	0
1	0	30					30	1
2	0	30	60				60	2
3	0	30	60	90			90	3
4	0	30	60	90	120		120	4
5	0	30	60	90	120	150	150	5

The entries in the above table are obtained as follows: As the five items can be loaded as W/w_1 = 5, when load is zero the value is 30 × 0 = 0, when load is 1, value 30 × 1 = 30 and so on. The maximum in the row is written in 8th column, *i.e.* 0, 30, 60, ….., 150. And the load for that weight is written in the last column. Similarly we can write for item number 2.

$f_2^*(s) = \underset{x_2}{\text{Max}}\,[80\, x_2 + f_1\,(s - 3\, x_2)$ as the weight of item 2 is 3.

Specimen calculations:

For zero load: [80 × 0 + f_1 (0 + 3 × 0) = 0

For load 1: [80 + 0] = 80

The load of item 2 that can be loaded is W/w_2 = 5/3 = 1. Hence in the table for x_2 only 0 and 1 are shown.

Value of $80\, x_2 + \; f_1^* \,(s - 3x_2)$

Optimum Soln.

$x_2 \rightarrow$ / $s \downarrow$	0	1					$f_2^*\,(s)$	x_2^*
s	-	-	-	-	-	-	-	-
0	0 + 0 = 0						0	0
1	0 + 30 = 30						30	0
2	0 + 60 = 60						60	0
3	0 + 90 = 90	80 + 0 = 80					90	0
4	0 + 120 = 120	80 + 30 = 110					120	0
5	0 = 150 = 150	80 + 60 = 140					150	0

As all the maximum values are due to item number 1, the item number 2 is not loaded into the cargo. Hence here $x_2 = 0$.

For stage 3, the items that can be loaded into cargo is $S/w_3 = 5/2 = 2$. Hence 0, 1, 2 are shown in the table.

$f_3^* = \text{Max}\,[65x_3 + \; f_2^*\,(s - 2x_3)$

Optimum Solution.

$x_3 \rightarrow$ / $s \downarrow$	0	1	2				$f_3^*\,(s)$	x_3^*
s	-	-	-	-	-	-	-	-
0	0 + 0 = 0						0	0
1	0 + 30 = 30						30	0
2	0 + 60 = 60	65 + 0 = 65					65	1
3	0 + 90 = 90	65 + 35 = 95					95	1
4	0 + 120 = 120	65 + 60 = 125	130 + 0 = 130				130	2
5	0 + 150 = 150	65 + 90 = 155	130 + 30 = 160				160	2

In the above table, $x_1 = 1$, $x_2 = 0$ and $x_3 = 2$ and the maximum value is 160, therefore answer is:

$\mathbf{x^*_1 = 1, x^*_3 = 2}$ **and** $\mathbf{f^*_3 = 160}$.

This problem may be done in another way as shown below:

Method 2

Maximize $30x_1 + 80x_2 + 65x_3$ s.t.

$1x_1 + 3x_2 + 2x_3 \leq 5$ and all x_is are ≥ 0. And maximum number (= W/w) of each item is $x_1 = 5/1 = 5$, $x_2 = 5/3 = 1$ and $x_3 = 5/2 = 2$ where 5 = maximum load (W) and denominators are item load (w).

First let us load item x_1 and item x_2

w_2	x_2	v_2	w_1	0	1	2	3	4	5
$\downarrow$	$\downarrow$	$\downarrow$	x_1	0	1	2	3	4	5
			v_1	**0** (0)	**30** (1)	60	**90** (3)	120	150
0	0	0		0	30	60	90	**120** (4)	**150** (5)
3	1	80		80 (3)	110 (4)	140 (5)			

Now we have maximum value for combination of x_1 and x_2. For this let us add x_3. In the above table, for $x_1 = 0$ and $x_2 = 0$ the weight is zero and value is zero shown in block letter. When $x_1 = 1$ and $x_2 = 0$, the value is 30 shown in block letter. Similarly for weights 3, 4, and 5 are shown in brackets and the maximum of the value is shown in block letter.

Combination of x_1, x_2 with x_3.

		w_{12}	0	1	2	3	4	5
		$x_1 x_2$	0	1	2	3	4	5
		v_{12}	0	30	60	90	120	150
w_3	x_3	v_3						
0	0	0	**0**(0)	30	60	90	120	150
2	1	65	**65 (2)**	**95** (3)	125 (4)	155 (5)		
4	2	130	**130 (4)**	**160** (5)				

From the table, $x_1 = 1$, $x_2 = 0$ and $x_3 = 2$ substituting in inequalities, we get $30 \times 1 + 80 \times 0 + 65 \times 2 = 160$ and $1 \times 1 + 3 \times 0 + 2 \times 2 = 5$. The condition required is satisfied.

Problem 11.4

In a cargo-loading problem, there are four items of different weight per unit and value as shown below. The maximum cargo load is restricted to 17 units. How many units of each item is loaded to maximize the value?

Item (i)	*Weight* (w_1)	*Value* (v_1)
1	1	1
2	3	5
3	4	7
4	6	11

Solution

Let x_1, x_2, x_3 and x_4 be the items loaded then we have to maximize sum of $x_i v_i$, *i.e.*

Maximize Z = $a_1 x_1 + a_2 x_2 + a_2 x_3 + a_4 x_4$ s.t.

$$a_1 x_1 + a_2 x_2 + a_2 x_3 + a_4 x_4 \leq 17 \text{ and all } a_i \text{ are } \geq 0.$$

For item number 1$f_1(x_1)$ = Max $[a_1 v_1]$ where the value of a_1 may be anything between maximum weight (W) allowed divided by item weight (w). Here $W/w = 17/1 = 17$.

For item number 2, $f_2(x_2) = \text{Max}\,[a_2 v_2 + f_1(x_2 - a_2 w_2)]$

For item number 3, $f_3(x_3) = \text{Max}\,[a_3 v_3 + f_2(x_3 - a_3 w_3)]$

For item number 4, $f_4(x_4) = \text{Max}\,[a_4 v_4 + f_3(x_4 - a_4 w_4)]$

In general, for item 'i' $f_i(x_i)$ = Max. [$av + f_{i-1}(x_i - a_i w_i)$ is the recursive equation.

In the given problem the recursive equations are:

1. $x_1 v_1$
2. $5x_2 + f_1(x_2 - 3x_2)$
3. $7x_3 + f_2(x_3 - 4x_3)$
4. $11x_4 + f_3(x_4 - 6x_4)$

Substituting the value stage by stage, the values are tabulated in the table given below: (Remember as there are 4 items, this is a 4-stage problem.

x_i	*Stage* 1	$= x_1 v$	*Stage* 2	$= 5x_2 + f_1(x_2 - 3x_2)$	*Stage* 3	$= 7x_3 + f_2(x_3 - 4x_3)$	*Stage* 4	$= 11x_4 + f_3(x_4 - 6x_4)$	
	$w_1 = 1$	$v_1 = 1$	$w_2 = 3$	$v_2 = 5$	$w_3 = 4$	$v_3 = 7$	$w_4 = 6$	$v_4 = 11$	$F_I^*(x_I)$
	x_1	$f_1(x_1)$	x_2	$f_2(x_2)$	x_3	$f(x_3)$	x_4	$f_4(x_4)$	
0	0	**0**	0		0		0		0
1	1	**1**	0		0		0		1
2	2	**2**	0		0		0		2
3	3	3	1	**5 + 0 = 5**	0		0		5
4	4	4	1	5 + 1 = 6	1	**7 + 0 = 7**	0		7
5	5	5	1`	5 + 2 = 7	1	**7 + 1 = 8**	0		8
6	6	6	2	10 + 0 = 10	1	**7 + 2 = 9**	1	**11 + 0 = 11**	11
7	7	7	2	1 0 + 1 = 11	1	**7 + 5 = 12**	1	**11 + 1 = 12**	12
8	8	8	2	10 + 2 = 12	2	**14 + 0 = 14**	1	11 + 2 = 13	14
9	9	9	3	15 + 0 = 15	2	14 = 1 = 15	1	**11 + 5 = 16**	16
10	10	10	3	15 + 1 = 16	2	14 + 2 = 16	1	**11 + 7 = 18**	18
11	11	11	3	15 + 2 = 17	2	**14 + 5 = 19**	1	**11 + 8 = 19**	19
12	12	12	4	20 + 0 = 20	3	21 + 0 = 21	2	**22 + 0 = 22**	22
13	13	13	4	20 + 1 = 21	3	21 + 1 = 22	2	**22 + 1 = 23**	23
14	14	14	4	20 + 2 = 22	3	21 + 2 = 23	2	**22 + 2 = 24**	24
15	15	15	5	25 + 0 = 25	3	21 + 5 = 26	2	**22 + 5 = 27**	27
16	16	16	5	25 + 1 = 26	4	28 + 0 = 28	2	**22 + 7 = 29**	29
17	17	17	5	25 + 2 = 27	4	28 + 1 = 29	2	**22 + 8 = 30**	30

For $x_4 = 17$, Optimal return = $f_4^*(17) = 30$ for $x_4^* = 2$

For $x_3 = 17 - 2 \times 6 = 5$, Optimal return = $f_3^*(5) = 8$ for $x_3^* = 1$

For $x_2 = 5 - (1 \times 4) = 1$, Optimal return = $f_2^*(1) = 0$ for $x_2 = 0$

For $x_1 = 1 - 0 = 1$, Optimal return = $f_1^*(1) = 1$ for $x_1 = 1$.

Answer: To maximize the value of the cargo load 1 unit of item 1, 1 unit of item 3 and 2 units of item 4. The maximum value of the cargo is 30.

This problem can also be done in the same manner as the previous one. The only difficulty here is that the maximum weight is 17, we will get a very big table. The above method is more easy when the given maximum weight is more.

Problem 11.5.

In a cargo-loading problem, there are four items of different unit weight and value. The maximum cargo load is 6 units. How many units of each item are loaded to maximize the value?

Item	*Weight* (w_i)	*Value per unit.*
1	1	1
2	3	3
3	4	5
4	4	4

Solution

The model is: Maximize $Z = 1a + 3b + 5c + 4d$ subject to

$1a + 3b + 4c + 4d \leq 6$ units. And a, b, c, d all ≥ 0.

Number of units of 'a' = W /w = 6 / 1 = 6

'b' = 6 / 3 = 2

'c' = 6 / 4 = 1

'd' = 6 / 4 = 1

Let us combine weights c and d first.

			W	0	4	8	12	16
			C	0	1	2	3	4
W	*d*	*Z*	*Z*	0	5	10	15	20
0	0	0		**0**	**0, 5**	8, 10		
4	1	4		4, 4	**4, 5**			
8	2	8		8, 8				

Here for $c = 1$ and $d = 1$ the element (4, 5) has selected instead of (8, 8) and (8, 10) because it is within the given limit of maximum load 6 units.

C	0	1	1
D	0	0	1
W	0	4	4
Z	0	5	5

For Table 2 let us combine 'a' and 'b'

			W	0	1	2	3	4	5	6
			a	0	1	2	3	4	5	6
W	*b*	*Z*	*Z*	0	1	2	3	4	5	6
0	0	0		**0**	1, 1	2, 2	3, 3	4, 4	5, 5	6, 6
3	1	3		**3, 3**	**3, 3**					
6	2	6		**6, 6**						

A	0	0	1	0	6
B	0	1	0	2	0
W	0	3	3	6	6
Z	0	3	3	6	6

Now combining, *a* and *b* with *c* and *d* we get.

				W	0	4	4
				c	0	1	1
				d	0	0	1
W	*a*	*b*	*Z*	*Z*	0	5	5
0	0	0	0		**0**	**4, 5**(0,0,1,0)	**4, 5**(0, 0, 1, 1)
3	0	1	3		3, 3(0,1,0,0)	1, 5(0, 1, 1, 0)	
3	1	0	3		(1, 0,0,0,)	(1, 0, 1, 0)	
6	0	2	6		**6, 6**(0, 2, 0, 0)		
6	6	0	6		(6, 1,0,0)		

Maximum weight = 6 units, $a = 6$, $b = 0$, $c = 0$ and $d = 0$ or $a = 0$, $b = 2$, $c = 0$ and $d = 0$

Substituting the values in the model we get, Maximize $Z = 1a + 3b + 5c + 4d$ and

$1a + 3b + 4c + 4d \leq 6$

$1 \times 6 + 0 \times 0 + 0 \times 0 + 0 \times 0 = 6$ or $0 \times 0 + 3 \times 2 + 0 \times 0 + 0 \times 0 = 6$ and

$1 \times 6 + 3 \times 0 + 4 \times 0 + 4 \times 0 = 6$

Problem 11.6

Determine the value of u_1 , u_2, and u_3 so as to Maximize $u_1.\ u_2.\ u_3$ subject to $u_1 + u_2 + u_3 = 10$ and u_1, u_2 and u_3 all ≥ 0.

Solution

This can be treated as a 3-stage problem, with the state variable x_i and the return $f_i(x_i)$, such that

At stage 3, $x_3 = u_1 + u_2 + u_3 = 10$,

at stage 2, $x_2 = x_3 - u_3 = u_1 + u_2$,

at stage 1, $x_1 = x_2 - u_2 = u_1$

and the returns are:

$$f_3(x_3) = \max_{u_3} [u_3 f_2(x_2)]$$

$$f_2(x_2) = \max_{u_2} [u_2 f_1(x_1)]$$

$$f_1(x_1) = u_1$$

Since $u_1 = (x_2 - u_2)$

$$f_2(x_2) = \max_{u_2} [u_2(x_2 - u_2)] = \max_{u_2} [u_2 x_2 - u_2^2]$$

Differentiating $[u_2 x_2 - u_2^2]$ w.r.t. u_2 and equating to zero (to find the maximum value)
$u_2 - 2u_2 = 0$ or $u_2 = (x_2/_2)$, therefore,
$f_2(x_2) = (x_2/2) \,.\, x_2 - (x_2/2)^2 = (x_2^2/4)$

Now, $f_3(x_3) = \max\limits_{u_3} [u_3 \,.\, f_2(x_2)] = \max\limits_{u_3} [u_3 \,.\, (x_2^2/4)] = \max\limits_{u_3} [u_3 \,.\, (x_3 - u_3)^2/4]$

Differentiating $[u_3 \,.\, (x_3 - u_3)^2 / 4]$ w.r.t. u_3 and equating to zero,
$(1/4)\,[u_3 \,.\, 2(x_3 - u_2)(-1) + (x_3 - u_3)^2] = 0$ or
$(x_3 - u_3)(-2u_3 + x_3 - u_3) = 0$ or $(x_3 - u_3)(x_3 - 3u_3) = 0$
Now, either $u_3 = x_3$, which is trivial as $u_1 + u_2 + u_3 = x_3$ or $u_3 = (x_3/3) = (10/3)$
Therefore, $u_2 (x_2/2) = (x_3 - u_3)/2 = (1/2)\,[10 - (10/3)] = (10/3)$
$u_1 = x_2 - u_2 = (20/3) - (10/3) = (10/3)$
Therefore, $u_1 = u_2 = u_3 = (10/3)$ and maximum $(u_1 \,.\, u_2 .\, u_3) = (1000/27)$

Problem 11.7.

Minimize $Z = a^2 + b^2 + c^2$ subject to
$a + b + c \geq 15$ and all a, b, c are ≥ 0

Solution

This is a three-stage problem and let the state variables for each stage be x, y and z respectively, such that,

$z = a + b + c$, $y = z - c = a + b$ and $x = y - b = a$. For this recursive equations are:

$f(z) = \min\limits_{c} [c^2 + f(y)]$

$f(y) = \min\limits_{b} [b^2 + f(a)]$ and $f^*(a) = \min(a^2) = a^2$

Since, $x = y - b$ and $f(x) = a^2$

$f(y) = \min\limits_{b} [b^2 + b^2 + (y - b)^2]$

Differentiating $[b^2 + b^2 + (y - b)^2]$ with respect to b and equating to zero,

$d\,f(y)/db = 2b + 2(y - b)(-1) = 0$ or $-2y + 4b = 0$ or $b = (y/2)$

Therefore, $f^*(y) = (y/2)^2 + [y - (y/2)]^2 = (y^2/2)$.

Now $f(z) = \min\limits_{z} [z^2 + f(y)]$

Since $y = z - c$ and $f(y) = (y^2/2)$,

$f(z) = \min\limits_{c} [c^2 + (z - c)^2/2]$, differentiating, $[c^2 + (z - c)^2/2]$ with respect to c and equating to zero,

$d\,f(z)/dc = 2c - (c - z) = 0$ or $c = (z/3)$.
Therefore, $f^*(z) = (z/3)^2 + \{[z - (z/3)]^2\}/2 = (z^2/3)$

Since $a + b + c \geq 15$, for minimization of $f(z)$, $a + b + c = 15$ or $z = 15$,
Therefore, $f^*(z) = (15^2/3) = 75$ and $c = z/3 = 13/3 = 5$
$b = (y/2) = (z - c)/2 = (15 - 5)/2 = 5$,
$a = y - b = 10 - 5 = 5$
Thus minimum value of $a + b + c = 75$ and $a = b = c = 5$.

Problem 11.8.

Minimize $a^2 + b^2 + c^2$, subject to $a + b + c = 10$ when (*i*) a, b, c are non-negative, (*ii*) a, b, c are non-negative integers.

Solution

When a, b, c are continuous non-negative variables the solution can be obtained in the same way as in the example 11.7 above and the minimum value of:

$a^2 + b^2 + c^2 = (10^2/3) = 100/3$ and $a = b = c = (10 / 3)$

When the variables a, b, c are non-negative integers, the problem can be easily solved by the tabular or enumeration method treating it as a three-stage problem, with state variables x, y, z respectively for 3 stages.

At stage 1, the state variable x can take any integer value from 0 to 10, with return $f(x)$ =

$\underset{0 \leq a < 10}{\text{Minimum}}\,(a^2) = a^2 f\, 0$

At stage 2, $f(y) = \underset{0 \leq b \leq 10}{\text{Minimum}}\,[b^2 + f^*(y - b)^2]$

At stage 3, $f(z) = \underset{0 \leq c \leq 10}{\text{Minimum}}\,[c + f^*(b)] = \underset{0 \leq c \leq 10}{\text{minimum}}\,[c^2 + f^*(z - c)^2]$

first combining a^2 and b^2 and entering elements in the table as usual and marking (block letters) the minimum value for each combination, we get the following table.

		a	0	1	2	3	4	5	6	7	8	9	10
b	b^2	a^2	0	1	4	9	16	25	36	49	64	81	100
0	0		**0**	**1**	4	9	16	25	36	49	64	81	100
1	1		**1**	**2**	**5**	10	17	26	37	50	65	82	
2	4		4	**5**	**8**	**13**	20	29	40	50	68		
3	9		9	10	**13**	**18**	**25**	34	45	58			
4	16		16	17	20	**25**	**32**	**41**	**50**				
5	25		25	26	29	34	**41**	61					
6	36		36	37	40	45	52						
7	49		49	50	53	58							
8	64		64	65	68								
9	81		81	82									
10	100		100										

Optimal values of the above table are:

$y = 0$	1	2	3	4	5	6	7	8	9	10
$f^*(y) = 0$	1	2	5	8	13	18	25	32	41	50

Now combining the out come of a^2 and b^2, shown above with c^2, we get the following table.

		$a, b = y$	0	1	2	3	4	5	6	7	8	9	10
		$f^*(y)$	0	1	2	5	8	13	18	25	32	41	50
c	c^2												
0	0		**0**	**1**	**2**	5	8	13	18	25	32	41	50
1	1		**1**	**2**	**3**	**6**	**9**	14	19	26	33	42	
2	4		4	5	**6**	**9**	**12**	**17**	**22**	29	36		
3	9		9	10	11	14	**17**	**22**	**27**	**34**			
4	16		16	17	18	21	24	29	**34**				
5	25		25	26	27	30	33	38					
6	36		36	37	38	41	44						
7	41		41	42	43	41							
8	64		64	65	66								
9	81		81	82									
10	100		100										

Now $f^*(z) = 34$ for which the optimal value of $c^* = 3$ or 4

If $c = 3$, $f^*(b) = 25$, for which $a = 3$ and $b = 4$, or $a = 4$ and $b = 3$

If $c = 4$, $f^*(b) = 18$, for which $a = 3$, and $b = 3$.

Therefore minimum value of 34 corresponds to $(a, b, c) = (3, 3, 4)$ or $(3, 4, 3)$ or $(4, 3, 3)$.

Problem 11.9.

A manufacturing firm has a contract to supply lathe chucks as per the following schedule. The product made during a month will be supplied at the end of the month. The setup cost is Rs. 1000/-, while the inventory carrying cost is Re. 1/- per piece per month. In which month should the batches be produced and of what size, so that the total of setup and inventory carrying cost are minimized?

Month	*Number of items*
January	100
February	200
March	300
April	400
May	400
June	3 00

Solution

This problem is considered as six-stage problem and scheduling of inventory is done in 6 stages by using dynamic programming technique, we can start from the last month.

6^{th} Stage: Month of June: To save the carrying cost, nothing should have been left at the end of the month of May and also nothing should be left at the end of 6^{th} month, *i.e.* June, as this is the last month.

Produce 300 units for which the setup cost is Rs. 1000/- and no inventory carrying cost. Hence the total cost is **Rs. 1000/-.**

5^{th} stage: Month of May: There are two alternatives.

First alternative: Produce 700 units in 5^{th} month and send 400 units and 300 parts will remain as inventory for one month. Hence the total cost = Set up cost + inventory carrying cost for one month =

Rs. 1000 + Rs. 300/- = Rs. 1300/-

Second alternative : Produce 400 units in 5^{th} month and 300 units in 6^{th} month when the total cost is and send the goods in the respective month so that there will be no inventory carrying cost. We have only two setup costs *i.e.* Rs. 1000 + Rs. 1000 = Rs. 2000/-

The first alternative is cheaper, hence instead of producing 400 units in 5^{th} month and 300 units in 6^{th} month produce 700 units in 5th month and send 400 units to market and maintain an inventory of 300 units.

Stage 4: 4^{th} month: There are three alternatives.

(a) Produce 1100 units in 4th month and send 400 units in April to market and maintain an inventory of 700 units for one month and another 300 units for a period of 2 months. For which total cost is Setup cost for 1100 units + 2 months' inventory carrying cost for 300 units + 1 month inventory cost for 400 units = Rs. 1000 + Rs. 700 + Rs. 300 = Rs. 2000.

(b) Produce 300 units in 6^{th} month and 800 units in 4^{th} month at a cost of setup cost of 6^{th} month and setup cost of 4^{th} month + inventory of 400 units for one month. = Rs. 1000 + Rs. 1000 + Rs. 400 = Rs. 2400.

(c) Produce 700 units in 5th month and 400 units in 4^{th} month at a cost of setup cost of 5^{th} and 4th months and inventory carrying cost for one month for 300 units for 6th month.
= Rs. 1000/- + Rs. 1000/- + Rs. 300/- = Rs. 2300/-

Out of all the three decisions, the first decision (a) is optimal. The firm has to produce 1100 units in the 4^{th} month at the cost of Rs. 2000/- .

Stage 3: 3^{rd} month: There are four alternatives.

(a) Produce 1400 units in the third month at a cost of Setup cost of Rs. 1000/- + Inventory carrying charges of Rs. 1100/- + 700/- + 300/- = Rs. 3100/-.

(b) Produce 300 units in 6th month and 1100 units in 3rd month at a cost of Setup cost of Rs. 1000 + Rs. 1000/-) + inventory carrying cost of Rs. 800/- + Rs. 400/- = Total Rs. 3200/-

(c) Produce 700 units in 5th month and 700 units in 3rd month at cost of Setup cost of Rs. 1000 + Rs. 1000) + (inventory carrying cost of RS. 300/- + RS. 400/-) = Total Rs. 2700/ -.

(d) Produce 1100 units in 5th month and 300 units in the 3rd month at cost of (Setup cost of Rs. 1000/- + Rs. 1000/- + Inventory carrying charges of Rs. 700/- + Rs. 300/- = Rs. 3000/-.

The optimal decision at this stage is to produce 700 units in 5th month and the cost of production and inventory maintenance is Rs. 2700/-.

Stage 2: At 2nd month. There are 5 alternatives and they are:

(a) Produce 1600 units in 2nd month at a cost of Setup cost of Rs. 1000/- + inventory carrying charges of Rs. 1400 + 1100 + 700 + 300 = Total Rs. 4500/-.

(b) Produce 300 in 6th month and 1300 units in 2nd month at cost of Rs. 1000 + 1000 + 1100 + 800 + 400 = Total Rs. 4300/-.

(c) Produce 700 units in 5th month and 900 units in 2nd month at cost of Rs. 1300 + 1000 + 700 + 400 = Rs. 3400/-.

(d) Produce 1100 units in 4th month, 500 units in 2nd month at cost of Rs. 2000/- + 1000 + 300 = Rs. 3300/-.

(e) Produce 700 units in 3rd month, 700 in 5th month and 200 in 2nd month at cost of Rs. 3000/- + Rs. 700/- = Total Rs. 3700/-.

The optimal decision rule is Produce 500 units in 2nd month and 1100 units in 4th month at cost of Rs. 3300/-

1st stage: Month 1: There are k6 alternatives. They are:

(*a*) Produce 1700 units at cost of Rs.1000/- 1600 + 1400 + 1100 + 700 + 300 = Rs. 6100/-.

(b) Produce 300 units in 6th month and 1400 units in 1st month and the cost is: Rs. 100/- + 1000/- + Rs. 1300/- + 1100/- + 800/- + 400/- = Total Rs. 5600/-

(c) Produce 700 units in 5th month and 1000 units in the 1st month and the cost is Rs. 1300 + 1000/- + 900/- + 700 + 400 = Total Rs. 4300/-.

(e) Produce 1100 units in 4th month and 600 units in 1st month and the cost is : Rs. 2000/- + 1000 + 500 + 300 = Total Rs. 3800/-

(f) Produce 700 units in 3rd month, and 700 in 5th month and 300 units in the 1st month at a cost of Rs. 2700/- + 1000 + 200 = Rs. 3900/-.

(g) Produce 500 units in the 2nd month and 1100 units in the 4th month and 100 units in the 1st month at a cost of Rs. 3300/- + Rs. 1000/- = Total Rs. 4300/-

The optimal decision rule is to Produce 600 units in 1st month and 1100 units in 4th month at a total cost of Rs. 3800/-.

Hence the minimum cost policy is to produce a batch of 600 units in January and a batch of 1100 units in April, which gives a minimum of setup and inventory carrying cost of Rs. 3800/-.

Problem 11.10.

Solve the following Linear Programming (L.P.) problem using Dynamic Programming (D.P.) technique.

Maximize $5x + 9y$ subject to

$$-x + 3y \leq 3,$$

$5x + 3y \leq 27$ and both x and y are ≥ 0.

Solution

Problem 11.8 is an integer-programming problem, which was solved by using dynamic programming method. The present problem is a linear programming problem, where we are concerned with non-negative integers, *i.e.* it allows for continuous values of variables.

Let us represent the given problem in L.P. way. We want to decide two items of products *A* and *B* (in the problem variable '*x*' represents product *A* and '*y*' represents product *B*. The profit per unit of *A* is Rs. 5/- and that of B is Rs. 9/-. The time required (in D.P. terms: it is weight of *A* and *B*) to produce are 5 hours and 3 hours respectively. The total time of producing *A* and *B* is 27 hours. It is also seen from the inequalities given that each unit of *B* requires 5 units of material and *A* does not require the material and for every unit we produce, we get one unit of material free. And the material on hand is 3 units. Products are represented by variables *x* and *y*. The first constraint, $-1x + 5y \leq 3$ describes that for every one unit of *B* we require 5 units of material and for every one unit of *x* we produce, we get one unit of material free. The inequality may be written as:

$5y \leq 3 + 1x$.

As there are two variables, this may be considered as 2-stage problem. To solve the problem, let us start from the last stage. One more stage to go implies that we are deciding on *y*. Let the R.H.S, *i.e.* capacities available for allocation at the beginning this stage be b_1 and b_2. (Remember, in L.P.P the R.H.S. capacities are generally represented by b_i *s* in general format.) Let b_1 and b_2 are associated with the first and second constraints respectively. The maximum value of capacities is specified in the R.H.S. of the two constraints as 3 and 27. It is evident from first constraint that we are taking a decision on *y* alone and the R.H.S. capacity available is b_1. Then 5*y* has to be less than or equal to b_1 *i.e.* $y \leq 5$. Similarly from inequality $-x + 5y \leq 3$, we have $y \leq b_2/3$. Both together would mean that *y* has to be less than or equal to the minimum of $(b_1/5)$ and $(b_2/3)$. Expressing these two mathematically,

$f_1 (b_1, b_2)$ = Maximum 9 [y] = 9 Minimum [$b_1/5, b_2 /3$] *i.e.* $y \leq$ minimum [$b_1/5, b_2/3$] ... (1)

Now let us go backwards by one stage. Two more stages to go implies that we have at our disposal the whole capacities as indicated by the right hand sides of the two constraints. We want to decide on the *x* that will maximize the overall objective function. Given that we start this stage with 3 and 27 capacities or two different things, if we decide on a value of *x*, then we will be left with $(3 + x)$ and $(27 - 5x)$ respectively for allocation in the subsequent stages. This is because the coefficients of *x* in first and second constraints are –1 and 5 respectively. Thus with $N = 2$ for second stage ($N = 1$ for the first stage, *N* denotes the number of stage) our starting state can be represented by the pair (3, 27), and the decision of *x* leaves us with the ending state represented by $(3 + x, 27 - 5x)$. The effect corresponding to the state, as given by the objective function is 5*x*. Finally from the two constraints, putting $y = 0$, we found that *x* can lie between –3 and 27/5, as negative values are not allowed, *x* can take the value between 0 and 27/5 (both inclusive). Expressing this in the usual notations, we have:

$$f_2 (3, 27) = \underset{0 \leq x \leq 27/5}{\text{Minimum}}[5x + f_1 (3 + x, 27 - 5x] \quad \ldots(2)$$

From (1) above, we know that $f_1 (b_1, b_2) = 9$ minimum [$b_1/5, b_2/3$], Therefore,

$f_1 (3 + x, 27 - 5x) = 9$ minimum [$(3 + x) / 5, (27 - 5x)/3$] ,

Thus, if $(3 + x)/5$ is $\leq 27 - 5x)/3$, then, $f_1 (3 + x, 27 - 5x) = 9 (3 + x) / 5$ …(3)

Otherwise, $f_1 (3 + x, 27 - 5x) = 9 (27 - 5x)/3$ ...(4)

We now find the range of *x* for which $(3 + x)/5, < (27 - 5x) / 3$.

Verify that to satisfy the condition, *x* should be less than 4.5. Replacing (3) and (4) in (2), we have:

$f_2 (3, 27)$ = Maximum [$5x + 9 (3 + x)/5$], if $x \leq 4.5$
= Maximum [$5x + 9 (27 - 5x)/3$] if $x > 4.5$

From the above, it is easy to verify that the maximum occurs at $x = 4.5$. The corresponding value of f_2 (3, 27) is the value of the objective function. The value of *y* may be obtained by working backwards.

$X = 4.5$ implies $f_1(3 + x, 27 - 5x) = f_1(7.5, 4.5)$.

From (1) we know that the optimal of y = Minimum [(7.5/5), (4.5/3)] = 1.5

Hence the required answer is $x = 4.5$, $y = 1.5$.

These problems can be solved more simply without involving mathematical complications as shown below:

Problem 11.11.

Solve the given L.P. Model by using dynamic programming technique.

Max $Z = a = 9b$ s.t. $2a + 1b \leq 25$, $0a + 1b \leq 11$ and both a and b are ≥ 0.

Solution

Given that $0a + 1b \leq 11$ and $2a + 1b \leq 25$. Let us assume these inequalities as equations as we do in graphical method, *i.e.*

$1b = 11$ and $1b = 25 - 2a$ as both are equal to $1b$, we can write as

$11 = 25 - 2a$ or $2a = 25 - 11 = 14$, or $a = 14 / 2 = 7$. Substituting this in the above we can write,

$25 - 2 \times 7 = b$ or $25 - 14 = b = 11$.

Hence $a = 7$, and $b = 11$

Problem 11.12.

Maximize $3a + 5b$ s.t.

$A \leq 4$, $b \leq 6$, $3a + 2b \leq 18$ and both a and b are³ 0

Solution

Given that $b \leq 6$

$2b \leq 18 - 3a$ or $b \leq (18 - 3a) / 9$ or $b \leq 9 - (3/2)\, a$

Solving these two $b = 6 = (18 - 3a)/2$.

$18 - 3a = 12$ or $3a = 18 - 12 = 6$ or $a = (6/3) = 2$

Checking this with condition $a \leq 4$, this holds good.

Substituting $a = 2$ in $b = 9 - (3/2)\, a = 9 - (3/2) \times 2 = 6$

Therefore, $a = 2$ and $b = 6$ and the maximum value $= 2 \times 3 + 5 \times 6 = 36$.

Problem 11.13.

Maximize $Z = 50x + 80y$ s.t.

$X \leq 80$, $y \leq 60$ and $5x + 6y \leq 600$, $x + 2y \leq 160$ and both x and y are ≥ 0.

Solution

Select the inequalities

$5x + 6y \leq 600$ and $x + 2y \leq 160$, this will give us

$x \leq (600 - 6y) /5$ and $x \leq 160 - 2y$, equating the two:

$(600 - y)/5 = 160 - 2y$ or

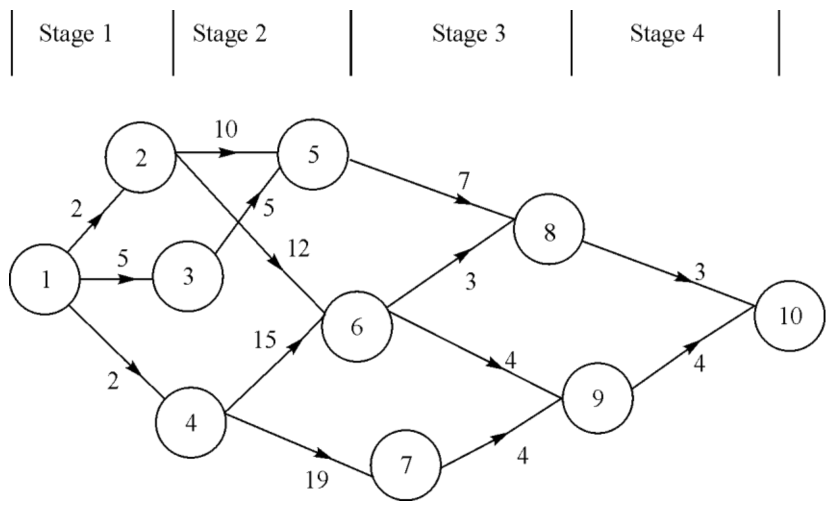
$600 - 6y = 800 - 10y$ or $4y = 200$ or $y = 50$.

Substituting the value of y we get $x = 60$

Answer : $x = 60$, $y = 50$.

Problem 11.14

Mr. Banerjee, a sales manager, has decided to travel from city 1 to city 10. He wants to plan for minimum distance programme and visit maximum number of branch offices as possible on the route. The route map of the various ways of reaching city 10 from city 1 is shown below. The numbers on the arrow indicates the distance in km. (× 100). Suggest a feasible minimum path plan to Mr. Banerjee.



Solution

The problem may be considered as 4-stage problem. In stage 1 the manager leaves from station 1 (node number 1) and can reach stations 2, 3, and 4 directly. Let us consider the distances 1 to 2 is 200 km, 1 to 3 is 500 km and 1 to 4 is 200 km. As this is minimization problem and the manager wants to visit more number of branch offices and travel less distance, we can show the routes which show the minimum distance in full line and the rest of the lines we can neglect or we can show in dotted lines. In this problem, lines 1 – 2 and 1 – 3 will be in full line and line 2 – 5 in dotted line. The distance covered up to that stage is written just above the node.

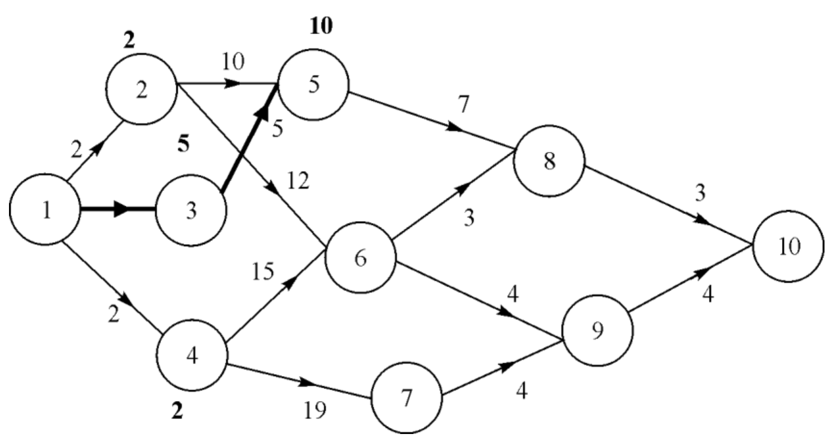


Figure 11.1

In the second stage, he can reach station 5 directly from 2 and 3, station 6 directly from 2 and 4 and station 4 from 4.

