

BM T63 – BIOMEDICAL SIGNAL PROCESSING**UNIT-I: DISCRETE TIME SIGNALS AND LINEAR SYSTEMS****PART- A****1. Distinguish between continuous signal and discrete signal. (May 2016)**

Continuous signals: The signals that are defined at all instants of time are known as continuous signals and are denoted by $x(t)$

Discrete signals: The signals that are defined at discrete instants of time are known as discrete signals and are denoted by $x(n)$.

2. What is called as quantization? (May 2016)

The process of converting a discrete time continuous amplitude signal $x(n)$ into a discrete time discrete amplitude signal.

3. Define discrete time signals and systems. (Nov 2016)

A discrete time signal is a function defined only at particular time instants. It is discrete in time but continuous in amplitude. An example is temperature recorded at regular intervals of time in a day.

A discrete time system is a device or algorithm that operates on a discrete time input signal $x(n)$, according to some well-defined rule, to produce another discrete time signal $y(n)$ called the output signal.

4. What is correlation? What are its types? (Nov 2016, May 2019)

Correlation is basically used to compare two signals. Correlation is a measure of the degree to which two signals are similar. The correlation of two signals is divided into

- (i) Cross – correlation
- (ii) Auto – correlation

5. State sampling theorem. (May 2017, Dec 2017, May 2019)

A band limited continuous time signal, with higher frequency f_m hertz, can be uniquely recovered from its samples provided the sampling rate $F \geq 2f_m$ samples per second.

6. Distinguish between power and energy signal with an example. (May 2017, Dec 2019)

The energy of a discrete time signal $x(n)$ is defined as

$$Energy = E \sum_{n=-\infty}^{\infty} |x(n)|^2$$

A signal $x(n)$ is called an energy signal if and only if the energy obeys the relation $0 < E < \infty$.

For an energy signal $P=0$.

The average power of a discrete time signal $x(n)$ is defined as

$$Power = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N |x(n)|^2$$

A signal $x(n)$ is called a power signal if and only if the power satisfies $0 < P < \infty$.

For a power signal $E=\infty$.

7. Compare FIR and IIR systems. (Dec 2017, Dec 2018)

FIR system: This type of system has an impulse response which is zero outside a finite time interval. Example: $h(n) = 0$, for $n < 0$ and $n \geq N$

IIR System: An IIR system exhibits an impulse response of an infinite duration.

8. Distinguish between stationary and non-stationary signals with an example. (May 2018)

- (i) A stationary signal is denoted by a sine-wave equation, which has a constant time period,

whereas a non-stationary signal would have a sine wave with a constantly changing time period.

- (ii) The frequency for a sine-wave equation remains constant whereas the frequency in the non-stationary signal varies with time.

- (iii) The spectral content for a stationary signal in sine wave equation is constant, while in case of a non-stationary equation is ever-changing with respect to time.

- (iv) Examples for stationary signals include white noise, single tone sine-wave with constant frequency and multitone sinewave with a constant frequency whereas Non-stationary signal examples include Speech signals and multitone sine wave with varied frequency.

9. What is Nyquist rate? (May 2018)

The frequency $2f_m$, which under sampling theorem must be exceeded by the sampling frequency is known as Nyquist rate.

10. What is meant by aliasing effect? (Dec 2018)

Let us consider a band limited signal $x(t)$ having no frequency component for $|\Omega| > \Omega_m$. If we sample the signal $x(t)$ with a sampling frequency $F < 2f_m$, the periodic continuation of $X(j\Omega)$ results in spectral overlap. In this case, the spectrum $X(j\Omega)$ cannot be recovered using a low pass filter. This effect is known as aliasing effect.

11. Define stable and unstable systems. (Dec 2019)

Any relaxed system is said to be bounded input bounded output (BIBO) stable if and only if every bounded input yields a bounded output. The necessary and sufficient condition for stability is $\sum_{n=-\infty}^{\infty} |h(n)| < \infty$. If the condition is not satisfied then the system is said to be unstable.

Part – B

1. Explain in detail about Reconstruction of analog signal from its samples. (May 2016)

From sampling theorem it is clear that if we sample band limited analog signal $x_a(t)$ above its Nyquist rate, then we can reconstruct $x_a(t)$ from its samples $x(n)$.

We can write Eq. as

$$TX(e^{j\Omega T}) = \sum_{k=-\infty}^{\infty} X_a \left(j\Omega + j\frac{2\pi k}{T} \right)$$

The above equation is a periodic extension of $X_a(j\Omega)$ and we know that, if $F \geq 2f_m$, we can recover $X_a(j\Omega)$ by passing $TX(e^{j\Omega T})$ through a lowpass filter with frequency response.

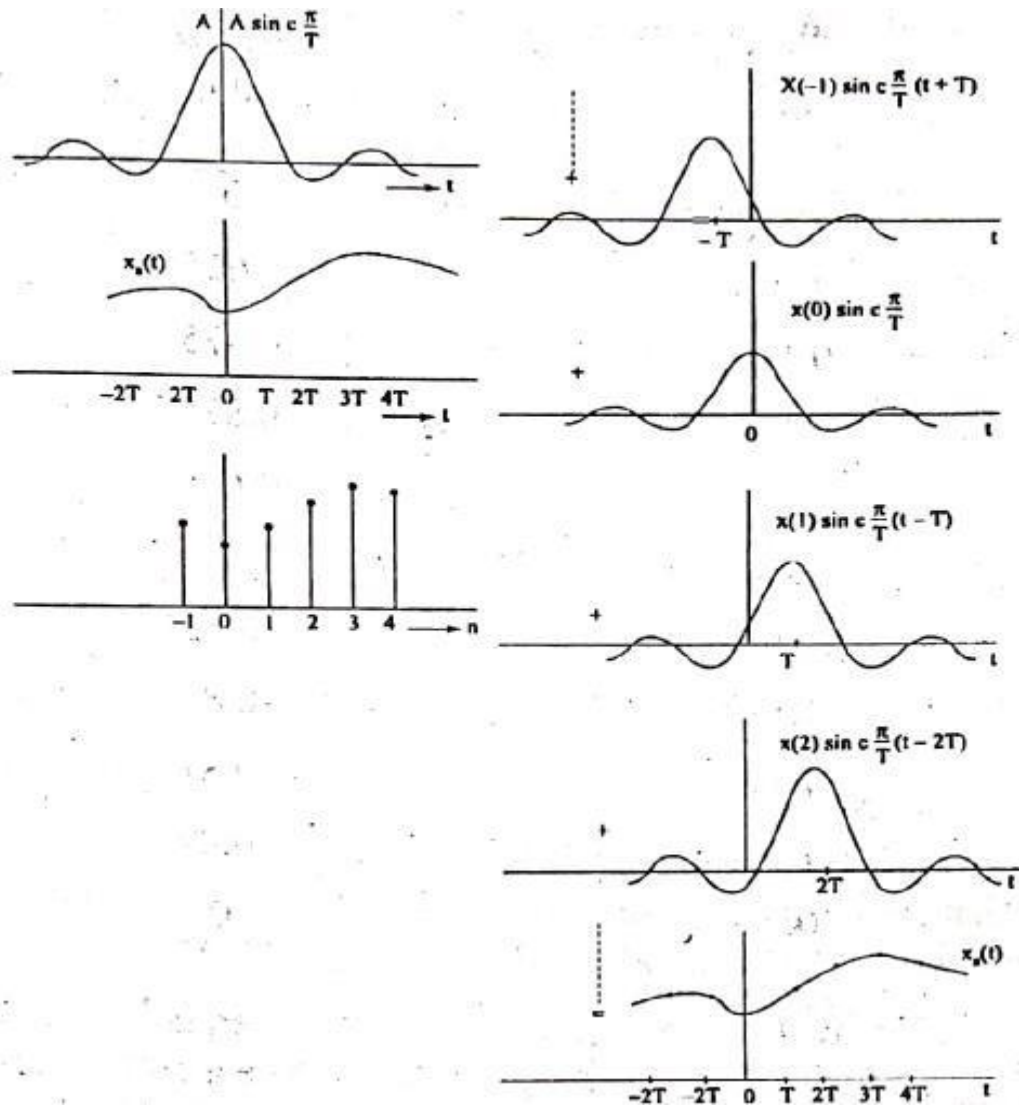
$$H(j\Omega) = \begin{cases} 1 & \text{for } |\Omega| \leq \frac{\Omega_s}{2} \\ 0 & \text{for } |\Omega| > \frac{\Omega_s}{2} \end{cases}$$

Now we can write

$$X_a(j\Omega) = H(j\Omega)TX(e^{j\Omega T})$$

Replace $X(e^{j\Omega T})$ in Eq. by its equivalent form i.e.,

$$X(e^{j\Omega T}) = \sum_{n=-\infty}^{\infty} x(nT)e^{-j\omega nT}$$



Reconstruction of band-limited signals from its samples

$$\text{then } X(j\Omega) = H(j\Omega)T \sum_{n=-\infty}^{\infty} x(nT)e^{-j\omega nT}$$

$$x_a(t) = F^{-1}[X_a(j\Omega)]$$

$$= T \sum_{n=-\infty}^{\infty} x(nT)F^{-1}\{H(j\Omega)e^{-j\omega nT}\}$$

$$F^{-1}[H(j\Omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(j\Omega)e^{j\Omega t} d\Omega$$

$$= \frac{1}{2\pi} \int_{-\Omega_s/2}^{\Omega_s/2} e^{j\Omega t} d\Omega = \frac{1}{\pi t} \sin \frac{\Omega_s t}{2}$$

From the time shifting property

$$F^{-1} [H(j\Omega)e^{-j\omega nT}] = \frac{1}{\pi} \frac{\sin \frac{\pi}{T}(t - nT)}{(t - nT)}$$

$$T \cdot F^{-1} [H(j\Omega)e^{-j\omega nT}] = \frac{\sin \frac{\pi}{T}(t - nT)}{\frac{\pi}{T}(t - nT)}$$

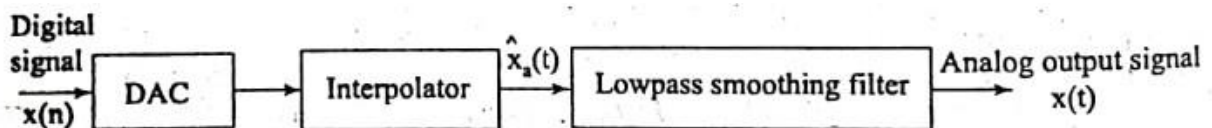
$$x_a(t) = \sum_{n=-\infty}^{\infty} x(nT) \frac{\sin \frac{\pi}{T}(t - nT)}{(t - nT)}$$

$$= \sum_{n=-\infty}^{\infty} x(nT) \operatorname{sinc} \frac{\pi}{T}(t - nT)$$

$$\therefore \operatorname{sinc} cx = \frac{\sin \pi x}{\pi x}$$

Notice that the response extends from $t = -\infty$ to $t = \infty$. This implies that there is a response for $t < 0$ to a unit impulse applied at $t = 0$. Hence such an ideal filter is physically not realizable.

In practice, we need a different approach to convert digital signal to analog signal. One such approach is shown in Fig. Here the digital to analog converter produces an output voltage that is proportional to the value of the binary word. The output of DAC is applied to an interpolator which converts the output samples of



Digital to Analog conversion

DAC into an analog-signal.