The distance from 2 to 5 is previous distance covered + present distance = 200 + 1000 = 1200,

Similarly, from 3 to 5 is 500 + 500 = 1000 km.

The distance from 2 to 6 = 200 + 1200 = 1400 km.

The distance from 4 to 6 = 200 + 1500 = 1700 km.

The distance from 4 to 7 is 200 + 1900 = 2100 km.

The minimum of all these is 100 km. *i.e.* **the manager travels from 1 to 3 and from 3 to 5 covering 100 Km.**

(Remember, in maximization problem, we consider the maximum distance.)

In third stage, the manager may be at station 5 or at station 6 or at station 7. From there he can directly go to station 8 or station 9.

Let us workout the minimum distance from 5, 6 and 7 to 8 and 9.

From 5 to 8 the distance = 1000 + 700 = 1700 km.

From 6 to 8 the distance is 1400 + 300 = 1700 km.

From 6 to 9 the distance is 1400 + 400 = 1800 km.

From 7 to 9 the distance is 2100 + 400 = 2500 km.

The minimum of all these is 1700 km. *i.e*. the manager can go from 5 to 8 or 6 to 8 the distance is 1700 km only.

In the 4th stage he can reach station 10 from station 8 or 9. The minimum distance from 8 and 9 to 10 is:

From 8 to 10 the distance is 1700 + 300 = 2000 km.

From 9 to 10 the distance is 1800 + 400 = 2200 km.

Hence the minimum distance from stations 1 to 10 on the path is **2000 km** on routes **1 – 3 – 5 – 8** – 10 and **1 – 2 – 6 – 8 – 10.** This is shown in figure 11.3.

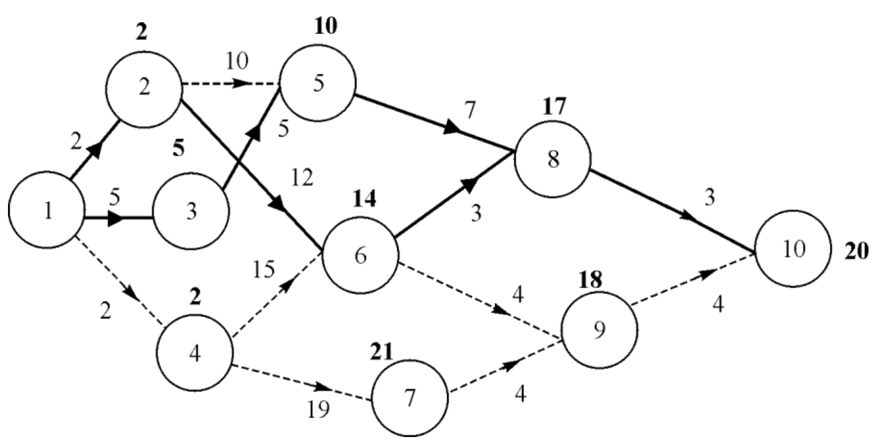


Figure 11.2

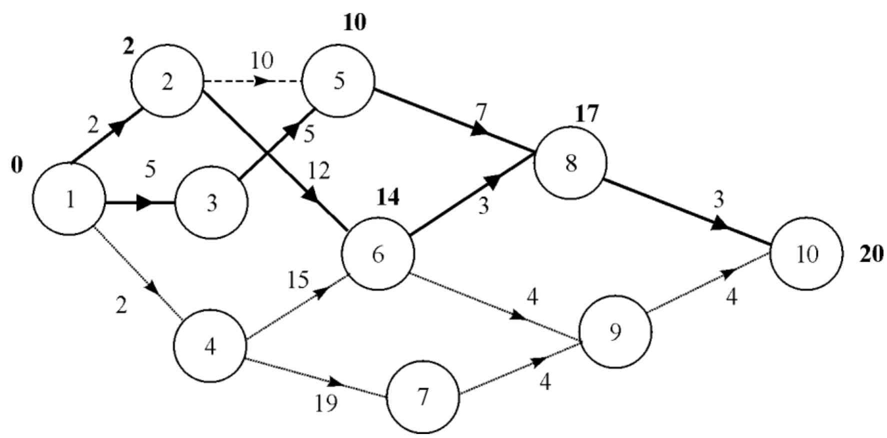


Figure 11.3

Problem 11.15.

The following figure (11.4) shows the route map of various branch offices of a company. The marketing executive of the company should like to start from Head office at *A* and reach the branch office at *B* by traveling shortest path and visiting as many as branch offices. Help him to plan his journey by using dynamic programming technique.

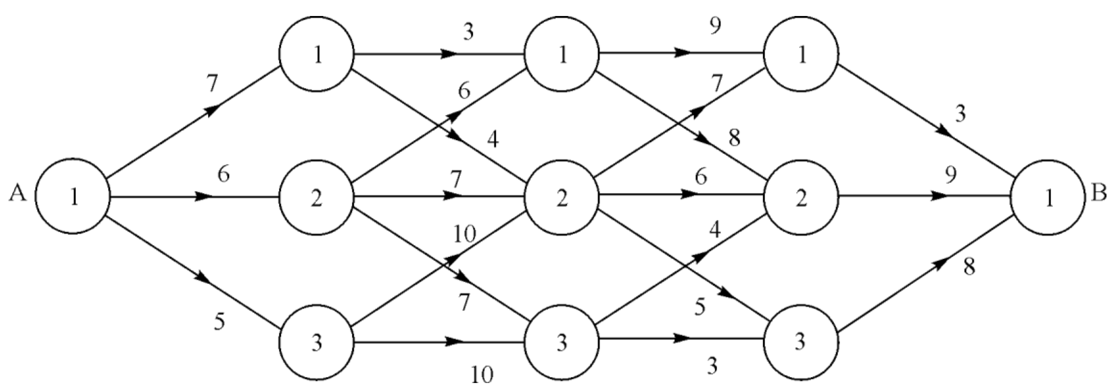


Figure 11. 4

Solution

First let us identify the stages and then plan for the journey of executive from stage to stage.

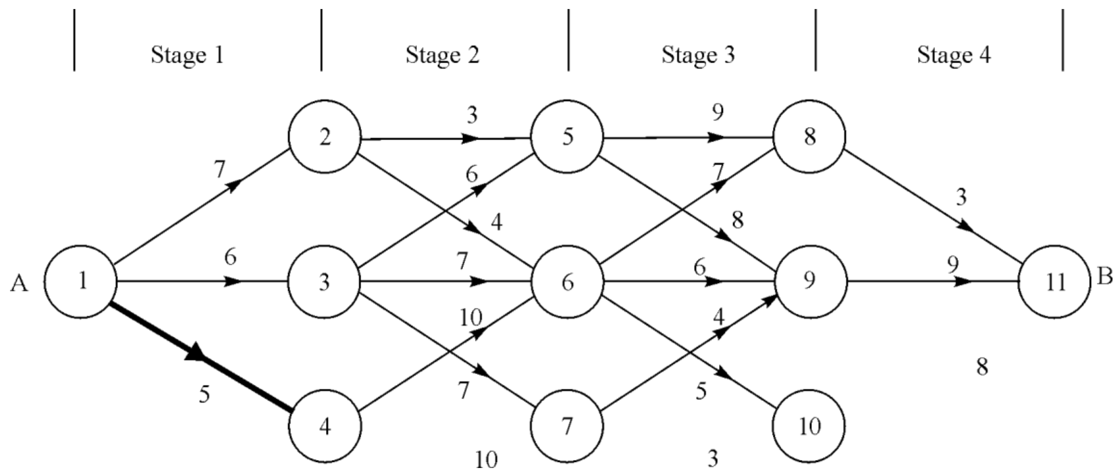


Figure 11.5

Let us renumber the branch offices from 1 to 11. Let the executive start from head office 1; from there he can reach the branch offices 2, 3 and 4 directly in stage 1. Let us workout the minimum distance.

From 1 to 2 minimum distance is 7 km. From 1 to 3 the minimum distance is 6 km and that from 1 to 4 is 5 km. Hence the executive travels from 1 to 4 first. Now he is at 4.

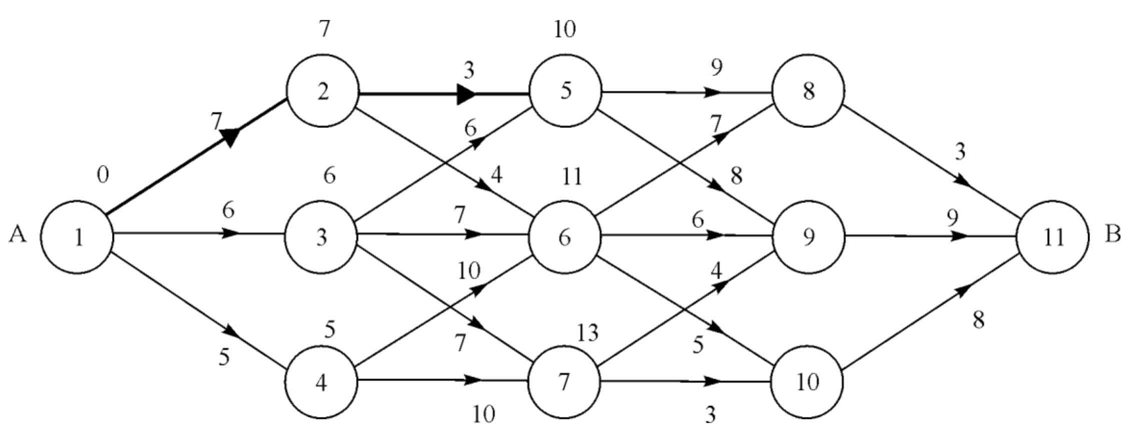


Figure 11.6

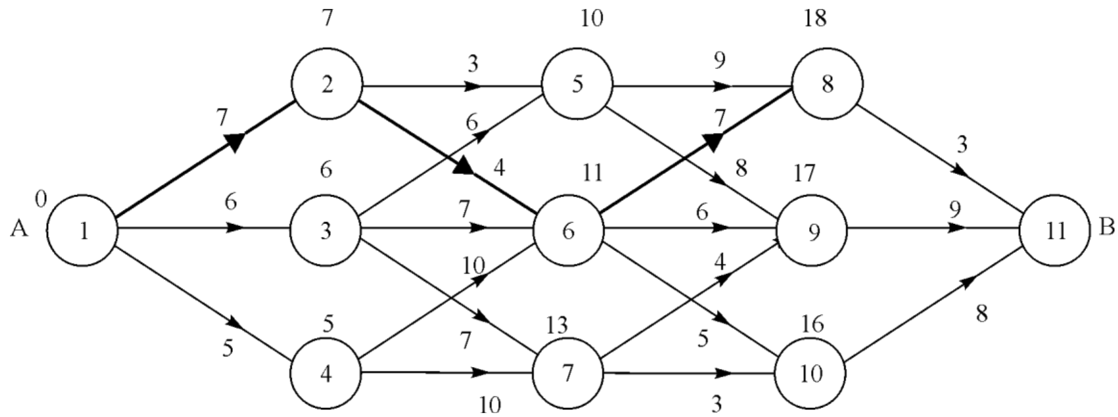


Figure 11.7

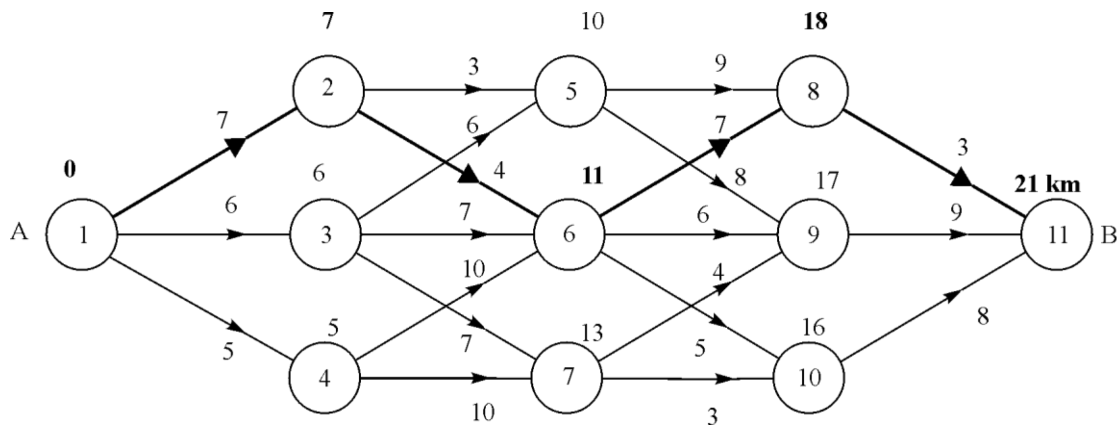


Figure. 11.8

In figure 11.6, the executive has to reach the branch office 5, 6, or 7 from 2, 3 and four. When we work out the minimum distance we find that route 1 – 2 – 6 will give 11 km, *i.e*. the executive travels to station 6. In figure 11.7, we find that the executive travels to branch office 8 where the minimum distance is 18 km. From there he can reach the last office, *i.e*. 11, and the total distance is 21 km. The optimal route is **1– 2 – 6 – 8 – 11** and the optimal distance is **21 km**. The optimal route is shown in thick lines.

QUESTIONS

1. A company has 9 salesmen, who have to be allocated to 3 marketing zones. The return from each zone depends on the number of salesmen working in that zone. The expected returns for different number of salesmen in different zones, as estimated from the past records are given below. Find the optimal allocation to maximize the return.

No. of salesmen	*Zone1*	*Zone 2*	*Zone 3*
	Return in Rs.	*Return in Rs.*	*Return in Rs.*
0	42	30	45
1	54	45	58
2	60	64	60
3	65	68	65
4	70	75	72
5	80	79	75
6	82	85	84
7	90	92	90
8	100	99	100
9	110	105	120

2. In a cargo-loading problem, there are four items of different per unit weight and value as given below. The minimum cargo load is restricted to 10 units. How many units of each item are loaded to maximize the value.

Item	*Weight* w_i	*Value per unit* v_i
1	2	2
2	3	5
3	4	7
5	5	9

3. Minimize $Z = a^2 + b^2 + c^2$ subject to

$$a + b + c \geq 10 \text{ and } a, b, c, \text{ all } \geq 0$$

4. Maximize $Z = 3a + 5b$ subject to

$$a \leq 4$$
$$b \leq 6$$
$$3a + 2b \leq 18 \text{ and } a, b \text{ both} \geq 0$$

MULTIPLE CHOICE QUESTIONS

1. In Dynamic Programming Problems, the decisions are made in
   (a) Single stage (*b*) 2-stages
   (*c*) Multi-stages (*d*) No decision making process ( )
2. In dynamic programming problems, the main problem is divided into subproblems. Each sub-problem is known as:
   (a) Part (*b*) Stage
   (*c*) State (*d*) Mini problem ( )

3. The technique of Dynamic Programming problem is developed by:

   (*a*) Taylor (*b*) Gilberth
   (*c*) Richard Bellman (*d*) Bellman and Clarke ( )

4. Another name used to Dynamic Programming is:

   (*a*) Multistage problem (*b*) Recursive optimization
   (*c*) State problems (*d*) No second name. ( )

5 If the outcome at any decision stage is unique and known for the problem, then the Dynamic programming problem is known as:

   (a) Probabilistic dynamic programming problem
   (b) Stochastic dynamic programming problem
   (c) Static dynamic programming problem
   (d) Deterministic dynamic programming problem. ( )

6. The possible decisions at any stage are known as:

   (a) States (*b*) Steps
   (*c*) Parts (*d*) None ( )

7. The rule which determines the decision at each stage is known as

   (a) State (*b*) Stage
   (*c*) Policy (*d*) Decision. ( )

ANSWERS

1. (*c*)	2. (*b*)	3. (*c*)	4. (*b*)
5. (*d*)	6. (*a*)	7. (*c*)	

CHAPTER – 12

Decision Theory

INTRODUCTION

The decisions are classified according to the *degree of certainty* as **deterministic models**, where the manager *assumes* complete certainty and each strategy results in a **unique payoff,** and **Probabilistic models**, where each strategy leads to *more than one payofs* and the manager attaches a *probability measure* to these payoffs. The scale of assumed certainty can range from complete certainty to complete uncertainty hence one can think of **decision making under certainty (DMUC) and decision making under uncertainty (DMUU)** on the two extreme points on a scale. The region that falls between these extreme points corresponds to the concept of probabilistic models, and referred as **decision-making under risk (DMUR)**. Hence we can say that most of the decision making problems fall in the category of decision making under risk and the assumed degree of certainty is only one aspect of a decision problem. The other way of classifying is: Linear or non-linear behaviour, static or dynamic conditions, single or multiple objectives. One has to consider all these aspects before building a model.

Decision theory deals with decision making under conditions of risk and uncertainty. For our purpose, we shall consider all types of decision models including deterministic models to be under the domain of decision theory. In management literature, we have several quantitative decision models that help managers identify optima or best courses of action.

Complete uncertainty	*Degree of uncertainty*	*Complete certainty*
Decision making	Decision making	Decision-making
Under uncertainty	**Under risk**	**Under certainty.**

Before we go to decision theory, let us just discuss the issues, such as (*i*) What is a decision? (*ii*) Why must decisions be made? (*iii*) What is involved in the process of decision-making? (*iv*) What are some of the ways of classifying decisions? This will help us to have clear concept of decision models.

WHAT IS A DECISION?

A **decision** is the conclusion of a process designed to weigh the relative utilities or merits of a set of available alternatives so that the most preferred course of action can be selected for implementation. Decision-making involves all that is necessary to identify the most preferred choice to satisfy the desired goal or objective. Hence decision-making process must involve a set of goals or objectives, a system of priorities, methods of enumerating the alternative courses of feasible and viable courses and a system of identifying the most favourable alternative. One must remember that the decisions are sequential in nature. It means to say that once we select an alternative, immediately another question arises. For example if you take a decision to purchase a particular material, the next question is how much. The next question is at what price. The next question is from whom… Like that there is no end.

WHY MUST DECISIONS BE MADE?

In management theory we study that the essence of management is to make decisions that commit resources in the pursuit of organizational objectives. Resources are limited and wants and needs of human beings are unlimited and diversified and each wants to satisfy his needs in an atmosphere, where resources are limited. Here the decision theory helps to take a certain decision to have most satisfactory way of satisfying their needs. Decisions are made to achieve these goals and objectives.

DECISION AND CONFLICT

When a group of people is working together in an organization, due to individual behaviour and mentality, there exists a conflict between two individuals. Not only that in an organization, each department has its own objective, which is subordinate to organizational goal, and in fulfilling departmental goals, there exists a conflict between the departments. Hence, any decision maker has to take all these factors into consideration, while dealing with a decision process, so that the effect of conflicts between departments or between subordinate goals is kept at minimum in the interest of achieving the overall objective of the organization.

TWO PHASES OF THE PROCESS OF DECISION-MAKING

The decision theory has assumed an important position, because of contribution of such diverse disciplines as philosophy, economics, psychology, sociology, statistics, political science and operations research to the area decision theory. In decision-making process we recognize two phases: (1) How to formulate goals and objectives, enumerate environmental constraints, identify alternative strategies and project relevant payoffs. (2) Concentration on the question of how to choose the optimal strategy when we are given a set of objectives, strategies, payoffs. We concentrate more on the second aspect in our discussion.

CLASSIFICATIONS OF DECISIONS

In general, decisions are classified as **Strategic decision**, which is related to the organization's outside environment, **administrative decisions** dealing with structuring resources and operational decisions dealing with day-to-day problems.

Depending on the nature of the problem there are **Programmed decisions**, to solve repetitive and well-structured problems, and **Non-programmed decisions**, designed to solve non-routine, novel, illstructured problems.

Depending on the scope, complexity and the number of people employed decision can be divided as **individual and managerial** decisions.

Depending on the sphere of interest, as **political**, **economic**, or **scientific** etc. decision can be divided as **static** decision requiring only one decision for the planning horizon and **dynamic decision** requiring a series of decisions for the planning horizon.

STEPS IN DECISION THEORY APPROACH

1. List the **viable alternatives (strategies)** that can be considered in the decision.
2. List all **future events** that can occur. These future events (not in the control of decision maker) are called as **states of nature**.
3. Construct a payoff table for each possible combination of alternative course of action and state of nature.
4. Choose the criterion that results in the largest payoff.

DECISION MAKING UNDER CERTAINTY (DMUC)

Decision making under certainty assumes that all relevant information required to make decision is certain in nature and is well known. It uses a deterministic model, with complete knowledge, stability and no ambiguity. To make decision, the manager will have to be quite aware of the strategies available and their payoffs and each strategy will have unique payoff resulting in certainty. The decision-making may be of single objective or of multiple objectives.

Problem 12.1.

ABC Corporation wants to launch one of its mega campaigns to promote a special product. The promotion budgets not yet finalized, but they know that some Rs. 55,00,000 is available for advertising and promotion.

Management wants to know how much they should spend for television spots, which is the most appropriate medium for their product. They have created five 'T.V. campaign strategies' with their projected outcome in terms of increase in sales. Find which one they have to select to yield maximum utility. The data required is given below.

Strategy	*Cost in lakhs of Rs.*	*Increased in sales in lakhs of Rs.*
A	1.80	1.78
B	2.00	2.02
C	2.25	2.42
D	2.75	2.68
E	3.20	3.24

Solution

The criteria for selecting the strategy (for maximum utility) is to select the strategy that yields for maximum utility *i.e*. highest ratio of outcome *i.e*. increase in sales to cost.

Strategy	*Cost in Lakhs of Rs.*	*Increase in Sales in Lakhs of Rs.*	*Utility or Payoffs*	*Remarks.*
A	1.80	1.78	1.78 / 1.80 = 0.988	
B	2.00	2.02	2.02 / 2.00 = 1.010	
C	2.25	2.42	2.42 / 2.25 = 1.075	**Maximum Utility**
D	2.75	2.68	2.68 / 2.75 = 0.974	
E	3.20	3.24	3.24 / 3.20 = 1.012	

The company will select the third strategy, *C*, which yields highest utility.

Now let us consider the problem of making decision with multiple objectives.

Problem 12.2.

Consider a M/s *XYZ* company, which is developing its annual plans in terms of three objectives: (1) Increased profits, (2) Increased market share and (3) increased sales. M/S *XYZ* has formulated three different strategies for achieving the stated objectives. The table below gives relative weightage of objectives and scores project the strategy. Find the optimal strategy that yields maximum weighted or composite utility.

Measure of → *Performance of Three objectives*	*ROI (Profit)*	*% Increase (Market share)*	*% Increase (Sales growth)*
Weights →	0.2	0.5	0.3
Strategy			
S_1	7	4	9
S_2	3	6	7
S_3	5	5	10

Solution

(The profit objective could be stated in and measured by absolute Rupee volume, or percentage increase, or return on investment (ROI). The market share is to be measured in terms of percentage of total market, while sales growth could be measured either in Rupees or in percentage terms. Now, in order to formulate the payoff matrix of this problem, we need two things. (*i*) We must assign relative weights to each of the three objectives. (*ii*) For each strategy we will have to project **a score** in each of the three dimensions, one for each objective and express these scores in terms of utilities. The Optimal strategy is the one that yields the maximum weighted or composite utility.)

Multiplying the utilities under each objective by their respective weights and then summing the products calculate the weighted composite utility for a given strategy. For example:

For strategy $S_1 = 7 \times 0.2 + 5 \times 0.5 + 9 \times 0.3 = 6.1$

Measure of → *Performance of Three objectives*	*ROI (Profit)*	*% Increase (Market share)*	*% Increase (Sales growth)*	*Weighted or Composite Utility (CU)*
Weights →	0.2	0.5	0.3	
Strategy				
S_1	7	4	9	$0.2 \times 7 + 0.5 \times 4 + 0.3 \times 9 = 6.1$
S_2	3	6	7	$0.2 \times 3 + 0.5 \times 6 + 0.3 \times 7 = 5.7$
S_3	5	5	10	$0.2 \times 5 + 0.5 \times 5 + 0.3 \times 10 = 6.6$ Maximum utility

DECISION MAKING UNDER RISK (DMUR)

Decision-making under risk (DMUR) describes a situation in which each strategy results in more than one outcomes or payoffs and the manager attaches a probability measure to these payoffs. **This model covers the case when the manager projects two or more outcomes for each strategy and he or she knows, or is willing to assume, the relevant probability distribution of the outcomes.** The following assumptions are to be made: (1) Availability of more than one strategies, (2) The existence of more than one states of nature, (3) The relevant outcomes and (4) The probability distribution of outcomes associated with each strategy. The optimal strategy in decision making under risk is identified by the strategy with **highest expected utility (or highest expected value).**

Problem 12.3.

In a game of head and tail of coins the player A will get Rs. 4/- when a coin is tossed and head appears; and will lose Rs. 5/- each time when tail appears. Find the optimal strategy of the player.

Solution

Let us apply the expected value criterion before a decision is made. Here the two monetary outcomes are + Rs. 4/- and – Rs. 5/- and their probabilities are ½ and ½. Hence the expected monetary value = $EMV = u_1 p_1 + u_2 p_2 = + 4 \times 0.5 + (-5) \times 0.5 = -0.50$. This means to say on the average the player will loose Rs. 0.50 per game.

Problem 12.4.

A marketing manager of an insurance company has kept complete records of the sales effort of the sales personnel. These records contain data regarding the number of insurance policies sold and net revenues received by the company as a function of four different sales strategies. The manager has constructed the conditional payoff matrix given below, based on his records. (The state of nature refers to the number of policies sold). The number within the table represents utilities. Suppose you are a new salesperson and that you have access to the original records as well as the payoff matrix. Which strategy would you follow?

State of nature	N_1	N_2	N_3
Probability	0.2	0.5	0.3
Strategy ↓	Utility	Utility	Utility
S_1 (1 call, 0 follow up)	4	6	10
S_2 (1 call, one follow up)	6	5	9
S_3 (1 call, two follow-ups)	2	10	8
S_4 (1 call, three follow-ups)	10	3	7

Solution

As the decision is to be made under risk, multiplying the probability and utility and summing them up give the expected utility for the strategy.

State of Nature	N_1	N_2	N_3	*Expected utility or expected payoffs*
Probability	0.2	0.5	0.3	
Strategy	Utility	Utility	Utility	
S_1	4	6	10	0.2 × 4 + 0.5 × 6 + 0.3 × 10 = 6.8
S_2	6	5	9	0.2 × 6 + 0.5 × 5 + 0.3 × 9 = 6.4
S_3	2	10	8	**0.2 × 2 + 0.5 × 10 + 0.3 × 8 = 7.8**
S_4	10	3	7	0.2 × 10 + 0.5 × 3 + 0.3 × 7 = 5.6

As the third strategy gives highest expected utility 1 call and 2 follow up yield highest utility.

Problem 12.5.

A company is planning for its sales targets and the strategies to achieve these targets. The data in terms of three sales targets, their respective utilities, various strategies and appropriate probability distribution are given in the table given below. What is the optimal strategy?

Sales targets (× *lakhs*)	50	75	100			
Utility	4	7	9			
	Prob.	Prob.	Prob.			
Strategies						
S_1	0.6	0.3	0.1			
S_2	0.2	0.5	0.3			
S_3	0.5	0.3	0.2			

Solution

Expected monetary value of a strategy = ∑ Sales target × Probability

Expected utility of a strategy = ∑ Utility × Probability.

Sales targets (× lakhs)	*50*	*75*	*100*	*Expected Monetary Value*	*Expected Utility*
Utility	4	7	9		
	Prob.	*Prob.*	*Prob.*		
Strategies.					
S_1	0.6	0.3	0.1	50 × 0.6 + 75 × 0.3 + 100 × 0.1 = 62.5	4 × 0.6 + 7 × 0.3 + 9 × 0.1 = 5.4
S_2	0.2	0.5	0.3	**50 × 0.2 + 75 × 0.5 + 100 × 0.3 = 77.5**	**4 × 0.2 + 7 × 0.5 + 9 × 0.3 = 7.0**
S_3	0.5	0.3	0.2	50 × 0.5 + 75 × 0.3 + 100 × 0.2 = 67.5	4 × 0.5 + 7 × 0.3 + 9 × 0.2 = 5.9

As both expected money value and expected utility of second strategy are higher than the other two, strategy two is optimal.

DECISION MAKING UNDER UNCERTAINTY

Decision making under uncertainty is formulated exactly in the same way as decision making under risk, only difference is that no probability to each strategy is attached. Let us make a comparative table to compare the three, *i.e*. decision making under certainty, risk, and uncertainty.

Decision making under certainty	*Decision making under risk*	*Decision making under Uncertainty.*
State of Nature	*State of Nature*	*State of Nature*
N	N_1 N_2 N_3 Probability p_1 p_2 p_3	N_1 N_2 N_3
Strategy Utility or Payoff	Strategy Utility or Payoff	Strategy Utility or Payoff.
S_1 u_1 S_2 u_2 S_3 u_3	S_1 u_{11} u_{12} u_{13} S_2 u_{21} u_{22} u_{23} S_3 u_{31} u_{32} u_{33}	S_1 u_{11} u_{12} u_{13} S_2 u_{21} u_{22} u_{23} S_3 u_{31} u_{32} u_{33}
* One state of nature * Single column matrix * Deterministic outcomes * Optimal strategy is the one with the highest utility.	* More than one states of nature * Multiple column matrix * Probabilistic outcomes (*i.e.* Probabilities are attached to Various states of nature) * Optimal strategy is identified by the use of expected value criterian.	*More than one states of nature. *Multiple column matrix. * Uncertain outcomes (*i.e.* probabilities are not attached to various states of nature). * Optimal strategy is identified using a number of different criterian.

In decision making under uncertainty, remember that no probabilities are attached to set of the states of nature. Sometimes we may have only positive elements in the given matrix, indicating that the company under any circumstances will have profit only. Sometimes, we may have negative elements, indicating potential loss. While solving the problem of decision making under uncertainty, we have two approaches, the first one is **pessimistic approach** and the second one is **optimistic approach**. Let us examine this by solving a problem.

Problem 12.6.

The management of *XYZ* company is considering the use of a newly discovered chemical which, when added to detergents, will make the washing stet, thus eliminating the necessity of adding softeners. The management is considering at present time, these three alternative strategies.

S_1 = Add the new chemical to the currently marketed detergent DETER and sell it under label 'NEW IMPROVED DETER'.

S_2 = Introduce a brand new detergent under the name of 'SUPER SOFT'

S_3 = Develop a new product and enter the softener market under the name 'EXTRA WASH'.

The management has decided for the time being that only one of the three strategies is economically feasible (under given market condition). The marketing research department is requested to develop a conditional payoff matrix for this problem. After conducting sufficient research, based on personal interviews and anticipating the possible reaction of the competitors, the marketing research department submits the payoff matrix given below. Select the optimal strategy.

	State of nature.		
Strategies.	N_1	N_2	N_3
	Utility of Payoffs.		
S_1	15	12	18
S_2	9	14	10
S_3	13	4	26

Solution

When no probability is given, depending upon risk, subjective values, experience etc., each individual may choose different strategies. These are selected depending on the *choice criterion*. That is why sometimes the decision making under uncertainty problems are labeled as **choice creation models**. Two criterians may be considered here. One is **Criterion of Optimism** and the other is **Criterion of Pessimism.**

CRITERION OF OPTIMISM

Here we determine the **best possible outcome** in each strategy, and then identify **the best of the best** outcome in order to select the optimal strategy. In the table given below the best of the best is written in the left hand side margin.

	State of nature.			*Best or Maximum outc*ome (*Row maximum*)
Strategies.	N_1	N_2	N_3	
	Utility of Payoffs.			
S_1	15	12	18	18
S_2	9	14	10	14
S_3	13	4	26	**26 Maximax.**

While applying the criterion of optimism, the idea is to choose the *maximum of the maximum values*; the choice process is also known as **Maximax**.

CRITERION OF PESSIMISM

When criterion of pessimism is applied to solve the problem under uncertainty, first determine *worst possible outcome in each strategy* (row minimums), and select **the best of the worst outcome in** order to select the optimal strategy. The worst outcomes are shown in the left hand side margin.

	State of nature.			*Worst or minimum out come* (*Row minimums*)
Strategies.	N_1	N_2	N_3	
	Utility of Payoffs.			
S_1	15	12	18	**12 Maximin**
S_2	9	14	10	9
S_3	13	4	26	4

Best among the worst outcome is 12, hence the manager selects the first strategy. Maximin assumes complete pessimism. Maximax assumes complete optimism. To establish a degree of optimism or pessimism, the manager may attach some weights to the best and the worst outcomes in order to reflect in degree of optimism or pessimism. Let us assume that manager attaches a coefficient of optimism of 0.6 and then obviously the coefficient of pessimism is 0.4. The matrix shown below shows how to select the best strategy when weights are given.

Strategy.	*Best or maximum Payoffs*	*Worst or minimum Payoffs*	*Weighted Payoffs.*
Weights.	0.6	0.4	
S_1	18	12	$0.6 \times 18 + 0.4 \times 12 = 15.6$
S_2	14	9	$0.6 \times 14 + 0.4 \times 9 = 12.0$
S_3	26	4	**$0.6 \times 26 + 0.4 \times 4 = 17.2$ Maximum.**

CRITERION OF REGRET

In this case, we have to determine the **regret matrix or opportunity loss matrix**. To find the opportunity loss matrix (column opportunity loss matrix), subtract all the elements of a column from the highest element of that column. The obtained matrix is known as *regret matrix*. While selecting the best strategy, we have to select such a strategy, whose opportunity loss is zero, *i.e.* zero regret. If we select any other strategy, then the regret is the element at that strategy. For the matrix given in problem 12.6 the regret matrix is

Strategies.	*State of nature.*		
	N_1	N_2	N_3
	Utility of Payoffs.		
S_1	0	2	8
S_2	6	0	16
S_3	2	10	0

Rule for getting the regret matrix: In each column, identify the highest element and then subtract all the individual elements of that column, cell by cell, from the highest element to obtain the corresponding column of the regret matrix.

To select the optimal strategy we first determine the *maximum regret* that the decision maker can experience for each strategy and then identify the *maximum of the maximum* regret values. This is shown in the table below:

Strategies.	*State of nature.*			
	N_1	N_2	N_3	
	Regret or Opportunity loss.			Maximum regret.
S_1	0	2	8	**8 minimax.**
S_2	6	0	16	16
S_3	2	10	0	10

Select the **minimum of the maximum regret (Minimax regret).** The choice process can be known as **minimax regret**. Suppose two strategies have same minimax element, then the manager needs additional factors that influence his selection.

EQUAL PROBABILITY CRITERION

As we do not have any *objective evidence* of a probability distribution for the states of nature, one can use *subjective criterion*. Not only this, as there is no objective evidence, we can assign *equal probabilities* to each of the state of nature. This subjective assumption of equal probabilities is known as **Laplace criterion,** or **criterion of insufficient reason** in management literature.

Once equal probabilities are attached to each state of nature, we revert to decision making under risk and hence can use the expected value criterion as shown in the table below:

	State of nature			
	N_1	N_2	N_3	*Expected monetary value (EMV)*
Probabilities→	1/3	1/3	1/3	
Strategy	*Utility or Payoffs.*			
S_1	15	12	18	**15 × 1/3 + 12 × 1/3 + 18 × 1/3 =15 Maximum**
S_2	9	14	10	9 × 1/3 + 14 × 1/3 + 10 × 1/3 = 11
S_3	13	4	26	13 × /3 + 4 × 1/3 + 26 × 1/3 = 14 1/3

As S_1 is having highest EMV it is the optimal strategy.

DECISION MAKING UNDER CONFLICT AND COMPETITION

In the problems discussed above, we have assumed that the manager has a finite set of strategies and he has to identify the optimal strategy depending on the condition of complete certainty to complete uncertainty. In all the models, the assumptions made are (1) Various possible future environments that the decision maker will face can be enumerated in a finite set of states of nature and (2) The complete payoff matrix is known. Now, let us consider that two rationale competitors or opponents are required to select optimal strategies, given a series of assumptions, including: (1) The strategies of each party are *known* to both opponents, (2) Both opponents choose their strategies simultaneously, (3) the loss of one party equals exactly to *gain* of the other party, (4) Decision conditions remain the same, and (5) It is a *repetitive* decision making problem (refer to Game theory).

Two opponents are considered as two **players**, and we adopt the convention that a **positive payoff** will mean a **gain** to the **row player *A*** or **maximizing player**, and a **loss** to the **column player *B*** or **minimizing player**. (Refer to 2 person zero sum game).

Consider the matrix given: maximin identifies outcome for player *A* and Minimax identifies the optimal strategy outcome for player *B*. This is because each player can adopt the policy, which is best to him. *A* wants to maximize his minimum outcomes and *B* wants to minimize his maximum loses.

		Player B				
		B_1	B_2	B_3	B_4	Row minimum
	A_1	8	12	7	3	3
Player *A*	A_2	$\boxed{9}$	14	10	16	**9**
	A_3	7	4	26	5	4
Column maximum		**9**	14	26	16	

A selects the second strategy as it guaranties him a minimum of 9 units of money and *B* chooses strategy 2 as it assures him a minimum loss of 9 units of money. This type of games is known as **pure strategy game**.The alement where minimax point and maximin point are same known as **saddle point**.

HURWICZ CRITERION (CRITERION OF REALISM)

This is also known as weighted average criterion, it is a compromise between the maximax and maximin decisions criteria. It takes both of them into account by assigning them weights in accordance with the degree of optimism or pessimism. The alternative that maximizes the sum of these weighted payoffs is then selected.

Problem 12.7.

The following matrix gives the payoff of different strategies (alternatives) *A*, *B*, and *C* against conditions (events) *W*, *X*, *Y* and *Z*. Identify the decision taken under the following approaches:
(i) Pessimistic, (*ii*) Optimistic, (*iii*) Equal probability, (*iv*) Regret, (*v*) Hurwicz criterion. The decision maker's degree of optimism (α) being 0.7.

Events

	W	*X*	*Y*	*Z*
	Rs.	*Rs.*	*Rs.*	*Rs.*
A	4000	–100	6000	18000
B	20000	5000	400	0
C	20000	15000	– 2000	1000

Solution (for i, ii, and iii)

	W	*X*	*Y*	*Z*	*Maximum regret. Rs.*
	Regret Rs.	*Regret Rs.*	*Regret Rs*	*Regret Rs*	—
A	16000	15100	0	0	16000
B	0	10000	5600	18000	18000
C	0	0	8000	17000	17000

	Pessimistic Maximin value	*Optimistic Maximax value*	*Equal Probability value*
A	– Rs. 100/-	Rs. 18000	Rs. ¼ (4000 – 100 + 6000 + 18000) = Rs. 6975/-
B	Rs. 0/-	Rs. 20000	Rs. ¼ (20000 + 5000 + 400 + 0) = Rs. 6350/-
C	– Rs. 2000	Rs. 20000	Rs. ¼ (20000 + 15000 – 2000 + 1000) = Rs. 8,500/-

Under pessimistic approach, *B* is the optimal strategy, under optimistic approach *B* or *C* are optimal strategies, and under equal probability approach *C* is the optimal strategy.

(iv) Given table represents the regrets for every event and for each alternative calculated by:

= i^{th} regret = (maximum payoff – i^{th} payoff) for the j^{th} event.

As strategy *A* shows minimal of the maximum possible regrets, it is selected as the optimal strat egy.

(v) For a given payoff matrix the minimum and the maximum payoffs for each alternative is given in the tablebelow:

Alternative	*Maximum payoff Rs*	*Minimum payoff. Rs*	*Payoff = α × maximum payoff + (1–α) minimum payoff , where α = 0.7 (Rs)*
A	18000	– 100	0.7 × 18000 – 0.3 × 100 = 12570
B	20000	0	0.7 × 20000 + 0.3 × 0 = 14000
C	20000	–2000	0.7 × 20000 – 0.3 × 2000 = 13400

Under Hurwicz rule, alternative *B* is the optimal strategy as it gives highest payoff.

Problem 12.8.

A newspaper boy has the following probabilities of selling a magazine. Cost of the copy is Rs. 0.30 and sale price is Rs. 50. He cannot return unsold copies. How many copies can he order?

No. of copies sold	*Probability*
10	0.10
11	0.15
12	0.20
13	0.25
15	0.30
Total	1.00

Solution

The sales magnitude of newspaper boy is 10, 11, 12, 13, 14 papers. There is no reason for him to buy less than 10 or more than 14 copies. The table below shows conditional profit table, *i.e*. the profit resulting from any possible combination of supply and demand. For example, even if the demand on some day is 13 copies, he can sell only 10 and hence his conditional profit is 200 paise. When stocks 11 copies, his profit is 220 paise on days when buyers request 11, 12, 13, and 14 copies. But on the day when he has 11 copies, and the buyers buy only 10 copies, his profit is 170 paisa, because one copy is unsold. Hence payoff = 20 × copies sold – 30 × copies unsold. Hence conditional profit table is:

Conditional Profit Table (paisa)
Possible Stock Action

Possible demand (number of copies)	*Probability*	*10 copies*	*11 copies*	*12 copies*	*13 copies*	*14 copies*
10	0.10	200	170	140	110	80
11	0.15	200	220	190	160	130
12	0.20	200	220	240	210	180
13	0.25	200	220	240	260	230
14	0.30	200	220	240	260	280

Expected Profit Table
Expected Profit from Stocking in Paisa

Possible demand	*Probability*	*10 copies*	*11 copies*	***12 copies***	*13 copies*	*14 copies*
10	0.10	20	17	14	11	8
11	0.15	30	33	28.5	24	19.5
12	0.20	40	44	48	42	36
13	0.25	50	55	60	65	57.5
14	0.30	60	66	72	78	84
Total Expected Profit in Paisa		200	215	**222.5**	220	205

The newsboy must therefore order 12 copies to earn the highest possible average daily profit of 222.5 paise. Hence optimal stock is 12 papers. This stocking will maximize the total profits over a period of time. Of course there is no guarantee that he will make a profit of 222.5 paise tomorrow. However, if he stocks 12 copies each day under the condition given, he will have average profit of 222.5 paisa per day. This is the best be can do because the choice of any of the other four possible stock actions will result in a lower daily profit.

(Note: The same problem may be solved by Expected Opportunity Loss concept as shown below)

EOL (Expected Opportunity Loss) can be computed by multiplying the probability of each of state of nature with the appropriate loss value and adding the resulting products.

For example: $0.10 \times 0 + 0.15 \times 20 + 0.20 \times 40 + 0.25 \times 60 + 0.30 \times 80 = 0 + 3 + 15 + 24 = 50$ paise.

Conditional Loss Table in Paisa
Possible Stock Action (Alternative)

Possible demand *Number of copies (event)*	*Probability*	*10 copies*	*11 copies*	*12 copies*	*13 copies*	*14 copies*
10	0.10	0	30	60	90	120
11	0.15	20	0	30	60	90
12	0.20	40	20	0	30	60
13	0.25	60	40	20	0	30
14	0.30	80	60	40	20	0

If the newspaper boy stocks 12 papers, his expected loss is less.

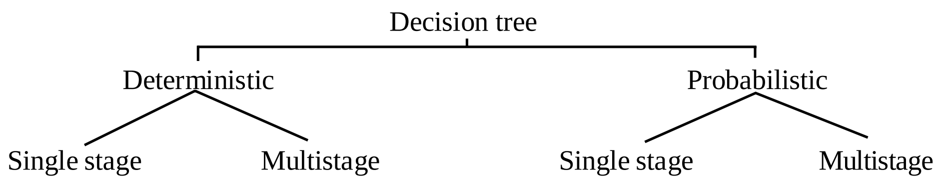
Possible demand Number of copies (event)	*Probability*	*10 copies*	*11 copies*	*12 copies*	*13 copies*	*14 copies*
10	0.10	0	3	6	9	12
11	0.15	3	0	4.5	9	13.5
12	0.20	8	4	0	6	12
13	0.25	15	10	5	0	7.5
14	0.30	24	18	12	6	0
EOL (Paisa)		50	35	27.5	30	45

DECISION TREES

All the decision-making problems discussed above are **single stage** decision-making problems. It is because in all the problems, an assumption is made that the available data regarding payoffs, strategies, states of nature, competitor's actions and probability distribution is not subject to revision and that the entire decision horizon is considered as a single stage. Only one decision is made and these single stage models are static decision models, because available data is not revised under the assumption that time does not change any basic facts, and that no new information is sought. There are, however, business situations where the manager needs to make not one, but a **sequence of decisions**. These problems then become **multistage problems;** because the outcome of one decision affects subsequent decisions. In situations, that require a sequence of decisions, the manager can utilize a simple but useful schematic device known as **decision tree**. **A decision tree is a schematic representation of a decision problem**.

A decision tree consists of **nodes, branches, probability estimates**, and **payoffs**. There are two types of nodes, one is **decision node** and other is **chance node**. A decision node is generally represented by a square, □ requires that a conscious decision be made to choose one of the branches that emanate from the node (*i.e.* one of the availed strategies must be chosen). The branches emanate from and connect various nodes. We shall identify two types of branches: **decision branch** and **chance branch.** A decision branch denoted by parallel lines (=) represents a strategy or course of action. Another type of branch is chance branch, represented by single line (—) represents a chance determined event. Indicated alongside the chance branches are their respective probabilities. When a branch marks the end of a decision tree *i.e.* it is not followed by a decision or chance node will be called as terminal branch. A terminal branch can represent a decision alternative or chance outcome.

The payoffs can be positive (profit or sales) or negative (expenditure or cost) and they can be associated with a decision branch or a chance branch. The payoffs are placed alongside appropriate branch except that the payoffs associated with the **terminal branches** of the decision tree will be shown at the end of these branches. The decision tree can be **deterministic** or **probabilistic** (stochastic), and it can he represent a single-stage (one decision) or a multistage (a sequence of decisions) problem.



The classification of decision tree is shown above.

12.17.1

A deterministic decision tree represents a problem in which each possible alternative and its outcome are known with certainty. That is, a deterministic tree does not contain any chance node. **A single stage deterministic decision tree is one that contains no chance nodes and involves the making of only one decision.**

Problem 12.9.

A business manager wants to decide whether to replace certain equipment in the first year or in the second year or not replace at all. The payoffs are shown below. Draw a decision tree to decide the strategy.

Profits or Payoffs in Rupees

Strategy	*First year*	*Second year*	*Total*
A Replace now	4000	6000	10000
B Replace after one year	5000	4000	9000
C Do not replace	5000	3000	8000

Solution: (Figure 12.1)

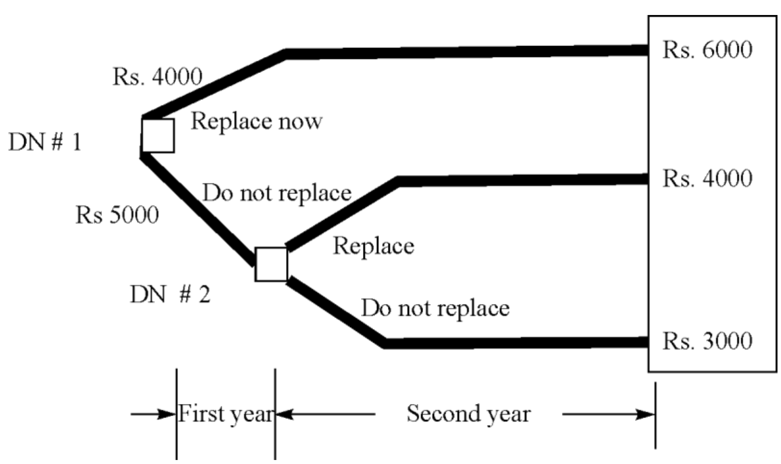


Figure 12.1

The optimal strategy is to replace the equipment now.

Stochastic Decision Trees

These are characterized by the presence of chance nodes. **A single-stage stochastic decision tree is one that contains at least one chance node and involves the making of only one decision.** Conceptually, any conditional payoff matrix can be represented as a single- stage stochastic decision tree, and vice versa. However, such problems (involving one decision) are best formulated and solved by the payoff matrix approach.

A multistage stochastic decision tree is one that contains at least one chance node and involves the making of a sequence of decisions. The decision tree approach is most useful in analyzing and solving the multistage stochastic decision problems.

Problem 12.10.

Basing on the recommendations of the strategic advisory committee of M/S Zing manufacturing company it has decided to enter the market with a new consumer product. The company has just established a corporate management science group with members drawn from research and development, manufacturing, finance and marketing departments. The group was asked to prepare and present an investment analysis that will consider expenditures for building a plant, sales forecasts for the new product, and net cash flows covering the expected life of the plant. After having considered several alternatives, the following strategies were presented to top management.

Strategy A: Build a large plant with an estimated cost of Rs. 200 crores.

This alternative can face two states of nature or market conditions: High demand with a probability of 0.70, or a low demand with a probability of 0.30. If the demand is high, the company can expect to receive an annual cash flow of Rs. 50,00,000 for 7 years. If the demand were low the annual cash flow would be only Rs. 10,00,000, because of large fixed costs and inefficiencies caused by small volume. As shown in figure, strategy *A* ultimately branches into two possibilities depending on whether the demand is high or low. These are identified as decision tree terminal points A_1 and A_2 (Figure 12.2).

Strategy B: Build a small plant with an estimated cost of Rs. 1 crore.

This alternative also faces two states of nature: High demand with a probability of 0.70, or a low demand with a probability of 0.30. If the demand is low and remains low for 2 years, the plant is not expanded. However, if initial demand is high and remains high for two years, we face another decision of whether or not to expand the plant. If it is assumed that the cost of expanding the plant at that time is Rs. 1.5 crore. Further, it is assumed that after this second decision the probabilities of high and low demand remains the same.

As shown in the figure 12.2 strategy *B* eventually branches into five possibilities. Identified by terminal points B_1 to B_5.

Estimate of the annual cash flow and probabilities of high demand and low demand are shown in figure 12.2.

What strategy should be selected?

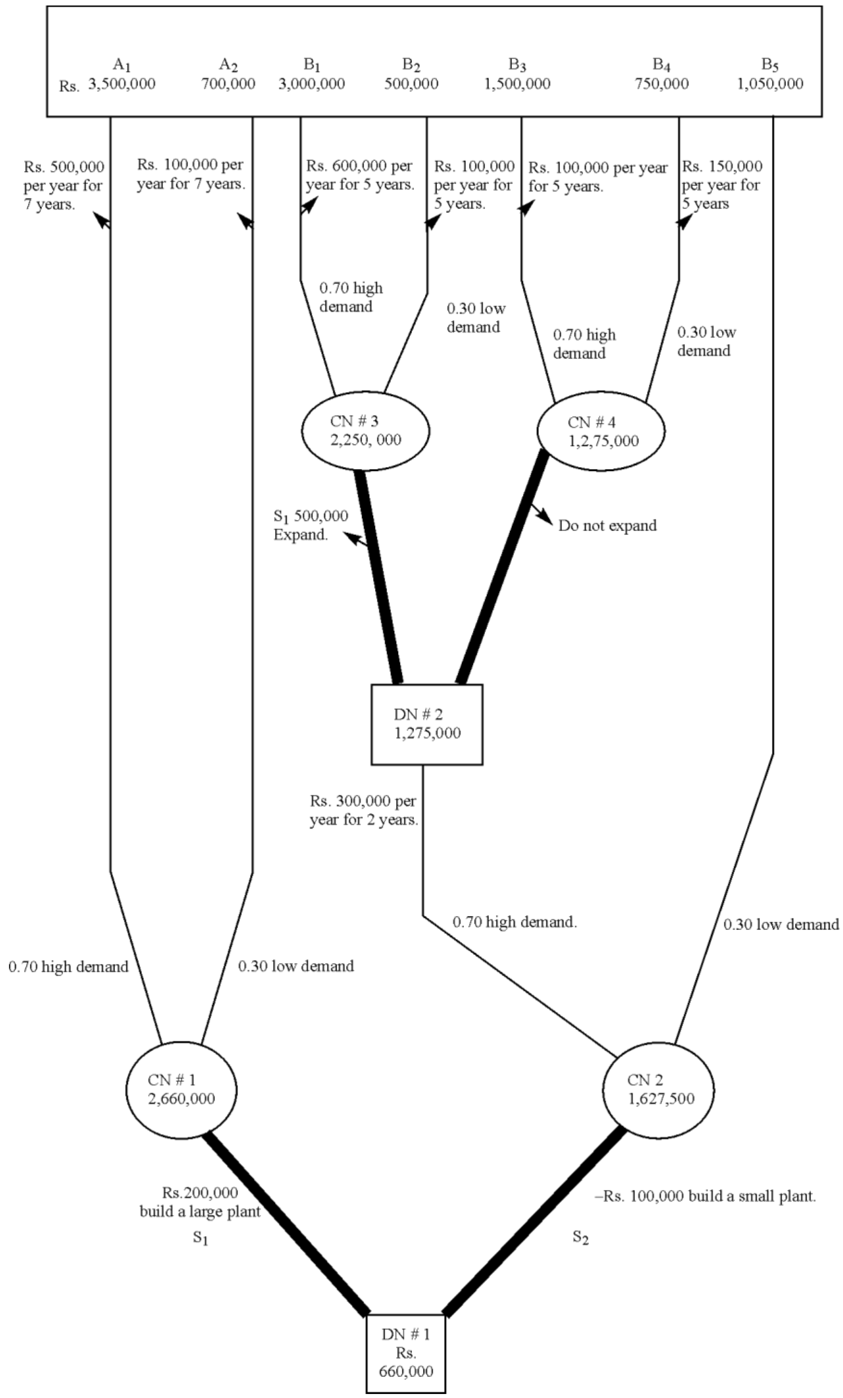


Figure 12.2. Multistage decision tree

Solution

The decision tree shown in figure 12.2 is drawn in such a manner that the starting point is a decision node and the progression is from left to right. The number of branches, and the manner, in which various decision and chance nodes are connected by means of branches, indicates various paths through the tree. Along the braches stemming from decision nodes, we write down the decision alternatives and/ or their monetary payoffs or costs, along the branches probabilities, and monetary payoff finally, at the extreme right hand side of the decision tree (terminal branches, after which no decision is made or a chance node is appeared) the relevant payoffs are shown. At the end of each terminal the related payoff is shown.

Once the relevant information regarding decision nodes, chance nodes, decision and chance branches, rewards or costs of decision branches probabilities and payoffs associated with chance braches are known, we can analyze the tree.

Analysis

The analysis of decision tree consists of calculating the **position value** of each node through the process or **roll back**. The concept of roll back implies that we start from the end of the tree, where the payoff is associated with the terminal branches as indicated and go back towards the first decision node (DN # 1) *i.e.* we proceed from right to left.

As we roll back, we can face either a chance node or a decision node. T**he position value of the chance node is simply the expected value of the payoffs represented by various branches that stem from the node.**

For example, the position value of Chance node 1 (CN # 1) is

= 0.7 (7 × 500,000) + 03 × (7 × 100,000) = Rs. 2,660,000

Position value of chance nodes 3 and 4 (CN # 3, CN # 4) are:

Position value of CN # 3 = 0.7 (5 × 600,000) + 0.3 (5 × 100,000) = Rs. 2, 250, 000.

Position value of CN # 4 = 0.7 (5 × 300,000) + 0.3 (5 × 150,000) = Rs. 1, 275, 000.

The position value of a *decision node* is the highest (assuming positive payoffs) of the position value of nodes, or the node, to which it is connected, *less* the cost involved in the specific branch leading to that node. For example, as we roll back to decision node 2 (DN # 2), we note that Rs. 150, 000 (cost of expansion) must be subtracted from the position value of chance node 3 (CN # 3) *i.e.* Rs. 225,000. That is the branch yields Rs. 2,250,000 – Rs. 1,750,000 =, Rs. 750,000. And this must be compared with the CN # 4 position value of Rs. 1,275,000. The higher of the two values i.e. Rs. 1, 275,000 is the position value of DN # 2. The position value of a node will be placed inside the symbol for the node.

Next, let us rollback to CN # 2, as in CN # 3 and CN # 4, the position value of CN # 2 is also calculated by the **expected value concept**. However, in the case of CN # 2, one of the branches emanating from it leads with a probability 0.7 to a decision node (the payoff for this branch is a total cash flow of Rs. 5,600,000 plus the position value of DN # 2) while the other is the terminal branch, having a probability of 0.3, with its own pay of Rs. 1,050, 000. Hence the position value of CN # 2 is:

0.7 (Rs. 600,000+ Rs. 1,275,000) + 0.3 (7 × 150,000) = Rs. 1,627,500.

We are now ready to roll back to DN # 1. As shown in figure 12.2, the position values of CN # 1 and CN # 2 that are connected to decision node 1 are already calculated. From the position value of

CN # 1, we subtract Rs. 2,000,000 (cost of building a large plant) and obtain 2,660, 000 – 2,000,000 = Rs. 660,000. From the position value of CN # 2, we subtract 1, 000,000, the cost of building a small plant and get 1, 627, 500 – 1, 000, 000 = Rs. 627, 500. Thus, when we compare the two decisions branches emanating from DN #1, we find that the strategy A, to build a large plant, yields the higher payoff. Hence the position value of DN # 1 is Rs. 660, 000. That the strategy A is the optimal strategy and its expected value is Rs. 660, 000.

When we summarize, the elements and concepts needed to consider a decision are:

- All decisions and chance nodes.
- Branches that connect various decision and chance nodes.
- Payoff (reward or cost) associated with branches emanating from decision nodes.
- Probability values associated with braches emanating from chance nodes.
- Payoffs associated with each of chance branches.
- Payoffs associated with each terminal branch at the no conclusion of each path that can be traced through various combinations that form the tree.
- Position values of Chance and Decision nodes.
- The process of roll back.

Our decision tree problem described above involves a sequence of only two decisions, and a chance node had only two branches. This is obviously a simplified example, designed only to show the concept, structure, and mechanics of the decision-tree approach. The following are only some of the refinements that can be introduced in order to get more reality.

- The sequence of decision can involve a larger number of decisions.
- At each decision node, we can consider a larger number of strategies.
- At each chance node, we can consider a larger number of chance branches. Actually, we can even assume continuous probability distribution at each chance node.
- We can introduce more sophisticated and more detailed projections of cash flows.
- We can use the concept discount that would take into account the fact that present rupee value worth more future value.
- We can also obtain an idea of the **quality of the risk** associated with relevant decision-tree paths. That is, in addition to calculating the *expected value*, we can calculate such parameters as **range** and **standard deviation** of the payoff distribution associated with each relevant path.
- We can conduct Bayesian analysis that permits introduction of new information and revision of probabilities.

Admittedly, neither the problems nor the decisions are that simple in real world. However, the attempts to analyze decision problems in a quantitative fashion yield not only some ''ball park'' figure, but also valuable qualitative insights into the entire decision environment.

Problem 12.11.

A client has an estate agent to sell three properties *A*, *B* and *C* for him and agrees to pay him 5% commission on each sale. He specifies certain conditions. The estate agent must sell property *A* first, and this he must do within 60 days. If and when *A* is sold the agent receives his 5% commission on that sale. He can then either back out at this stage or nominate and try to sell one of the remaining two properties within 60 days. If he does not succeed in selling the nominated property in that period, he is not given opportunity to sell the third property on the same conditions. The prices, selling costs (incurred by the estate agent whenever a sale is made) and the estate agent's estimated probability of making a sale are given below:

Property	*Price of Property in Rs.*	*Selling Costs in Rs.*	*Probability of Sales*
A	12,000	400	0.70
B	25,000	225	0.60
C	50,000	450	0.50

(1) Draw up an appropriate decision tree for the estate agent.

(2) What is the estate agent's best strategy under Expected monitory value approach (EMV)?

Solution

The estate agent gets 5% commission if he sells the properties and satisfies the specified condition. The amount he receives as commission on the sale of properties *A*, *B* and *C* will be Rs. 600/-, RS. 1250/- and Rs. 2500 respectively. Since selling costs incurred by him are Rs. 400/-, Rs. 225/- and Rs. 450/-, his conditional profits from sale of properties *A*, *B* and *C* are Rs. 200/-, Rs. 1025/- and Rs. 2050/- respectively. The decision tree is shown in figure 12.3.

EMV of node *D* = Rs. $(0.5 \times 2050 + 0.5 \times 0)$ = Rs. 1025.

EMV of node *E* = Rs. $(0.6 \times 1025 + 0.4 \times 0)$ = Rs. 615.

EMV of node 3 = Maximum of Rs. (1025, 0) = Rs. 1025.

EMV of node 4 = Maximum of Rs. (615, 0) = Rs. 615.

EMV of node *B* = Rs. $[0.6 (1025 + 1025) + 0.4 \times 0]$ = Rs. 1230.

EMV of node *C* = Rs. $[0.5 (2050 + 615) + 0.5 \times 0]$ = RS. 1332.50.

Therefore, EMV of node 2 = Rs. 1332.50, higher among EMV at *B* and *C*.

Therefore, EMV of node *A* = Rs. $[0.7(200 + 1332.50) + 0.3 \times 0]$ = Rs. 1072.75

Therefore, EMV of node 1 = Rs. 1072.75.

The optimal strategy path is drawn in bold lines. Thus, the optimum strategy for the estate agent is to sell *A*; if he sells *A* then try to sell *C* and if he sells *C* then try to sell *B* to get an optimum, expected amount of Rs. 1072.50.

Figure 12.3 shows Decision tree for problem 12.11.

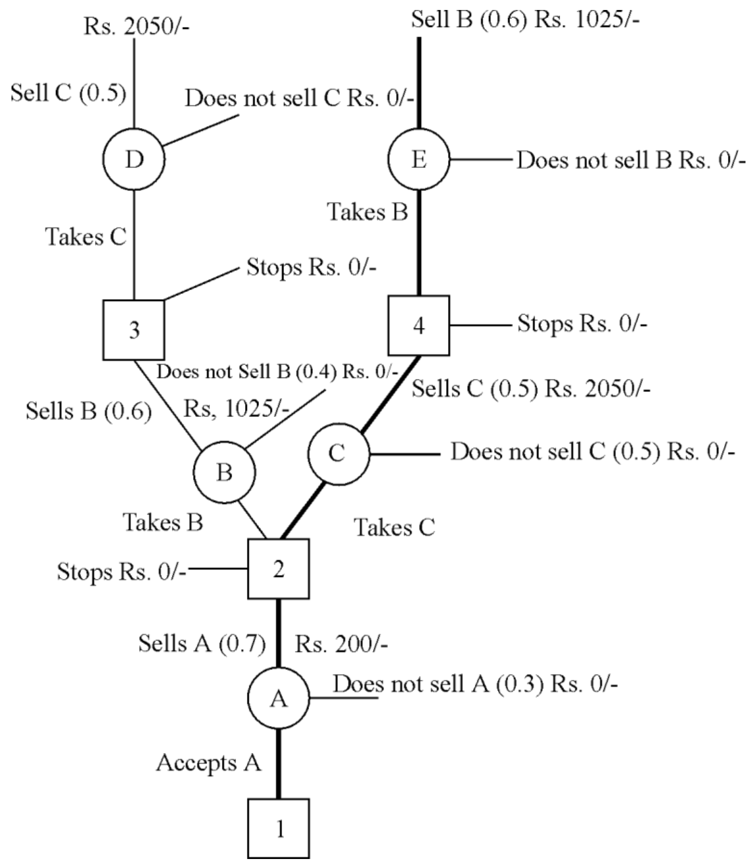


Figure 12. 3.

Problem 12.12.

Mr. Sinha has to decide whether or not to drill a well on his farm. In his village, only 40% of the wells drilled were successful at 200 feet of depth. Some of the farmers who did not get water at 200 feet drilled further up to 250 feet but only 20% struck water at 250 feet. Cost of drillings is Rs. 50/- per foot. Mr. Sinha estimated that he would pay Rs. 18000/- during a 5-year period in the present value terms, if he continues to buy water from the neighbour rather than go for the well which would have life of 5 years. Mr. Sinha has three decisions to make: (*a*) Should he drill up to 200 feet? (*b*) If no water is found at 200 feet, should he drill up to 250 feet? (*c*) Should he continue to buy water from his neighbour? Draw up an appropriate decision tree and determine its optimal decision.

Solution

Decision tree is shown in figure 12.4. The cost associated with each outcome is written on the decision tree.

EMV of node B = Rs. [0.2 × 0 + 0.8 × 18000] = Rs. 14,400/-

Therefore EMV of node 2 = Rs. 16,900/- lesser of the two values of Rs. 16900 and Rs. 18000/-

Therefore EMV of node A = Rs. [0.40 × 0 + 0.6 × 16900/-] = Rs. 10,140/-

EMV of node 1 = Rs. 18000/- lesser of the two values Rs. 20,140/- and Rs. 18000/-

The optimal least cost course of action for Mr. Sinha is not to drill the well and pay Rs. 18000/- for water to his neighbour for five years.

Figure 12.4 shows Decision tree for problem No. 12.12.

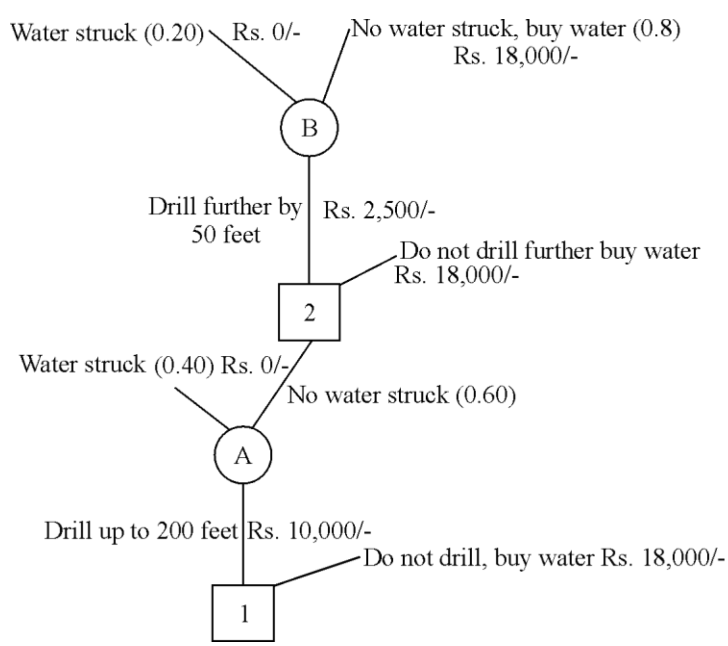


Figure 12.4.

QUESTIONS

1. What is a decision? Differentiate between programmed and non-programmed decisions.
2. Define the term Decision theory. Describe decision models based on the criterion of degree of certainty.
3. Explain the concept of expected value. Give general formula for calculating the expected value when we are a finite number of outcomes.

4. Three strategies and three states of nature are given and payoffs represent profits. (*i*) What is the optimal strategy if we apply the criterion of pessimism? (*ii*) Develop a regret matrix and apply the minimax regret criterion to identify the optimal strategy.

State of nature.

Strategy	N_1	N_2	N_3
S_1	47	49	33
S_2	32	25	41
S_3	51	30	14

5. A complex airborne navigating system incorporates a sub-assembly, which unrolls a map of the flight, plan synchronously with the movement of the aeroplane. This subassembly is bought on very good terms from a subcontractor, but is not always in perfect adjustment on delivery. The subassemblies can be readjusted on delivery to guarantee accuracy at a cost of Rs. 50/- persubassembly. It is not, however, possible to distinguish visually those sub-assemblies that need adjustment.

   Alternatively, the sub-assemblies can each be tested electronically at a cost of Rs. 10/- per subassembly tested. Past experience shows that about 30 % of those supplied are defective; the probability of the test indicating a bad test indicates a good adjustment when the sub-assembly is found to be faulty when the system has its final check, the cost of subsequent rectification will be Rs. 140/-.

   Draw up an appropriate decision tree to show the alternatives open to the purchaser and use it to determine its appropriate course of action.

7. A large steel manufacturing company has three options with regard to production (a) Produce commercially (*b*) Build pilot plant and (*c*) Stop producing steel. The management has estimated that their pilot plant, if built, has 0.8 chance of high yield and 0.2 chance of low yield. If the pilot plant does show a high yield, management assigns a probability of 0.75 that the commercial plant will also have a high yield. If the pilot plant shows a low yield, there is only a 0.1 chance that the commercial plant will show a high yield. Finally, management's best assessment of the yield on a commercial-size plant without building a pilot plant first has a 0.6 chance of high yield. A pilot plant will cost Rs. 3,00,000/. The profits earned under high and low yield conditions are Rs. 1,20,00,000/- and – Rs. 12,00,000/- respectively. Find the optimum decision for the company.

MULTIPLE-CHOICE QUESTIONS

1. The conclusion of a process designed to weigh the relative utilities of a set of available alternatives to select most preferred one is known as:

   (*a*) Concluding session, (*b*) Conclusion,
   (*c*) End of the process, (*d*) Decision. ( )

2. The body of knowledge that deals with the analysis of making of decisions is known as

   (*a*) Decisions (*b*) Knowledge base
   (*c*) Decision theory (*d*) Decision analysis. ( )

3. Decisions that are meant to solve repetitive and well structured problems are known as:
   (*a*) Repetitive decisions, (*b*) Structured decisions,
   (*c*) Programmed decisions, (*d*) Linear programming. ( )
4. Decisions that handle non-routine, novel, and ill structured problems are known as:
   (*a*) Non-programmed decisions, (*b*) Programmed decisions,
   (*c*) Ill-structured decisions (*d*) Non-linear programming. ( )

ANSWERS

1. (*d*) 2. (*c*) 3. (*c*) 4. (*a*)

CHAPTER – 13

Simulation

INTRODUCTION

Simulation is the most important technique used in analyzing a number of complex systems where the methods discussed in previous chapters are not adequate. There are many real world problems which cannot be represented by a mathematical model due to stochastic nature of the problem, the complexity in problem formulation and many values of the variables are not known in advance and there is no easy way to find these values.

Simulation has become an important tool for tackling the complicated problem of managerial decision-making. Simulation determines the effect of a number of alternate policies without disturbing the real system. Recent advances in simulation methodologies, technical development and software availability have made simulation as one of the most widely and popularly accepted tool in Operation Research. Simulation is a quantitative technique that utilizes a computerized mathematical model in order to represent actual decision-making under conditions of uncertainty for evaluating alternative courses of action based upon facts and assumptions.

John Von Newmann and Stainslaw Ulam made first important application of simulation for determining the complicated behaviour of neutrons in a nuclear shielding problem, being too complicated for mathematical analysis. After the remarkable success of this technique on neutron problem, it has become more popular and has many applications in business and industry. The development of digital computers has further increased the rapid progress in the simulation technique.

Designers and analysts have long used the techniques of simulation by physical sciences. Simulation is the representative model of real situation. Fore example, in a city, a children's park with various signals and crossing is a simulated model of city traffic. A planetarium is a simulated model of the solar system. In laboratories we perform a number of experiments on simulated model to predict the behaviours of the real system under true environment. For training a pilot, flight simulators are used. The simulator under the control of computers gives the same readings as the pilot gets in real flight. The trainee can intervene when there is signal, like engine failure etc. Simulation is the process of generating values using random number without really conducting experiment. Whenever the experiments are costly or infeasible or time-consuming simulation is used to generate the required data.

DEFINITION

1. Simulation is a representation of reality through the use of model or other device, which will react in the same manner as reality under a given set of conditions.

2. Simulation is the use of system model that has the designed characteristic of reality in order to produce the essence of actual operation.
3. According to Donald G. Malcolm, simulation model may be defined as one which depicts the working of a large scale system of men, machines, materials and information operating over a period of time in a simulated environment of the actual real world conditions.
4. According to Naylor, et al. simulation is a numerical technique for conducting experiments on a digital computer, which involves certain types of mathematical and logical relationships necessary to describe the behaviour and structure of a complex real world system over extended period of time.

There are two types of simulation, they are:

1. Analog Simulation: Simulating the reality in physical form (*e.g.*: Children's park, planetarium, etc.) is known as analog simulation.
2. Computer Simulation: For problems of complex managerial decision-making, the analogue simulation may not be applicable. In such situation, the complex system is formulated into a mathematical model for which a computer programme is developed. Using high-speed computers then solves the problem. Such type of simulation is known as computer simulation or system simulation.

CLASSIFICATION OF SIMULATION MODELS

Simulation models are classified as:

(a) Simulation of Deterministic models:

In the case of these models, the input and output variables are not permitted to be random variables and models are described by exact functional relationship.

(b) Simulation of Probabilistic models:

In such cases method of random sampling is used. The techniques used for solving these models are termed as Monte-Carlo technique.

(c) Simulation of Static Models:

These models do not take variable time into consideration.

(d) Simulation of Dynamic Models:

These models deal with time varying interaction.

ADVANTAGES OF SIMULATION

Simulation is a widely accepted technique of operations research due to the following reasons:

* It is straightforward and flexible.
* It can be used to analyze large and complex real world situations that cannot be solved by conventional quantitative analysis models.
* It is the only method sometimes available.
* It studies the interactive effect of individual components or variables in order to determine which ones are important.
* Simulation model, once constructed, may be used over and again to analyze all kinds of different situations.

* It is the valuable and convenient method of breaking down a complicated system into subsystems and their study. Each of these subsystems works individually or jointly with others.

LIMITATIONS OF SIMULATION TECHNIQUE

* Since simulation model mostly deals with uncertainties, the results of simulation are only reliable approximations involving statistical errors, optimum results cannot be produced by simulation.
* In many situations, it is not possible to identify all the variables, which affect the behaviour of the system.
* In very large and complex problems, it is very difficult to make the computer program in view of the large number of variables and the involved inter–relationship among them.
* For problems requiring the use of computer, simulation may be comparatively costlier and time consuming in many cases.
* Each solution model is unique and its solutions and inferences are not usually transferable to other problems, which can be solved by other techniques.