2. Describe in detail about the representation and classification of discrete signals.

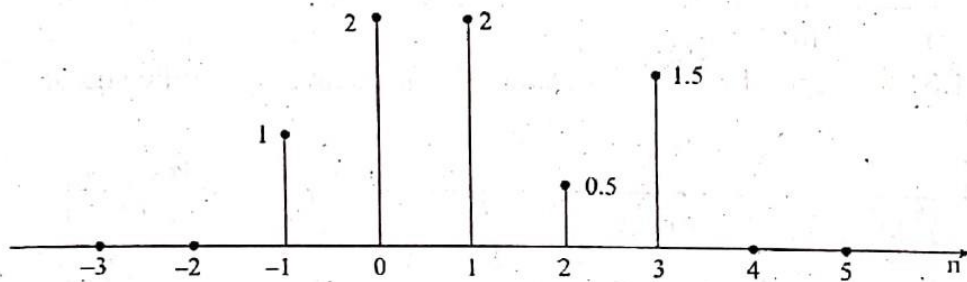
(May 2016)

There are different types of representations for discrete-time signals. They are

1. Graphical representation
2. Functional representation
3. Tabular representation
4. Sequence representation

Graphical representation

Let us consider a signal $x(n]$ with values $x(-1) = 1; x(0) = 2; x(1) = 2; x(2) = 0.5$ and $x(3) = 1.5$. This discrete-time signal can be represented graphically as shown in Fig.



Graphical representation of a discrete-time signal.

Functional representation

The discrete-time signal shown in Fig. can be represented using functional representation as below.

$$x(n) = \begin{cases} 1 & \text{for } n = -1 \\ 2 & \text{for } n = 0, 1 \\ 0.5 & \text{for } n = 2 \\ 1.5 & \text{for } n = 3 \\ 0 & \text{otherwise} \end{cases}$$

Tabular representation

The discrete-time signal can also be represented as

n	-1	0	1	2	3
$x(n)$	1	2	2	0.5	1.5

Sequence representation

A finite duration sequence with time origin ($n = 0$) indicated by the symbol \uparrow is represented as

$$x(n) = \{1, 2, 2, 0.5, 1.5\}$$

\uparrow

An infinite duration sequence can be represented as

$$x(n) = \{\dots 0.2, 1, -1, 3, 2 \dots\}$$

\uparrow

A finite duration sequence that satisfies the condition $x(n) = 0$ for $n < 0$ can be represented as

$$x(n) = \{2, 4, 6, 8, -3\}$$

Classification of Discrete Time Signals:**(a) Energy Signals and Power Signals:**

The energy of a discrete time signal $x(n)$ is defined as

$$Energy = E \sum_{n=-\infty}^{\infty} |x(n)|^2$$

A signal $x(n)$ is called an energy signal if and only if the energy obeys the relation $0 < E < \infty$.

For an energy signal $P=0$.

The average power of a discrete time signal $x(n)$ is defined as

$$Power = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N |x(n)|^2$$

A signal $x(n)$ is called a power signal if and only if the power satisfies $0 < P < \infty$.

For a power signal $E=\infty$.

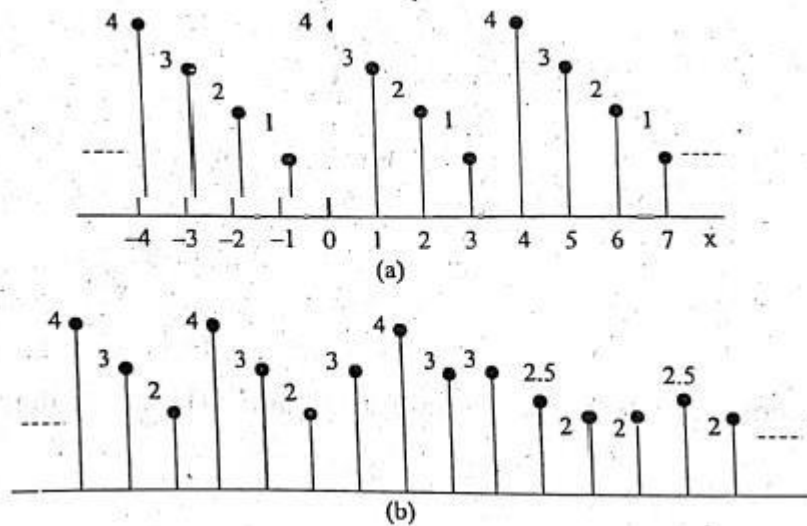
(b) Periodic and aperiodic signals:

A discrete-time signal $x(n)$ is said to be periodic with period N if and only if

$$x(N + n) = x(n) \text{ for all } n$$

The smallest value of N for which Eq. holds is known as fundamental period. If Eq. does not satisfy even for one value of n , then the discrete-time signal is aperiodic. An example for periodic and aperiodic signals are shown in Fig. The periodic sequence shown in Fig. is repeating for every 4 samples.

Therefore the period of the signal $N=4$.



In section we studied that all continuous-time sinusoidal signals with fundamental frequency Ω are periodic. In contrast, all the discrete-time sinusoidal sequences are not periodic. Consider a discrete-time signal given by

$$x(n) = A \sin(\omega_0 n + \theta)$$

where A is amplitude, ω_0 and θ are frequency and phase shift respectively. The unit of ω_0 and θ are radians per sample and radians respectively. A discrete-time sequence is periodic if it satisfies the condition

$$x(n + N) = x(n)$$

From Eq.

$$\begin{aligned} x(n + N) &= A \sin(\omega_0(n + N) + \theta) \\ &= A \sin(\omega_0 n + \omega_0 N + \theta) \end{aligned}$$

For $x(n)$, Eq. satisfies if and only if $\omega_0 N$ is an integer multiple of 2π , i.e.,

$$\omega_0 N = 2\pi m$$

from which

$$\omega_0 = 2\pi \left(\frac{m}{N} \right)$$

or

$$N = 2\pi \left(\frac{m}{\omega_0} \right)$$

(c) Symmetric (Even) and Asymmetric (Odd) signals:

A discrete-time signal $x(n)$ is said to be a symmetric (even) signal if it satisfies the condition

$$x(-n) = x(n) \quad \text{for all } n.$$

Example: $x(n) = \cos \omega n$

The signal is said to be an odd signal if it satisfies the condition

$$x(-n) = -x(n) \quad \text{for all } n.$$

Example: $A \sin \omega n$

If $x(n)$ is odd then $x(0) = 0$

A signal $x(n)$ can be expressed as the sum of even and odd components. That is,

$$x(n) = x_e(n) + x_o(n)$$

where $x_e(n)$ is even component of the signal and $x_o(n)$ is odd component of the signal.

Replace n by $-n$ in Eq. and we get

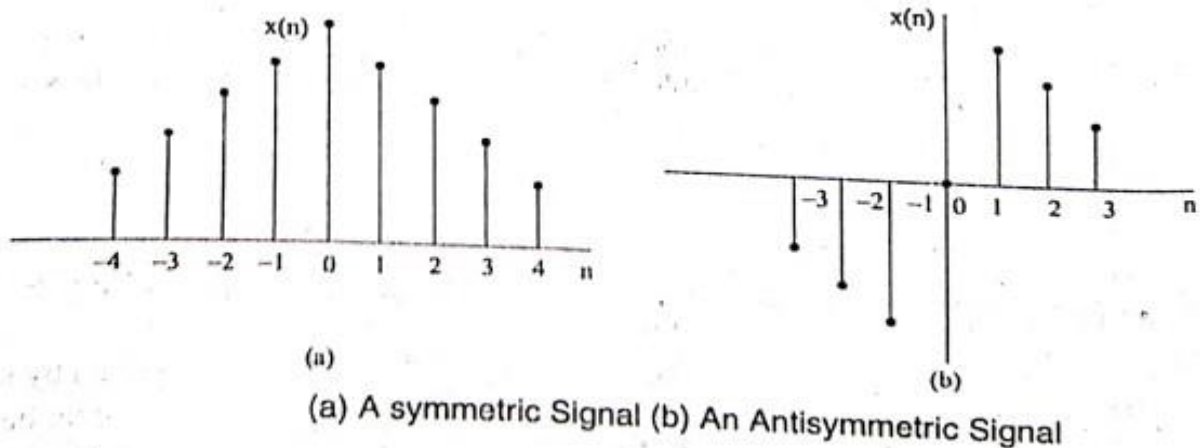
$$x(-n) = x_e(-n) + x_o(-n) = x_e(n) - x_o(n)$$

Adding Eq. yields

$$\begin{aligned} 2x_e(n) &= x(n) + x(-n) \\ \Rightarrow x_e(n) &= \frac{1}{2}[x(n) + x(-n)] \end{aligned}$$

Similarly, we can get

$$x_o(n) = \frac{1}{2}[x(n) - x(-n)]$$



(d) Causal and Non-causal signals:

A signal $x(n]$ is said to be causal if its value is zero for $n < 0$. Otherwise the signal is noncausal.

Examples for causal signals:

$$x_1(n) = a^n u(n)$$

$$x_2(n) = \{1, 2, -3, -1, 2\}$$

Examples for noncausal signals:

$$x_1(n) = a^n u(-n + 1)$$

$$x_2(n) = \{1, -2, 1, 4, 3\}$$

A signal that is zero for all $n \geq 0$ is called an anticausal signal.

3. Check whether the following systems are linear or not. (Dec 2016)

(a) $y(n) = x^2(n)$

Given $y(n) = x^2(n)$

$$y(n) = T[x(n)] = x^2(n)$$

For an input $x_1(n)$, the output $y_1(n)$ is

$$y_1(n) = T[x_1(n)] = x_1^2(n)$$

For an input $x_2(n)$, the output $y_2(n)$ is

$$y_2(n) = T[x_2(n)] = x_2^2(n)$$

The output due to weighted sum of inputs is

$$y_3(n) = T[ax_1(n) + bx_2(n)] = [ax_1(n) + bx_2(n)]^2$$

The weighted sum of outputs is given by

$$\begin{aligned} ay_1(n) + by_2(n) &= ax_1^2(n) + bx_2^2(n) \\ \Rightarrow y_3(n) &= ax_1^2(n) + bx_2^2(n) \end{aligned}$$

Therefore, the system is nonlinear.

$$(b) \quad y(n) = nx(n)$$

Given $y(n) = nx(n)$

$$y(n) = T[x(n)] = nx(n)$$

For an input $x_1(n)$, the output $y_1(n)$ is

$$y_1(n) = T[x_1(n)] = nx_1(n)$$

For an input $x_2(n)$, the output $y_2(n)$ is

$$y_2(n) = T[x_2(n)] = nx_2(n)$$

The output due to weighted sum of inputs is

$$y_3(n) = T[ax_1(n) + bx_2(n)] = n[ax_1(n) + bx_2(n)] = anx_1(n) + bnx_2(n)$$

The weighted sum of outputs is given by

$$\begin{aligned} ay_1(n) + by_2(n) &= anx_1(n) + bnx_2(n) \\ \Rightarrow y_3(n) &= ay_1(n) + by_2(n) \end{aligned}$$

Therefore, the system is linear.

4. Determine and sketch the magnitude and phase response of the following systems. (Dec

2016, May 2017, May 2019)

$$(a) \quad y(n) = \frac{1}{3}[x(n) + x(n-1) + x(n-2)]$$

Given,

$$y(n) = \frac{1}{3}[x(n) + x(n-1) + x(n-2)]$$

Taking Fourier Transform on both sides, we have

$$y(e^{j\omega}) = \frac{1}{3}[X(e^{j\omega}) + e^{-j\omega} X(e^{j\omega}) + e^{-j2\omega} X(e^{j\omega})]$$

$$y(e^{j\omega}) = X(e^{j\omega}) \frac{[1 + e^{-j\omega} + e^{-j2\omega}]}{3}$$

The frequency response

$$\begin{aligned} H(e^{j\omega}) &= \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1 + e^{-j\omega} + e^{-j2\omega}}{3} \\ &= e^{-j\omega} \frac{[e^{j\omega} + 1 + e^{-j\omega}]}{3} \\ &= e^{-j\omega} \frac{[1 + 2\cos\omega]}{3} \end{aligned}$$

The magnitude response

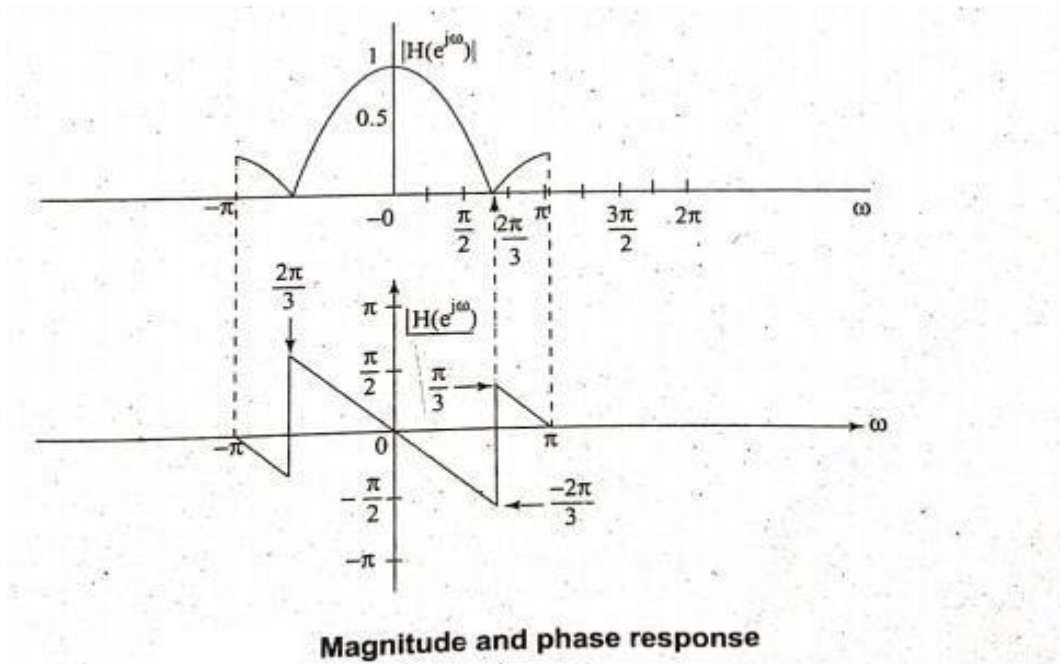
$$|H(e^{j\omega})| = \frac{|1 + 2\cos\omega|}{3} \quad (|e^{-j\omega}| = 1)$$

$$\text{Angle of } H(e^{j\omega}) = -\omega \text{ for } H(e^{j\omega}) > 0$$

$$= -\omega + \pi \text{ for } H(e^{j\omega}) < 0$$

ω	0	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	π
$H(e^{j\omega})$	1	0.8047	0.667	0.333	0	-0.138	-0.333
$ H(e^{j\omega}) $	1	0.8047	0.667	0.333	0	0.138	0.333
$\angle H(e^{j\omega})$	0	$-\frac{\pi}{4}$	$-\frac{\pi}{3}$	$-\frac{\pi}{2}$	$-\frac{2\pi}{3}$	$\frac{\pi}{4}$	0

The sketch of magnitude and phase response are shown in Fig.



$$(b) \ y(n) = \frac{1}{2}[x(n) + x(n-1)]$$

Given,

$$y(n) = \frac{1}{2}[x(n) + x(n-1)]$$

Taking Fourier Transform on both sides, we have

$$Y(e^{j\omega}) = \frac{1}{2}[X(e^{j\omega}) + e^{-j\omega} X(e^{j\omega})]$$

$$Y(e^{j\omega}) = X(e^{j\omega}) \frac{[1 + e^{-j\omega}]}{2}$$

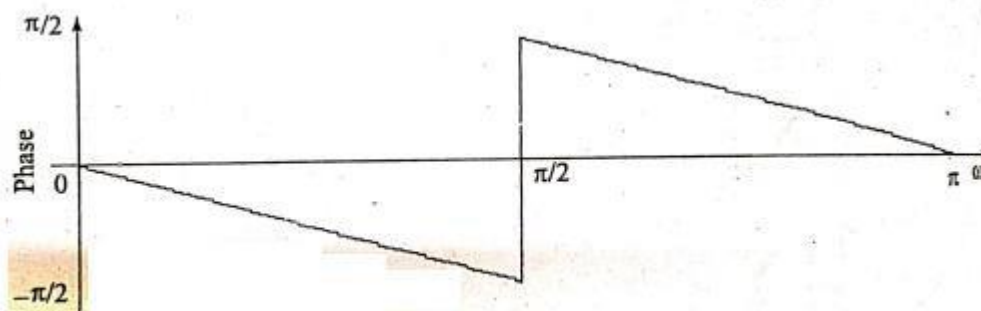
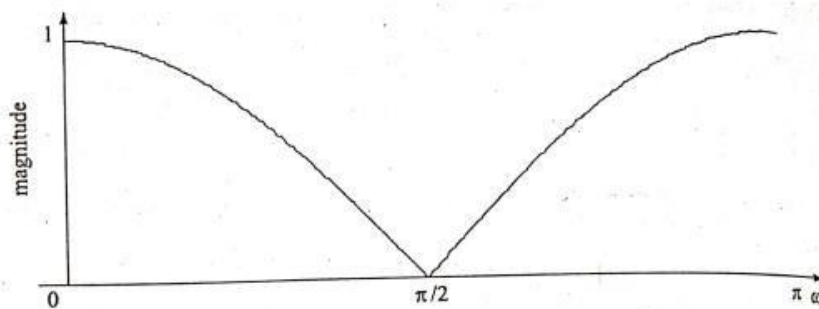
$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{[1 + e^{-j\omega}]}{2} = \frac{1 + \cos\omega - j\sin\omega}{2}$$

We know $|H(e^{j\omega})|$ is symmetric and angle of $H(e^{j\omega})$ is anti-symmetric with a period of 2π in ω . We only need to know $H(e^{j\omega})$ over the interval $0 \leq \omega \leq \pi$.

$$\begin{aligned} H(j\omega) &= \frac{1}{2} \sqrt{(1 + \cos\omega)^2 + \sin^2 \omega} = \frac{1}{2} \sqrt{1 + \cos^2 \omega + 2\cos\omega + \sin^2 \omega} \\ &= \frac{1}{2} \sqrt{2(1 + \cos\omega)} = \frac{1}{2} \sqrt{2 \cdot 2 \cos^2 \omega} = \cos\omega \end{aligned}$$

$$\text{Angle of } H(e^{j\omega}) = \tan^{-1} \left(\frac{-\sin \omega}{1 + \cos \omega} \right) = -\omega$$

ω	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	π
$H(e^{j\omega})$	1	0.812	0.707	0.5	0	-0.5	-0.707	-1
$ H(e^{j\omega}) $	1	0.812	0.707	0.5	0	0.5	0.707	1
$\angle H(e^{j\omega})$	0	$-\pi/6$	$-\pi/4$	$-\pi/3$	$-\pi/2$	$\pi/3$	$\pi/4$	0°



Magnitude response & phase response of the example

$$(c) \quad y(n) - \frac{1}{2}y(n-1) = x(n)$$

Given,

$$y(n) - \frac{1}{2}y(n-1) = x(n)$$

Taking Fourier Transform on both sides, we have

$$Y(e^{j\omega}) - \frac{1}{2}[e^{-j\omega}Y(e^{j\omega})] = X(e^{j\omega})$$

$$Y(e^{j\omega})[1 - \frac{e^{-j\omega}}{2}] = X(e^{j\omega})$$

The frequency response

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{2}{2 - e^{-j\omega}}$$

5. For each impulse response determines the system is (a) stable (b) causal. (May 2017, May 2019)

$$(i) \quad h(n) = \sin \left(\frac{n\pi}{2} \right)$$

$h(n) \neq 0$ for $n < 0$. So the system is non-causal.

For stability

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty = \sum_{n=-\infty}^{\infty} \left| \sin \frac{n\pi}{2} \right|$$

For odd values of n the value, $\left| \sin \frac{n\pi}{2} \right| = 1$

$$\sum_{n=-\infty}^{\infty} |h(n)| = \infty$$

The system is unstable.

$$(ii) \quad h(n) = \delta(n) + \sin n\pi$$

For all integer values of n , $\sin n\pi = 0$.

The system is causal as $h(n) = 0$ for $n < 0$.

For the system to be stable it has to satisfy the condition

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty.$$

In this case

$$\begin{aligned} \sum_{n=-\infty}^{\infty} |h(n)| &= |h(-\infty)| + \dots + |h(-1)| + |h(0)| + |h(1)| + \dots + |h(\infty)| \\ &= 0 + \dots + 0 + 1 + 0 + \dots + 0 \\ &= 1 < \infty \end{aligned}$$

Therefore, the system is stable.

$$(iii) \quad h(n) = 2^n u(-n)$$

$h(n) \neq 0$ for $n < 0$. So the system is non-causal.

For stability

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

$$\sum_{n=-\infty}^{\infty} 2^n u(-n) = \sum_{n=-\infty}^0 2^n = \sum_{n=0}^{\infty} 2^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

$$= 1 + \frac{1}{2} + \frac{1}{2^2} + \dots \infty$$

$$= \frac{1}{1 - \frac{1}{2}} = 2 < \infty$$

So the system is stable.

6. (a) Compute the convolution of the signals $x(n) = \{1, 2, 3, 4, 5, 3, -1, -2\}$ and $h(n) = \{3, 2, 1, 4\}$ using tabulation method. (Dec 2017)

$x(n)/h(n)$	1	2	3	4	5	3	-1	-2
3	3	6	9	12	15	9	-3	-6
2	2	4	3	8	10	6	-2	-4
1	1	2	3	4	5	3	-1	-2
4	4	8	12	16	20	12	-4	-8

$$y(n) = x(n) * h(n)$$

Therefore, $y(n) = \{3, 8, 14, 21, 34, 35, 24, 15, 7, -6, -8\}$.

(b) Check whether the following systems are static or dynamic, linear or non-linear, time variant or invariant, Causal or non-causal, stable or unstable. (Dec 2017)

(i) $y(n) = \cos[x(n)]$

(a) The output at any instant depends on the input at that instant. Therefore, the system is static.

(b) For an input $x_1(n)$

$$y_1(n) = \cos[x_1(n)]$$

For an input $x_2(n)$

$$y_2(n) = \cos[x_2(n)]$$

The output due to weighted sum of inputs is

$$\begin{aligned} y_3(n) &= T[ax_1(n) + bx_2(n)] \\ &= \cos[ax_1(n) + bx_2(n)] \end{aligned}$$

The weighted sum of outputs is given by

$$\begin{aligned} ay_1(n) + by_2(n) &= a \cos[x_1(n)] + b \cos[x_2(n)] \\ y_3(n) &\neq ay_1(n) + by_2(n) \end{aligned}$$

Hence the system is non-linear.

(c) The output depends on present inputs only. Hence the system is causal.

(d) Given $y(n) = T[x(n)] = \cos[x(n)]$

The output due to delayed input

$$y(n, k) = T[x(n - k)] = \cos[x(n - k)]$$

The delayed output is

$$\begin{aligned} y(n - k) &= \cos[x(n - k)] \\ y(n, k) &= y(n - k) \end{aligned}$$

Therefore, the system is time invariant.

(e) $y(n) = \cos[x(n)]$

If the input $x(n)$ is bounded, $|x(n)| < \infty$

Then $|y(n) = \cos[x(n)]| < \infty$ is also bounded. Hence the system is stable.

The system is static, non-linear, causal, time-invariant and stable.

(ii) $y(n) = x(2n)$

(a) The output at any instant not depending on the input at that instant. Therefore, the system is dynamic.

(b) For an input $x_1(n)$

$$y_1(n) = x_1(2n)$$

For an input $x_2(n)$

$$y_2(n) = x_2(2n)$$

The output due to weighted sum of inputs is

$$\begin{aligned} y_3(n) &= T[ax_1(n) + bx_2(n)] \\ &= ax_1(2n) + bx_2(2n) \end{aligned}$$

The weighted sum of outputs is given by

$$\begin{aligned} ay_1(n) + by_2(n) &= ax_1(2n) + bx_2(2n) \\ y_3(n) &= ay_1(n) + by_2(n) \end{aligned}$$

Hence the system is linear.