MONTE-CARLO SIMULATION

The Monte-Carlo method is a simulation technique in which statistical distribution functions are created using a series of random numbers. Working on the digital computer for a few minutes we can create data for months or years. The method is generally used to solve problems which cannot be adequately represented by mathematical models or where solution of the model is not possible by analytical method. Monte-Carlo simulation yields a solution, which should be very close to the optimal, but not necessarily the exact solution. But this technique yields a solution, which converges to the optimal solution as the number of simulated trials tends to infinity. The Monte-Carlo simulation procedure can be summarized in the following steps:

Step 1: Clearly define the problem:

(a) Identify the objectives of the problem.

(b) Identify the main factors, which have the greatest effect on the objective of the problem.

Step 2: Construct an approximate model:

(a) Specify the variables and parameters of the mode.

(b) Formulate the appropriate decision rules, *i.e*. state the conditions under which the experiment is to be performed.

(c) Identify the type of distribution that will be used. Models use either theoretical distributions or empirical distributions to state the patterns of the occurrence associated with the variables.

(d) Specify the manner in which time will change.

Problem 13.1.

With the help of a single server queuing model having inter-arrival and service times constantly 1.4 minutes and 3 minutes respectively, explain discrete simulation technique taking 10 minutes as the simulation period. Find from this average waiting time and percentage of idle time of the facility of a customer. Assume that initially the system is empty and the first customer arrives at time t = 0.

Solution

Data: System is initially empty. Service starts as soon as first customer arrives. First customer arrives at t = 0.

The departure time of first customer = 0 + 3 *i.e*. arrival time + service time = 3 minutes (Dep) in the table. The second customer arrives at 1.4 minutes and third arrives at 2.8 minutes (Arr). Until the first customer leaves the system, second and third customers have to wait for service. We can calculate waiting time for second customer by taking the difference of time of departure of first customer and the time of arrival of second customer *i.e*. 3 – 1.4 = 1.6 minutes. The procedure is shown in the table below:

Time	*Event Arr = arrival Dep = departure*	*Customer Number*	*Waiting time.*
0.0	Arr.	1	--
1.4	Arr.	2	-
2.8	Arr.	3	--
3.0	Dep	1	3.00 – 1.40 = 1.6 min. for customer 2.
4.2	Arr.	4	
5.6	Arr.	5	
6.00	Dep	2	6.00 – 2.80 = 3.2 min. for customer 3.
7.00	Arr.	6	
8.4	Arr.	7	
9.00	Dep.	3	9.00 – 4.20 = 4.8 min. for customer 4
9.80	Arr.	8	
10.00	End of given time	-	10.00 – 5.60 = 4.4 min. for customer 5
			10.00 – 7.00 = 3.0 min. for customer 6
			10.00 – 8.4 = 1.6 min. for customer 7
			10.00 – 9.80 = 0.2 min. for customer 8.

Average waiting time per customer for those who must wait = Sum of waiting time of all customers / number of waiting times taken = (1.4 + 2.8 + 4.2 + 5.6 + 7.0 + 8.4 + 9.8) / 7 = 18.8 / 7 = 2.7 minutes.

Percentage of idle time of server = Sum of idle time of server / total time = 0%.

RANDOM NUMBERS

Random number is a number in a sequence of numbers whose probability of occurrence is the same as that of any other number in that sequence.

Pseudo-Random Numbers

Random numbers which are generated by some deterministic process but which satisfy statistical test for randomness are called **Pseudo-random numbers.**

Generation of Random Numbers

Using some arithmetic operation one can generate Pseudo-random numbers. These methods most commonly specify a procedure, where starting with an initial number called **seed** is generates the second number and from that a third number and so on. A number of recursive procedure are available, the most common being the **congruence method** or the **residue method.** This method is described by the expression:

$$\boldsymbol{r_{i+1} = (ar_{i+b})\ (\text{modulo } m),}$$

Where a, b and m are constants, r_i and r_{i+1} are the ith and (i + 1)th random numbers.

The expression implies multiplication of a by r_i and addition of b and then division by m. Then r_{i+1} is the remainder or residue. To begin the process of random number generation, in addition to a, b and m, the value of r_0 is also required. It may be any random number and is called **seed**.

Problem 13.2

With the help of an example explain the additive multiplicative and mixed types of the congruence random number generators.

Solution

The ongruence random number generator is described by the recursive expression

$\boldsymbol{r_{i+1} = (ar_{i+b})\ (\text{modulo } m),}$

Where a, b and m are constants. The selection of these constants is very important as it determines the starting of random number, which can be obtained by this method. The above expression is for a **mixed type congruential method** as it comprises both multiplication of a and r_i and addition of ar_i and b.

If a = 1, the expression reduces to $r_{i+1} = (r_{i+b})$ (modulo m). This is known as **additive type expression.**

When b = 0, the expression obtained is $r_{I+1} = (ar\ I)$ (modulo m), this is known as **multiplicative method.**

To illustrate the different types of the congruence methods, let us take a = 16, b = 18 and m = 23 and let the starting random number or seed be r_0 = 1.

(a) Mixed Congruential method: $\boldsymbol{r_{i+1} = (ar_i + b)\ (\text{modulo } m),}$ therefore,

r_i	$r_{i+1} = (ar_i + b)$ (modulo m),	=	Residue
r_1	(16 × 1 + 18) / 23	34 / 23	1 residue 11
r_2	(16 × 11 + 18) /23	194 / 23	8 + residue 10
r_3	(16 × 10 + 18) / 23	178 / 23	7 + residue 17
r_4	(16 × 17 + 18) / 23	290 / 23	12 + residue 14
r_5	(16 × 14 + 18) / 23	242 / 23	10 + residue 12
r_6	(16 × 12 + 18) / 23	210 / 23	9 + residue 3
r_7	(16 × 3 + 18) / 23	66 / 23	2 + residue 20
r_8	(16 × 20 + 18) / 23	338 / 23	14 + residue 16
r_9	(16 × 16 + 18) / 23	274 / 23	11 + residue 21
r_{10}	(16 × 21 + 18) / 23	354 / 23	15 + residue 9
r_{11}	(16 × 9 + 18) / 23	162 / 23	7 + residue 1

The random numbers generated by this method are: 1, 11, 10, 17, 14, 12, 3, 20, 16, 21, and 9.

(b) Multiplicative Congruential Method: $r_{i+1} = ar_i$ (modulo m)

r_i	$r_{i+1} = ar_i$ (modulo m)	Random Number
r_1	(16 × 1) / 23	0 + Residue 16
r_2	(16 × 16) / 23	11 + Residue 3
r_3	(16 × 3) / 23	2 + Residue 2
r_4	(16 × 2) / 23	1 + Residue 9
r_5	(16 × 9) / 23	6 + Residue 6
r_6	(16 × 6) / 23	4 + Residue 4
r_7	(16 × 4) / 23	2 + Residue 18
r_8	(16 × 18) / 23	12 + Residue 12
r_9	(16 × 12) / 23	8 + Residue 8
r_{10}	(16 × 8) / 23	5 + Residue 13
r_{11}	(16 × 13) / 23	9 + residue 1

The string of random numbers obtained by multiplicative congruential method is 1, 16, 3, 2, 9, 6, 4, 18, 12, 8, and 13.

(c) Additive Congruential Method: $r_{i+1} = (r_{i+b})$ (modulo m).

r_i	$r_{i+1} = (r_i + b)$ (modulo m)	*Random Number*
r_1	(1 + 18) / 23	0 + residue 19
r_2	(19 + 18) / 23	1 + residue 14
r_3	(14 + 18) / 23	1 + residue 9
r_4	(9 + 18) / 23	1 + residue 4
r_5	(4 + 18) / 23	0 + residue 22
r_6	(22 + 18) / 23	1 + residue 17
r_7	(17 + 18) / 23	1 + residue 12
r_8	(12 + 18) / 23	1 + residue 7
r_9	(7 + 18) / 23	1 + residue 2
r_{10}	(2 + 18) / 23	0 + residue 20
r_{11}	(20 + 18) / 23	1 + residue 15

The random numbers generated are: 1, 19, 14, 19, 4, 22, 17, 12, 7, 2, 20, and 15.

Problem 13.3.

The distribution of inter-arrival time in a single server model is	$T = 1$	2	3
	$f(T) = ¼$	½	¼
And the distribution of service time is	$S = 1$	2	3
	$F(S) = ½$	¼	¼

Complete the following table using the two digit random numbers as 12, 40, 48, 93, 61, 17, 55, 21, 85, 68 to generate arrivals and 54, 90, 18, 38, 16, 87, 91, 41, 54, 11 to generate the corresponding service times.

Arrival Number	Random Number	Arrival Time	Time Service Begins	Random number	Time Servic ends	Waiting time in Queue

Solution

The distribution of inter-arrival times and the two-digit random numbers assigned to different values of T is as below:

T	*f(T)*	$\sum f(T)$	*Random numbers*
1	0.25	0.25	00 to 24
2	0.50	0.75	25 to 74
3	0.25	1.00	75 to 99

Inter-arrival times corresponding to random numbers 12, 40, 48, 93, 61, 17, 55, 21, 85 and 68 are Given 1, 2, 2, 3, 2, 1, 2, 1, 3, 2 respectively. Similarly, the distribution of service times and two-digit random numbers assigned to different values of S are as follows:

S	*f (s)*	$\sum f(s)$	*Random number*
1	0.50	0.50	00 to 49
2	0.25	0.75	25 to 74
3	0.25	1.00	75 to 99

The simulation is done as follows:

Arrival number	*Random number*	*Arrival time*	*Time Service Begins in Mins.*	*Random number*	*Time Service Ends in Mins.*	*Waiting Time in Queue.*
1	12	1	1	54	3	-
2	40	3	3	90	6	-
3	48	5	6	18	7	1
4	93	8	8	38	9	-
5	61	10	10	16	11	-
6	17	11	11	87	14	-
7	55	13	14	91	17	1
8	21	14	17	41	18	3
9	85	17	18	54	20	1
10	68	19	20	11	21	1

The working of the above table is as below: The simulation of the single-server system starts at zero time. First customer arrives at 1 time unit after that and the service immediately begins. Since the service time for the first customer is 2 time units, service ends at 3 time units. The second customer arrives after an inter-arrival time of 2 time units and goes to service immediately at 3 time units. The third customer who arrives at 5 time units has to wait till the service of 2nd customer ends at 6 units of time. The other entries are also filled on the same logic.

Problem 13.4.

A coffee house in a busy market operates counter service. The proprietor of the coffee house has approached you with the problem of determining the number of bearers he should employ at the counter. He wants that the average waiting time of the customer should not exceed 2 minutes. After recording the data for a number of days, the following frequency distribution of inter-arrival time of customers and the service time at the counter are established. Simulate the system for 10 arrivals of various alternative number of bearers and determine the suitable answer to the problem.

Inter-arrival time in mins.	*Frequency (%)*	*Service time in mins.*	*Frequency (%)*
0	5	1.0	5
0.5	35	2.0	25
1.0	25	3.0	35
1.5	15	4.0	20
2.0	10	5.0	15
2.5	7		
3.0	3		

Solution

It is queuing situation where customers arrive at counter for taking coffee. Depending upon the number of bearers, the waiting time of the customers will vary. It is like a single queue multi-channel system and waiting customer can enter any of the service channel as and when one becomes available. By taking two-digit random number interarrival and interservice times are as follows:

Random number for arrivals:

Inte-arrival time in minutes	*Frequency*	*Cumulative frequency*	*Random numbers*
0	5	5	00 to 04
0.5	35	40	05 to 39
1.0	25	65	40 to 64
1.5	15`	80	65 to 79
2.0	10	90	80 to 89
2.5	7	97	90 to 96
3.0`	3	100	97 to 100

Random number for Service:

Service time in minutes	*Frequency*	*Cumulative frequency*	*Random number*
1.0	5	5	00 to 04
2.0	25	30	05 to 29
3.0	35	65	30 to 64
4.0	20	85	65 to 84
5.0	15	100	85 to 99

Arrival Number	*Random Number*	*Inter Arrival Time*	*Random Number*	*Service Time*	*Arrival Time*	*Bearer One Service Begins*	*Bearer One Service Ends*	*Bearer Two Service Begins*	*Bearer Two Service Ends*	*Customer Waiting Time*
1	–	–	31	3	0	0.0	3.0			0
2	48	1.0	46	3	1.0			1.00	4.00	0
3	51	1.0	24	2	2.0	3.0	5.0			1.0
4	06	0.5	54	3	2.5			4.00	7.00	1.5
5	22	0.5	63	3	3.0	5.0	8.00			2.0
6	80	2.01	82	4	5.0			7.00	11.00	2.0
7	56	1.0	32	3	6.0	8.0	11.00			2.0
8	06	0.5	14	2	6.5			11.00	13.00	4.5
9	92	2.5	63	3	9.0	11.0	14.00			2.0
10	51	1.0	18	2	10.0			13.00	15.00	3.0

The customer waiting time with two servers is sometimes greater than 2 minutes. Hence let us try with one more bearers. The table below shows the waiting time of customers with three bearers.

With two bearers, total waiting time is 18 minutes. Hence average waiting time is 18 / 10 = 1.8 minutes.

Arrival Number	*Server One Service Begins*	*Server One Service Ends*	*Server Two Service Begins*	*Server Two Service Ends*	*Server Three Service Begins*	*Service Three Service Ends*	*Customer Waiting Time*
1	0.0	3.0					0
2			1.0	4.0			0
3					2.0	4.0	0
4	3.0	6.0					0.5
5			4.0	7.0			1.0
6					5.0	9.0	0
7	6.0	9.0					0
8			7.0	9.0			0
9					9.0	12.0	0
10	10.0	12.0					0

With three bearers, the total waiting time is 1.5 minutes. Average waiting time is 0.15 minutes. Similarly, we can also calculate the average waiting time of the bearers.

CHAPTER – 14

Introduction to Non-linear Programming

INTRODUCTION

Non-Linear Programming is a mathematical technique for determining the optimal solution to many business problems. Knowledge of differential calculus is essential to do computational work in solving the problems. In linear programming problems, we use to deal with linear objective functions and constraints to find the optimal solution. The constraints we have used in linear programming technique is of ≤ or ≥ type or a combination of these two. It is also assumed in linear programming that the cost of production, or unit profit contribution or problem constraints do not vary for the planning period and also at different levels of production. But it is only an assumption to simplify the matter. But in real world problem the profit, requirement of resources by competing candidate all will vary at different levels of production. Also due to many economic behaviours of demand, cost etc. the objective function tends to be non-linear many a time.

GENERAL NON-LINEAR PROGRAMMING PROBLEM

Let z be a real valued function of n variables defined by:

(a) $z = f(x_1, x_2, \ldots\ldots x_n) \longrightarrow$ **Objective function.**

Let $(b_1, b_2, \ldots\ldots b_m)$ be a set of constraints, such that:

$$\left.\begin{array}{ll} g^1(x_1, x_2, \ldots\ldots x_n) & [\leq \text{ or } \geq \text{ or } =]\ b_1 \\ g^2(x_1, x_2, \ldots\ldots x_n) & [\leq \text{ or } \geq \text{ or } =]\ b_2 \\ g^3(x_1, x_2, \ldots\ldots x_n) & [\leq \text{ or } \geq \text{ or } =]\ b_3 \\ g^m(x_1, x_2, \ldots\ldots x_n) & [\leq \text{ or } \geq \text{ or } =]\ b_m \end{array}\right] \longrightarrow \textbf{Structural constraints.} \quad (b)$$

Where g^1 s are real valued functions of n variables, $x_1, x_2, \ldots\ldots x_n$. Finally, let

(c) $x_j \geq 0$ where $j = 1, 2, \ldots. n. \longrightarrow$ **Non-negativity constraint.**

If either $f(x_1, x_2, \ldots\ldots, x_n)$ or some $g^1(x_1, x_2, \ldots\ldots x_n)$ or both are non-linear, then the problem of determining the n-type $(x_1, x_2, \ldots. x_n)$ which makes z a minimum or maximum and satisfies both (b) and (c), above is called a **general non-linear programming problem (GNLPP)**.

General Non-Linear Programming Problem can be solved by a method very similar to Simplex algorithm. Also there are many methods have been developed to get the solution since the appearance of the fundamental theoretical paper by **Kuhn** and **Tucker** (1915). In the coming discussion some methods of solution to general non-linear programming problem are discussed.

In Linear Programming, the objective function and also the constraints were linear in decision variable. Though this linearity is justified in many real life situations, there do arise such problems in business and industry that the relationship between the decision variables and the objective function itself may contain non-linear expression of the decision variables. One way seems to be to approximate the non-linear relationship is by replacing approximated linear relationships and view the given problem as a perturbed version of the ideal problem. But the conclusion in such a situation may not be valid for the given problem or present solutions which may not give the required optimality. Unfortunately, there is no known algorithm to effectively and efficiently solve a given general non-linear programming problem. A method that is found to be useful in one problem may not be useful in another. This is one of the reasons why all the non-linear programming problems cannot be grouped under the same title. To approximate the difficulty in the approach let us distinguish between the factors that make Linear Programming Problem (LPP) more attractive and a Non-Linear Programming Problem (NLPP) as more complex.

* The algorithm for solving LPP is based on the property that optimal solutions are to be found at the extreme points of the convex polyhedron. This implieds that we limit our search to corner points and this could be completed in a finite number of iterations. But in NLPP the optimal solution can be anywhere along the boundaries of the feasible region or anywhere within the feasible region.
* Linear relationship between the decision variables is very easily amendable to linear algebraic transformation but non-linear relationship should be dealt with extreme care resulting in complex situations.
* The non-linear nature of relationship results in distinction between local solutions and global solutions. This means that any solution that is locally optimal has to be tested for its optimality over the entire feasible region and not only at the extreme points as has been possible in an LPP. This also means that Simplex type algorithms do not suffice to solve NLPPs.

Let us take a small numerical example and try to understand the difference between LPP and NLPP.

Example

Minimize $Z = [(x_1 - 8)^2 + (x_2 - 4)^2]^{1/2}$

Subject to $x_1 + x_2 \leq 8$

$-3x_1 + 2x_2 \leq 6$ and

Both x_1 and $x_2 \geq 0$

Solving graphically, we have to find a feasible point that lies at the shortest distance away from the points (8,4).

The optimal solution $x_1 = 6$ and $x_2 = 2$ where the indifference circle $Z = 2.828$ is tangent to the boundary of the feasible region. The optimum solution does not lie at an extreme point and thus a simplex type algorithm could not solve the problem. The feasible existence of local optima might not give an optimal solution for the same region. It may also be possible in some NLPPs that the feasible region may consist of two or more entirely disconnected sets of points.

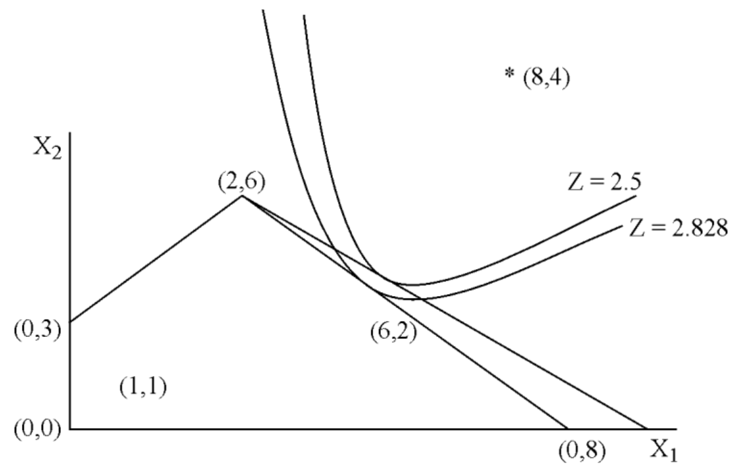


Figure 14.1.

MATHEMATICAL FORMULATION OF THE PROBLEM

Let us take a numerical example to understand the formulation of the problem.

Problem 14.1.

A company manufactures two products *A* and *B* on two machines. Whatever is manufactured is sold in the market, as the market for the product is good. The capacities of the machines are limited to produce daily 80 units of product *A* and 60 units of product *B*. The raw material supply required for the product is limited to produce 600 units per day. The labour required is 160 man-days and the organization has 160 men on the roll. The production of *A* requires one man day hour of labour and that of product *B* is 2 man day hour. The company's objective is to maximize the total profit if the sales–price relationships are as given below.

Product	*Unit price*	*Quantity demand*	*Cost function*
A	P_1	$1500 - 5P_1$	$200\, a + 0.1\, a^2$
B	P_2	$3800 - 10P_2$	$300\, b + 0.1\, b^2$

In the above table *a* and *b* are the number of units of *A* and *B* produced, respectively. This can also be written as:

$a = 1500 - 5\, P_1$ and $b = 3800 - 10\, P_2$

Solution

Let *R* be the revenue on sales and *C* the cost of production, so that Profit = Revenue – Cost.

$R = P_1\, a + P_2\, b = (300 - 0.2\, a)\, a + (380 - 0.1b)\, b = 300a - 0.2a^2 + 380b - 0.1b^2$ and

$C = (200a + 0.1a^2) + (300b + 0.1b^2)$

Therefore, Maximize $Z = R - C = 100a + 0.3a^2 + 80b - 0.2b^2$

To check P_1 and P_2, computed in the problem let us construct

$A \leq 80$, $b < 60$, $5a + 6b \leq 600$, $a + 2b \leq 160$ and both a and b are ≥ 0.

The mathematical form of NLPP is shown in Fig. 14.2.

GLOBAL MINIMA AND LOCAL MINIMA OF A FUNCTION

One of the major difficulties one has to face in solving an NLPP is the determination of the solution point, which gives not only optimal solution for the objective function at the point but also optimizes the function over the complete solution space.

Definition of Global Minimum: A function $f(x)$ has a **global minimum** at a point x^o of a set of points K if an only if $f(x^\circ) \leq f(x)$ for all x in K.

Definition of Local Minimum: A function $f(x)$ has the **local minimum** point x^o of a set of points K if and only if there exists a positive number such that $f(x^o) \leq f(x)$ for all x in K at which $\| x^0 - x \| < \subset$-

Remarks: In case of functions of a single variable, a necessary condition for a particular solution $x = x^*$ to be either a minimum or maximum is that $d/dx\, f(x) = 0$ at $x = x^*$. That is x^* must be local minimum (local maximum) if $f(x)$ is strictly convex (strictly concave) within a neighbourhood of x^*. That is, second derivative of x^* is positive (negative). If second derivatives are zero then we move to higher derivatives, to find the global minimum (global maximum) and identify that one of them yields the smallest (largest) value of $f(x)$.

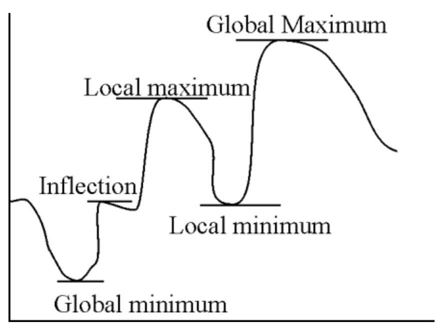


Figure 14.2.

For functions involving several variables we carry out the analysis using partial derivatives. Though non-linear optimization problems require determination of a global minimum, the computational procedures will, in general, lead to a solution which is only a local minimum. There is no general procedure to determine whether the local minimum is really a global minimum. In contrast the simplex procedure of an LPP gives a local minimum, which is also a global minimum. This is the reason why we have to develop some new mathematical concepts to deal with NLPP.

LAGRANGE MULTIPLIERS

Here the optimization problem of continuous functions is discussed. As the non-linear programming problem is composed of some differentiable objective function and equality side constraints, the optimization may be achieved by the use of **Lagrange multipliers** (a way of generating the necessary condition for a stationary point).

A Lagrange multiplier measures the sensitivity of the optimal value of the objective function to change in the given constraints b_i in the problem. Consider the problem of determining the global optimum of

$Z = f(x_1, x_2, \ldots x_n)$ subject to the 'm' constraints $g_i(x_1, x_2, \ldots x_n) = b_i$, $i = 1, 2, \ldots m$. Let us first formulate the Lagrange function L defined by:

$L(x_1, x_2, \ldots\ldots x_n, \ \lambda_1, \ \lambda_2, \ \ldots \lambda_n) - \Sigma \ \lambda_i [g_i(x_1, x_2, \ldots x_n) = b_i$ where $i = 1, 2, \ldots m$ and $\lambda_1, \lambda_2, \ldots \lambda_n$ are called as **Lagrange Multipliers**. The optimal solution to the Lagrange function is determined by taking partial derivatives of the function L with respect to each variable (including Lagrange multipliers and setting each partial derivative to zero and finding the values that make the partial derivatives zero. Then the solution will turn out to be the solution to the original problem.

Problem 14.2

Find the extreme value of $Z = f(x_1, x_2) = 2\,x_1 x_2$

Subject to $x_1^{\,2} + x_2^{\,2} = 1$

Solution

Let $L(x_1, x_2, \lambda) = 2\,x_1 x_2 + \lambda(1 - x_1^{\,2} + x_2^{\,2})$. Then

$$0 = \frac{\partial L}{\partial x_1} = 2\,x_2 - 2\,x_1\lambda$$

$$0 = \frac{\partial L}{\partial x_2} = 2\,x_1 - 2\,x_2\lambda$$

$$0 = \frac{\partial L^2}{\partial \lambda} = 1 - x_1^{\,2} - x_2^{\,2}$$

By solving, we get, $x_2 = x_1\lambda$, $x_1 = x_2\lambda$ and therefore, $x_2 = x_1\lambda = x_2\lambda^2$.

Therefore, either $x_2 = 0$ or $\lambda \pm 1$. But if $x_2 = 0$, then the $x_1 = 0$ and the constraint $x_1^2 + x_2^2 = 1$, is not satisfied. Thus x_2 is not equal to 0 and either $\lambda = 1$ or $\lambda = -1$.

Case 1: $\lambda = 1$, $x_1 = x_2$, then $2x_1^2 = 1$ and $x_1 = \pm(\sqrt{2}/2)$

Case 2: $\lambda = -1$, $x_2 = -x_1$ and $x_1 = \pm(\sqrt{2}/2)$.

Hence the solutions for the problem are:

$\{[(\sqrt{2}/2), (\sqrt{2}/2), 1], [-\sqrt{2}/2, -\sqrt{2}/2, 1], [\sqrt{2}/2, -\sqrt{2}/2, -1], [-\sqrt{2}/2,$ @2/2, $-1]\}$ for $[x_1, x_2, \lambda]$ (here denotes transpose).

Since the set $K = [x / (x_1^2 + x_2^2) = 1]$ is closed and bounded and $f(x) = 2x_1 x_2$ is continuous, f has both maxima and minima over K. Thus Max f over K is 1 and occurs at both $[\sqrt{2}/2, \sqrt{2}/2]$ and $[-\sqrt{2}/2, -\sqrt{2}/2]$.

The minimum value of f is -1 and occurs at both $[-\sqrt{2}/2, \sqrt{2}/2]$ and $[\sqrt{2}/2, -\sqrt{2}/2]$.

KUHN – TUCKER CONDITIONS

If the constraints of a Non-linear Programming Problem are of inequality form, we can solve them by using Lagrange multipliers, which are slightly modified. Let us consider a problem.

Maximize $Z = f(x_1, x_2, x_3, \ldots\ldots x_n)$, subject to the constraints

$G(x_1, x_2, x_3, \ldots\ldots x_n) \leq c$ and $x_1, x_2, \ldots x_n \geq 0$ and c is a constant.

The constraints can be modified to the form $h(x_1, x_2, \ldots x_n) \leq 0$ by introducing a function $h(x_1, x_2, \ldots x_n = g(x_1, x_2, \ldots x_n) - c$

Maximize $Z = f(x)$

Subject to $h(x) \leq 0$ and $x \geq 0$ where, $x \in R^n$.

This problem can be slightly modified by introducing a new variable S. Define $S^r = -h(x)$ or $h(x) + S^2 = 0$, S can be interpreted as slack variable. It appears as its square in the constraint equation so as to ensure its being non-negative.

The problem can be restated as Optimize $Z = f(x) \cdot x \in R^n$

Subject to constraints $h(x) + S^2 = 0$ and $x \geq 0$
which is the problem of constrained optimization in $(n + 1)$ variables with a single equation constraint and can be solved by Lagrange multiplier method.

To determine the stationary points, consider the Lagrange function as $L(x, S, \lambda) = f(x) - \lambda [h(x) + S^2]$, where λ is Lagrange multiplier. Necessary conditions for stationary points are:

$$\frac{\partial L}{\partial x_j} = \frac{\partial f}{\partial x_j} - \lambda \frac{\partial h}{\partial x_j} = 0 \quad \text{for } j = 1 \text{ to } n \qquad \ldots 1$$

$$\frac{\partial L}{\partial \lambda} = -[h(x) + S^2] = 0. \qquad \ldots 2$$

$$\frac{\partial L}{\partial S} = -2S\lambda = 0. \qquad \ldots 3$$

Equation 3 gives $\frac{\partial L}{\partial S} = 0$ which receives either $\lambda = 0$ or $S = 0$. If $S = 0$ equation 2 implies $h(x) = 0$, thus 2 and 3 together imply

$$\lambda\, h(x) = 0 \text{ of } S = 0 \qquad \ldots 4$$

The slack variable was introduced to convert the unequal constraints to an equal one, so it may be discarded and as $S^2 \geq 0$, equation 2 gives:

$$h(x) \leq 0 \qquad \ldots 5$$

whenever $h(x)$, 0 from equation 4, we get $\lambda = 0$, whenever $\lambda > 0$ $h(x) = 0$. λ is unrestricted in sign whenever $h(x) \leq 0$ and the problem reduces to the problem of equation constraint.

The necessary conditions for the point x to be a point of maximum are stated as:

$$f_j - \lambda h_j = 0 \; (j = 1, 2, 3, \ldots n)$$

$\lambda h = 0$ maximum f

$h \leq 0$ subject to the constraint $\lambda \geq 0$ and $h \leq 0$.

(a) General case of the constrained optimization of nonlinear function in n variables under m $(< n)$ inequality constraint:

Consider NLPP Maximize $Z = f(x)$ $x \in R^n$

Subject to constraint $g^i(x) \leq c_i$ $i = 1, 2, \ldots m$ and $x \geq 0$

Introducing the function $h^i(x) = g^i(x) - c_i$ for all $i = 1, 2, \ldots m$ the inequality constraint can be written as

$h^i(x) \le 0$ for $i = 1, 2, \ldots m$.

By introducing the slack variables $S_t t = 1, 2, \ldots m$ defined by $h^i(x) + S_i^2 = 0$, $i = 1, 2, \ldots m$.

The inequality constraints are converted to equality ones. The stationary value of x can thus be obtained by Lagrangian multiplier method. The Lagrangian function is

$L(x, S, \lambda) = f(x) - \Sigma \lambda_i [h^i(x) + S_i^c]$ where $\lambda = (\lambda_1, \lambda_2 \ldots \lambda_m)$ Lagrangian multipliers.

Necessary conditions for $f(x)$ to be the maximum are:

1. $\dfrac{\partial L}{\partial x_j} = \dfrac{\partial f}{\partial x_j} - \sum_{j=1}^{m} \lambda_j \dfrac{\partial h^i}{\partial x_j}$ for $j = 1, 2, \ldots n$

2. $\dfrac{\partial L}{\partial \lambda_i} = h^i + S_{i=0}^{\;2}$ for all $i = 1, 2, \ldots m$

3. $\dfrac{\partial L}{\partial s_i} = -2S_i\lambda_i = 0$ for $i = 1, 2, \ldots m$

$L = L(x, S, \lambda)$ $f = f(x)$ $h^i = h^i(x)$ from equation 3 either $\lambda_i = 0$ or $S_i = 0$.

Using the same argument as in the single in equality case, conditions (3) and (2) together are replaced by the conditions (5), (6) and (7) below:

(5) $\lambda_i h_i = 0$ for $i = 1, 2, \ldots m$

(6) $h\, i \le 0$ for $i = 1, 2, \ldots m$

(7) $\lambda_i \ge 0$ for $i = 1, 2, \ldots m$

Kuhn – Tucker conditions for maximum are restated as:

$$f_i = \sum_{i=1}^{m} \lambda_j h_j^{\,i} \quad (j = 1, 2, \ldots m)$$

$$\lambda_i h^i \le 0 \quad (i = 1, 2, \ldots m)$$

$$\lambda_i \ge 0 \quad (i = 1, 2, \ldots m)$$

where $h^i{}_j = \dfrac{\partial h^i}{\partial x_j}$

Maximize f subject to $h^i \le 0$, $i = 1, 2, \ldots, m$

SUFFICIENCY OF KUHN – TUCKER CONDITION

The Kuhn – Tucker conditions for a maximization NLPP of maximizing $f(x)$ subject to the constraints $h(x) \le 0$ and $x \ge 0$ are sufficient conditions for a maximum of $f(x)$, if $f(x)$ and $h(x)$ are convex.

Proof

Let us consider function $L(x, S, \lambda) = f(x) - \lambda\,[h(x) + S^2]$ where S is defined by $h(x) + S^2 = 0$ is concave in X under the given conditions. In that case the stationary point obtained from Kuhn -

Tucker conditions must be global maximum point.

Since $h(x) + S^2 = 0$ and from the necessary conditions $\lambda S^2 = 0$.

Since $h(x)$ is convex and $\lambda \geq 0$, it follows that $\lambda h(x)$ is also convex and $-h(x)\lambda$ is concave. So we can conclude

$f(x) - \lambda h(x)$ and hence $f(x) - \lambda [h(x) + S^2] = L(x, S, \lambda)$ is concave in x.

Problem 14.1.

Maximize $Z = 3.6 x_1 - 0.4 x_1^2 + 1.6x_2 - 0.2 x_2^2$ subject to constraints

$2x_1 + x_2 \leq 10$ and both x_1 and x_2 are ≥ 0.

Solution

Here $f(x) = 3.6 x_1 - 0.4 x_1^2 + 1.6 x_2 - 0.2 x_2^2$

$$g(x)\ 2x_1 + x_2,\ c = 10$$

$$h(x) = g(x) - c = 2x_1 + x_2 - 10$$

The Kuhn – Tucker conditions are:

$$\frac{\partial(x)}{\partial x_1} - \lambda \frac{\partial h(x)}{\partial x_1} = 0$$

$$\frac{\partial f(x)}{\partial x_2} - \lambda \frac{\partial h(x)}{\partial x_2} = 0$$

$$\lambda h(x) = 0$$

$$h(x) = 0$$

$\lambda \geq 0$ where λ is Lagrangian multipliers.

$$i.e.\ 3.6 - 0.8 x_1 = 2\lambda \quad \ldots(1)$$

$$1.6 - 0.4 x_2 = \lambda \quad \ldots(2)$$

$$\lambda\ [2x_1 + x_2 - 10] = 0 \quad \ldots(3)$$

$$2x_1 + x_2 - 10 \leq 0 \quad \ldots(4)$$

$$\lambda \geq 0 \quad \ldots(5)$$

From equation (3) either $\lambda = 0$ or $2x_1 + x_2 - 10 = 0$

Let $\lambda = 0$, then (2) and (5) yield $x_1 = 4.5$ and $x_2 = 4$, with these values of x_1 and x_2, equation (4) cannot be satisfied. The optima solution cannot be obtained for $\lambda = 0$.

Let λ is not equal to zero, which implies that $2x_1 + x_2 - 10 = 0$. This together with (1) and (2) yields the stationary value.

$x_0 = (x_1, x_2) = (3.5, 3.0)$

Therefore, $h(x)$ is convex in x and $f(x)$ is concave in x.

Thus, Kuhn–Tucker conditions are sufficient conditions for maximum.

Therefore, $x_0 = (3.5, 3.0)$ is solution to given NLPP. The maximum value of $Z = 10$.

***CHAPTER* – 15**

Programme Evaluation and Review Technique and Critical Path Method (PERT and CPM)

INTRODUCTION

Programme Evaluation and Review Technique (PERT) and Critical Path Method (CPM) are two techniques that are widely used in planning and scheduling the large projects. A project is a combination of various activities. For example, Construction of a house can be considered as a project. Similarly, conducting a public meeting may also be considered as a project. In the above examples, construction of a house includes various activities such as searching for a suitable site, arranging the finance, purchase of materials, digging the foundation, construction of superstructure etc. Conducting a meeting includes, printing of invitation cards, distribution of cards, arrangement of platform, chairs for audience etc. In planning and scheduling the activities of large sized projects, the two network techniques — PERT and CPM — are used conveniently to estimate and evaluate the project completion time and control the resources to see that the project is completed within the stipulated time and at minimum possible cost. Many managers, who use the PERT and CPM techniques, have claimed that these techniques drastically reduce the project completion time. But it is wrong to think that network analysis is a solution to all bad management problems. In the present chapter, let us discuss how PERT and CPM are used to schedule the projects.