(c) $y(n) = x(2n)$

$$\text{For } n = -1, y(-1) = x(-2)$$

$$n = 0, y(0) = x(0)$$

$$n = 1, y(1) = x(2)$$

For positive values of n the output $y(n)$ depends on future inputs. Hence the system is non-causal

(d) Given $y(n) = T[x(n)] = x(2n)$

The output due to delayed input

$$y(n, k) = T[x(n - k)] = x[2n - k]$$

The delayed output is

$$y(n-k) = x(2(n-k)) = x(2n-2k)$$

$$y(n,k) \neq y(n-k)$$

Therefore, the system is time variant.

(e) $y(n) = x(2n)$

If the input $x(n)$ is bounded, $|x(n)| < \infty$

Then $|y(n) = x(2n)| < \infty$ is also bounded. Hence the system is stable.

The system is dynamic, linear, non-causal, time-variant and stable.

(iii) $y(n) = x(-n+2)$

(a) The output at any instant not depending on the input at that instant. Therefore, the system is dynamic.

(b) For an input $x_1(n)$

$$y_1(n) = x_1(-n+2)$$

For an input $x_2(n)$

$$y_2(n) = x_2(-n+2)$$

The output due to weighted sum of inputs is

$$y_3(n) = T[ax_1(n) + bx_2(n)]$$

$$= ax_1(-n+2) + bx_2(-n+2)$$

The weighted sum of outputs is given by

$$ay_1(n) + by_2(n) = ax_1(-n+2) + bx_2(-n+2)$$

$$y_3(n) = ay_1(n) + by_2(n)$$

Hence the system is linear.

(c) $y(n) = x(-n+2)$

For $n = -1$, $y(-1) = x(3)$

$n = 0$, $y(0) = x(2)$

$n = 1$, $y(1) = x(1)$

For positive values of n the output $y(n)$ depends on future inputs. Hence the system is non-causal

(d) Given $y(n) = T[x(n)] = x(-n + 2)$

The output due to delayed input

$$y(n, k) = T[x(n - k)] = x[-n - k + 2]$$

The delayed output is

$$y(n - k) = x(-(n - k) + 2) = x(-n + k + 2)$$

$$y(n, k) \neq y(n - k)$$

Therefore, the system is time variant.

(e) $y(n) = x(-n + 2)$

If the input $x(n)$ is bounded, $|x(n)| < \infty$

Then $|y(n) = x(-n + 2)| < \infty$ is also bounded. Hence the system is stable.

The system is dynamic, linear, non-causal, time-variant and stable.

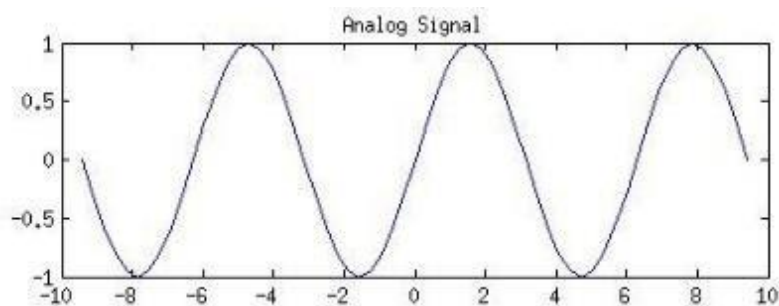
7. (a) Explain the classification of signals. (Dec 2017)

Signal is defined as any physical quantity that varies with time, space or any other independent variable.

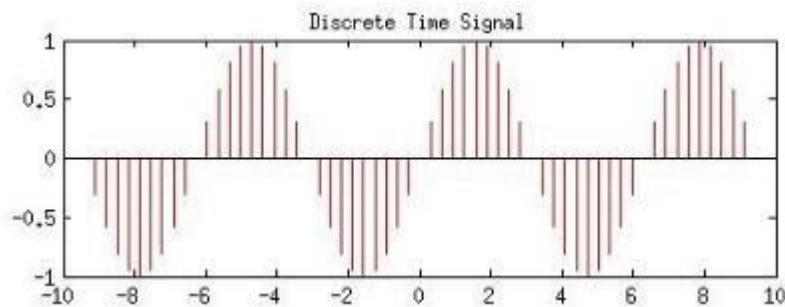
Basically the signals classified into,

- (i) Continuous time signals
- (ii) Discrete time signals
- (iii) Digital signals

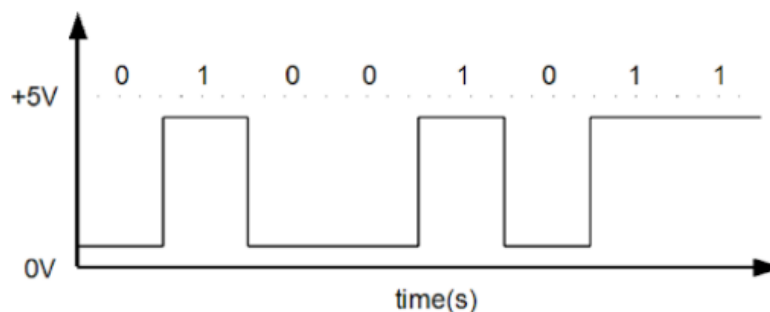
Continuous time signals: The signals that are defined at all instants of time are known as continuous signals and are denoted by $x(t)$



Discrete time signals: The signals that are defined at discrete instants of time are known as discrete signals and are denoted by $x(n)$.



Digital signals: The signals those are discrete in time and quantized in amplitude are digital signals.

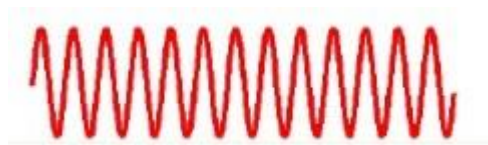


Classification of signals:

The signals are classified according to their characteristics. Some of them are

(1) Deterministic and random signals:

Deterministic signal is a signal exhibiting no uncertainty of value at any given instant of time. Its instantaneous value can be accurately predicted by mathematical equation. Ex: $A \sin \Omega t$



Random signal is a signal characterized by uncertainty before its actual occurrence. Ex: Noise signal.

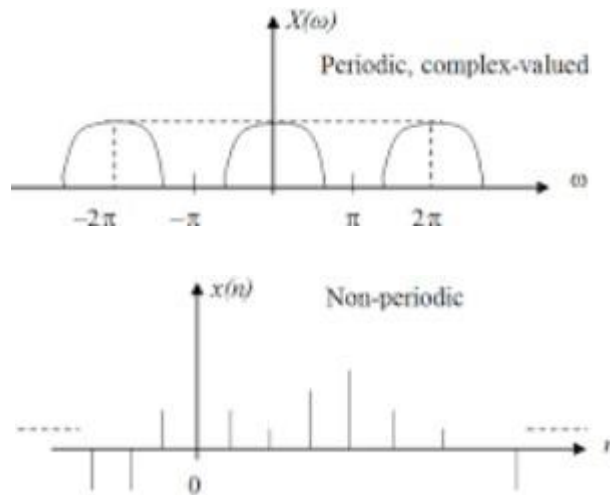


(2) Periodic and aperiodic signals:

A continuous time signal $x(t)$ is said to be periodic if it satisfies the condition

$$x(t + T) = x(t) \text{ for all } t$$

The smallest value of T satisfies the above condition is known as fundamental period. A signal is said to be aperiodic if the condition is not satisfied for atleast one value of t .

**(3) Symmetric (Even) and Anti-symmetric (Odd) signals:**

A continuous time signal $x(t)$ is said to be symmetric (even) signal if it satisfies the condition

$$x(-t) = x(t) \text{ for all } t$$

Example of a symmetric signal is $x(t) = A \cos t$

A continuous time signal $x(t)$ is said to be antisymmetric (odd) signal if it satisfies the condition

$$x(-t) = -x(t) \text{ for all } t$$

Example of a symmetric signal is $x(t) = A \sin t$

(4) Energy and Power signals:

The energy of a discrete time signal $x(n)$ is defined as

$$\text{Energy} = E \sum_{n=-\infty}^{\infty} |x(n)|^2$$

A signal $x(n)$ is called an energy signal if and only if the energy obeys the relation $0 < E < \infty$.

For an energy signal $P=0$.

The average power of a discrete time signal $x(n)$ is defined as

$$Power = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N |x(n)|^2$$

A signal $x(n)$ is called a power signal if and only if the power satisfies $0 < P < \infty$.

For a power signal $E=\infty$.

(5) Causal and non-causal signals:

A continuous time signal $x(t)$ is said to be causal if $x(t) = 0$ for $t < 0$, otherwise the signal is noncausal. For an anticausal signals $x(t) = 0$ $t > 0$

(b) Find the normalized energy and power of the signal. (Dec 2017)

(i) $x(t) = e^{-10t}u(t)$

Given $x(t) = e^{-10t}u(t)$

The energy
$$E = \lim_{T \rightarrow \infty} \int_{-T}^T |x(t)|^2 dt$$

$$E = \lim_{T \rightarrow \infty} \int_{-T}^T (e^{-10t})^2 dt$$

$$E = \lim_{T \rightarrow \infty} \int_{-T}^T (e^{-20t}) dt = -\frac{1}{20}[-1] = \frac{1}{20} \text{ Joules}$$

Power
$$P = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(t)|^2 dt$$

$$P = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T (e^{-10t})^2 dt = \frac{1}{\infty} = 0$$

Energy of the signal is finite and power is zero. Therefore the signal is energy signal.

(ii) $x(t) = e^{-at}u(t), a > 0$

Given $x(t) = e^{-at}u(t)$

The energy
$$E = \lim_{T \rightarrow \infty} \int_{-T}^T |x(t)|^2 dt$$

$$E = \lim_{T \rightarrow \infty} \int_{-T}^T (e^{-at})^2 dt$$

$$E = \lim_{T \rightarrow \infty} \int_{-T}^T (e^{-2at}) dt = -\frac{1}{2a} [e^{-2aT} - e^{2aT}]$$

Power

$$P = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(t)|^2 dt$$

$$P = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T (e^{-at})^2 dt = \frac{1}{\infty} = 0$$

Therefore the signal is neither energy nor power signal.

8. Find the convolution of the two signals. (May 2018)

$$(i) \quad x(n) = \left(\frac{1}{3}\right)^{-n} u(-n-1); h(n) = u(n-1)$$

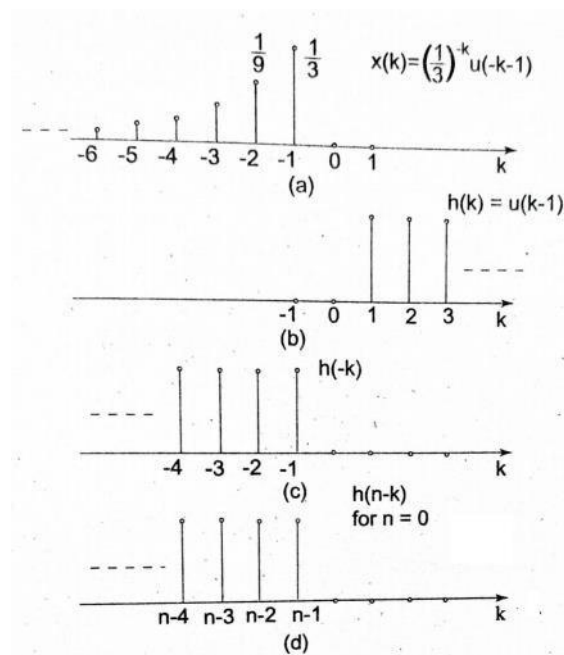
$$\text{Given } x(n) = \left(\frac{1}{3}\right)^{-n} u(-n-1)$$

$$h(n) = u(n-1)$$

$$y(n) = x(n) * h(n)$$

Let

$$= \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$



From fig we can find that for $n-1 \leq -1$

$$x(k) = 0 \text{ for } k > -1$$

$$h(n-k) = 0 \text{ for } k > n-1$$

For $n-1 < -1$, the interval on summation is from $k = -\infty$ to $n-1$

$$y(n) = \sum_{k=-\infty}^{n-1} 3^k \text{ for } n-1 \leq -1 \text{ or for } n \leq 0$$

For $n-1 > -1$ the interval on summation is from $k = -\infty$ to -1

Therefore
$$y(n) = \sum_{k=-\infty}^{-1} 3^k \text{ for } n > 0$$

For $n \leq 0$

$$\begin{aligned} y(n) &= \sum_{k=-\infty}^{n-1} 3^k \\ y(n) &= 3^{n-1} + 3^{n-2} + \dots \\ &= 3^{n-1} \left[1 + \frac{1}{3} + \dots \right] \\ &= 3^{n-1} \left[\frac{1}{1 - \frac{1}{3}} \right] = 0.5(3)^n \end{aligned}$$

For $n > 0$

$$y(n) = \sum_{k=-\infty}^{-1} 3^k = \sum_{k=1}^{\infty} \left(\frac{1}{3} \right)^k = \frac{\frac{1}{3}}{1 - \frac{1}{3}} = 0.5$$

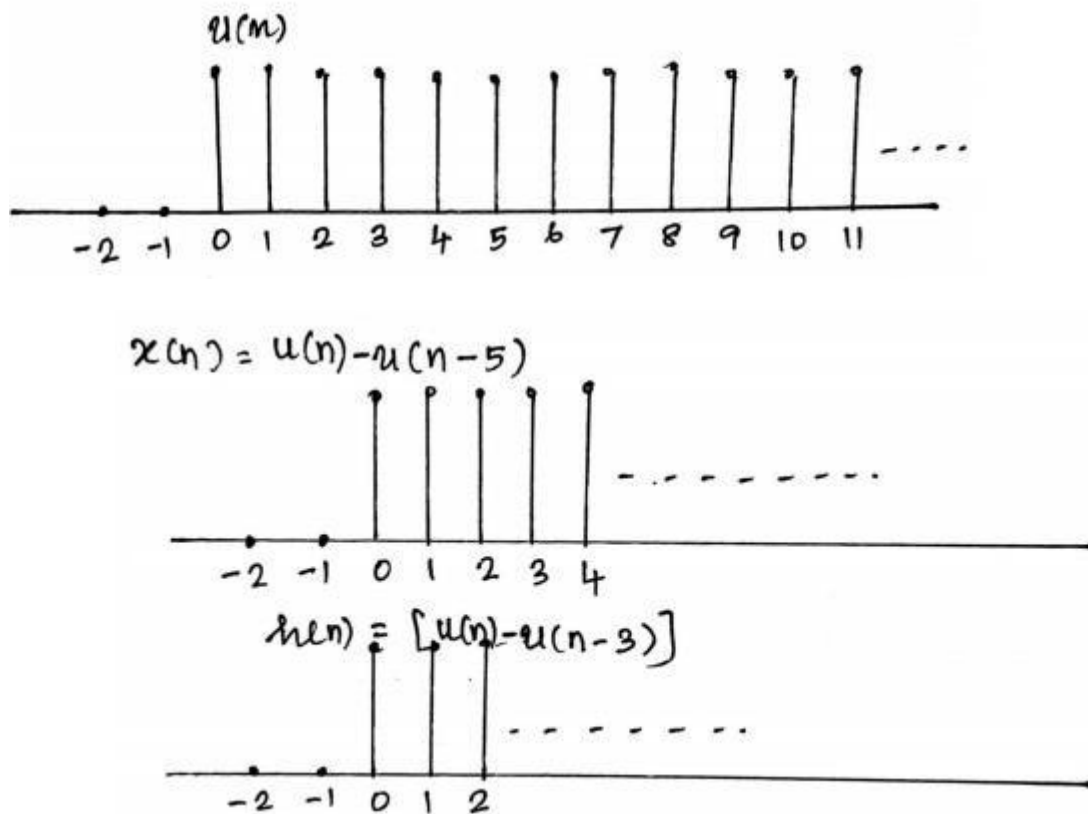
(ii) $x(n) = u(n) - u(n-5)$

$$h(n) = 2[u(n) - u(n-3)]$$

$$y(n) = x(n) * h(n)$$

Let

$$= \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$



From the fig, the $x(n)$ and $h(n)$ sequence are

$$x(n) = \{1, 1, 1, 1, 1\}$$

$$h(n) = \{2, 2, 2\} \quad (\because h(n) = 2[u(n) - u(n-3)])$$

$h(n)/x(n)$	1	1	1	1	1
2	2	2	2	2	2
2	2	2	2	2	2
2	2	2	2	2	2

$$y(n) = x(n) * h(n)$$

Therefore,

$$y(n) = \{2, 4, 6, 6, 6, 4, 2\}$$

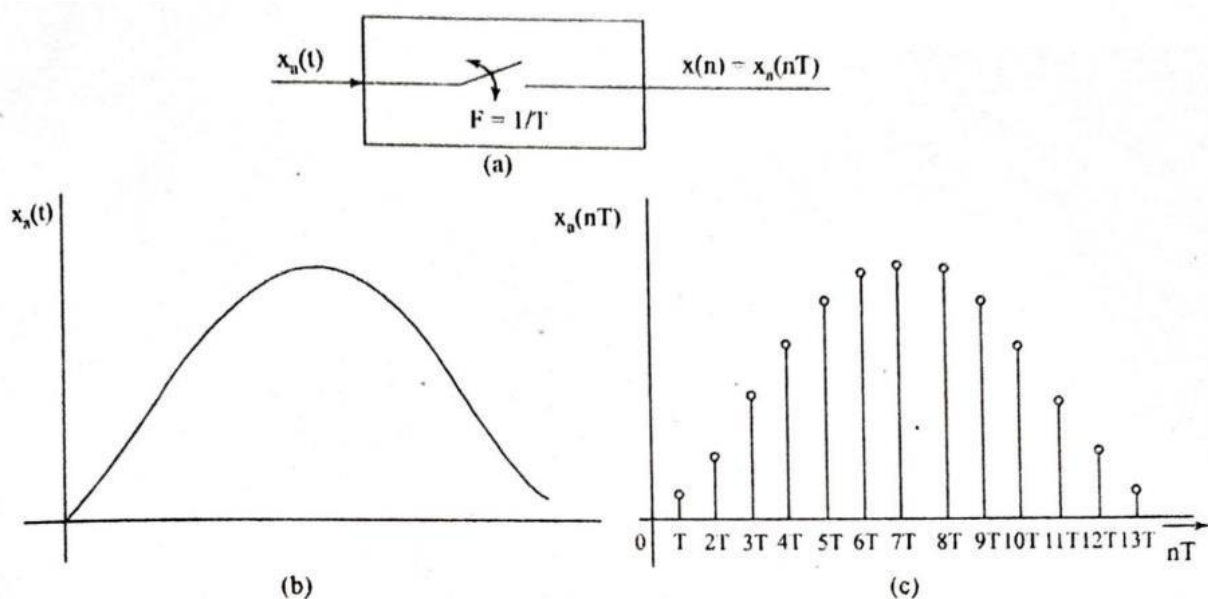
9. Discuss in detail about the following. (May 2018)

(a) Low pass sampling

Sampling is a process of converting a continuous time signal into discrete time signal. It is convenient to represent the sampling operation by a fictitious switch shown in fig. The switch closes for a very short interval of time τ , during which the signal presents at the output.

The time interval between successive samples is T seconds and the sampling frequency is given by $F = \frac{1}{T}$ hertz. The discrete time signal obtained by sampling is processed and then

converted into a continuous time signal. In this section, we consider the relation between the analog signal and the sampled discrete time in the time and frequency domains.

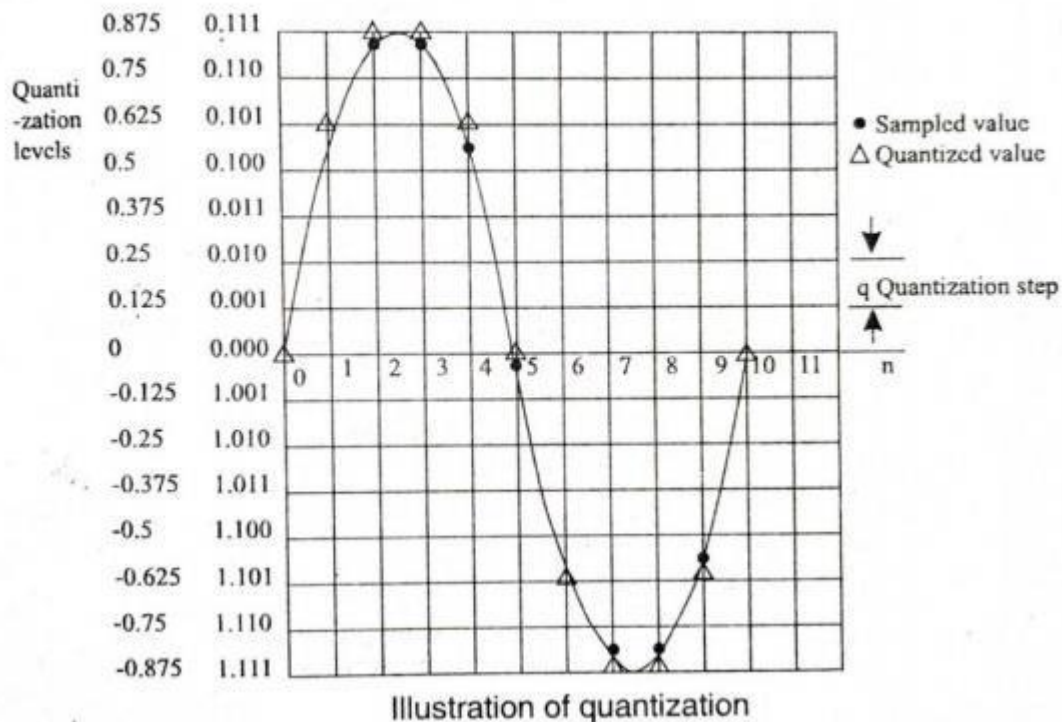


(b) Quantization error

The process of converting a discrete-time continuous amplitude signal $x(n)$ into a discrete-time discrete amplitude signal $x_q(n)$ is known as quantization. This is done by rounding off each sample in $x(n)$ to the nearest quantization level. Then each sample in $x_q(n)$ is represented by a finite number of digits using a coder. If a signal with amplitude range R is represented by an $b + 1$ bit word (including sign bit) then the number of values, or quantization levels, that can be represented is 2^{b+1} . The difference between adjacent levels, or the quantization step in terms of the range of the signal is

$$q = \frac{\text{range of signal}}{\text{Number of quantization levels}} = \frac{R}{2^{b+1}}$$

with fixed point representation (for details see section of fractional number. If the range of the signal exceeds ± 1 , it is necessary to scale the signal.