Initially, projects were represented by **milestone chart** and **bar chart**. But they had little use in controlling the project activities. **Bar chart** simply represents each activity by bars of length equal to the time taken on a common time scale as shown in figure 15. l. This chart does not show interrelationship between activities. It is very difficult to show the progress of work in these charts. An improvement in bar charts is **milestone chart**. In milestone chart, key events of activities are identified and each activity is connected to its preceding and succeeding activities to show the logical relationship between activities. Here each key event is represented by a node (a circle) and arrows instead of bars represent activities, as shown in figure 15.2. The extension of milestone chart is PERT and CPM network methods.

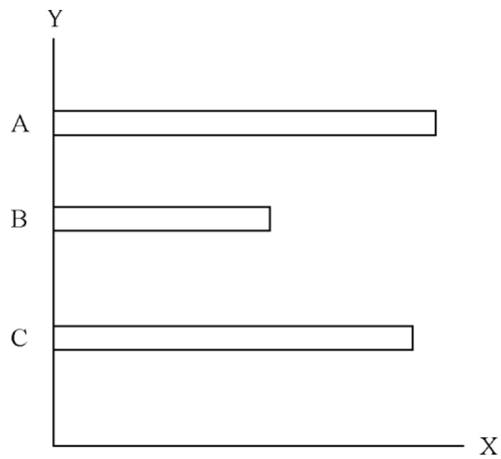


Figure 15.l. Bar chart.

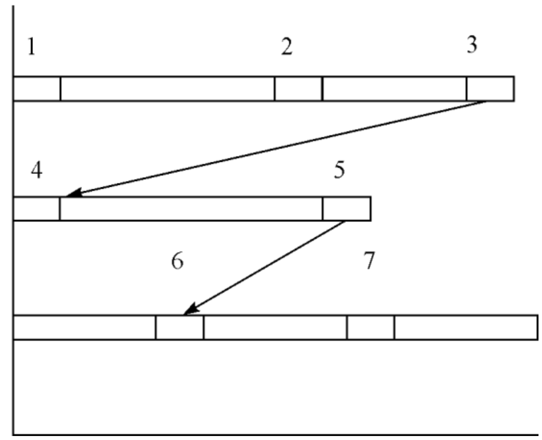


Figure 15.2. Milestone chart.

PERT AND CPM

In PERT and CPM the milestones are represented as ***events***. Event or node is either starting of an activity or ending of an activity. Activity is represented by means of an arrow, which is resource consuming. Activity consumes resources like time, money and materials. Event will not consume any resource, but it simply represents either starting or ending of an activity. Event can also be represented by rectangles or triangles. When all activities and events in a project are connected logically and sequentially, they form a ***network***, which is the basic document in network-based management. The basic steps for writing a network are:

(a) List out all the activities involved in a project. Say, for example, in building construction, the activities are:

 (i) Site selection,
 (ii) Arrangement of Finance,
 (iii) Preparation of building plan,
 (iv) Approval of plan by municipal authorities,
 (v) Purchase of materials,
 (vi) Digging of foundation,
 (vii) Filling up of foundation,
 (viii) Building superstructure,
 (ix) Fixing up of doorframes and window frames,
 (x) Roofing,
 (xi) Plastering,
 (xii) Flooring,
 (xiii) Electricity and water fittings,
 (xiv) Finishing.

(b) Once the activities are listed, they are arranged in sequential manner and in logical order. For example, foundation digging should come before foundation filling and so on.

(c) After arranging the activities in a logical sequence, their time is estimated and written against each activity. For example: Foundation digging: 10 days, or 1½ weeks.

(d) Some of the activities do not have any logical relationship, in such cases; we can start those activities simultaneously. For example, foundation digging and purchase of materials do not have any logical relationship. Hence both of them can be started simultaneously. Suppose foundation digging takes 10 days and purchase of materials takes 7 days, both of them can be finished in 10 days. And the successive activity, say foundation filling, which has logical relationship with both of the above, can be started after 10 days. Otherwise, foundation digging and purchase of materials are done one after the other; filling of foundation should be started after 17 days.

(e) Activities are added to the network, depending upon the logical relationship to complete the project network.

Some of the points to be remembered while drawing the network are

(a) There must be only one beginning and one end for the network, as shown in figure 15.3.

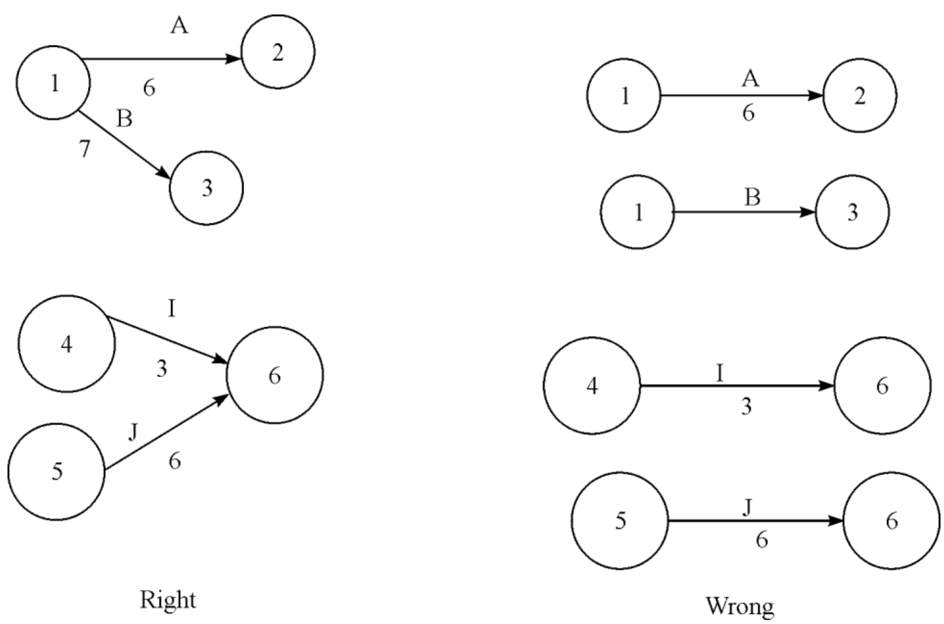


Figure 15. 3. Writing the network.

(b) Event number should be written inside the circle or node (or triangle/square/rectangle etc). Activity name should be capital alphabetical letters and would be written above the arrow. The time required for the activity should be written below the arrow as in figure 15. 4

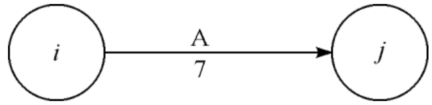


Figure 15.4. Numbering and naming the activities.

(c) While writing network, see that activities should not cross each other. And arcs or loops as in figure 15.5 should not join Activities.

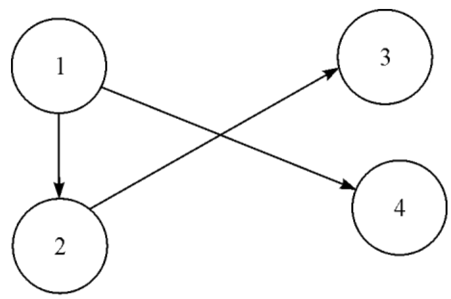


Figure 15.5. Crossing of activities not allowed.

(d) While writing network, looping should be avoided. This is to say that the network arrows should move in one direction, *i.e*. starting from the beginning should move towards the end, as in figure 15.6.

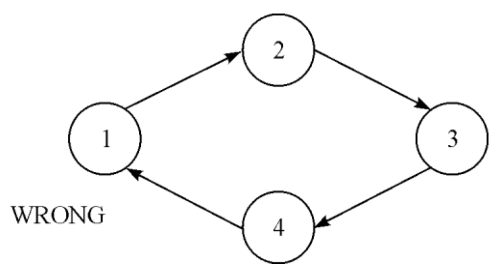


Figure 15. 6. Looping is not allowed.

(e) When two activities start at the same event and end at the same event, they should be shown by means of a **dummy activity** as in figure 15.7. Dummy activity is an activity, which simply shows the logical relationship and does not consume any resource. It should be represented by a dotted line as shown. In the figure, activities *C* and *D* start at the event 3 and end at event 4. *C* and *D* are shown in full lines, whereas the dummy activity is shown in dotted line.

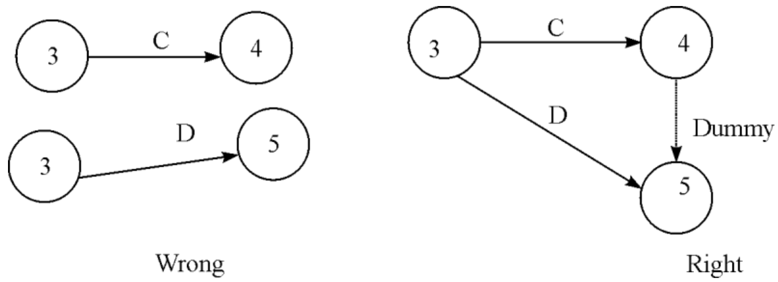


Figure 15.7. Use of Dummy activity.

(f) When the event is written at the tail end of an arrow, it is known as **tail event**. If event is written on the head side of the arrow it is known as **head event**. A tail event may have any number of arrows (activities) emerging from it. This is to say that an event may be a tail event to any number of activities. Similarly, a head event may be a head event for any number of activities. This is to say that many activities may conclude at one event. This is shown in figure 15.8.

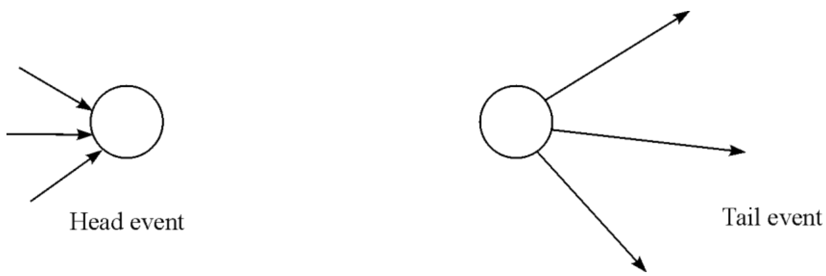


Figure 15.8. Tail event and Head event.

The academic differences between PERT network and CPM network are:

(*i*) PERT is event oriented and CPM is activity oriented. This is to say that while discussing about PERT network, we say that Activity 1-2, Activity 2-3 and so on. Or event 2 occurs after event 1 and event 5 occurs after event 3 and so on. While discussing CPM network, we say that Activity *A* follows activity *B* and activity *C* follows activity *B* and so on. Referring to the network shown in figure 9, we can discuss as under.

PERT way: Event 1 is the predecessor to event 2 or event 2 is the successor to event 1. Events 3 and 4 are successors to event 2 or event 2 is the predecessor to events 3 and 4.

CPM way: Activity 1-2 is the predecessor to Activities 2-3 and 2-4 or Activities 2-3 and 2-4 are the successors to activity 1-2.

(*ii*) PERT activities are probabilistic in nature. The time required to complete the PERT activity cannot be specified correctly. Because of uncertainties in carrying out the activity, the time cannot be specified correctly. Say, for example, if you ask a contractor how much time it takes to construct the house, he may answer you that it may take 5 to 6 months. This is because of his expectation of uncertainty in carrying out each one of the activities in the construction of the house. Another example is if somebody asks you how much time you require to reach railway station from your house, you may say that it may take 1 to 1½ hours. This is because you may think that you may not get a transport facility in time. Or on the way to station, you may come across certain work, which may cause delay in your journey from house to station. Hence PERT network is used when the activity times are probabilistic.

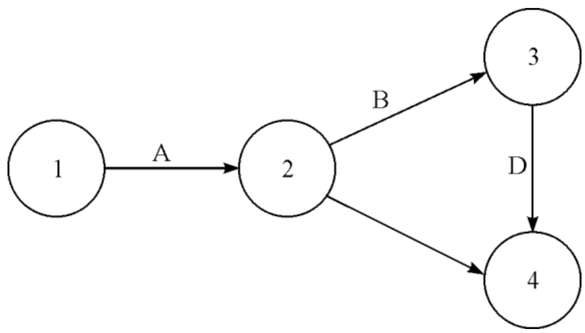


Figure 15.9. Logical relationship in PERT and CPM.

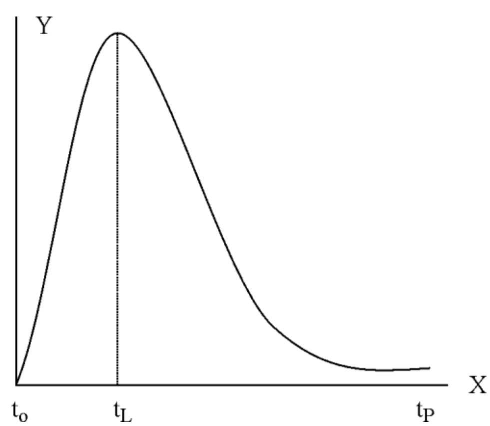


Figure 15.10. Three Time estimates.

There are three time estimates in PERT, they are:

(a) **OPTIMISTIC TIME:** Optimistic time is represented by $\mathbf{t_O}$. Here the estimator thinks that everything goes on well and he will not come across any sort of uncertainties and estimates lowest time as far as possible. He is optimistic in his thinking.

(b) **PESSIMISTIC TIME:** This is represented by $\mathbf{t_P}$. Here estimator thinks that everything goes wrong and expects all sorts of uncertainties and estimates highest possible time. He is pessimistic in his thinking.

(c) **LIKELY TIME:** This is represented by t_L. This time is in between optimistic and pessimistic times. Here the estimator expects he may come across some sort of uncertainties and many a time the things will go right.

So while estimating the time for a PERT activity, the estimator will give the three time estimates. When these three estimates are plotted on a graph, the probability distribution that we get is closely associated with **Beta Distribution curve**. For a Beta distribution curve as shown in figure 6.10, the characteristics are:

Standard deviation = $(t_P - t_O)/_6 = \sigma$, $t_P - t_O$ is known as range.

Variance = $\{(t_P - t_O)/_6\}^2 = \sigma^2$

Expected Time or Average Time = $t_E = (t_O + 4t_L + t_P) / 6$

These equations are very important in the calculation of PERT times. Hence the student has to remember these formulae.

Now let us see how to deal with the PERT problems.

(g) **Numbering of events:** Once the network is drawn the events are to be numbered. In PERT network, as the activities are given in terms of events, we may not experience difficulty. Best in case of CPM network, as the activities are specified by their name, is we have to number the events. For numbering of events, we use D.R. Fulkerson's rule. As per this rule:

An initial event is an event, which has only outgoing arrows from it and no arrow enters it. Number that event as 1.

Delete all arrows coming from event 1. This will create at least one more initial event. Number these initial events as 2, 3 etc.

Delete all the outgoing arrows from the numbered element and which will create some more initial events. Number these events as discussed above.

Continue this until you reach the last event, which has only incoming arrows and no outgoing arrows.

While numbering, one should not use negative numbers and the initial event should not be assigned 'zero'. When the project is considerably large, at the time of execution of the project, the project manager may come to know that some of the activities have been forgotten and they are to be shown in the current network. In such cases, if we use **skip numbering**, it will be helpful. Skip numbering means, skipping of some numbers and these numbers may be made use to represent the events forgotten. We can skip off numbers like 5, 10, 15 etc. or 10, 20 and 30 or 2, 12, 22 etc. Another way of numbering the network is to start with 10 and the second event is 20 and so on. This is a better way of numbering the events.

Let now see how to write network and find the project completion time by solving some typical problems.

Problem 15.1.

A project consists of 9 activities and the three time estimates are given below. Find the project completion time (T_E).

1. Write the network for the given project and find the project completion time?

Activities		*Days*		
i	*j*	T_O	T_L	T_P
10	20	5	12	17
10	30	8	10	13
10	40	9	11	12
20	30	5	8	9
20	50	9	11	13
40	60	14	18	22
30	70	21	25	30
60	70	8	13	17
60	80	14	17	21
70	80	6	9	12

Solution

In PERT network, it is easy to write network diagram, because the successor and predecessor event relationships can easily be identified. While calculating the project completion time, we have to calculate t_e *i.e.* expected completion time for each activity from the given three-time estimates. In case, we calculate project completion time by using t_O or t_L or t_P separately, we will have three completion times. Hence it is advisable to calculate t_E expected completion time for each activity and then the project completion time. Now let us work out expected project completion time.

Predecessor event	*Successor event*	*Time in days*			T_E = $(t_O + 4t_L + t_P)/6$	*Range* $t_P - t_O$	*S.D* (σ) $(t_P - t_O)/6$	*Variance* σ^2
10	20	5	12	17	9.66 (10)	12	2	4
10	30	8	10	13	10.17 (10)	5	0.83	0.69
10	40	9	11	12	10.83 (11)	3	0.5	0.25
20	30	5	8	9	7.67 (8)	4	0.66	0.44
20	50	9	11	13	11.00 (11)	4	0.66	0.44
40	60	14	18	22	18.00 (18)	8	1.33	1.78
30	70	21	25	30	25.18 (25)	9	1.5	2.25
60	70	8	13	17	12.83 (13)	9	1.5	2.25
50	80	14	17	21	17.17 (17)	7	1.16	1.36
70	80	6	9	12	9.00 (9)	6	1.0	1.0

For the purpose of convenience the t_E got by calculation may be rounded off to nearest whole number (the same should be clearly mentioned in the table). The round off time is shown in sbrackets. In this book, in the problems, the decimal, will be rounded off to nearest whole number.

To write the network program, start from the beginning *i.e.* we have 10 – 20, 10 – 30 and 10 – 40. Therefore from the node 10, three arrows emerge. They are 10 – 20, 10 – 30 and 10 – 40. Next from the node 20, two arrows emerge and they are 20 – 30 and 20 – 50. Likewise the network is constructed. The following convention is used in writing network in this book.

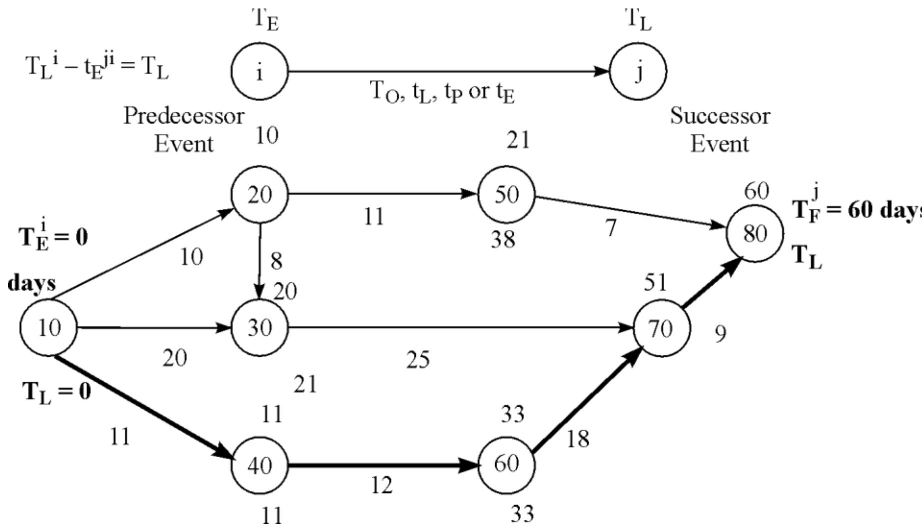


Figure 15.11. Network for Problem 15.1

Let us start the event 10 at 0th time *i.e.* expected time $T_E = 0$. Here T_E represents the occurrence time of the event, whereas t_E is the duration taken by the activities. T_E belongs to event, and t_E belongs to activity.

$$T_E^{10} = 0$$
$$T_E^{20} = T_E^{10} + t_E^{10-20} = 0 + 10 = 10 \text{ days}$$
$$T_E^{30} = T_E^{10} + t_E^{10-30} = 0 + 10 = 10 \text{ days}$$
$$T_E^{30} = T_E^{20} + t_E^{20-30} = 10 + 8 = 18 \text{ days}$$

The event 30 will occur only after completion of activities 20–30 and 10–30. There are two routes to event 30. In the **forward pass** *i.e.* when we start calculation from 1st event and proceed through last event, we have to workout the times for all routes and select the **highest one** and the **reverse** is the case of the **backward pass** *i.e.* we start from the last event and work back to the first event to find out the occurrence time.

$$T_E^{40} = T_E^{10} + t_E^{10-40} = 0 + 11 = 11 \text{ days}$$
$$T_E^{50} = T_E^{20} + t_E^{20-30} = 10 + 11 = 21 \text{ days}$$
$$T_E^{60} = T_E^{40} + t_E^{40-60} = 11 + 18 = 29 \text{ days}$$
$$T_E^{70} = T_E^{30} + t_E^{30-70} = 18 + 25 = 43 \text{ days}$$
$$T_E^{70} = T_E^{60} + t_E^{60-70} = 29 + 13 = 42 \text{ days}$$
$$T_E^{80} = T_E^{70} + t_E^{70-80} = 43 + 9 = 52 \text{ days}$$
$$T_E^{80} = T_E^{50} + t_E^{50-80} = 21 + 17 = 38 \text{ days}$$

T_E^{80} = 52 days. Hence the project completion time is 52 days. The path that gives us 52 days is known as

Critical path. Hence 10–20–30–70–80 is the critical path. Critical path may be represented by double line (⟹) or thick line (⟶) or hatched line (⫽⟶). In this book thick line is used. All other parts *i.e.* 10–40–60–70–80, 10–20–50–80 and 10–30–70–80 are known as **non-critical paths**. All activities on critical path are **critical activities**.

The significance of critical path is delay in completion of critical activities which will increase the project completion time.

Now in the above project, the project completion time is 52 days. In case everything goes correctly the project will be completed in 52 days. Suppose the manager may want to completed the project in 50 days, then what is the probability of completing the project in 50 days? To find the answer for this, let us recollect what is discussed earlier.

Activity $i - j$ is given three time estimates *i.e.* t_O, t_L, and t_P and assumed that the distribution of these time estimates follows ß distribution curve. The approximate mean time for each activity is given by

$$t_E = (t_O + 4t_L + t_P)/6$$

The meaning of this expected time is that there is a fifty-fifty chance of completing the activities in a time duration $t_{\mathbf{E}}$ as shown in the curve.

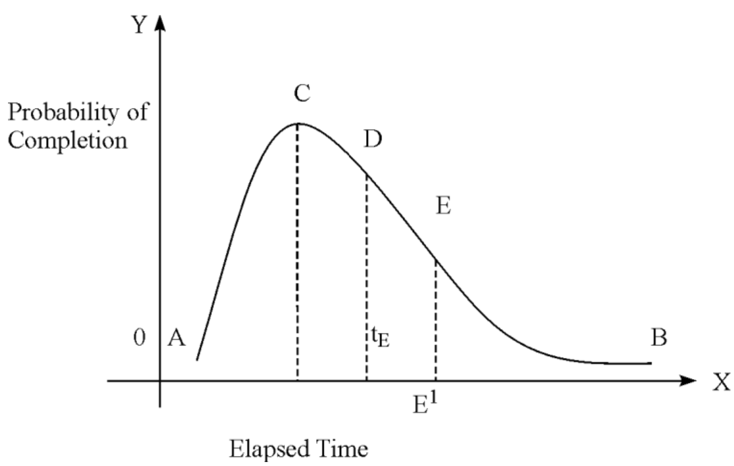


Figure 15.12

The vertical line at *D* represents t_E and the chance of completing activities at t_E is ½ Suppose we want to find out probabilities of completing the activities at *EE* ?,

The probability = Area under ACE / Area under ACB

While calculating the probability of completing the project (having number of activities), the following procedure is applied (here, we apply central limit theorem).

Step 1: Identify critical path and critical activities

Step 2: Find variance (σ^2) for critical activities.

$$\sigma^2_{ij} = [(t^{ij}_p - t^{ij}_o)/6]^2$$

Step 3: List out critical activities and their σ^2

Step 4: Find the sum of variance of critical activities *i.e.* $\Sigma\sigma^2$

Step 5: Find the square root of sum of variance *i.e.* $\sqrt{\Sigma\sigma^2}$

Step 6: Find the difference between the contractual time (T_L) *i.e.*, time by which the project is to be completed and project completion time T_E, i.e. $T_L - T_E$

Depending on the value of T_L, this may be +ve or 0 or –ve number. That is

If $T_L = T_E$ then $T_L - T_E = 0$

If $T_L > T_E$ then $T_L - T_E$ = Positive Number

If $T_L < T_E$ then $T_L - T_E$ = Negative Number

Step 7: Find the ratio $(T_L - T_E)/\sqrt{\Sigma\sigma^2} = Z$, this is the length of ordinate at T_L on the curve.

Step 8: Refer to Table 15.1, which gives the height of Z and the probability of completing the project.

If $T_L = T_E$ the probability is ½.

If $T_L > T_E$ then Z is Positive, the probability of completing the project is higher than 0.5

If $T_L < T_E$ then Z is Negative, the probability of completing the project is lower than 0.5

Table: 15.I
Standard Normal Distribution Function

Z (+)	*Probability P_r (%)*	*Z (–)*	*Probability (P_r) (%)*
0	50.0	0	50.0
+0.1	53.98	–0.1	46.02
+0.2	57.95	–0.2	42.07
+0.3	61.79	–0.3	38.21
+0.4	65.54	–0.4	34.46
+0.5	69.15	–0.5	30.85
+0.6	72.57	–0.6	27.43
+0.7	75.80	–0.7	24.20
+0.8	78.81	–0.8	21.19
+0.9	81.59	–0.9	18.41
+1.0	84.13	–1.0	15.87
+1.1	86.43	–1.1	13.57
+1.2	88.49	–1.2	11.51
+1.3	90.32	–1.3	9.68
+1.4	91.92	–1.4	8.08
+1.5	93.32	–1.5	6.68
+1.6	94.52	–1.6	5.48
+1.7	95.54	–1.7	4.46
+1.8	96.41	–1.8	3.59
+1.9	97.13	–1.9	2.87
+2.1	98.21	–2.1	1.79
+2.2	96.61	–2.2	1.39
+2.3	98.93	–2.3	1.07
+2.4	99.19	–2.4	0.82
+2.5	99.38	–2.5	0.62
+2.6	99.53	–2.6	0.47
+2.7	99.65	–2.7	0.35
+2.8	99.74	–2.8	0.26
+2.9	99.81	–2.9	0.19
+3.0	99.87	–3.0	0.13

Now coming to the problem Number 15.1, given that $T_L = 52$ days.

Critical activities		σ^2
I	J	
10	20	4.00
20	30	0.44
30	70	2.25
70	80	1.00
$\Sigma\sigma^2$		7.69

$$\sqrt{\Sigma\sigma^2} = \sqrt{7.69} = 2.77$$

$$T_L - T_E = 50 - 52 = -2$$

$$(T_L - T_E) / \sqrt{\Sigma\sigma^2} = -2 / 2.77 = -0.722 = Z = \text{Normal deviate.}$$

–0.722 falls at between probability 22.7. The probability is very low. Hence the manager should not accept to complete the project in 50 days.

Say for example given that $T_L = 58$ days then 58 – 55 = +3

$$(T_L - T_E) / \sqrt{\Sigma\sigma^2} = 3 / 2.77 = 1.08 = Z = \text{Normal deviate.}$$

1.08 falls at 85% probability. The probability of completing the project is high. The manager can accept the offer.

Let us continue further discussion on problem no. 15.1

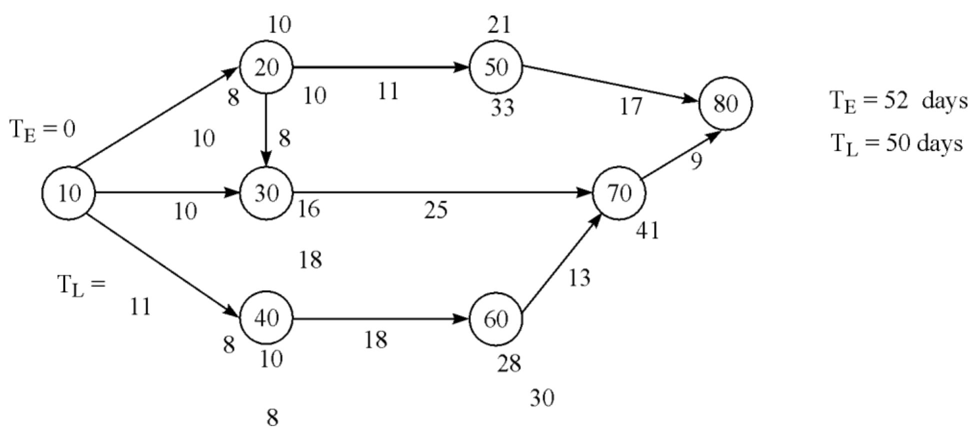


Figure 15.13.

Let us assume that the contractual time = 50 days.

This is written at the end event. Now let us work back to find out when the project should be started if the delivery time is 52 days.

$$T_L^{80} = 50 \text{ days}$$

$$T_L^{70} = T_L^{80} - t_E^{70-80} = 50 - 9 = 41 \text{ days this we write below the node}$$

$$T_L^{50} = T_L^{80} - t_E^{50-80} = 50 - 17 = 33 \text{ days}$$

$$T_L^{60} = T_L^{70} - t_E^{60-70} = 41 - 13 = 28 \text{ days}$$
$$T_L^{40} = T_L^{60} - t_E^{40-60} = 28 - 22 = 6 \text{ days}$$
$$T_L^{20} = T_L^{50} - t_E^{20-50} = 33 - 11 = 22 \text{ days}$$
$$T_L^{30} = T_L^{70} - t_E^{30-70} = 41 - 25 = 16 \text{ days}$$
$$T_L^{20} = T_L^{30} - t_E^{20-30} = 16 - 8 = 8 \text{ days}$$

T_L^{20} has two values *i.e.* 22 days and 8 days. Here as we are going back to find out when the project is to be started, take lowest of the two *i.e.* $T_L^{20} = 8$ days

$$T_L^{10} = T_L^{20} - t_E^{10-20} = 8 - 12 = -4 \text{ days}$$
$$T_L^{10} = T_L^{30} - t_E^{10-30} = 16 - 10 = 6 \text{ days}$$
$$T_L^{10} = T_L^{40} - t_E^{10-40} = 10 - 11 = -1 \text{ days}$$

Take $T_L = -1$ days which is lowest. Hence the project is to be started 1 day before the scheduled starting time.

Now at the critical events calculate $(T_L - T_E)$. For all critical events it is –1 day.

This $(T_L - T_E)$ is known as slack and is represented by Greek letter 'τ'. On the critical path τ remains to be same. In fact slack is the breathing time for the contractor. If $(T_L - T_E)$, slack for all critical events is zero. If $(T_L > T_E)$ it is a positive number and if $(T_L < T_E)$ it will be a negative number for all critical events. For non-critical activities this difference between T_L and T_E *i.e.* $(T_L - T_E)$ shows the breathing time available to the manager at that activity. For example take the event 50

For this event T_L = 33days and T_E = 21 days *i.e.* 33 – 21 = 12 days of time available for this manager. In case of any inconvenience he can start the activity 50-80 any day between 21st day and 33rd day. Now let us work out some more examples.

Problem 15. 2.

Steps involved in executing an order for a large engine generator set are given below in a jumbled manner. Arrange them in a logical sequence, draw a PERT network and find the expected execution time period.

Activities (not in logical order)	*Time in weeks*		
	t_O	t_L	t_P
Order and receive engine	1	2	3
Prepare assembly drawings	1	1	1
Receive and study order	1	2	3
Apply and receive import license for generator	3	5	7
Order and receive generator	2	3	5
Study enquiry for engine generator set	1	2	3
Fabricate switch board	2	3	5
Import engine	1	1	1
Assemble engine generator	1	2	3
Submit quotation with drawing and full	1	2	3
Prepare base and completing	2	3	4
Import generator	1	1	1
Order and receive meters, switch gears for switch board	2	3	4
Test assembly	1	1	1

Solution

As the activities given in the problem are not in logical order, first we have to arrange them in a logical manner.

S.No.	*Activities*	*Time in weeks*		
		t_o	t_L	t_p
A	Study enquiry for engine generator set	1	2	3
B	Prepare assembly drawings	1	1	1
C	Submit quotation with drawing and full	1	2	3
D	Receive and study order	1	2	3
E	Apply and receive import license for generator	3	5	7
F	Order and receive engine	1	2	3
G	Order and receive generator	2	3	5
H	Inspect engine	1	1	1
I	Order and receive meters, switch gears for switch board	2	3	4
J	Prepare base	2	3	4
K	Complete assemble engine generator	1	2	3
L	Fabricate switch board	2	3	5
M	Test assembly	1	1	1

Solution

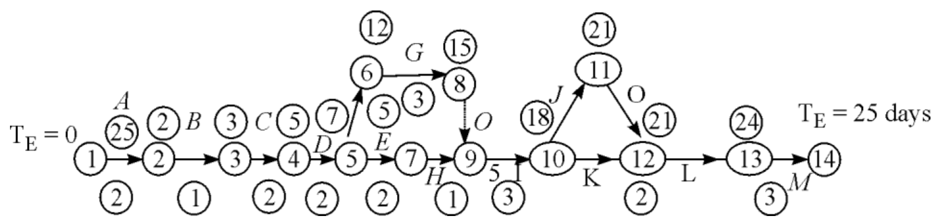


Figure 15.14.

The second step is to write network and number the events

Activities	*Predecessor event*	*Successor Event*	*Weeks*			$t_E = \frac{t_O + 4t_L + t_P}{6}$	$\sigma = (t_P - t_O)/6$	σ^2
			t_O	t_L	t_P			
A	1	2	1	2	3	2	1/3 = 0.33	0.102
B	2	3	1	1	1	1	0	0
C	3	4	1	2	3	2	0.33	0.102
D	4	5	1	2	3	2	0.33	0.102
E	4	6	3	5	7	5	0.66	0.44
F	4	7	1	2	3	2	0.33	0.102
G	6	9	2	3	5	3	0.5	0.25
H	7	10	1	1	1	1	0	0
I	5	8	2	3	4	3	0.33	0.102
J	10	11	2	3	4	3	0.33	0.102
K	11	12	1	2	3	2	0.33	0.102
L	8	12	2	3	5	3	0.5	0.25
M	12	13	1	1	1	1	0	0

CRITICAL PATH = 1 – 2 –3 – 4 – 6 – 8 – 9 – 10 – 11 – 12 – 13

T_E = 25 Weeks

Problem 15.3.

A small project is composed of 7 activities whose time estimates are listed below. Activities are being identified by their beginning (*i*) and ending (*j*) node numbers.

Activities		*Time in weeks*		
i	*j*	t_o	t_l	t_p
1	2	1	1	7
1	3	1	4	7
1	4	2	2	8
2	5	1	1	1
3	5	2	5	14
4	6	2	5	8
5	6	3	6	15

1. Draw the network
2. Calculate the expected variances for each
3. Find the expected project completed time
4. Calculate the probability that the project will be completed at least 3 weeks than expected
5. If the project due date is 18 weeks, what is the probability of not meeting the due date?

Solution

Activities		*Weeks*			$t_E = t_E + 4t_L + t_P/6$	t_E	$\sigma = (t_p - t_o)/6$	σ^2
i	*j*	t_O	t_L	t_P				
1	2	1	1	7	2	6	1	1
1	3	1	4	7	6	6	1	**1**
1	4	2	2	8	3	6	1	1
2	5	1	1	1	1	0	0	0
3	5	2	5	14	6	12	2	**4**
4	6	2	5	8	5	6	1	1
5	6	3	6	15	7	12	2	**4**

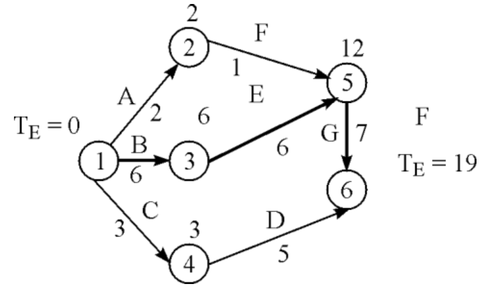


Figure 15.15.

Critical activities	*Variance*
1 – 3	1
3 - 5	4
5 - 6	4
$\Sigma\sigma^2$	9

$$\sqrt{\Sigma\sigma^2} = \sqrt{9} = 3$$

4. Probability of completing the project at least 3 weeks earlier i.e. 16 in weeks

$$T_L = 16 \text{ weeks}, T_E = 19 \text{ weeks.}$$

$$T_L - T_E = -3 \text{ weeks}$$

$$Z = (T_L - T_E) / \sqrt{\Sigma\sigma^2} = -3 / 3 = -1$$

From table the probability of completing the project = 15.9%

5. if T_L = 18 weeks. Probability of completing in 11 weeks is (18 – 19) / 3 = –1/3
   From table the probability = 38.2%

Probability of not meeting due date = 100 – 38.2 = 61.8%

i.e. 61.8% of the time the manager cannot complete the project by due date.