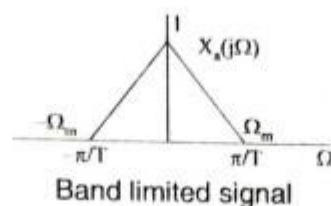


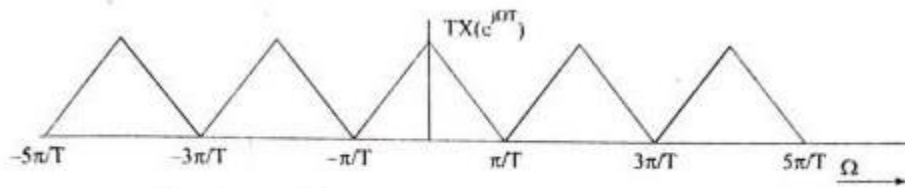
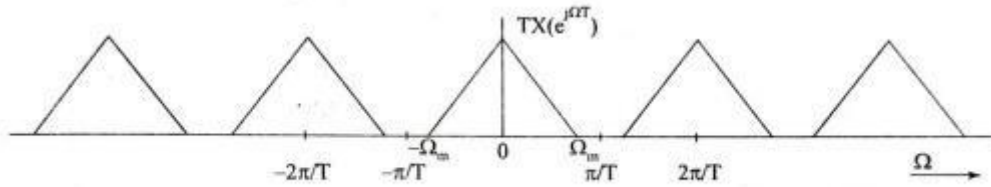
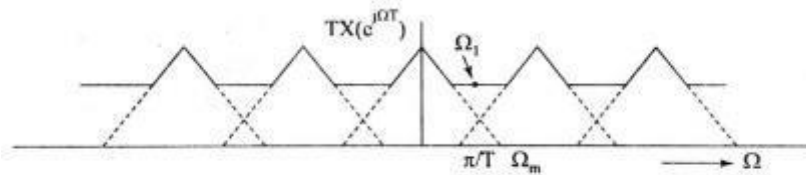
The process of quantization is shown in Fig. The time axis of the discrete-time signal is labelled with sample number ($n = 0, 1, 2 \dots$). Corresponding to different values of sample number n , the discrete time continuous amplitude signal is shown in Fig. We can represent the sample values by a sequence

$$x(n) = \{0, 0.620, 0.85, 0.85, 0.575, -0.03, -0.625, -0.85, -0.85, -0.575, 0\}$$

(c) Aliasing effect

Let us consider a band limited signal $x(t)$ having no frequency component for $|\Omega| > \Omega_m$. If we sample the signal $x(t)$ with a sampling frequency $F < 2f_m$, the periodic continuation of $X(j\Omega)$ results in spectral overlap. In this case, the spectrum $X(j\Omega)$ cannot be recovered using a low pass filter. This effect is known as aliasing effect.



Spectrum of Sample Sequence when $\Omega = \pi/T$ Spectrum of Sample Sequence when $\Omega < \pi/T$ Aliasing version of frequency spectrum when $\Omega > \pi/T$

If we define sampling frequency $F = \frac{1}{T}$ we have

$$\Omega_m \leq \pi F \leq \frac{\Omega_s}{2}$$

$$2\pi f_m \leq \pi F$$

$$\text{or } F \geq 2f_m$$

From Eq. we find that to avoid aliasing the sampling frequency must be greater than twice the highest frequency present in the signal.

10. Check whether the following discrete time systems are linear, time-invariant, stable or not? (Dec 2018)

(a) $y(n) = nx(n)$

(i) For an input $x_1(n)$

$$y_1(n) = nx_1(n)$$

For an input $x_2(n)$

$$y_2(n) = nx_2(n)$$

The output due to weighted sum of inputs is

$$\begin{aligned}y_3(n) &= T[ax_1(n) + bx_2(n)] \\&= anx_1(n) + bnx_2(n)\end{aligned}$$

The weighted sum of outputs is given by

$$\begin{aligned}ay_1(n) + by_2(n) &= anx_1(n) + bnx_2(n) \\y_3(n) &= ay_1(n) + by_2(n)\end{aligned}$$

Hence the system is linear.

(ii) Given $y(n) = T[x(n)] = nx(n)$

The output due to delayed input

$$y(n, k) = T[x(n - k)] = nx(n - k)$$

The delayed output is

$$\begin{aligned}y(n - k) &= (n - k)x(n - k) \\y(n, k) &\neq y(n - k)\end{aligned}$$

Therefore, the system is time variant.

(iii) $y(n) = nx(n)$

If the input $x(n)$ is bounded, $|x(n)| < \infty$

Then $|y(n) = nx(n)| < \infty$ is also bounded. Hence the system is stable.

The system is linear, time-variant and stable.

(b) $y(n) = Ax(n) + B$

(i) For an input $x_1(n)$

$$y_1(n) = Ax_1(n) + B$$

For an input $x_2(n)$

$$y_2(n) = Ax_2(n) + B$$

The output due to weighted sum of inputs is

$$y_3(n) = T[ax_1(n) + bx_2(n)]$$

$$\begin{aligned}
 &= A[ax_1(n) + bx_2(n)] + B \\
 &= Aax_1(n) + Abx_2(n) + B
 \end{aligned}$$

The weighted sum of outputs is given by

$$\begin{aligned}
 y_3(n) &= T[ax_1(n) + bx_2(n)] \\
 &= a[Ax_1(n) + B] + b[Ax_2(n) + B] \\
 &= aAx_1(n) + aB + bAx_2(n) + bB \\
 ay_1(n) + by_2(n) &= anx_1(n) + bnx_2(n) \\
 y_3(n) &\neq ay_1(n) + by_2(n)
 \end{aligned}$$

Hence the system is non-linear.

(ii) Given $y(n) = T[x(n)] = Ax(n) + B$

The output due to delayed input

$$y(n, k) = T[x(n - k)] = Ax(n - k) + B$$

The delayed output is

$$\begin{aligned}
 y(n - k) &= Ax(n - k) + B \\
 y(n, k) &= y(n - k)
 \end{aligned}$$

Therefore, the system is time invariant.

(iii) $y(n) = Ax(n) + B$

If the input $x(n)$ is bounded, $|x(n)| < \infty$

Then $|y(n) = Ax(n) + B| < \infty$ is also bounded. Hence the system is stable.

The system is non-linear, time invariant and stable.

11. Find the cross correlation for the sequence $x(n) = \{1, 2, 3, 4\}$ and $y(n) = \{4, 3, 2, 1\}$ (Dec 2018)

$$x(l) = \{1, 2, 3, 4\}; \quad y(l) = \{4, 3, 2, 1\}; \quad y(-l) = \{1, 2, 3, 4\}$$

$$\gamma_{xy}(l) = x(l) * y(-l)$$

$x(l)/y(-l)$	1	2	3	4
1	1	2	3	4
2	2	4	6	8
3	3	6	9	12
4	4	8	12	16

$$\gamma_{xy}(l) = \{1, 4, 10, 20, 25, 24, 16\}$$

12. Determine whether the discrete time system $y(n) = x(n) \cos(\omega n)$. (Dec 2019)

- (a) Memoryless
- (b) Stable
- (c) Causal
- (d) Linear
- (e) Time Invariant

Given $y(n) = x(n) \cos(\omega n)$

(a) The output at any instant not depending on the input at that instant. Therefore, the system is dynamic or Memory

(b) $y(n) = x(n) \cos(\omega n)$

If the input $x(n)$ is bounded, $|x(n)| < \infty$

Then $|y(n) = x(n) \cos(\omega n)| < \infty$ is also bounded. Hence the system is stable.

(c) $y(n) = x(n) \cos(\omega n)$

For $n = -1$, $y(-1) = x(-1) \cos(-\omega)$

$n = 0$, $y(0) = x(0) \cos(0)$

$n = 1$, $y(1) = x(1) \cos(\omega)$

For positive values of n the output $y(n)$ depends on future inputs. Hence the system is non-causal.

(d) $y(n) = x(n) \cos(\omega n)$

$$y(n) = T[x(n)] = x(n) \cos(\omega n)$$

For the two inputs $x_1(n)$ and $x_2(n)$ the outputs are given by

$$y_1(n) = T[x_1(n)] = x_1(n) \cos(\omega n)$$

$$y_2(n) = T[x_2(n)] = x_2(n) \cos(\omega n)$$

The weighted sum of outputs is

$$ay_1(n) + by_2(n) = ax_1(n) \cos(\omega n) + bx_2(n) \cos(\omega n)$$

The output due to weighted sum of inputs is given by

$$y_3(n) = T[ax_1(n) + bx_2(n)] = ax_1(n) \cos(\omega n) + bx_2(n) \cos(\omega n)$$

$$y_3(n) = ay_1(n) + by_2(n)$$

Therefore, the system is linear.

(e) Given $y(n) = T[x(n)] = x(n) \cos(\omega n)$

The output due to delayed input

$$y(n, k) = T[x(n - k)] = x(n - k) \cos(\omega n)$$

The delayed output is

$$y(n - k) = x(n - k) \cos(\omega(n - k))$$

$$y(n, k) \neq y(n - k)$$

Therefore, the system is time variant.

The system is dynamic, stable, non-causal, linear and time variant.

13. (a) Determine the power and RMS value of the signal $x(t) = A \cos(\Omega_0 t + \theta)$. (Dec 2019)

Given $x(t) = A \cos(\Omega_0 t + \theta)$

The power

$$P = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(t)|^2 dt$$

$$\begin{aligned}
 P &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(t)|^2 dt \\
 &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T e^{-6t} dt \\
 &= \lim_{T \rightarrow \infty} \left[\frac{1}{-6} e^{-6t} \right]_{-T}^T = \lim_{T \rightarrow \infty} \frac{1}{12T} = 0
 \end{aligned}$$

The energy of the signal is finite, and the power is 0. Therefore the signal $x(t)$ is an energy signal.

BM T63 – BIOMEDICAL SIGNAL PROCESSING**UNIT-II: DTFT AND Z TRANSFORM****PART- A****1. What are the different types of IIR systems? (May 2016)**

The different types of IIR systems are

- Direct Form Structures Form I and Form II.
- Cascade Form Structure.
- Parallel Form Structure.
- Lattice and Lattice-Ladder structure.

2. What is known as stability criterion? (May 2016)

The system having system function is stable if and only if all poles of $H(z)$ are inside the unit circle. That the necessary and sufficient condition for a causal LTI system to be BIBO stable is

$$\sum_{n=0}^{\infty} |h(n)| < \infty$$

For a causal system

$$H(z) = \sum_{n=0}^{\infty} h(n)z^{-n}$$

$$|H(z)| = \sum_{n=0}^{\infty} |h(n)z^{-n}| \leq \sum_{n=0}^{\infty} |h(n)| |z^{-n}|$$

3. What is meant ROC? (Nov 2016, May 2019)

The region of convergence (ROC) of $X(z)$ is the set of all values of z for which $X(z)$ attains a finite value.

4. What is the advantage and disadvantage of direct form structure? (Nov 2016)

The IIR Direct Form I and Form II Transposed structures implement forward coefficients first. The Form I Transposed and Form II structures implement reverse coefficients first. Using Form I and Form II and their transposed structures has the same advantages and disadvantages as using the FIR Direct Form and FIR Direct Form Transposed structures. The IIR Direct Form structures usually require few mathematical operations. However, the sensitivity to finite word length effects limits the use of this form in fixed-point implementations. Use the IIR Cascaded Second-Order Sections Form structures to alleviate finite word length effects.

5. What are the properties of ROC? (May 2017)

- The ROC is a ring or disk in the z -plane centered at the origin.
- The ROC cannot contain any poles.
- The ROC of an LTI stable system contains the unit circle.

- The ROC must be a connected region.

6. What are the conditions of stability of a causal system? (May 2017, May 2019)

For a causal system

$$H(z) = \sum_{n=0}^{\infty} h(n)z^{-n}$$

$$|H(z)| = \sum_{n=0}^{\infty} |h(n)z^{-n}| \leq \sum_{n=0}^{\infty} |h(n)| |z^{-n}|$$

Evaluated on the unit circle

$$|H(z)| \leq \sum_{n=0}^{\infty} |h(n)| < \infty$$

For a stable system the ROC of the system function includes the unit circle. But for a causal system the ROC is exterior of some circle of radius r . Consequently, a causal and stable system must have a system function for $|z| > r > 1$. Since ROC cannot contain any poles of $H(z)$, a causal linear time invariant system is BIBO stable if and only if all the poles of $H(z)$ are inside the unit circle.

7. What are the different methods for evaluating inverse z-transform? (Dec 2017)

The four methods that are often used for the evaluation of the inverse z-transform

1. Long division method
2. Partial fraction method
3. Residue method
4. Convolution method

8. Distinguish between fourier series and fourier transform. (Dec 2017)

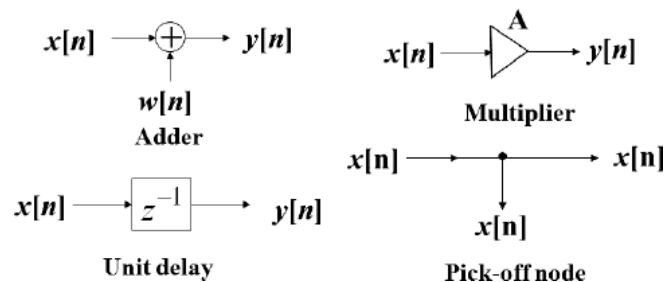
S.No	Fourier Series	Fourier Transform
1	Fourier series is for periodic signals	Fourier transform is for aperiodic signals
2	Used to decompose signals into basic elements (complex exponentials)	Used to analyze signal in another domain (e.g from time to frequency or vice versa)
3	Gives the harmonic content of a periodic time function	Gives the frequency information for an aperiodic signal
4	Discrete frequency spectrum	Continuous frequency spectrum

9. Write the properties of z transform. (May 2018)

Property	Time Domain	z- Domain	Region of Convergence
Notation	$h[n]$	$H[z]$	R_h
	$g[n]$	$G[z]$	R_g
Linearity	$\alpha h[n] + bg[n]$	$\alpha H[z] + bG[z]$	At least $R_h \cap R_g$
Time shifting	$h[n - n_0]$	$z^{-n_0} H(z)$	R_h , except $z=0$ if $n_0 > 0$ and $z=0$ if $n_0 < 0$
Time reversal	$h[-n]$	$H(z^{-1})$	$1/R_h$
Convolution	$h[n] * g[n]$	$H[z]G[z]$	At least $R_h \cap R_g$

10. Recall the basic building blocks of digital filter and sketch the FIR filter structure. (May 2018)

Basic building blocks:



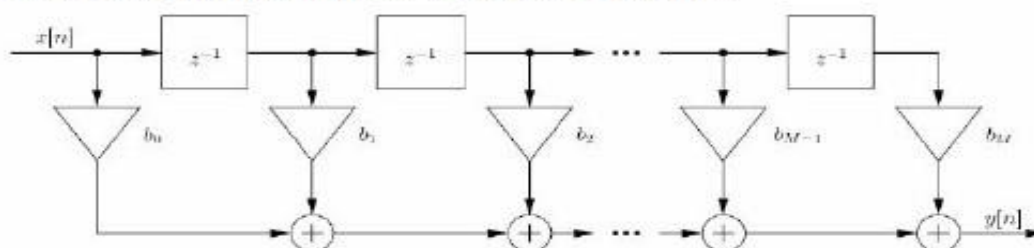
A realization is **canonic** if the realization uses minimum number of delay units. Two realizations are equivalent if they have the same transfer function.

Transpose operation generates an equivalent structure from a given realization by the following steps:

- Interchange input and output nodes
- Reverse all the paths
- Replace pick – off nodes with adders and vice – versa.

Direct – Form/Transversal/Tapped Delay Line Structure

From the above equation, the direct form structure is as follows

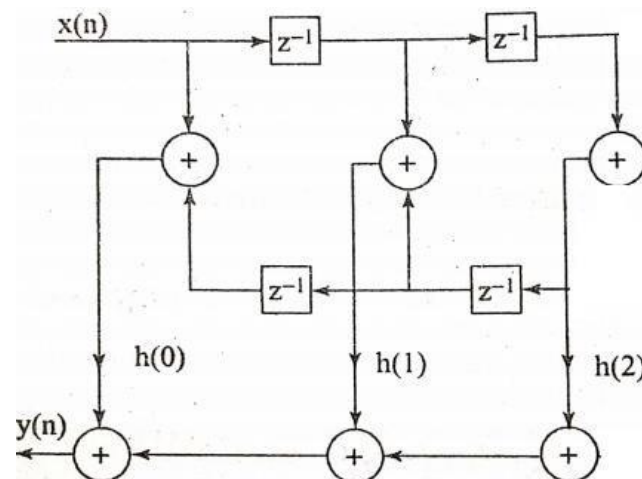
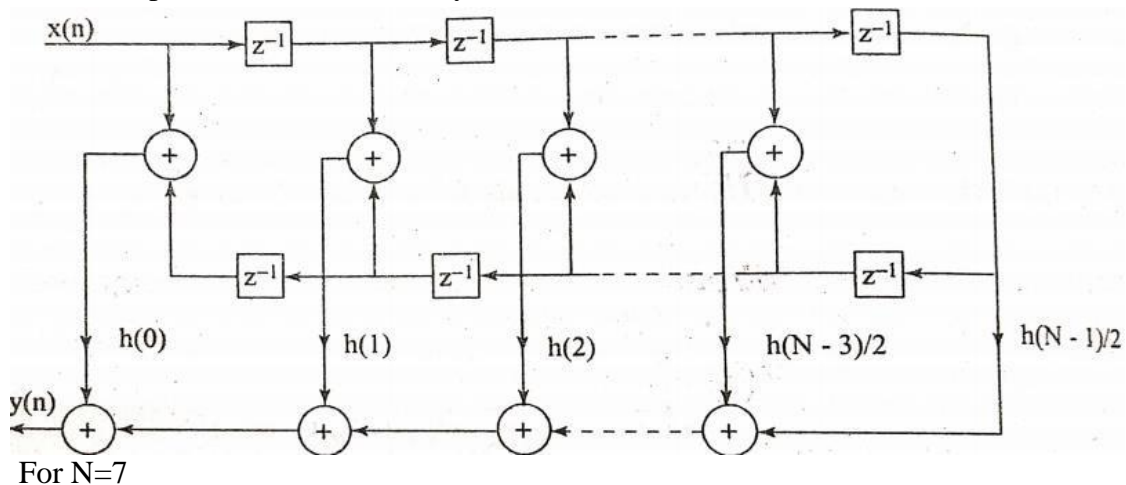


11. Write any four properties of discrete time fourier transform. (Dec 2018)

Property	$x[n] \xleftrightarrow{DTFT} X(e^{j\omega}) \quad y[n] \xleftrightarrow{DTFT} Y(e^{j\omega})$
Linearity	$ax[n] + by[n] \xleftrightarrow{DTFT} aX(e^{j\omega}) + bY(e^{j\omega})$
Time Shift	$x[n - n_0] \xleftrightarrow{DTFT} e^{-j\omega n_0} X(e^{j\omega})$
Frequency Shift	$e^{j\gamma n} x[n] \xleftrightarrow{DTFT} X(e^{j(\omega - \gamma)})$
Differentiation in Frequency	$-jnx[n] \xleftrightarrow{DTFT} \frac{d}{d\omega} X(e^{j\omega})$
Convolution-Multiplication	$\sum_{k=-\infty}^{\infty} x[k]y[n - k] \xleftrightarrow{DTFT} X(e^{j\omega})Y(e^{j\omega})$

12. Draw the linear phase structure of FIR systems for N=7. (Dec 2018)

The linear phase structure of FIR system for N is odd



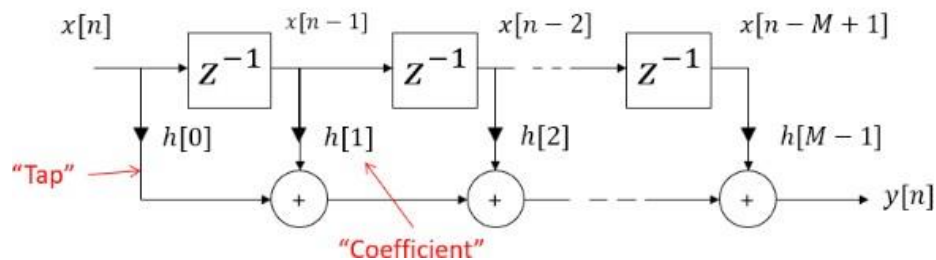
13. What is linear phase response? (Dec 2019)

For a linear phase filter $\theta(\omega) \propto \omega$, the linear phase filter does not alter the shape of the original signal. If the phase response of the filter is nonlinear the output signal may be distorted one. In many cases a linear phase characteristic is required throughout the passband of the filter to preserve the shape of a given signal within the passband. An IIR filter cannot produce a linear phase. The FIR filter can give linear phase, when the impulse response of the filter is symmetric about its mid-point.

14. What is the general form of FIR filter? (Dec 2019)

FIR system is easily implemented directly from convolution summation.

$$y[n] = \sum_{k=0}^{M-1} h[k]x[n-k]$$

**PART – B****1. Explain in detail about direct form I and II with examples. (May 2016)****Direct Form I:**

Consider a system transfer function

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s^2 + 3s + 2}{s^2 + 2s + 2}$$

To realize the above system, first divide the numerator and denominator with s^2 (highest power of s). Then we obtain

$$\frac{Y(s)}{X(s)} = \frac{1 + \frac{3}{s} + \frac{2}{s^2}}{1 + \frac{2}{s} + \frac{2}{s^2}}$$

$$\Rightarrow \left(1 + \frac{2}{s} + \frac{2}{s^2}\right) Y(s) = X(s) \left[1 + \frac{3}{s} + \frac{2}{s^2}\right]$$

$$Y(s) + \frac{2Y(s)}{s} + \frac{2Y(s)}{s^2} = X(s) + \frac{3X(s)}{s} + \frac{2X(s)}{s^2}$$

Let

$$X(s) + \frac{3X(s)}{s} + \frac{2X(s)}{s^2} = W(s)$$

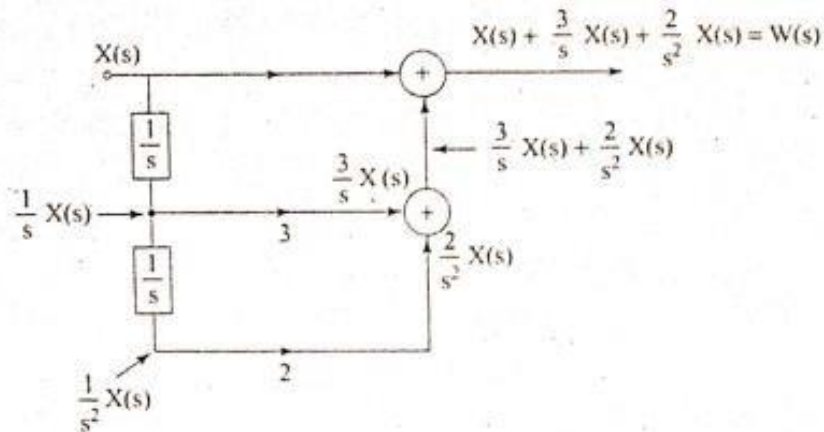
Eq. (1)

Then

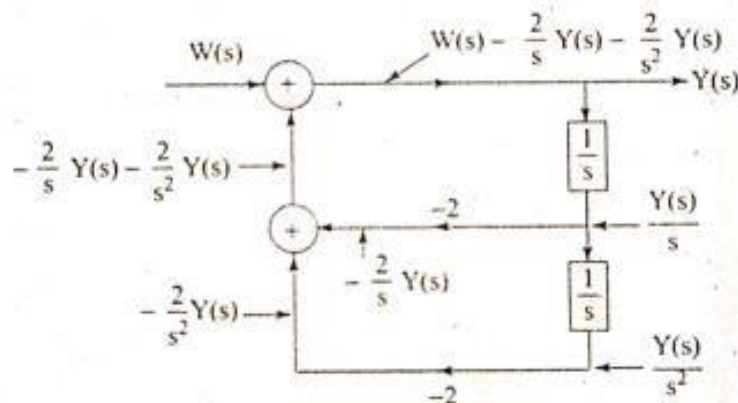
$$Y(s) + \frac{2Y(s)}{s} + \frac{2Y(s)}{s^2} = W(s)$$

$$\Rightarrow Y(s) = W(s) - \frac{2}{s} Y(s) - \frac{2}{s^2} Y(s) \quad \text{Eq. (2)}$$

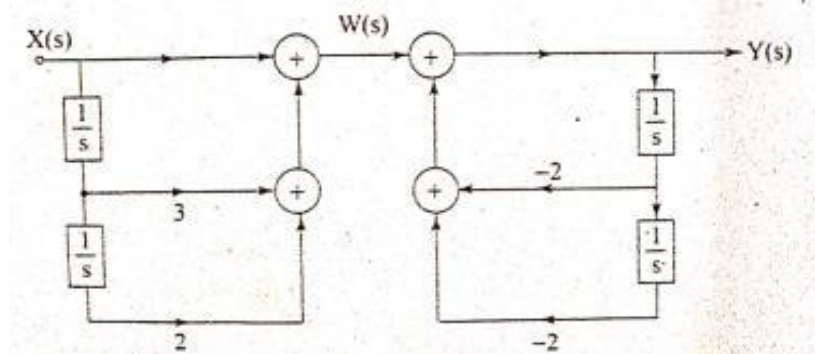
First we realize Eq.(1) using integrators, as shown in fig.