Example 15.4

There are seven activities in a project and the time estimates are as follows

Activities	*Time in weeks*		
	t_O	t_L	t_P
A	2	6	10
B	4	6	12
C	2	3	4
D	2	4	6
E	3	6	9
F	6	10	14
G	1	3	5

The logical of activities are:

1. Activities *A* and *B* start at the beginning of the project.
2. When *A* is completed *C* and *D* start.
3. *E* can start when *B* and *D* are finished.
4. *F* can start when *B*, *C* and *D* are completed and is the final activity.
5. *G* can start when *F* is finished and is final activity the.

(a) What is the expected time of the duration of the project?

(b) What is the probability that project will be completed in 22 weeks?

Solution

First we use to establish predecessor and successor relationship and then find standard deviation σ , variance σ^2 and expected time of completing activities, t_E.

Activities	*Predecessor Event*	*Weeks*			$t_E =$ $t_O + 4t_L + t_P/6$	$\sigma =$ $(t_P - t_O)/6$	σ^2
		t_O	t_L	t_P			
A	–	2	6	10	6	8/6 = 1.33	1.77
B	–	4	6	12	10	8/6 = 1.33	1.77
C	A	2	3	4	3	2/6 = 0.33	0.11
D	A	2	4	6	4	4/6 = 0.66	0.44
E	B, D	3	6	9	5		1.
F	B, C, D	6	10	14	10	8/6 = 1.33	1.77
G	F	1	3	5	3	4/6=0.66	0.44

Now to write network the logical (predecessor) relationship is considered.

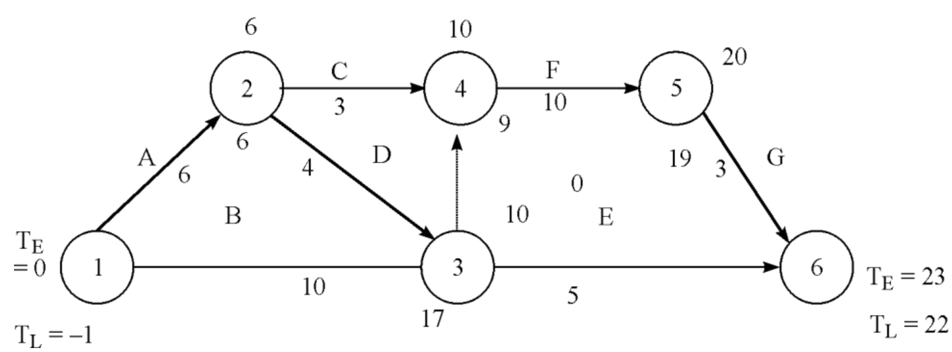


Figure. 15.16.

After writing the network, numbering of events and t_E is entered on the network. Next the project completion time is worked out. The project completion time T_E = 23 weeks. This project has two critical paths i.e. *A – D – F – G* and *B – F – G*.

(1)

Critical path	*Variance* σ^2
A	1.77
D	0.44
F	1.77
G	0.44
$\Sigma\sigma^2$	4.42

$$\sqrt{\Sigma\sigma^2} = \sqrt{4.42} = 2.10$$

(2)

Critical path	*Variance* σ^2
B	1.77
F	1.77
G	0.44
$\Sigma\sigma^2$	3.98

$$\sqrt{\Sigma\sigma^2} = \sqrt{3.98} = 1.99$$

In the problem T_L is given as 22 weeks. Therefore $T_L - T_E = 22 - 23 = -1$

Therefore probability of completing the project in 22 weeks

$$-1 / 2.10 = -0.476 \text{ OR } -1 / 1.99 = 0.502$$

The probability of completing the project is approximately 49%.

CRITICAL PATH METHOD (CPM) FOR CALCULATING PROJECT COMPLETION TIME

In critical path method, the time duration of activity is deterministic in nature *i.e.* there will be a single time, rather than three time estimates as in PERT networks. The network is activity oriented. The three ways in which the CPM type of networks differ from PERT networks are

CPM	*PERT*
(*a*) Network is constructed on the basis of jobs or activities (activity oriented).	(*a*) Network is constructed basing on the events (event oriented)
(*b*) CPM does not take uncertainties involved in the estimation of times. The time required is deterministic and hence only one time is considered.	(*b*) PERT network deals with uncertainties and hence three time estimations are considered (Optimistic Time, Most Likely Time and Pessimistic Time)
(*c*) CPM times are related to cost. That is can be by decreasing the activity duration direct costs increased (crashing of activity duration is possible)	(*c*) As there is no certainty of time, activity duration cannot be reduced. Hence cost cannot be expressed correctly. We can say expected cost of completion of activity (crashing of activity duration is not possible)

Writing the CPM Network

First, one has to establish the logical relationship between activities. That is predecessor and successor relationship, which activity is to be started after a certain activity. By means of problems let us see how to deal with CPM network and the calculations needed.

Problem 15.5.

A company manufacturing plant and equipment for chemical processing is in the process of quoting tender called by public sector undertaking. Help the manager to find the project completion time to participate in the tender.

S.No.	*Activities*		*Days*
1	A	–	3
2	B	–	4
3	C	A	5
4	D	A	6
5	E	C	7
6	F	D	8
7	G	B	9
8	H	E, F, G	3

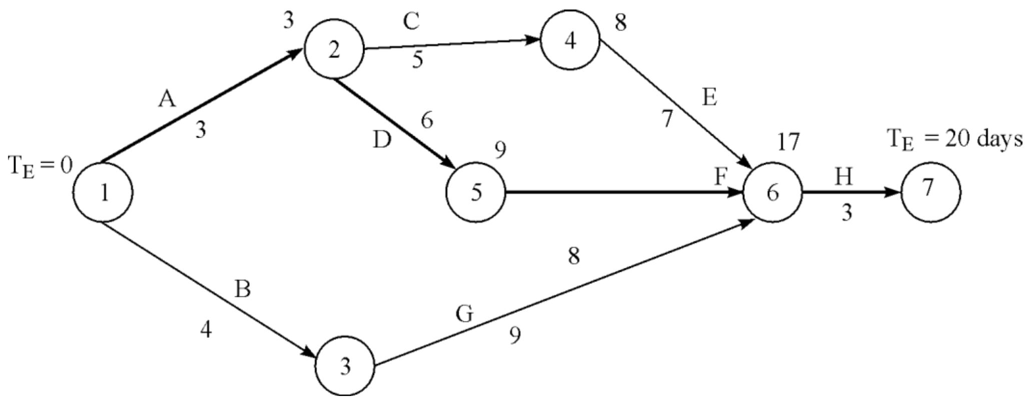


Figure 15.17.

(1) Write the network referring to the data
(2) Number the events as discussed earlier.
(3) Calculate T_E as done in PERT network $T^j_E = (T^i_E + T^{ij})_E$
(4) Identify the critical path

Project completion time = 20 weeks and the critical path = $A - D - F - H$.

Problem 15. 6.

A small project has 7 activities and the time in days for each activity is given below:

Activity	*Duration in days*
A	6
B	8
C	3
D	4
E	6
F	10
G	3

Given that activities *A* and *B* can start at the beginning of the project. When *A* is completed *C* and *D* can start. *E* can start only when *B* and *D* are finished. *F* can start when *B, C* and *D* are completed and is the final activity. *G* can start when *E* is finished and is the final activity. Draw the network and find the project completion time.

Activity	*Immediate predecessor*	*Time in days*
A	–	6
B	–	8
C	A	3
D	A	4
E	B , D	6
F	B, C or D	10
G	E	3

Draw the network and enter the times and find T_E .

Solution

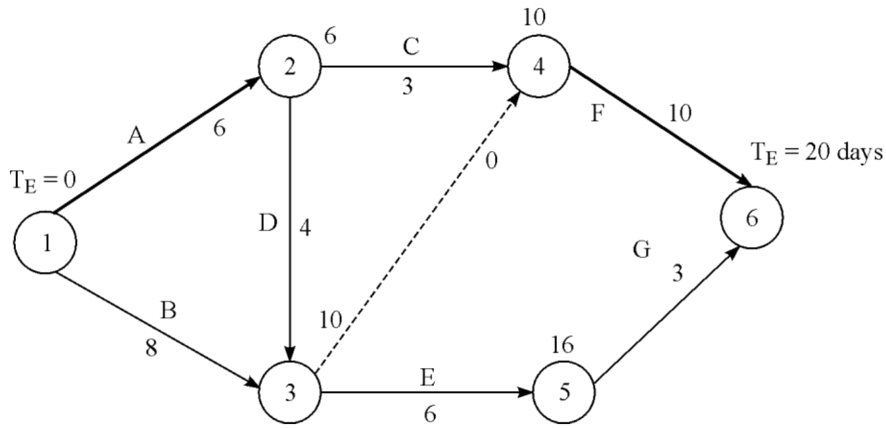


Figure 15.18

Project completion time = 20 days and critical path is $A - D - F$.

Time Estimation in CPM

Once the network is drawn the nextwork is to number the events and enter the time duration of each activity and then to calculate the project completion time. As we know, the CPM activities have single time estimates, and no uncertainties are concerned, the system is deterministic in nature. While dealing with CPM networks, we came across the following times.

(a) Earliest Event Time: We have defined event as either starting or ending of an activity. Earliest event time means what is the earliest time by which that event occurs. Let us consider a small example to understand this. Consider the figure 6.20.

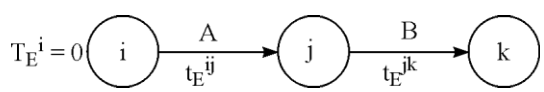


Figure 15.19

In figure, the network has two activities A and B. Activity A *i.e.* '*ij*' is predecessor to activity B *i.e.* activity *jk*. The time taken by activity A is t_E^{ij} and that of B is t_E^{jk}. If the event '*i*' occurs at time 0, then event '*j*' occurs at the earliest at $0 + t_E^{ij}$ *i.e.* $t_E^i + t_E^{ij} = t_E^j$ and the earliest time by which event '*k*' occurs is $T_E^K = t_E^j + t_E^{jk}$. But when various lines as shown in the figure 15.23 connect a node, the procedure is as follows.

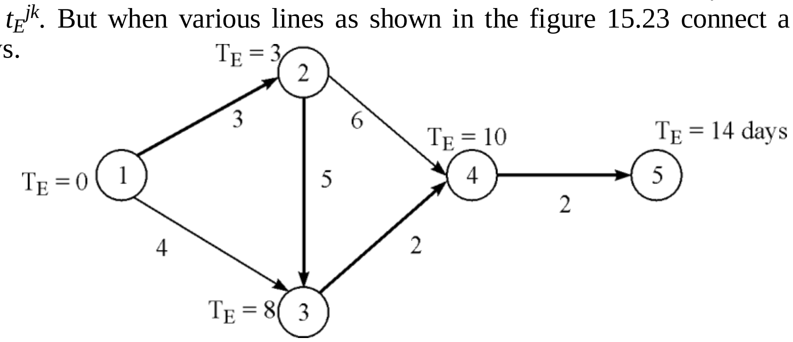


Figure 15.20

Event 3 is having two routes 1 – 2 – 3 and 1 – 3

$$T_E^1 = 0$$
$$T_E^2 = T_E^1 + t_E^{12} = 0 + 3 = 3$$
$$T_E^3 = T_E^1 + t_E^{13} = 0 + 4 = 4 \text{ also } T_E^3 = T_E^2 + t_E^{23} = 3 + 5 = 8$$

As the rule says that event 3 occurs only after the completion of activities 1-4 and 2-3. Activity 1-3 ends on 4th day and event 2-3 ends on 8th day. Hence event 3 occurs on 8^{th} day. This means the formula for finding T_E is

$$T_E^j = (T_E^i + t_E^{ij})_{max}$$

When the event has more routes, we have to calculate T_E for all routes and take the maximum of all the routes.

Problem 15. 7.

Find the slack of each event

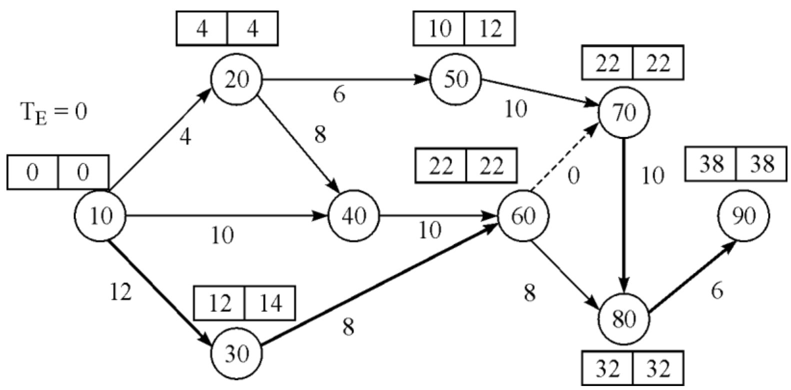


Figure 15.21

T_E^{90} = 38 days, T_L^{90} = 38 days

Critical path = 10 – 20 – 40 – 60 – 70 – 80 – 90

Event 'i'	*Event 'j'*	*Duration T^{ij} days*	$t_E^j = t_E^i + t^{ij}\uparrow$	T_E^j	$T_L^i \downarrow$	T_L^j	*Slack = $T = T_L - T_E$*
90	80	6	**38**	38	**32**	38	0
80	70	10	**32**	32	**22**	32	0
80	60	8	30	32	24	32	0
70	60	0	**22**	22	**22**	22	0
70	50	10	20	22	**12**	22	0
60	40	10	**22**	22	**12**	22	0
60	30	8	20	22	14	22	0
50	20	6	**10**	10	6	12	+2
40	20	8	**12**	12	4	12	0
40	10	10	10	12	2	12	0
30	10	12	12	12	2	14	+2
20	10	4	4	**4**	**0**	4	0

Thick numbers are maximums

Thin numbers are minimums

Latest Allowable Occurrence Time

The next one is the Latest Allowable Occurrence time represented by T_L^i. This is illustrated by a simple example.

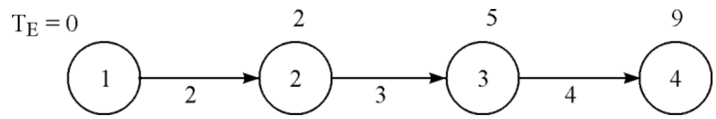


Figure 15.22

Earliest occurrence time of event 4 = 9 days. As the activities 3 - 4 take 4 days, the latest time by which activity starts is $T_L^4 - t_E^{3-4} = 9 - 4 = 5$th day, which is also the Earliest Occurrence time of event 3. Similarly, Latest Time by which event 2 occurs in $T_L^3 - t_E^{2-3} = 5 - 3 = 2$ and so on. If a node is connected by number of paths then we have to find Latest Allowable Occurrence time as discussed below.

Consider the figure given below:

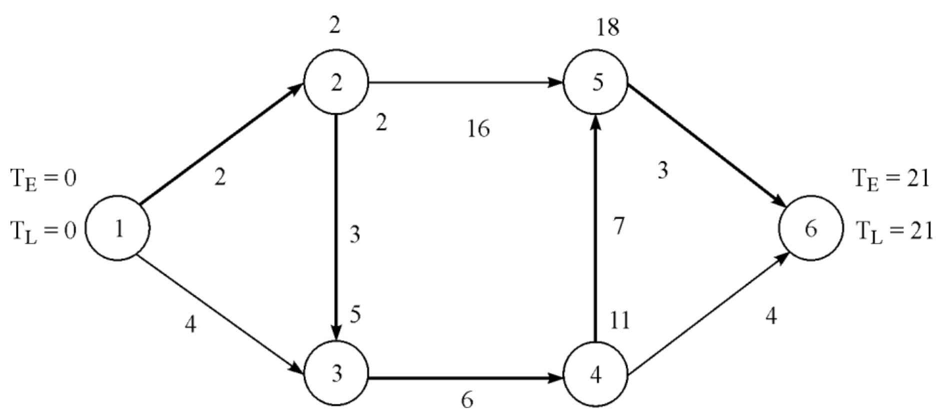


Figure 15.23

The earliest occurrence time of event 6 is 21 days. As activities 4 – 6 take 4 days the earliest occurrence time is

$T_L^6 - t_E^{46} = 21 - 4 = 17$ days. But there is another route 6 – 5 – 4. If we consider this route $T_L^5 = T_L^6 - t_E^{56} = 21 - 3 = 18$th day, $T_L^4 = T_L^5 - t_E^{54} = 18 - 7 = 11$ days. As the latest allowable occurrence time for event 4 is 17th day and 11th day, the event 4 will not occur until activities 6 – 4 and 6 –5 are completed. As 11th day is the smallest, the event 4 occurs on 11th day. Hence the formula for $T_L^i = (T_L^i - t_E^{ij})_{\text{minimum}}$.

Slack time

Slack time is difference between latest event and the earliest event time *i.e.* $T_L^i - t_E^i = \tau^i$

Float: Now let us define the times for activities $i - j$.

(i) Earliest start time: - This is the earliest occurrence time for the event from which the activity arrow originates and is represented by T_E^i

(ii) Earliest finish time: - This is the earliest occurrence time of the event from which the activity arrow originates plus the duration of the activity. $T_E^i + t_E^{ij}$

(iii) Latest start time:- This is the latest occurrence time for the node at which the activity arrow terminates minus the duration of activity *i.e.* $T_L^j - t_E^{ij}$

(iv) Latest finish time:- This is the latest occurrence time for the node at which the activity arrow terminates, represented by T_L^j

(v) Maximum time available for activity is $T_L^j - T_E^i$

(vi) Total float: - If the job $i - j$ requires time t^{ij} units, the actual float for jobs $i - j$ is the difference between the maximum time available for the job and the actual time.

Total float for $i - j$ $= (T_L^j - t_E^i) - t^{ij}$

$= (T_L^j - t^{ij}) - T_E^i$

This is the latest time for the activity minus the earliest start time.

(vii) Free float: - Free float for an activity is based on the possibility that all events occur at their earliest times that means all activities start as early as possible. If you have two activities $i - j$ and $j - k$ *i.e.*, activity $j - k$ is a successor activity to activity $i - j$

Let T_E^i = Earliest Occurrence Time for event 'i'.

T_E^j = Earliest Occurrence Time for event 'j'.

This means that the earliest possible start time for activity $i - j$ is T_E^i and for the activity $j - k$ is T_E^j. Let the activity duration be t_E^{ij}. In case T_E^j is greater than $T_E^i + t_E^{ij}$ activity $j - k$ cannot start until T_E^j. The difference between $T_E^j - (T_E^i + t_E^{ij})$ is known as Free Float. Therefore, Free Float for activity $i - j = T_E^j - (T_E^i + t_E^{ij})$. But $(T_E^i + t_E^{ij})$ is earliest finish time for activity $i - j$. Hence free float = T_E^j – Earliest Finish Time of $i - j$. Free float for activity $i - j$ is the difference between its Earliest Finish Time and Earliest Start time of its successor activity.

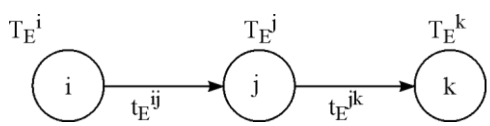


Figure 15.24

(viii) Another type of float is ''Independent Float''. Referring to figure 15.25. Consider the activity $i–j$. Activity $h - i$ is predecessor to $i - j$ and activity $j - k$ is a successor to activity $i - j$ and T_L^i is to latest finish time of activity $h - i$.

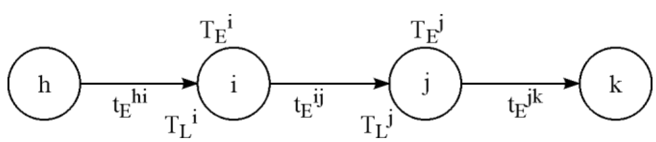


Figure 15.25

And activity *j-k* starts at the earliest possible moment i.e. $T_E{}^j$. This means activity $i – j$ can take time duration between $T_E{}^i$ to $T_E{}^j – T_L{}^i$, without affecting the networks. The difference between the $T_E{}^j – T_L{}^i$ and $t_E{}^{ij}$ is known as Independent Float.

Independent float for $i – j = (T_E{}^j – T_L{}^i) – t_E{}^{ij}$

(ix) Another type of float is Interference Float. Interference Float is the difference between Total Float and the Free Float. In fact it is head event slack.

Interference float = Total Float – Free Float

$$F_{IT} = F_T – F_F$$
$$F_T = (T_L{}^j – T_E{}^j) – t_E{}^{ij}$$
$$F_F = (T_E{}^j – T_E{}^i) – t_E{}^{ij}$$
$$F_{IT} = (T_L{}^j – T_E{}^i – t_E{}^{ij}) – (T_E{}^j – T_E{}^i – t_E{}^{ij})$$
$$F_{IT} = (T_L{}^j – T_E{}^j) = \text{Head event slack.}$$

Summary of float

S.No.	*Type of float*		*Formulae*
1.	Total float (FT)	Excess of maximum available time over the activity time.	$F_T = (T_L{}^j – T_E{}^j) – t_E{}^{ij}$
2.	Free float (FF)	Excess of available time over the activity time when all jobs start as early as possible.	$F_F = (T_E{}^j – T_E{}^i) – t_E{}^{ij}$
3.	Independent float F_{ID}	Excess of maximum available time over the activity time.	$F_{ID} = (T_E{}^j – T_L{}^i) – t_E{}^{ij}$
4.	Interfering float (F_{IT})	Difference between total float and free float	$F_{iT} = F_T – F_F$

PROJECT COST ANALYSIS

So far we have dealt with how to find project completion time in PERT and CPM networks. In CPM network, when the time required by an activity is deterministic in nature, we may come across a situation that we may have to reduce the activity duration. This is not possible in PERT activity; because activity duration is probabilistic in nature and we have three time estimates. Which time (either t_O, t_L or t_P) is to be reduced is a question. Hence activity time crashing is possible in critical path network only.

Before crashing the activity duration, we must understand the costs associated with an activity.

Direct Cost

Direct costs are the costs that can be identified with activity. For example, labour costs, material cost etc. When an activity whose duration is to be reduced (crashed), we have to supply extra resources, specially manpower. Let us say an activity takes 7 days with 2 men. If 4 men works it can be done in 4 days. The cost of 2 workmen increases. As we go on reducing the activity time, cost goes on increasing as shown in figure 15.26.

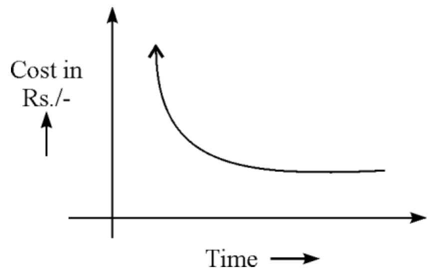


Figure 15.26 Direct Cost.

Indirect Cost

These are the costs, which cannot be identified with the activity. Say the salary of a manager, who is in-charge of many projects. Exact amount of his salary that should be charged to a particular project cannot be estimated correctly as it is very difficult to say how much time he has spent on each project. We can express all indirect costs put together in terms of an amount per time period, for example say Rs. 100/- per day, as the indirect costs are expressed as so much of amount per time period, as the duration of project goes on reducing the indirect cost also goes on decreasing as in figure 15.27.

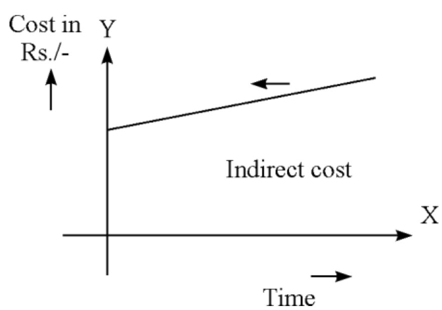


Figure 15.27 Indirect Cost.

Total cost

The total cost which is the sum of direct cost and indirect cost is shown in figure 15.28. As the project duration goes on reducing the total cost reduces from *B* to *C* and if duration is still crashed the total cost increases to *A*. Hence our problem here is to find out the optimal duration of the project and optimal cost.

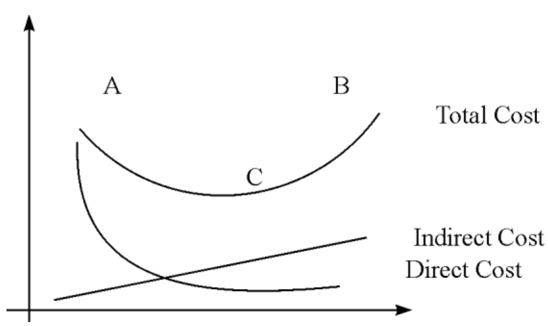


Figure15.28 Total cost curve.

Cost Slope

Consider a small portion of total cost curve and enlarge it. It appears like a straight line as shown in figure 15.29.

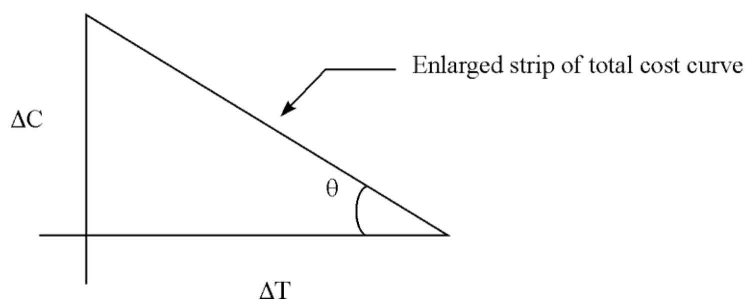


Figure 15.29 Cost Slope.

If θ is the inclination then tan θ = ΔC / Δt. This indicates how much cost increases for crashing a unit of time period.

In other words cost slope is the slope of the direct cost curve, approximated as a straight line. It is given by

$$\text{Cost Slope} = \frac{\text{Crash cost} - \text{Normal cost}}{\text{Normal time} - \text{Crash time}} = \frac{\Delta C}{\Delta t}$$

Where ΔC = increase in cost, Δt = is decrease in time.

Problem 15.8.

A project consists of 4 activities. Their logical relationship and time taken is given along with crash time and cost details. If the indirect cost is Rs. 2000/- per week, find the optimal duration and optimal cost.

Activity	*Predecessor*	*Normal*		*Crash*	
		Time in days	*Cost in Rs/-*	*Time in days*	*Cost in Rs/-*
A	-	4	4,000	2	12,000
B	A	5	3,000	2	7,500
C	A	7	3,600	5	6,000
D	B	4	5,000	2	10,000
		TOTAL	15,600		35,500

Solution

Slopes

(1) Find ΔC = Crash cost – Normal cost

(2) Find Δt = Normal time – Crash time

(3) Find $\Delta C / \Delta t$ = cost slope.

(4) Identify the critical path and underline the cost slopes of the critical activities.

(5) As the direct cost increases and indirect cost reduces, crash such activities whose cost slopes are less than the indirect cost given.
(6) Select the lowest cost slope and crash it first, then next highest and so on.
(7) Do not crash activities on non-critical path until they become critical activities in the process of crashing.
(8) In case any non-critical activity becomes critical activity at the time of crashing consider the cost slopes of both the critical activities, which have same time span and the costs slopes of both activities.
(9) Crashing should be continued until the cost slope becomes greater than the indirect cost.
(10) Do not crash such activities whose cost slope is greater than the indirect cost.
(11) Crashing is done on a graph sheet with squared network drawn to scale.

Activity	*Predecessor*	*Normal*		*Crash*		ΔC	Δt	$\frac{\Delta C}{\Delta t}$
		Time in days	*Cost in Rs./-*	*Time in days*	*Cost in Rs./-*			
A	-	4	4,000	2	12,000	8,000	2	**4,000**
B	A	5	3,000	2	7,500	4,500	3	**1,500**
C	A	7	3,600	5	6,000	2,400	2	1,200
D	B	4	5,000	2	10,000	5,000	2	**2,500**
		TOTAL	15,600		35,500			

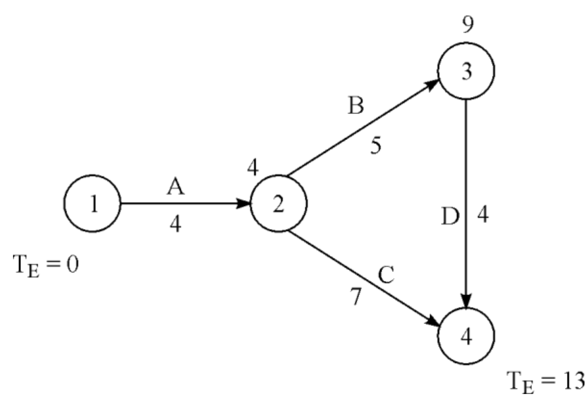


Figure 15.30

Now activities *A*, *B* and *D* are critical activities. Activity *B* is the only activity whose cost slope is less than indirect cost. Hence we can crash only activity *B*. For crashing we have to write the squared network. While writing squared network critical activities are shown on a horizontal line and non-critical activities are shown as in the figure *i.e*. above and / or below the critical path as the case may be. That is non-critical paths above critical path are shown above vice versa.

Though the activity *B* can be crashed by 3 days, only 2 days are crashed because after crashing 2 days at 11th day, activity 2-4 (C) also becomes critical activity. At this stage if we want to crash one more day we have to crash activity 2–4 *i.e*. C also along with 2 – 3. Now the cost slopes of activities *B* and *C* are to be considered which will be greater than indirect cost. Hence no crashing can be done. 11 days is the optimal period and optimal cost is Rs. 39, 100/-.

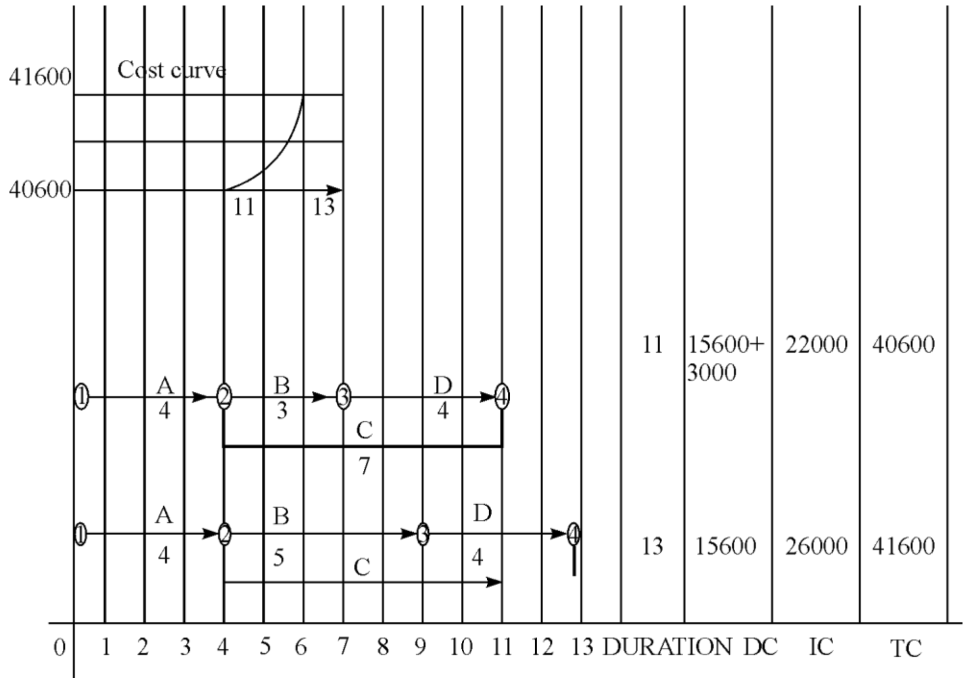


Figure 15.31

Problem 15.9.

(a) A maintenance project has following estimates of times in hours and cost in rupees for jobs. Assuming that jobs can be done either at normal or at fast pace, but not any pace in between. Plot the relationship between project completion time and minimum project cost.

(b) Assuring a relationship between job duration and job cost and with overhead cost of Rs. 25/- per hour, plot the cost – time relationship.

Jobs	*Predecessor*	*Normal*		*Crash*	
		Time in hrs	*Cost in Rs/-*	*Time in hrs*	*Cost in Rs/-*
A	-	8	80	6	100
B	A	7	40	4	94
C	A	12	100	5	184
D	A	9	70	5	102
E	B, C, D	6	50	6	50
		TOTAL	300		530

Solution

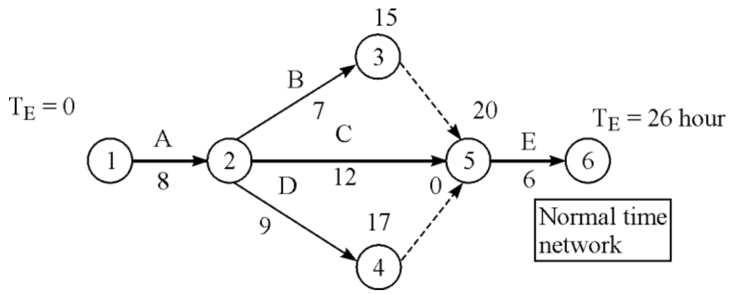


Figure 15.32.

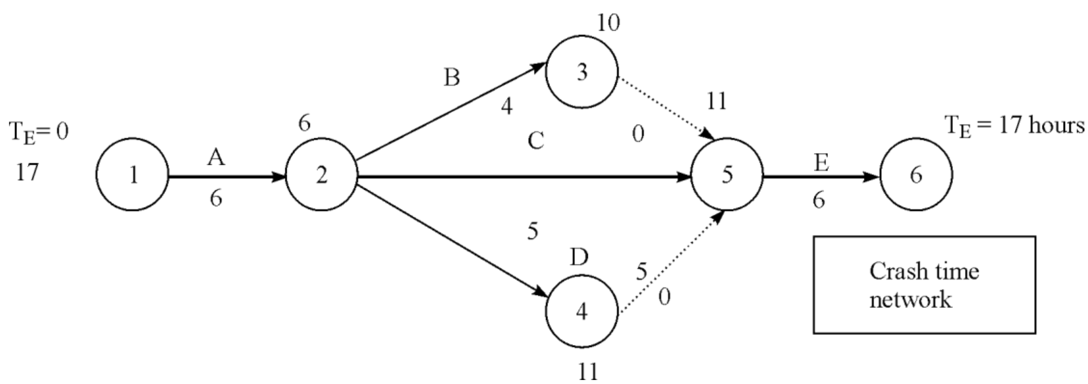


Figure 15.33.

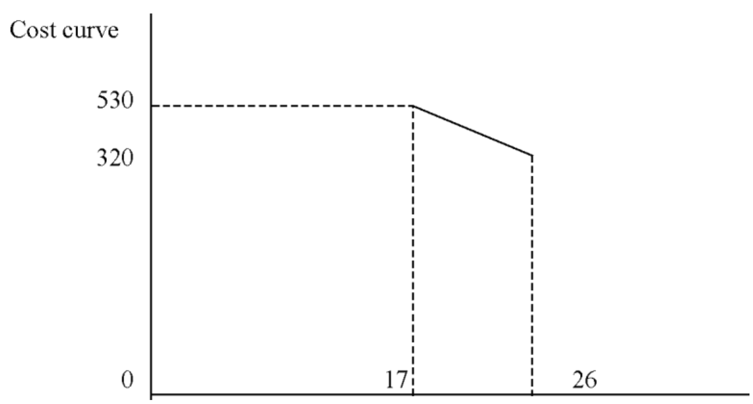


Figure 15.34.

Indirect cost = Rs. 25/- per hour

Jobs	*Predecessor*	*Normal*		*Crash*		ΔC	Δt	$\frac{\Delta C}{\Delta t}$
		Time in hrs	*Cost in Rs./-*	*Time in hrs*	*Cost in Rs./-*			
A	-	8	80	6	100	2	20	10
B	A	7	40	4	94	3	54	18
C	A	12	100	5	184	7	84	12
D	A	9	70	5	102	4	32	8
E	B, C, D	6	50	6	50	-	-	-
		Total	300		530			

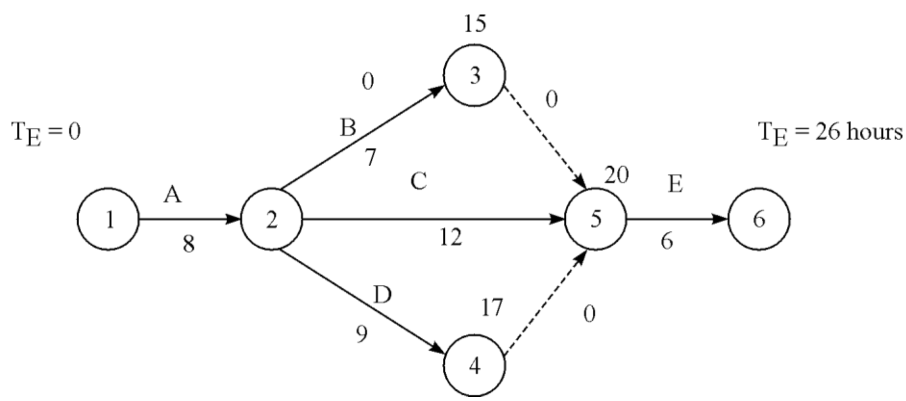


Figure 15.35

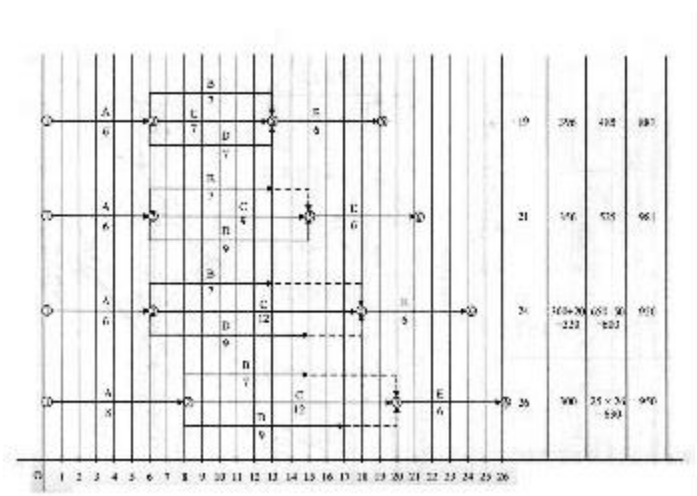


Figure 15.36 (a) Squared Network for Problem 6.8

(a) Figure 15.36 (*a*) shows the squared network.

(b) As critical activity A has got cost slope of Rs. 10/-, which is less than the indirect cost it is crashed by 2 days.

Hence duration is 24 hrs.

Direct Cost = Rs. 300 + 2 × 10 = Rs. 320

Indirect Cost = Rs. 650 – 2 × 25 = Rs. 600

Total Cost = Rs. 920

(c) Next, critical activity *C* has got a cost slope 12, which is less than 25. This is crashed by 3 days, though it can be crashed 7 days. This is because, if we crash further, activity *D* becomes critical activity, hence its cost slope also to be considered.

Duration is 21 hrs.

Direct Cost = Rs. 320 + 3 × 12 = Rs. 356

Indirect Cost = Rs. 600 – 3 × 25 = Rs. 525

Total Cost = Rs. 881 (Rs. 356 + Rs. 525)

(d) Now cost slope of activities *C* and *D* put together = Rs.12 + 8 = Rs. 20 which is less than Rs. 25/-, indirect cost, both are crashed by 2 hrs

Duration is 19 hrs.

Direct Cost = Rs. 356 + 2 × 20 = Rs. 396

Indirect Cost = Rs. 525 – 2 × 25 = Rs. 475

Total Cost = Rs. 871 (Rs. 396 + 475)

As we see from the network, no further crashing can be done. Optimal time = 19 hrs and optimal cost = Rs. 871/-

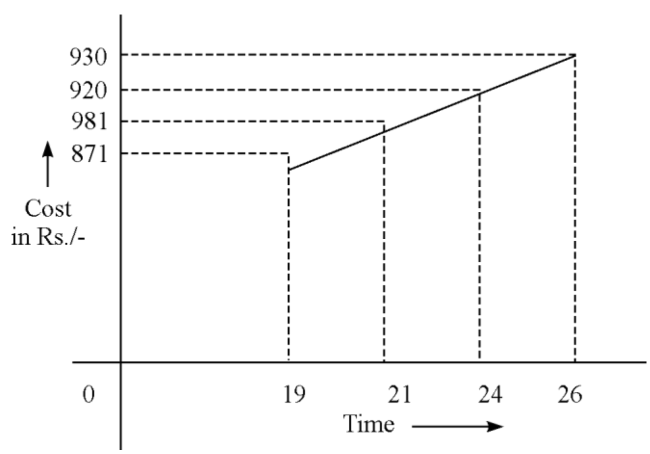


Figure 15.37

Problem 15.9.

The following details pertain to a job, which is to be scheduled to optimal cost.

Jobs	*Predecessor*	*Normal*		*Crash*		ΔC	Δt	$\frac{\Delta C}{\Delta t}$
		Timein hrs	*Cost in Rs./-*	*Time in hrs*	*Cost in Rs./-*			
A	–	3	1,400	2	2,100	700	1	700
B	C	6	2,150	5	2,750	600	1	600
C	–	2	1,600	1	2,400	800	1	800
D	A, B	4	1,300	3	1,800	500	1	500
E	C	2	1,700	1	2,500	800	1	800
F	D	7	1,650	4	2,850	400	3	133
G	E, F	4	2,100	3	2,900	800	1	800
H	D	3	1,100	2	1,800	500	1	500
		TOTAL	13,000		18,900			

We can enter Δt in last column.

Assume that indirect cost is Rs. 1100/- per day. Draw least cost schedule. The related network is shown below:

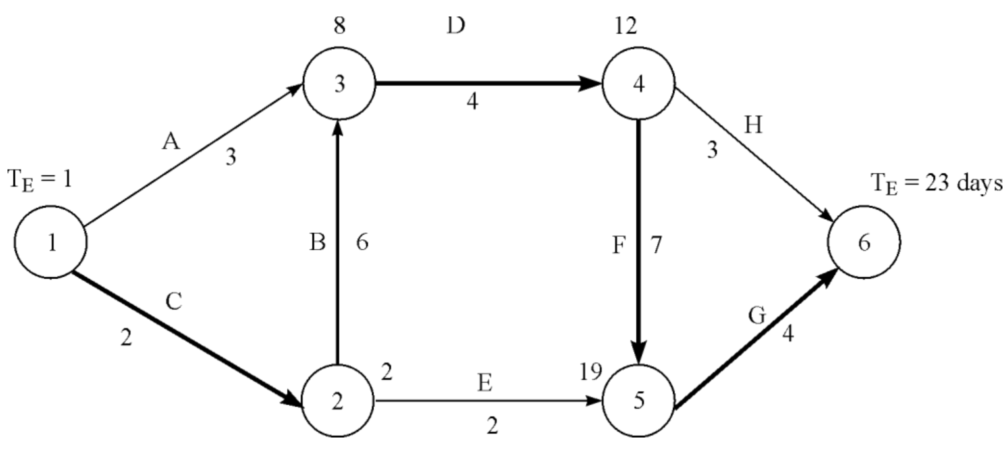


Figure 15.38

Project completion time is 23 days. Critical path is $C – B – D – F – G$. Lowest cost slope is Rs. 133 for critical activity F, this can be crashed by 3 days.

Next cost slope is Rs.500/- for activity D. This can be crashed by 1 day.

Next cost slope is Rs.600/- for activity B. This can be crashed by 1 day. Next lower cost slope is Rs. 800/- for critical activities C and G. C can be crashed by 1 day and G can be crashed by 1 day.

All critical activities have been crashed and non-critical activities have slack. Hence they are not to be crashed. Hence optimal cost is Rs. 33699 and optimal time is 16 days.

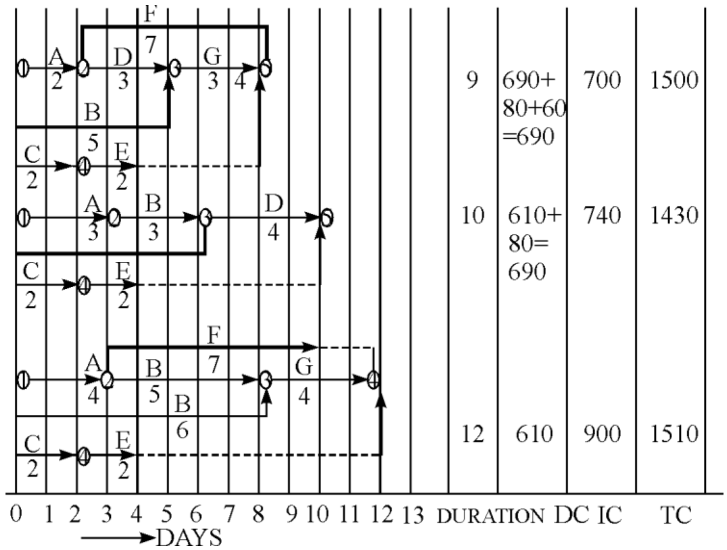


Figure 15.39 Squared network for problem 15.9

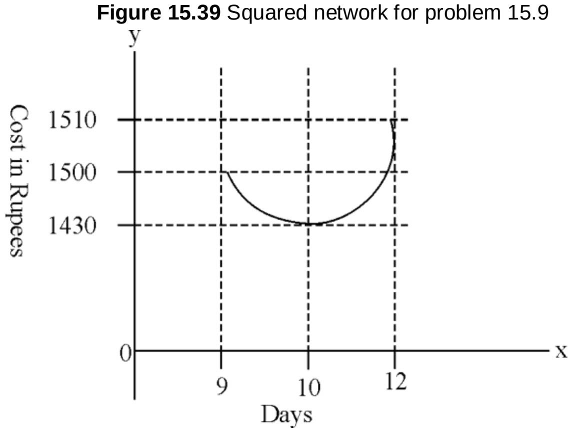


Figure 15.40

Problem 15.10.

Given below are network data and time–cost trade off data for small maintenance work.

Jobs	*Predecessor*	*Normal*		*Crash*	*Cost slope*
		Time in hrs	*Cost in Rs./-*	*Time in hrs*	*Rs. / day* ΔC
A	–	3	50	2	**50**
B	–	6	140	4	60
C	–	2	50	1	30
D	A	5	100	2	**40**
E	C	2	55	2	–
F	A	7	115	5	30
G	B, D	4	100	2	**70**
		TOTAL	610		

Assume that the indirect cost including the cost of lost production and associated costs to be as given below:

Project duration in days	12	11	10	9	8	7
Indirect cost in Rs./-	900	820	740	700	660	620

Work out the minimum total cost for various project duration and suggest the duration for minimum total cost.

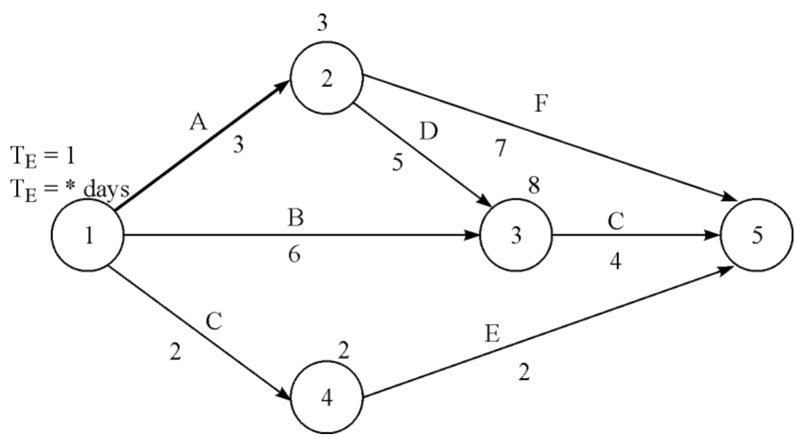


Figure 15.41.

$A - D - G$ = critical path, T_E = 12 days.

(1) Activity *D* has lowest cost slope. It can be crashed by 3 days. It is less than the cost slope for 12 days *i.e*. Rs. 980/-. By crashing activity *D* by 3 days, activities *F* and *B* have become critical activities.

(2) Next we can crash activity *A*, whose cost slope is Rs. 50/-, it can be crashed by 1 day, but we have to crash activity *B* also along with *A*.

Total Cost slope = Rs. 110/- per day. Here the total cost increases. Hence optimal duration is 10 days and optimal cost is Rs. 1430/-.

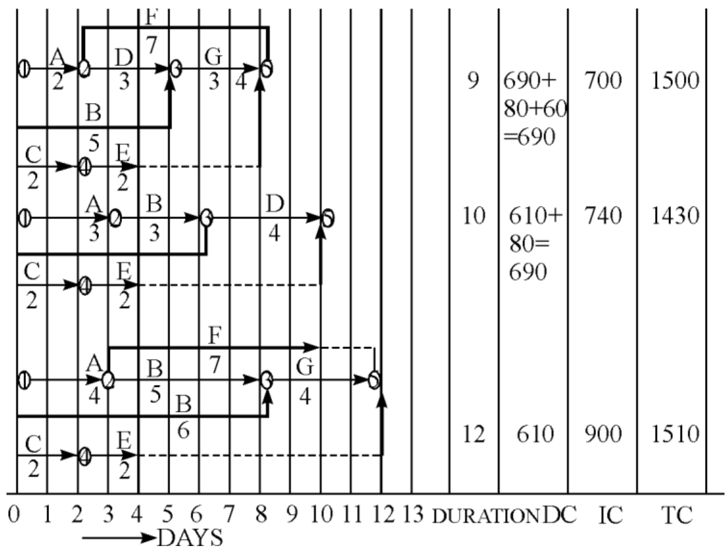


Figure 15. 42 Squared network for problem 15.10

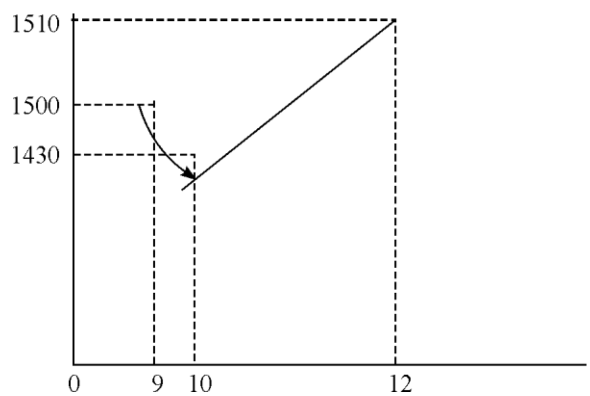


Figure 15.43

Before concluding this chapter, it is better to introduce the students to further developments or advanced topics in network techniques.

1. **Updating the network:** In large project works, as the project progresses, we may come across situation like

   (a) The time estimates made before may be wrong that a particular activity may take less or more time. And we may also sense that we have forgotten certain activities. In such cases, we have to **update the project**

   Leaving the executed activities, remaining activities may have to be modified and the remaining network is redrawn. This is known as updating the network.

2. **Resource leveling and Resource smoothing:** When we have to manage project with available resources, we have two options. First one is resource leveling. Here when the resources availability is less than the maximum resources required for an activity, then delay the job having largest float and divert the resources to critical activities. When two or more jobs compete for same resource, first try to allocate to an activity, which is of short duration and next to the activity which having next highest duration. Here available resource is **a constraint**. The **project duration time may increase** during the process.

3. **Resource smoothing:** Here total project duration is **maintained to the minimum level**. By shifting the activities having floats the demand for resources are smoothened. Here main **constraint is project duration time**.

QUIZ PAPER

Unit – I

Historical Development and Resource Allocatin Model

1. Operations Research is the outcome of
   (a) National emergency (*b*) Political problems
   (*c*) Combined efforts of talents of all fields (*d*) Economics and Engineering. ( )
2. O.R. came into existence during
   (a) World War I, (*b*) India and Pakistan War,
   (*c*) World War II, (*d*) None of the above. ( )
3. The name of the subject Operations Research is due to the fact that
   (a) Problems can be solved by war approach
   (b) The researchers do the operations
   (c) The war problems are generally known as operations and inventing a new way of solving such problems.
   (d) Mathematical operations are used in solving the problems. ( )
4. The first country to use Operations Research method to solve problems is
   (a) India, (*b*) China,
   (*c*) U.K., (*d*) U.S.A. ( )
5. The name Operations Research is first coined in the year
   (*a*) 1945, (*b*) 1935,
   (*c*) 1940, (*d*) 1950 ( )
6. The person who coined the name Operations Research is:
   (a) Bellman, (*b*) Newman,
   (*c*) McClosky and Trefrhen, (*d*) None of the above ( )
7. O.R. Society of India is founded in the year
   (*a*) 1965, (*b*) 1970,
   (*c*) 1959, (*d*) 1972, ( )
8. The objective of Operations Research is:
   (a) To find new methods of solving Problems,
   (b) To derive formulas
   (c) Optimal utilization of existing resources
   (d) To utilize the services of scientists. ( )

9. Operations Research is the art of giving
   (a) Good answers for war problems,
   (b) Bad answers to war problems,
   (c) Bad answers to problems where otherwise worse answers are given.
   (d) Good answers to problems where otherwise bad answers are given, ( )
10. Operations Research is
   (a) Independent thinking approach, (*b*) Group thinking approach
   (*c*) Inter-disciplinary team approach, (*d*) None of the above. ( )
11. The first step in solving Operations Research problem is
   (a) Model building, (*b*) Obtain alternate solutions,
   (*c*) Obtain basic feasible solutions, (*d*) Formulation of the problem. ( )
12. The model, which gives physical or visual representation of the problem, is
   (a) Analogue model, (*b*) Static model,
   (*c*) Iconic model, (*d*) Symbolic model. ( )
13. One of the properties of O.R. model is
   (a) Model should be complicated,
   (b) Model is structured to suit O.R. techniques,
   (c) Model should be structured to suit all the problems we come across,
   (d) Model should be easy to derive. ( )
14. The problem, which is used to disburse the available limited resources to activities, is known as
   (*a*) O.R. Model, (*b*) Resources Model,
   (*c*) Allocation Model, (*d*) Activities model. ( )
15. 15. A wide class of allocation models can be solved by a mathematical technique know as:
   (a) Classical model, (*b*) Mathematical Model,
   (*c*) Descriptive model, (*d*) Linear Programming model. ( )
16. 16. One of the properties of Linear Programming Model is
   (a) It will not have constraints,
   (b) It should be easy to solve,
   (c) It must be able to adopt to solve any type of problem,
   (d) The relationship between problem variables and constraints must be linear. ( )
17. The constraints of Maximisation problem are of
   (a) Greater than or equal type, (*b*) Less than or equal type,
   (*c*) Less than type, (*d*) Greater than type, ( )
18. The slack variables indicate
   (a) Excess resource available, (*b*) Shortage of resource available,
   (*c*) Nil resources, (*d*) Idle resource. ( )
19. In graphical solution of solving Linear Programming problem to convert inequalities into equations, we
   (a) Use Slack variables,
   (b) Use Surplus variables,

(c) Use Artificial surplus variables,
(d) Simply assume them to be equations. ( )

20. To convert ≤ type of inequality into equations, we have to
(a) Assume them to be equations, (*b*) Add surplus variables,
(*c*) Subtract slack variables. (*d*) Add slack variables. ( )

21. To convert ≥ type of inequality into equations, we have to
(a) Add slack variable, (*b*) Subtract slack Variable,
(*c*) Subtract surplus variable (*d*) Add surplus variable. ( )

22. In Graphical solution of maximisation problem, the line, which we move from origin to the extreme point of the polygon is :
(a) Any one side of the polygon, (*b*) Iso cost line,
(*c*) Iso profit line, (*d*) An imaginary line, ( )

23. The key row indicates
(a) Incoming variable, (*b*) outgoing variable,
(*c*) Slack variable, (*d*) Surplus variable, ( )

24. The key column indicates
(a) Outgoing variable, (*b*) Incoming variable,
(*c*) Independent variable, (*d*) Dependent variable, ( )

25. The penalty for not taking correct decision is known as
(a) Fine, (*b*) Loss,
(*c*) Cost, (*d*) Opportunity cost.

26. To transfer the key row in simplex table we have to
(a) Add the elements of key row to key number,
(b) Subtract the elements of key row from topmost no key row,
(c) Divide the elements of key row by key number,
(d) None of the above. ( )

27. The solution of the Linear programming problem in graphical solution lies in
(a) First quadrant, (*b*) Second quadrant,
(*c*) Third quadrant, (*d*) Fourth quadrant, ( )

28. When we solve maximization problem by simplex method the elements of net evaluation row of optimal solution must be (when we use opportunity cost concept)
(*a*) Either zeros or positive numbers, (*b*) Either zeros or negative numbers,
(*c*) All are negative numbers, (*d*) All are zeros. ( )

29. 29. When all the elements of replacement ratio column are equal, the situation is known as
(*a*) Tie, (*b*) Degeneracy,
(*c*) Break, (*d*) None of the above. ( )

30. 30. When the elements of net evaluation row of simplex table are equal, the situation is known as
(*a*) Tie, (*b*) Degeneracy,
(*c*) Break, (*d*) Shadow price. ( )

31. The number at the intersection of key row and key column is known as
    (a) Column number, (*b*) Row number,
    (*c*) Key number, (*d*) Cross number. ( )
32. Dual of a Duel is
    (*a*) Primal, (*b*) Dual,
    (*c*) Prima dual, (*d*) None of the above. ( )
33. Primal of a Primal is :
    (*a*) Primal, (*b*) Dual,
    (*c*) Prima primal, (*d*) duo primal. ( )
34. Dual of a Dual of Dual is
    (*a*) Dual, (*b*) Primal,
    (*c*) Double dual, (*d*) Single dual. ( )
35. Primal of a dual is
    (*a*) Primal, (*b*) Dual,
    (*c*) Prime dual, (*d*) Prime primal. ( )
36. If Dual has a solution, then the primal will
    (*a*) Not have a solution, (*b*) Have only basic feasible solution,
    (*c*) Have a solution (*d*) None of the above. ( )
37. If Primal Problem is a maximisation problem, then the dual will be
    (*a*) Maximisation Problem, (*b*) Minimisation Problem,
    (*c*) Mixed Problem, (*d*) None of the above. ( )
38. To get the Replacement ration column elements we have to
    (*a*) Divide Profit column elements by key number,
    (*b*) The first column elements of identity is divided by key number
    (*d*) Divide the capacity column elements by key number. ( )
39. The cost coefficient of slack variable is
    (a) Zero, (*b*) One,
    (*c*) > than one, (*d*) < than one, ( )
40. The cost coefficient of artificial surplus variable is
    (*a*) 0, (*b*) 1,
    (*c*) M (*d*) > than 1. ( )
41. If the primal has an unbounded solution, then the dual has
    (a) Optimal solution, (*b*) No solution,
    (*c*) Bound solution, (*d*) None of the above. ( )

ANSWERS

1. (*c*)	2. (*c*)	3. (*d*)	4. (*c*)	5. (*c*)	6. (*c*)
7. (*c*)	8. (*c*)	9. (*c*)	10. (*c*)	11. (*d*)	12. (*c*)
13. (*b*)	14. (*c*)	15. (*d*)	16. (*d*)	17. (*a*)	18. (*d*)
19. (*d*)	20. (*d*)	21. (*c*)	22. (*c*)	23. (*b*)	24. (*b*)
25. (*d*)	26. (*c*)	27. (*a*)	28. (*b*)	29. (*b*)	30. (*a*)
31. (*c*)	32. (*a*)	33. (*a*)	34. (*b*)	35. (*a*)	36. (*a*)
37. (*b*)	38. (*c*)	39. (*a*)	40. (*c*)	41. (*b*)	

QUIZ PAPERS

Unit – II

Transprotation Model and Assignment Model

1. Transportation problem is basically a
   (a) Maximisation model, (*b*) Minimisation model,
   (*c*) Transshipment problem, (*d*) Iconic model. ( )
2. The column, which is introduced in the matrix to balance the rim requirements, is known as:
   (a) Key column, (*b*) Idle column,
   (*c*) Slack column, (*d*) Dummy Column. ( )
3. The row, which is introduced in the matrix to balance the rim requirement, is known as:
   (a) Key row, (*b*) Idle row,
   (*c*) Dummy row, (*d*) Slack row. ( )
4. One of the differences between the Resource allocation model and Transportation Model is
   (a) The coefficients of problem variables in Resource allocation model may be any number and in transportation model it must be either zeros or ones,
   (b) The coefficients of problem variable in Resource allocation model must be either zeros or ones and in Transportation model they may be any number,
   (c) In both models they must be either zeros or ones only,
   (d) In both models they may be any number. ( )
5. To convert the transportation problem into a maximisation model we have to
   (a) write the inverse of the matrix,
   (b) Multiply the rim requirements by –1,
   (c) To multiply the matrix by –1,
   (d) We cannot convert the transportation problem into a maximisation problem, as it is basically a minimisation problem. ( )
6. In a transportation problem where the demand or requirement is equal to the available resource is known as
   (a) Balanced transportation problem,
   (b) Regular transportation problem,
   (c) Resource allocation transportation problem,
   (d) Simple transportation model. ( )

7. The total number of allocation in a basic feasible solution of transportation problem of $m \times n$ size is equal to
   (*a*) $m \times n$, (*b*) $(m / n) - 1$,
   (*c*) $m + n + 1$ (*d*) $m + n - 1$. ( )
8. When the total allocations in a transportation model of m x n size is not equals to $m + n - 1$ the situation is known as
   (a) Unbalanced situation, (*b*) Tie situation,
   (*c*) Degeneracy, (*d*) None of the above. ( )
9. The opportunity cost of a row in a transportation problem is obtained by:
   (a) Deducting the smallest element in the row from all other elements of the row,
   (b) Adding the smallest element in the row to all other elements of the row,
   (c) Deducting the smallest element in the row from the next highest element of the row,
   (d) Deducting the smallest element in the row from the highest element in that row. ( )
10. In Northwest corner method the allocations are made
    (a) Starting from the left hand side top corner,
    (b) Starting from the right hand side top corner,
    (c) Starting from the lowest cost cell,
    (d) Starting from the lowest requirement and satisfying first. ( )
11. VAM stands for:
    (a) Value added method, (*b*) Value assessment method,
    (*c*) Vogel Adam method, (*d*) Vogel's approximation method. ( )
12. MODI stands for
    (a) Modern distribution, (*b*) Mendel's distribution method,
    (*c*) Modified distribution method, (*d*) Model index method. ( )
13. In the optimal solution, more than one empty cells have their opportunity cost as zero, it indicates
    (a) The solution is not optimal; (*b*) The problem has alternate solution,
    (*c*) Something wrong in the solution, (*d*) The problem will cycle, ( )
14. In case the cost elements of one or two cells are not given in the problem, it means:
    (a) The given problem is wrong, (*b*) We can allocate zeros to those cells,
    (c) Allocate very high cost element to those cells,
    (d) To assume that the route connected by those cells are not available. ( )
15. To solve degeneracy in the transportation problem we have to:
    (a) Put allocation in one of the empty cells as zero,
    (b) Put a small element epsilon in any one of the empty cells,
    (c) Allocate the smallest element epsilon in such a cell, which will not form a closed loop with other loaded cells,
    (d) Allocate the smallest element epsilon in such a cell, which will form a closed loop with other loaded cells. ( )

16. A problem where the produce of a factory is stored in warehouses and then they are transported to various demand points as and when the demand arises is known as
    (a) Transshipment problem, (*b*) Warehouse problem,
    (*c*) Storing and transport problem, (*d*) None of the above. ( )
17. Implied Cost in transportation problem sets (in the existing program):
    (a) The lowest limit for the empty cell beyond which it is not advisable to include in the programme,
    (b) The highest limit for the empty cell beyond which it is not advisable to include in the programme,
    (c) The opportunity cost of the empty cell,
    (d) None of the above. ( )
18. In transportation model, the opportunity cost is given by
    (a) Implied cost + Actual cost of the cell, (*b*) Actual cost of the cell – Implied cost,
    (*c*) Implied cost – Actual cost of the cell, (*d*) Implied cost × Actual cost of the cell. ( )
19. If u_i and v_j are row and column numbers respectively, then the implied cost is given by:
    (a) $u_i + v_j$, (*b*) $u_i - v_j$,
    (*c*) $u_i \times v_j$, (*d*) u_i / v_j . ( )
20. If a transportation problem has an alternate solution, then the other alternate solutions are derived by:
    (Given that the two matricides of alternate solutions are *A* and *B*, and *d* is any positive fraction number)
    (*a*) $A + (1 - d) \times B$, (*b*) $A\,(1 - d) + B$,
    (*c*) $dA + dB$, (*d*) $dA + (1 - d) \times B$. ( )
21. Assignment Problem is basically a
    (a) Maximisation Problem, (*b*) Minimisation Problem,
    (*c*) Transportation Problem, (*d*) Primal problem. ( )
22. The Assignment Problem is solved by
    (a) Simplex method, (*b*) Graphical method,
    (*c*) Vector method, (*d*) Hungarian method. ( )
23. In Index method of solving assignment problem
    (a) The whole matrix is divided by smallest element,
    (b) The smallest element is subtracted from whole matrix,
    (c) Each row or column is divided by smallest element,
    (d) The whole matrix is multiplied by -1. ( )
24. In Hungarian method of solving assignment problem, the row opportunity cost matrix is obtained by:
    (a) Dividing each row by the elements of the row above it,
    (b) Subtracting the elements of the row from the elements of the row above it,
    (c) Subtracting the smallest element from all other elements of the row,
    (d) Subtracting all the elements of the row from the highest element in the matrix. ( )

25. In Flood's technique of solving assignment problem the column opportunity cost matrix is obtained by:
    (a) Dividing each column by the elements of a column which is right side of the column,
    (b) By subtracting the elements of a column from the elements of the column which is right side of the column,
    (c) By subtracting the elements of the column from the highest element of the matrix,
    (d) By subtracting the smallest elements in the column from all other elements of the column. ( )
26. The property of total opportunity cost matrix is
    (a) It will have zero as elements of one diagonal,
    (b) It will have zero as the elements of both diagonals,
    (c) It will have at least one zero in each column and each row,
    (d) It will not have zeros as its elements. ( )
27. The horizontal and vertical lines drawn to cover all zeros of total opportunity matrix must be:
    (a) Equal to each other,
    (b) Equal to $m \times n$ (where m and n are number of rows and columns),
    (c) $m + n$ (where m and n are number of rows and columns),
    (d) Number of rows or columns. ( )
28. The assignment matrix is always a
    (a) Rectangular matrix, (*b*) Square matrix,
    (*c*) Identity matrix, (*d*) None of the above. ( )
29. To balance the assignment matrix we have to:
    (a) Open a Dummy row,
    (b) Open a Dummy column,
    (c) Open either a dummy row or column depending on the situation,
    (d) You cannot balance the assignment matrix. ( )
30. In cyclic traveling salesman problem the elements of diagonal from left top to right bottom are
    (a) Zeros, (*b*) All negative elements,
    (*c*) All infinity, (*d*) all ones. ( )
31. To convert the assignment problem into a maximization problem
    (a) Deduct smallest element in the matrix from all other elements,
    (b) All elements of the matrix are deducted form the highest element in the matrix,
    (c) Deduct smallest element in any row from all other elements of the row,
    (d) Deduct all elements of the row from highest element in that row. ( )
32. The similarity between Assignment Problem and Transportation Problem is:
    (a) Both are rectangular matrices,
    (b) Both are square matrices,
    (c) Both can be solved by graphical method,
    (d) Both have objective function and non-negativity constraints. ( )

33. The following statement applies to both transportation model and assignment model
    (a) The inequalities of both problems are related to one type of resource,
    (b) Both use VAM for getting basic feasible solution,
    (c) Both are tested by MODI method for optimality,
    (d) Both have objective function, structural constraint and non-negativity constraints. ( )
34. To test whether allocations can be made or not (in assignment problem), minimum number of horizontal and vertical lines are drawn. In case the lines drawn is not equal to the number of rows (or columns), to get additional zeros, the following operation is done:
    (a) Add smallest element of the uncovered cells to the elements to the line,
    (b) Subtract smallest element of uncovered rows from all other elements of uncovered cells,
    (c) Subtract the smallest element from the next highest number in the element,
    (d) Subtract the smallest element from the element at the intersection of horizontal and vertical lines. ( )
35. The total opportunity cost matrix is obtained by doing:
    (a) Row operation on row opportunity cost matrix,
    (b) Column operation on row opportunity cost matrix,
    (c) Column operation on column opportunity cost matrix,
    (d) None of the above. ( )
36. Flood's technique is a method used for solving
    (a) Transportation problem, (*b*) Resource allocation model,
    (*c*) Assignment mode, (*d*) Sequencing model. ( )
37. The assignment problem will have alternate solutions
    (a) when total opportunity cost matrix has at least one zero in each row and column,
    (b) When all rows have two zeros,
    (c) When there is a tie between zero opportunity cost cells,
    (d) If two diagonal elements are zeros. ( )
38. The following character dictates that assignment matrix is a square matrix:
    (a) The allocations in assignment problem are one to one,
    (b) Because we find row opportunity cost matrix,
    (c) Because we find column opportunity matrix,
    (d) Because make allocations, one has to draw horizontal and Vertical lines. ( )
39. When we try to solve assignment problem by transportation algorithm the following difficulty arises:
    (a) There will be a tie while making allocations,
    (b) The problem will get alternate solutions,
    (c) The problem degenerates and we have to use epsilon to solve degeneracy,
    (d) We cannot solve the assignment problem by transportation algorithm. ( )

ANSWERS

1. (*b*)	2. (*d*)	3. (*c*)	4. (*a*)
5. (*c*)	6. (*a*)	7. (*d*)	8. (*c*)
9. (*c*)	10. (*a*)	11. (*d*)	12. (*c*)
13. (*b*)	14. (*d*)	15. (*c*)	16. (*a*)
17. (*b*)	18. (*c*)	19. (*a*)	20. (*d*)
21. (*b*)	22. (*d*)	23. (*c*)	24. (*c*)
25. (*d*)	26. (*c*)	27. (*d*)	28. (*b*)
29. (*c*)	30. (*c*)	31. (*b*)	32. (*d*)
33. (*d*)	34. (*b*)	35. (*b*)	36. (*c*)
37. (*c*)	38. (*a*)	93. (*c*)	

QUIZ PAPERS

Unit – III

Sequencing Model

1. The objective of sequencing problem is
   (a) To find the order in which jobs are to be made,
   (b) To find the time required for completing all the jobs on hand,
   (c) To find the sequence in which jobs on hand are to be processed to minimize the total time required for processing the jobs,
   (d) To maximize the effectiveness. ( )
2. The time required for printing of four books *A*, *B*, *C* and *D* is 5, 8, 10 and 7 hours. While its data entry requires 7, 4, 3 and 6 hours respectively, the sequence time that minimizes total elapsed time is
   (*a*) ACBD, (*b*) ABCD,
   (*c*) ADCB, (*d*) CBDA. ( )
3. If there are 'n' jobs and 'm' machines, there will be............sequences of doing the jobs.
   (*a*) $n \times m$, (*b*) $m \times n$,
   (*c*) n^m, (*d*) $(n\,!)^m$. ( )
4. In general, sequencing problem will be solved by using ………….
   (*a*) Hungarian Method, (*b*) Simplex method,
   (*c*) Johnson and Bellman method, (*d*) Flood's technique. ( )
5. In solving 2 machines and 'n' jobs, which of the following assumptions is wrong?
   (a) No passing is allowed,
   (b) Processing times are known,
   (c) Handling time is negligible,
   (d) The time of processing depends on the order of machining.
6. The following is the assumption made in the processing of 'n' jobs on 2 machines:
   (a) The processing time of jobs is exactly known and is independent of order of processing,
   (b) The processing times are known and they depend on the order of processing the job,
   (c) The processing time of a job is unknown and it is to be worked out after finding the sequence,
   (d) The sequence of doing jobs and processing times is inversely proportional. ( )

7. The following is one of the assumptions made while sequencing 'n' jobs on 2 machines
   (a) Two jobs must be loaded at a time on any machine,
   (b) Jobs are to be done alternatively on each machine,
   (c) The order of completing the jobs has high significance,
   (d) Each job once started on a machine is to be performed up to completion on that machine. ( )
8. This is not allowed in sequencing of 'n' jobs on two machines
   (a) Passing, (*b*) loading,
   (c) Repeating the job,
   (d) Once loaded on the machine it should be completed before removing from the machine. ( )
9. Write the sequence of performing the jobs for the problem given below:

Jobs	*A*	*B*	*C*	*D*	*E*
Time of machining on Machine X	6	8	5	9	1

   (a) They can be processed in any order,
   (b) As there is only one machine, sequencing cannot be done,
   (c) This is not a sequencing problem,
   (d) None of the above. ( )
10. Johnson Bellman rule states that
   (a) If the smallest processing time occurs under the first machine, do that job first,
   (b) If the smallest processing time occurs under the second machine, do that job first,
   (c) If the smallest processing time occurs under the first machine, do that job last,
   (d) If the smallest processing time occurs under the second machine keep the processing pending. ( )
12. To convert 'n' jobs and 3-machine problem into 'n' jobs and 2-machine problem, the following rule must be satisfied
   (a) All the processing times of second machine must be same,
   (b) The maximum processing time of 2nd machine must be ≤ the minimum processing times of first and third machines,
   (c) The maximum processing time of 1st machine must be ≤ the minimum processing times of other two machines,
   (d) The minimum processing time of 2nd machine must be ≤ the minimum processing times of first and third machines. ( )
13. If two jobs J_1 and J_2 have same minimum process time under first machine but processing time of J_1 is less than that of J_2 under second machine, then J_1 occupies
   (a) First available place from the left,
   (b) Second available place from the left,
   (c) First available place from the right,
   (d) Second available place from the right. ( )

14. If Jobs *A* and *B* have same processing times under machine I and Machine II, then prefer
    (a) Job *A*, (*b*) Job *B*,
    (*c*) Both *A* and *B*, (*d*) Either *A* or *B*. ( )
15. The given sequencing problem will have multiple optimal solutions when the two jobs have same processing times under:
    (a) First machine, (*b*) Both machines,
    (*c*) Second machine, (*d*) None of the above. ( )
16. If a job is having minimum processing time under both the machines, then the job is placed in:
    (a) Any one (first or last) position, (*b*) Available last position,
    (*c*) Available first position, (*d*) Both first and last positions. ( )
17. FIFO is most applicable to sequencing of
    (a) One machine and '*n*' jobs, (*b*) 2 machines and '*n*' jobs,
    (*c*) 3 machines '*n*' jobs, (*d*) '*n*' machines and 2 jobs. ( )
18. At a petrol Bunk, when '*n*' vehicles are waiting for service then this service rule is used:
    (a) LIFO, (*b*) FIFO,
    (*c*) Service in random order, (*d*) Service by highest profit rule. ( )
19. Consider the following sequencing problem, and write the optimal sequence:

Jobs:		1	2	3	4	5
Processing	M/C X	1	5	3	10	7
Time in Hrs.						
	M/C Y	6	2	8	4	9

    (*a*) 1 2 3 4 5 (*b*) 1 3 5 4 2
    (*c*) 5 4 3 2 1 (*d*) 1 4 3 5 2 ( )
20. In a 3 machines and 5 jobs problem, the least of processing times on machines *A*, *B* and *C* are 5, 1, and 3 hours and the highest processing times are 9, 5, and 7 respectively, then Johnson and Bellman rule is applicable if order of the machine is:
    (a) B-A-C, (*b*) A-B-C,
    (*c*) C-B-A, (*d*) Any order. ( )
21. In maximization case of sequencing problem of 2 machines and '*n*' jobs, the job is placed at available left first position if it has .... process time under machine ....
    (a) Least, first, (*b*) Highest, first,
    (*c*) Least, second, (*d*) Highest, second. ( )
22. The fundamental assumption of Johnson's method of sequencing is:
    (a) No passing rule, (*b*) Passing rule,
    (*c*) Same type of machines are to be used, (*d*) Non zero process time. ( )
23. If a job has zero process time for any machine, the job must
    (a) Possess first position only, (*b*) Possess last position only,
    (*c*) Possess extreme position, (*d*) Be deleted from the sequencing. ( )

24. The assumption made in sequencing problems *i.e*. No passing rule means:
    (a) A job once loaded on a machine should not be removed until it is completed,
    (b) A job cannot be processed on second machine unless it is processed on first machine,
    (c) A machine should not be started unless the other is ready to start,
    (d) No job should be processed unless all other machines are kept ready to start. ( )
25. The technological order of machine to be operated is fixed in a problem having:
    (a) 1 machine and '*n*' jobs, (*b*) 2 machines and '*n*' jobs,
    (*c*) 3 machines and '*n*' jobs, (*d*) '*n*' machines and 2 jobs. ( )
26. A sequencing problem is infeasible in case of:
    (a) 1 machine and '*n*' jobs, (*b*) 2 machines and '*n*' jobs,
    (*c*) 3 machines and '*n*' jobs, (*d*) 2 jobs and '*n*' machines. ( )
27. In a 2 jobs and '*n*' machines problem a lie at 45º represents:
    (a) Job 2 is idle, (*b*) Job 1 is idle,
    (*c*) Both jobs are idle, (*d*) Both jobs are under processing. ( )
28. In a 2 jobs and '*n*' machines problem, the elapsed time for job 1 is calculated as (Job 1 is represented on X-axis).
    (a) Process time for Job 1 + Total length of vertical line on graph,
    (b) Process time for Job 2 + Idle time for Job 1,
    (c) Process time for job 1 + Total length of horizontal line on graph,
    (d) Process time for job 2 – Idle time for job 1. ( )
29. In a 2 jobs and '*n*' machines sequencing problem the horizontal line on a graph indicates:
    (a) Processing time of Job 1,
    (b) Idle time of Job 1,
    (c) Idle time of both jobs,
    (d) Processing time of both jobs. ( )
30. In a 2 jobs, '*n*' machines sequencing problem, the vertical line on the graph indicates:
    (a) Processing time of Job 1, (*b*) Processing time of Job 2,
    (*c*) Idle time of Job 2, (*d*) Idle time of both jobs. ( )
31. In a 2 jobs and '*n*' machines sequencing problem we find that:
    (a) Sum of processing times of both the jobs is same,
    (b) Sum of idle times of both the jobs is same,
    (c) Sum of processing times and idle time of both the jobs is same,
    (d) Sum of processing times and idle time of both the jobs is different. ( )

ANSWERS

1. (*c*)	2. (*d*)	3 (*d*)	4. (*c*)
5. (*d*)	6. (*c*)	7. (*d*)	8. (*a*)
9. (*a*)	10. (*a*)	11. (*a*)	12. (*b*)
13. (*b*)	14. (*d*)	15. (*c*)	16. (*a*)
17. (*a*)	18. (*a*)	19. (*d*)	20. (*b*)
21. (*b*)	22. (*b*)	23. (*c*)	24. (*b*)
25. (*d*)	26. (*c*)	27. (*d*)	28. (*a*)
29. (*a*)	30. (*b*)	31. (*c*)	

Unit – IV
Replacement Model and Game Theory

REPLACEMENT MODEL

1. Contractual maintenance or agreement maintenance with manufacturer is suitable for equipment, which is
   (*a*) In its infant state, (*b*) When machine is old one,
   (*c*) Scrapped, (*d*) None of the above. ( )
2. When money value changes with time at 10 %, then PWF for first year is :
   (*a*) 1, (*b*) 0.909,
   (*c*) 0.852, (*d*) 0.9. ( )
3. Which of the following maintenance policies is not used in old age stage of a machine?
   (a) Operate up to failure and do corrective maintenance,
   (b) Reconditioning,
   (c) Replacement,
   (d) Scheduled preventive maintenance. ( )
4. When money value changes with time at 20%, the discount factor for 2nd year is:
   (*a*) 1 (*b*) 0.833
   (*c*) 0 (*d*) 0.6955 ( )
5. Which of the following replacement policies is considered to be dynamic in nature?
   (a) Time is continuous variable and the money value does not change with time,
   (b) When money value does not change with time and time is a discrete variable,
   (c) When money value changes with time,
   (d) When money value remains constant for some time and then goes on changing with time. ( )
6. When the probability of failure reduces gradually, the failure mode is said to be:
   (a) Regressive, (*b*) Retrogressive,
   (*c*) Progressive, (*d*) Recursive. ( )
7. The following replacement model is said to be probabilistic model:
   (a) When money value does not change with time and time is a continuous variable,
   (b) When money value changes with time,

(*c*) When money value does not change with time and time is a discrete variable,
(*d*) Preventive maintenance policy. ( )

8. A machine is replaced with average running cost
(*a*) Is not equal to current running cost,
(*b*) Till current period is greater than that of next period,
(*c*) If current period is greater than that of next period,
(*d*) If current period is less than that of next period. ( )

9. The curve used to interpret machine life cycle is
(*a*) Bath tub curve, (*b*) Time curve,
(*c*) Product life cycle, (*d*) Ogive curve. ( )

10. Decreasing failure rate is usually observed in ....stage of the machine
(*a*) Infant, (*b*) Youth,
(*c*) Old age, (*d*) Any time in its life. ( )

11. Which cost of the following is irrelevant to replacement analysis?
(*a*) Purchase cost of the machine,
(*b*) Operating cost of the machine,
(*c*) Maintenance cost of the machine,
(*d*) Machine hour rate of the machine. ( )

12. The type of failure that usually occurs in old age of the machine is
(*a*) Random failure, (*b*) Early failure,
(*c*) Chance failure, (*d*) Wear-out failure. ( )

13. Group replacement policy is most suitable for:
(*a*) Trucks, (*b*) Infant machines,
(*c*) Street light bulbs, (*d*) New cars. ( )

14. The chance failure that occurs on a machine is commonly found on a graph of time Vs failure rate (on X and Y axes respectively as
(*a*) Parabolic, (*b*) Hyperbolic,
(*c*) Line nearly parallel to X-axis, (*d*) Line nearly parallel to Y-axis. ( )

15. Replacement of an item will become necessary when
(a) Old item becomes too expensive to operate or maintain,
(b) When your operator desires to work on a new machine,
(c) When your opponent changes his machine in his unit,
(d) When company has surplus funds to spend. ( )

16. The production manager will not recommend group replacement policy
(a) When large number of identical items are to be replaced,
(b) In case Low cost items are to be replaced, where record keeping is a problem,
(c) For items that fail completely,
(d) For Reparable items. ( )

17. In replacement analysis the maintenance cost is a function of:
(a) Time, (*b*) Function,
(*c*) Initial investment, (*d*) Resale value. ( )

18. Which of the following is the correct assumption for replacement policy when money value does not change with time?
    (a) No Capital cost, (*b*) No scrap value,
    (*c*) Constant scrap value, (*d*) Zero maintenance cost. ( )
19. Which one of the following does not match the group?
    (a) Present Worth Factor (PWF), (*b*) Discounted rate (DR),
    (*c*) Depreciation value (DV), (*d*) Mortality Tables (MT). ( )
20. Reliability of an item is
    (a) Failure Probability, (*b*) 1 / Failure probability,
    (*c*) 1 - failure probability, (*d*) Life period / Failure rate. ( )
21. The following is not discussed in group replacement policy:
    (a) Failure Probability, (*b*) Cost of individual replacement,
    (*c*) Loss due to failure, (*d*) Present worth factor series. ( )
22. It is assumed that maintenance cost mostly depends on:
    (a) Calendar age, (*b*) Manufacturing date,
    (*c*) Running age, (*d*) User's age. ( )
23. Group replacement policy applies to:
    (a) Irreparable items, (*b*) Reparable items.
    (*c*) Items that fail partially, (*d*) Items that fail completely. ( )
24. If a machine becomes old, then the failure rate expected will be:
    (a) Constant, (*b*) Increasing,
    (*c*) Decreasing, (*d*) We Cannot be said. ( )
25. Replacement is said to be necessary if
    (a) Failure rate is increasing, (*b*) Failure cost is increasing,
    (*c*) Failure probability is increasing, (*d*) Any of the above. ( )
26. In this stage, the machine operates at highest efficiency and its production rate will be high.
    (a) Infant stage, (*b*) Youth stage,
    (*c*) Old age, (*d*) None of the above. ( )
27. Replacement decision is very much common in this stage:
    (a) Infant stage, (*b*) Old age,
    (*c*) Youth, (*d*) In all the above. ( )
28. The replacement policy that is imposed on an item irrespective of its failure is
    (a) Group replacement, (*b*) Individual replacement,
    (*c*) Repair spare replacement, (*d*) Successive replacement. ( )
29. When certain symptoms indicate that a machine is going to fail and to avoid failure if maintenance is done it is known as:
    (a) Symptoms maintenance, (*b*) Predictive maintenance,
    (*c*) Repair maintenance, (*d*) Scheduled maintenance. ( )
30. In retrogressive failures, the failure probability------------------- with time.
    (a) Increases, (*b*) Remains constant,
    (*c*) Decreases, (*d*) None of the above. ( )

GAME THEORY

1. If the value of the game is zero, then the game is known as:
   (a) Fair strategy, (*b*) Pure strategy,
   (*c*) Pure game, (*d*) Mixed strategy. ( )
2. The games with saddle points are :
   (a) Probabilistic in nature, (*b*) Normative in nature,
   (*c*) Stochastic in nature, (*d*) Deterministic in nature. ( )
3. When Minimax and Maximin criteria matches, then
   (a) Fair game exists, (*b*) Unfair game is exists,
   (*c*) Mixed strategy exists, (*d*) Saddle point exists. ( )
4. When the game is played on a predetermined course of action, which does not change throughout game, then the game is said to be
   (a) Pure strategy game, (*b*) Fair strategy game,
   (*c*) Mixed strategy game, (*d*) Unsteady game. ( )
5. If the losses of player *A* are the gins of the player *B*, then the game is known as:
   (a) Fair game, (*b*) Unfair game,
   (*c*) Nonzero sum game, (*d*) Zero sum game. ( )
6. Identify the wrong statement:
   (a) Game without saddle point is probabilistic,
   (b) Game with saddle point will have pure strategies,
   (c) Game with saddle point cannot be solved by dominance rule,
   (d) Game without saddle point uses mixed strategies. ( )
7. In a two-person zero sum game, the following does not hold correct:
   (a) Row player is always a loser, (*b*) Column player is always a winner,
   (*c*) Column player always minimizes losses, (*d*) If one loses, the other gains. ( )
8. If a two-person zero sum game is converted to a Linear Programming Problem,
   (a) Number of variables must be two only,
   (b) There will be no objective function,
   (c) If row player represents Primal problem, Column player represents Dual problem,
   (d) Number of constraints are two only. ( )
9. In case there is no saddle point in a game then the game is
   (a) Deterministic game, (*b*) Fair game,
   (*c*) Mixed strategy game, (*d*) Multiplayer game. ( )
10. When there is dominance in a game then
    (a) Least of the row ≥ highest of another row,
    (b) Least of the row ≤ highest of another row,
    (c) Every element of a row ≥ corresponding element of another row,
    (d) Every element of the row ≤ corresponding element of another row. ( )

11. When the game is not having a saddle point, then the following method is used to solve the game:
(a) Linear Programming method, (*b*) Minimax and maximin criteria,
(*c*) Algebraic method, (*d*) Graphical method. ( )

12. Consider the matrix given, which is a pay off matrix of a game. Identify the dominance in it.

		B		
		X	Y	Z
	P	1	7	3
A	Q	5	6	4
	R	7	2	0

(a) P dominates Q (*b*) Y dominates Z
(*c*) Q dominates R (*d*) Z dominates Y ( )

13. Identify the unfair game:

(*a*)

	C	D
A	0	0
B	0	0

(*b*)

	C	D
A	1	–1
B	–1	1

(*c*)

	C	D
A	–5	+5
B	+10	–10

(*d*)

	C	D
A	1	0
B	0	1

( )

14. If there are more than two persons in a game then the game is known as:
(a) Nonzero sum game (*b*) Open game
(*c*) Multiplayer game (*d*) Big game ( )

15. For the pay off matrix the player *A* always uses:

		B	
		I	II
A	I	–5	–2
	II	10	5

(*a*) First strategy (*b*) Mixed strategy of both II and I
(*c*) Does not play game (*d*) Second strategy. ( )

16. For the pay off matrix the player prefers to play

		B	
		I	II
A	I	–7	6
	II	–10	8

(a) Second strategy (*b*) First strategy
(*c*) Keep quite (*d*) Mixed strategy. ( )

17. For the game given the value is:

		B I	B II
A	I	2	3
	II	–5	5

(*a*) 3, (*b*) –5
(*c*) 5 (*d*) 2 ( )

19. In the game given the saddle point is:

		B I	B II	B III
A	I	2	–4	6
	II	0	–3	–2
	III	3	–5	4

(*a*) –2 (*b*) 0
(*c*) –3 (*d*) 2 ( )

20. A competitive situation is known as:
(*a*) Competition, (*b*) Marketing,
(*c*) Game, (*d*) None of the above. ( )

21. One of the assumptions in the game theory is:
(*a*) All players act rationally and intelligently,
(*b*) Winner alone acts rationally,
(*c*) Loser acts intelligently,
(*d*) Both the players believe in luck. ( )

22. A play is played when:
(*a*) The manager gives green signal,
(*b*) Each player chooses one of his courses of action simultaneously,
(*c*) The player who comes to the place first says that he will start the game,
(*d*) The late comer says that he starts the game. ( )

23. The list of courses of action with each player is
(*a*) Finite,
(*b*) Number of strategies with each player must be same,
(*c*) Number of strategies with each player need not be same,
(*d*) None of the above. ( )

24. A game involving '*n*' persons is known as:
(*a*) Multimember game, (*b*) Multiplayer game,
(*c*) *n*-person game, (*d*) Not a game. ( )

25. Theory of Games and Economic Behaviour is published by:
    (a) John Von Neumann and Morgenstern
    (b) John Flood
    (c) Bellman and Neumann
    (d) Mr. Erlang, ( )
26. In the matrix of a game given below the negative entries are:

$$\begin{array}{cc|cc} & & \multicolumn{2}{c}{B} \\ & & I & II \\ \hline A & I & 1 & 1 \\ & II & -1 & 1 \end{array}$$

    (a) Payments from A to B
    (b) Payments from B to A
    (c) Payment from players to organisers
    (d) Payment to players from organisers. ( )

ANSWERS

1. REPLACEMENT MODEL (1 TO 30)

1. (*a*)	2. (*b*)	3. (*d*)	4. (*b*)
5. (*c*)	6. (*b*)	7. (*d*)	8. (*d*)
9. (*a*)	10. (*a*)	11. (*d*)	12. (*d*)
13. (*c*)	14. (*c*)	15. (*a*)	16. (*d*)
17. (*a*)	18. (*c*)	19. (*d*)	20. (*c*)
21. (*d*)	22. (*c*)	23. (*d*)	24. (*b*)
25. (*d*)	26. (*b*)	27. (*b*)	28. (*a*)
29. (*b*)	30. (*c*)		

2. GAME THEORY: (1 TO 26)

1. (*c*)	2. (*d*)	3. (*d*)	4. (*a*)
5. (*d*)	6. (*c*)	7. (*a*)	8. (*c*)
9. (*c*)	10. (*d*)	11.(*b*)	12. (*d*)
13. (*d*)	14. (*c*)	15. (*d*)	16. (*b*)
17. (*d*)	18. (*c*)	19. (*c*)	20. (*c*)
21. (*a*)	22. (*b*)	23. (*c*)	24. (*c*)
25. (*a*)	26. (*a*)		

Unit – V

Inventory Management and Waiting Line Models

INVENTORY MODELS

1. One of the important basic objectives of Inventory management is:
   (a) To calculate EOQ for all materials in the organisation,
   (b) To go in person to the market and purchase the materials,
   (c) To employ the available capital efficiently so as to yield maximum results,
   (d) Once materials are issued to the departments, personally check how they are used. ( )
2. The best way of improving the productivity of capital is:
   (a) Purchase automatic machines, (*b*) Effective labour control,
   (c) To use good financial management,
   (d) Productivity of capital is to be increased through effective materials management. ( )
3. Materials management is a body of knowledge, which helps manager to:
   (a) Study the properties of materials, (*b*) Search for needed material,
   (c) Increase the productivity of capital by reducing the cost of material,
   (d) None of the above. ( )
4. The stock of materials kept in the stores in anticipation of future demand is known as:
   (a) Storage of materials, (*b*) Stock of materials,
   (*c*) Inventory, (*d*) Raw materials. ( )
5. The stock of animals reared in anticipation of future demand is known as:
   (a) Live stock inventory, (*b*) Animal inventory,
   (*c*) Flesh inventory, (*d*) None of the above. ( )
6. The working class of human beings is a class of inventor known as:
   (a) Live stock, (*b*) Human inventory,
   (*c*) Population, (*d*) Human resource inventory. ( )
7. In general, the percentage of materials cost in product is approximately equal to:
   (*a*) 40 to 50 % (*b*) 5 to 10 %
   (*c*) 2 to 3 % (*d*) 90 to 95% ( )

8. Materials management brings about increased productivity of capital by:
   (a) Very strict control over use of materials,
   (b) Increasing the efficiency of workers,
   (c) Preventing large amounts of capital locked up for long periods in the form of inventory.
   (d) To apply the principles of capital management, ( )
9. We can reduce the materials cost by:
   (*a*) Using systematic inventory control techniques,
   (*b*) Using the cheap material,
   (*c*) Reducing the use of materials,
   (*d*) Making hand to mouth purchase, ( )
10. The basis for ABC analysis is
   (a) Interests of Materials manager, (*b*) Interests of the top management,
   (*c*) Pareto's 80-20 rule, (*d*) None of the above. ( )
11. ABC analysis depends on the:
   (a) Quality of materials,
   (b) Cost of materials,
   (c) Quantity of materials used,
   (d) Annual consumption value of materials. ( )
12. '*A*' class materials consume:
   (a) 10% of total annual inventory cost, (*b*) 30% of total annual inventory cost,
   (*c*) 70 to 75% of total inventory cost, (*d*) 90% of total annual inventory cost. ( )
13. '*B*' class of materials consumes ....% of annual inventory cost.
   (*a*) 60 to 70% (*b*) 20 to 25%
   (*c*) 90 to 95% (*d*) 5 to 8% ( )
14. '*C*' class of materials consume................% of annual inventory cost.
   (*a*) 5 to 10 % (*b*) 20 to 30%
   (*c*) 40 to 50% (*d*) 70 to 80% ( )
15. The rent for the stores where materials are stored falls under:
   (a) Inventory carrying cost, (*b*) Ordering cost,
   (*c*) Procurement cost, (*d*) Stocking cost. ( )
16. Insurance charges of materials cost fall under:
   (a) Ordering cost, (*b*) Inventory carrying cost,
   (*c*) Stock out cost (*d*) Procurement cost. ( )
17. As the volume of inventory increases, the following cost will increase:
   (a) Stock out cost, (*b*) Ordering cost,
   (*c*) Procuring cost, (*d*) Inventory carrying cost. ( )
18. As the order quantity increases, this cost will reduce:
   (a) Ordering cost, (*b*) Insurance cost,
   (*c*) Inventory carrying cost, (*d*) Stock out cost. ( )

19. Procurement cost may be clubbed with:
    (a) Inventory carrying charges, (*b*) Stock out cost,
    (*c*) Loss due to deterioration, (*d*) Ordering cost. ( )
20. The penalty for not having materials when needed is:
    (*a*) Loss of materials cost, (*b*) Loss of ordering cost,
    (*c*) Stock out cost, (*d*) General losses. ( )
21. 21. Losses due to deterioration, theft and pilferage come under,
    (*a*) Inventory carrying charges, (*b*) Losses due to theft,
    (*c*) Not any cost, (*d*) Consumption cost. ( )
22. 22. Economic Batch Quantity is given by (where, C_1 = Inventory carrying cost, C_3 = Ordering cost, r = Demand for the product)
    (*a*) $(2C_1/C_3)^{1/2}$, (*b*) $(2\,C_3/C_1\,r\,)^{1/2}$,
    (*c*) $2C_3 r\,/\,C_1$, (*d*) $(2C_3\mathrm{r}\,/\,C_1)^{1/2}$. ( )
23. If λ is the annual demand, C_1 = Inventory carrying cost, i = rate of inventory carrying charges, p = unit cost of material in Rs., then EOQ =
    (*a*) $(2C_3\,\lambda\,/ip)^{1\,/\,2}$, (*b*) $2C_3\,\lambda\,/ip$,
    (*c*) $(2\,C_3/ip\,\lambda\,)^{1/2}$, (*d*) $(2\,\lambda\,/C_3\,ip)^{1/2}$. ( )
24. If C_1 = carrying cost, C_3 is the ordering cost, r = demand for the product, then the optimal period for placing an order is given by:
    (*a*) $(2\,C_3/C_1\,r)^{1/2}$ (*b*) $(2C_1\,C_3/r\,)^{1\,/\,2}$
    (*c*) $(\,2C_3\,r/C_1)^{1\,/2}$ (*d*) $(\,2C_1\,C_3\,r)^{1/2}$ ( )
25. 25. When C_1 = Inventory carrying cost, C_3 = ordering cost, r = demand for the product, the total cost of inventory is given by:
    (*a*) $(2C_1\,C_3\,r)$ (*b*) $(2C_1\,C_3)^{1/2}$
    (*c*) $(2C_3 r/C_1)^{1/2}$ (*d*) $(2C_1\,C_3\,r\,)^{1\,/2}$ ( )
26. When load is the annual demand for the material, p = unit price of the material in Rs., C_3 is the ordering cost, q = order quantity, then the total cost including the martial cost is given by:
    (*a*) $(q/2)\,ip + \lambda\,/q\,C_3 + \lambda\,p$ (*b*) $2C_3\,\lambda\;\;ip + \lambda\,p$
    (*c*) $(q/2)\,ip + \lambda\,p$ (*d*) $(\,2C_3\,q\;\;\lambda\,ip)^{1/2}$ ( )
27. In VED analyses, the letter V stands for:
    (a) Very important material, (*b*) Viscous material
    (*c*) Weighty materials, (*d*) Vital materials. ( )
28. In VED analysis, the letter D strands for:
    (a) Dead stock, (*b*) Delayed material,
    (*c*) Deserved materials, (*d*) Diluted materials. ( )
29. The VED analysis depends on:
    (a) Annual consumption cost of materials,
    (b) Unit price of materials,
    (c) Time of arrival of materials,
    (d) Criticality of materials. ( )

30. In FSN analysis the letter S stands for:
    (a) Slack materials, (*b*) Stocked materials,
    (*c*) Slow moving materials, (*d*) Standard materials. ( )
31. In FSN analysis, the letter N stands for:
    (a) Nonmoving materials, (*b*) Next issuing materials,
    (*c*) No materials, (*d*) None of the above. ( )
32. FSN analysis depends on:
    (a) Weight of the material, (*b*) Volume of the material,
    (*c*) Consumption pattern, (*d*) Method of moving materials. ( )
33. MRP stands for:
    (a) Material Requirement Planning, (*b*) Material Reordering Planning,
    (*c*) Material Requisition Procedure, (*d*) Material Recording Procedure. ( )
34. A system where the period of placing the order is fixed is known as:
    (a) *q*-system, (*b*) Fixed order system,
    (*c*) *p*-system, (*d*) Fixed quantity system. ( )
35. A system in which quantity for which order is placed is constant is known as:
    (a) *q*-System, (*b*) *p*-system,
    (*c*) Period system, (*d*) Bin system. ( )
36. LOB stands for:
    (a) Lot of Bills, (*b*) Line of Batches,
    (*c*) Lot of Batches, (*d*) Line of Balance. ( )
37. High reliability spare parts in inventory are known as:
    (a) Reliable spares, (*b*) Insurance spares,
    (*c*) Capital spares, (*d*) Highly reliable spares. ( )
38. The property of capital spares is:
    (*a*) They have very low reliability;
    (*b*) These can be purchased in large quantities, as the price is low,
    (*c*) These spares have relatively higher purchase cost than the maintenance spares,
    (*d*) They are very much similar to breakdown spares. ( )
39. Re-usable spares are known as:
    (a) Multi use spares, (*b*) Repeated useable stores,
    (*c*) Scrap materials, (*d*) Rotable spares. ( )
40. JIT stands for:
    (a) Just In Time Purchase, (*b*) Just In Time production,
    (*c*) Just In Time use of materials, (*d*) Just In Time order the material. ( )
41. The cycle time, selected in balancing a line must be:
    (a) Greater than the smallest time element given in the problem,
    (b) Less than the highest time element given in the problem,
    (c) Slightly greater than the highest time element given in the problem,
    (d) Left to the choice of the problem solver. ( )

42. The lead-time is the time:
    (a) To place orders for materials,
    (b) Of receiving materials,
    (c) Between receipt of material and using materials,
    (d) Between placing the order and receiving the materials. ( )
43. The PQR classification of inventory depends on:
    (a) Unit price of the material, (*b*) Annual consumption value of the material,
    (*c*) Criticality of the material, (*d*) Shelf life of the materials. ( )
44. The classification made on the weight of the materials is known as:
    (a) PQR analysis, (*b*) VED analysis,
    (*c*) XYZ analysis, (*d*) FSN analysis. ( )
45. At EOQ
    (a) Annual purchase cost = Annual ordering cost,
    (b) Annual ordering cost = Annual carrying cost,
    (c) Annual carrying cost = Annual shortage cost,
    (d) Annual shortage cost = Annual purchase cost. ( )
46. If shortage cost is infinity,
    (a) No shortages are allowed; (*b*) No inventory carrying cost is allowed,
    (*c*) Ordering cost is zero, (*d*) Purchase cost = Carrying cost. ( )
47. The most suitable system for a retail shop is
    (a) FSN Analysis, (*b*) ABC analysis,
    (*c*) VED analysis, (*d*) GOLF analysis. ( )
48. The inventory maintained to meet unknown demand changes is known as
    (a) Pipeline inventory, (*b*) Anticipatory inventory,
    (*c*) De coupling inventory, (*d*) Fluctuatory inventory. ( )
49. The most suitable inventory system for a Petrol bunk is
    (*a*) *P*-System, (*b*) 2 Bin system,
    (*c*) *Q*-System, (*d*) Probabilistic model. ( )
50. 50.The water consumption from a water tank follows
    (*a*) *P*-system, (*b*) PQ-system
    (*c*) *Q*-System, (*d*) EOQ System. ( )
51. 51.Which of the following inventories is maintained to meet expected demand fluctuations?
    (*a*) Fluctuatory Inventory, (*b*) Buffer stock,
    (*c*) De-coupling inventory, (*d*) Anticipatory inventory. ( )
52. 52.Which of the following increases with quantity ordered per order?
    (*a*) Carrying cost, (*b*) Ordering cost,
    (*c*) Purchase cost, (*d*) Demand. ( )
53. 53.The ordering cost per order and average unit carrying cost are constant, and demand suddenly falls by 75% then EOQ will:
    (*a*) Decrease by 50% (*b*) Not change
    (*c*) Increase by 50% (*d*) Decrease by 40% ( )

54. In JIT system, the following is assumed to be zero.
(*a*) Ordering cost, (*b*) Transportation cost,
(*c*) Carrying cost, (*d*) Purchase cost. ( )

55.55Which of the following analyses neither considers cost nor value?
(*a*) ABC, (*b*) XYZ,
(*c*) HML, (*d*) VED. ( )

ANSWERS

1. (*c*)	2. (*d*)	3. (*c*)	4. (*c*)
5. (*a*)	6. (*d*)	7. (*a*)	8. (*c*)
9. (*a*)	10. (*c*)	11. (*d*)	12. (*c*)
13. (*b*)	14. (*a*)	15. (*a*)	16. (*b*)
17. (*d*)	18. (*a*)	19. (*d*)	20. (*c*)
21. (*a*)	22. (*b*)	23. (*c*)	24. (*a*)
25. (*d*)	26. (*a*)	27. (*d*)	28. (*c*)
29. (*d*)	30. (*c*)	31. (*a*)	32. (*c*)
33. (*a*)	34. (*c*)	35. (*a*)	36. (*d*)
37. (*b*)	38. (*c*)	39. (*d*)	40. (*b*)
41. (*c*)	42. (*d*)	43. (*c*)	44. (*d*)
45. (*b*)	46. (*b*)	47. (*a*)	48. (*d*)
49. (*c*)	50. (*a*)	51. (*d*)	52. (*a*)
53. (*c*)	54. (*c*)	55. (*d*)	

WAITING LINE MODELS OR QUEUING THEORY

1. As per queue discipline the following is not a negative behavior of a customer:
(a) Balking, (*b*) Reneging,
(*c*) Boarding, (*d*) Collusion. ( )

2. The expediting or follow up function in production control is an example of
(*a*) LIFO, (*b*) FIFO,
(*c*) SIRO, (*d*) Preemptive. ( )

3. In M/M/S N/FIFO the following does not apply
(*a*) Poisson arrival, (*b*) Limited service,
(*c*) Exponential service, (*d*) Single server. ( )

4. The dead bodies coming to a burial ground is an example of:
(*a*) Pure Birth Process, (*b*) Pure Death Process,
(*c*) Birth and Death Process, (*d*) Constant Rate of Arrival. ( )

5. The system of loading and unloading of goods usually follows:
   (a) LIFO, (*b*) FIFO,
   (*c*) SIORO, (*d*) SBP. ( )
6. A steady state exists in a queue if:
   (*a*) $\lambda > \mu$, (*b*) $\lambda < \mu$,
   (*c*) $\lambda = \mu$, (*d*) $\lambda = \mu$. ( )
7. If the operating characteristics of a queue are dependent on time, then it is said to be:
   (a) Transient state, (*b*) Busy state,
   (*c*) Steady state, (*d*) Explosive state. ( )
8. A person who leaves the queue by losing his patience to wait is said to be:
   (a) Reneging, (*b*) Balking,
   (*c*) Jockeying, (*d*) Collusion. ( )
9. The characteristics of a queuing model is independent of:
   (a) Number of service stations, (*b*) Limit of length of queue,
   (*c*) Service Pattern, (*d*) Queue discipline. ( )
10. The unit of traffic intensity is:
    (a) Poisson, (*b*) Markow,
    (*c*) Erlang, (*d*) Kendall. ( )
11. In (M /M /1) : (∞ / FCFS) model, the length of the system L_s is given by:
    (*a*) $p^2/1/p$ (*b*) $p/1-\lambda$
    (*c*) $\lambda^2/(\mu - \lambda)$ (*d*) $\lambda^2/\mu(\mu - \lambda)$ ( )
12. In (M/M/1) : (∞/ FIFO) model, $1/(\mu - \lambda)$ represents:
    (a) L_s, Length of the system, (*b*) L_q length of the queue,
    (*c*) W_q Waiting time in queue, (*d*) W_s Waiting time in system. ( )
13. The queue discipline in stack of plates is:
    (a) SIRO, (*b*) Non-Pre-emptive,
    (*c*) FIFO, (*d*) LIFO. ( )
14. Office filing system follows:
    (a) LIFO, (*b*) FIFO,
    (*c*) SIRO, (*d*) SBP. ( )
15. SIRO discipline is generally found in:
    (a) Loading and unloading,
    (b) Office filing,
    (c) Lottery draw,
    (d) Train arrivals at platform. ( )
16. The designation of Poisson arrival, Exponential service, single server and limited queue selected randomly are represented by:
    (*a*) (M/E/S) : (∞/SIRO), (*b*) (M/M/1) : (∞/SIRO),
    (*c*) (M/M/S) : (N/SIRO), (*d*) (M/M/1) : (N/SIRO). ( )

17. For a simple queue (M/M/1), $\rho = \lambda/\mu$ is known as:
   (a) Poisson busy period, (*b*) Random factor,
   (*c*) Traffic intensity, (*d*) Exponential service factor. ( )
18. With respect to simple queuing model which on of the given below is wrong:
   (a) $L_q = \lambda\ W_q$ (*b*) $\lambda = \mu\ \beta$
   (*c*) $W_s = W_q + \mu$ (*d*) $L_s = L_q + \beta$ ( )
19. When a doctor attends to an emergency case leaving his regular service is called:
   (a) Reneging,
   (b) Balking,
   (c) Pre-emptive queue discipline,
   (d) Non-Pre-Emptive queue discipline. ( )
20. A service system, where customer is stationary and server is moving is found with:
   (a) Buffet Meals,
   (b) Outpatient at a clinic,
   (c) Person attending the breakdowns of heavy machines,
   (d) Vehicle at petrol bunk. ( )
21. In a simple queuing model the waiting time in the system is given by:
   (*a*) $(L_q/\lambda) + (1/\mu)$ (*b*) $1/(\mu - \lambda)$
   (*c*) $\mu/(\mu - \lambda)$ (*d*) $W_q + \mu$ ( )
22. This department is responsible for the development of queuing theory:
   (a) Railway station, (*b*) Municipal office,
   (*c*) Telephone department, (*d*) Health department. ( )
23. If the number of arrivals during a given time period is independent of the number of arrivals that have already occurred prior to the beginning of time interval, then the new arrivals follow ------------distribution.
   (a) Erlang, (*b*) Poisson,
   (*c*) Exponential, (*d*) Normal, ( )
24. Arrival $\rightarrow$ Service $\rightarrow$ Service $\rightarrow$ Service $\rightarrow$ Out
   The figure given represents:
   (a) Single channel single phase system,
   (b) Multichannel single-phase system,
   (c) Singlechannel multiphase system,
   (d) Multichannel multiphase system. ( )
25. In queue designation *A*/*B*/*S* : (*d*/*f*), what does *S* represent?
   (a) Arrival Pattern, (*b*) Service Pattern
   (*c*) Number of service channels, (*d*) Capacity of the system. ( )
26. When the operating characteristics of the queue system is dependent on time, the it is said to be:
   (a) Steady state, (*b*) Explosive state,
   (*c*) Transient state, (*d*) Any one of the above ( )

27. The distribution of arrivals in a queuing system can be considered as a:
    (a) Death Process, (*b*) Pure birth Process,
    (*c*) Pure live process, (*d*) Sick process. ( )
28. Queuing models measure the effect of:
    (a) Random arrivals,
    (b) Random service,
    (c) Effect of uncertainty on the behaviour of the queuing system,
    (d) Length of queue. ( )
29. Traffic intensity is given by:
    (a) Mean arrival rate / Mean service rate,
    (b) $\lambda \times \mu$,
    (c) μ / λ,
    (d) Number present in the queue/Number served. ( )
30. Variance of queue length is :
    (*a*) $\rho = \lambda / \mu$, (*b*) $\rho^2 / 1 - \rho$,
    (*c*) $\lambda / \mu - \lambda$, (*d*) $\rho / (1 - \rho)^2$. ( )

ANSWERS

1. (*c*)	2. (*d*)	3. (*d*)	4. (*a*)	5. (*a*)
6. (*c*)	7. (*a*)	8. (*a*)	9. (*d*)	10. (*c*)
11. (*b*)	12. (*c*)	13. (*d*)	14. (*a*)	15. (*c*)
16. (*d*)	17. (*c*)	18. (*c*)	19. (*d*)	20. (*c*)
21. (*a*)	22. (*c*)	23. (*b*)	24. (*c*)	25. (*a*)
26. (*c*)	27. (*b*)	28. (*c*)	29. (*a*)	30. (*d*)