The realization of Eq. (2) is



Combine the above two fig. to get the realization of the transfer function $H(s)$.



Direct Form II:

Consider an LTI system with transfer function

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s^2 + 4s + 2}{s^2 + 5s + 2}$$

Divide the transfer function into two parts

$$\frac{Y(s)}{X(s)} = \frac{Y(s)}{W(s)} \bullet \frac{W(s)}{X(s)}$$

$$\text{Where } \frac{W(s)}{X(s)} = \frac{1}{s^2 + 5s + 3} \quad \text{Eq. (1)}$$

$$\text{and } \frac{Y(s)}{W(s)} = s^2 + 4s + 2 \quad \text{Eq. (2)}$$

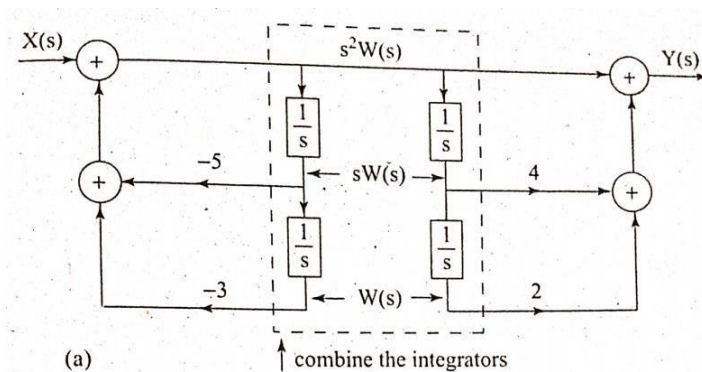
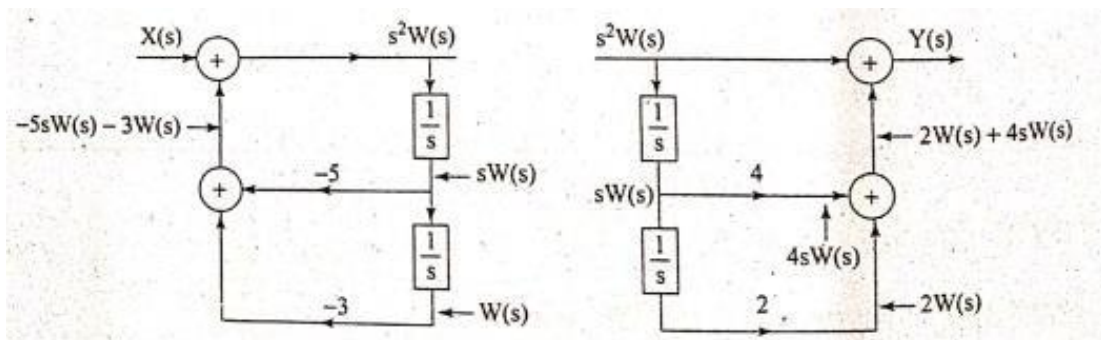
From eq.(1) we get

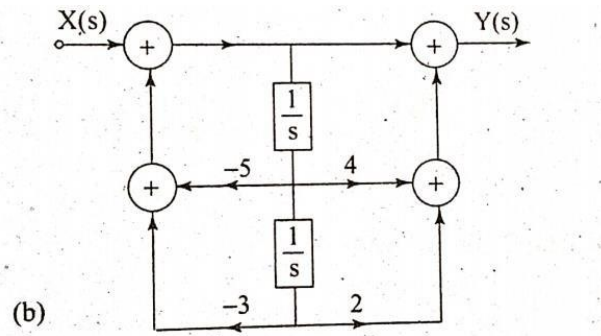
$$\begin{aligned} X(s) &= s^2 W(s) + 5s W(s) + 3W(s) \\ \Rightarrow s^2 W(s) &= X(s) - 5s W(s) - 3W(s) \end{aligned} \quad \text{Eq. (3)}$$

From eq.(1) we get

$$Y(s) = s^2 W(s) + 4s W(s) + 2W(s) \quad \text{Eq. (4)}$$

To realize Eq. (3) assume $s^2 W(s)$ is available. Pass this signal to successive integrators to get signals $sW(s)$ and then realize Eq.(3) as shown in fig. Similarly Eq. (4) shown in fig.





2. Describe in detail about Discrete Fourier Transform and its properties. (May 2016)

The DFT is the most important discrete transform, used to perform Fourier analysis in many practical applications. In digital signal processing, the function is any quantity or signal that varies over time, such as the pressure of a sound wave, a radio signal, or daily temperature readings, sampled over a finite time interval (often defined by a window function). In image processing, the samples can be the values of pixels along a row or column of a raster image. The DFT is also used to efficiently solve partial differential equations, and to perform other operations such as convolutions or multiplying large integers.

Since it deals with a finite amount of data, it can be implemented in computers by numerical algorithms or even dedicated hardware. These implementations usually employ efficient fast Fourier transform (FFT) algorithms; so much so that the terms "FFT" and "DFT" are often used interchangeably. Prior to its current usage, the "FFT" initialism may have also been used for the ambiguous term "finite Fourier transform".

DFT is a powerful computation tool that allows us to evaluate the Fourier Transform $X(e^{j\omega})$ on a digital computer. Let us consider a sequence $x(n)$ of length N . The discrete time fourier transform of the sequence $x(n)$ is given by,

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x(n)e^{-j\omega n}$$

The fourier transform $X(e^{j\omega})$ is continuous and periodic in ω , with period 2π . If we want to compute the fourier transform on a digital computer, we can compute only on a finite number points of ω in $X(e^{j\omega})$. These points can be obtained by sampling the fourier transform $X(e^{j\omega})$ in the interval of $0 \leq \omega < 2\pi$. If the number of points selected in this interval is N , then the spacing between two successive points is $\frac{2\pi}{N}$. These N points can be chosen as,

$$\omega_k = \frac{2\pi}{N} k \quad \text{for } k = 0, 1, 2, \dots, N-1.$$

Now we define

$$X(k) = X(e^{j\omega}) \quad \omega = \frac{2\pi k}{N}$$

$$= \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \quad k = 0, 1, 2, 3, \dots, N-1 \quad \text{Eq.(1)}$$

Both indices n and k are ranging from 0 to $N-1$. The Eq. (1) is called the discrete fourier transform (DFT) of $x(n)$.

To get the sequence $x(n)$ from $X(k)$, we have to multiply on both sides of Eq.(1) with $e^{j(\frac{2\pi}{N})mk}$ and sum the product from $k = 0$ to $N-1$.

That is

$$\begin{aligned} \sum_{k=0}^{N-1} X(k) e^{j(\frac{2\pi}{N})mk} &= \sum_{k=0}^{N-1} \sum_{n=0}^{N-1} x(n) e^{j2\pi km/N} e^{-j2\pi kn/N} \\ &= \sum_{k=0}^{N-1} \sum_{n=0}^{N-1} x(n) e^{-j2\pi k(n-m)/N} \\ &= \sum_{n=0}^{N-1} x(n) \sum_{k=0}^{N-1} e^{-j2\pi k(n-m)/N} \end{aligned}$$

Using the relation

$$\begin{aligned} \sum_{k=0}^{N-1} e^{-j2\pi k(n-m)/N} &= N \text{ if } n-m = 0, \pm N, \pm 2N \dots \\ &= 0 \text{ otherwise} \end{aligned}$$

We can rewrite above equation,

$$\begin{aligned} \sum_{k=0}^{N-1} X(k) e^{j(\frac{2\pi}{N})mk} &= x(m) \cdot N \\ \Rightarrow x(m) &= \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j(\frac{2\pi}{N})mk} \end{aligned}$$

Changing the index m to n , we get

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi nk/N}$$

Which is known as inverse Discrete Fourier Transform of $X(k)$

The expressions of DFT and IDFT are

DFT

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$$

IDFT

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N}$$

Properties of the DFT

Property	Time Domain	Frequency Domain
1. Linearity	$ax_1[n] + bx_2[n]$	$aX_1[k] + bX_2[k]$
2. Time-shifting	$x[n - m]$	$e^{-j2\pi km/N} X(k)$
3. Frequency-shifting (modulation)	$e^{-j2\pi k_0 n/N} x[n]$	$X(k - k_0)$
4. Time reversal	$x[-n]$	$X(-k)$
5. Conjugation	$x^*[n]$	$X^*(-k)$
6. Time-convolution	$x_1[n] \otimes x_2[n]$	$X_1[k] X_2[k]$
7. Frequency-convolution	$x_1[n] x_2[n]$	$\frac{1}{N} X_1[k] \otimes X_2[k]$

3. Find the inverse Z-transform of $H(z) = \frac{4z^2}{(z-2)^2}$ ROC: $|z| > 2$. (Nov 2016, May 2019)

Given

$$H(z) = \frac{4z^2}{(z-2)^2}$$

Inverse z-transform can be obtained using partial fraction method

$$H(z) = \frac{4z^2}{(z-2)^2}$$

$$\frac{H(z)}{z} = \frac{4z}{(z-2)^2}$$

Apply partial fraction,

$$\frac{4z}{(z-2)^2} = \frac{A}{(z-2)^2} + \frac{B}{(z-2)} \quad \text{Eq. (1)}$$

$$A = \cancel{(z-2)}^2 \frac{4z}{\cancel{(z-2)}^2}, \text{ where } z = 2$$

$$\mathbf{A = 8}$$

$$B = (z-2) \frac{4z}{(z-2)^2}, \text{ can be solved using differentiation technique}$$

$$B = \frac{d}{dz}(4z) = 4$$

$$\text{Therefore, } \mathbf{B = 4}$$

Eq. (1) becomes,

$$\frac{4z}{(z-2)^2} = \frac{8}{(z-2)^2} + \frac{4}{(z-2)}$$

$$Z^{-1}\left(\frac{4z}{(z-2)^2}\right) = Z^{-1}\left(\frac{8}{(z-2)^2} + \frac{4}{(z-2)}\right)$$

$$h(n) = 8n \bullet 2^n u(n) + 4 \bullet 2^n u(n)$$

$$h(n) = 4 \bullet 2^n [2n + 1] u(n)$$

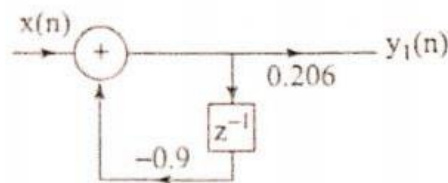
4. Realize the system in parallel form.

$$y(n) = -0.1y(n-1) + 0.72y(n-2) + 0.7x(n) - 0.252x(n-2) \quad (\text{Nov 2016, May 2019})$$

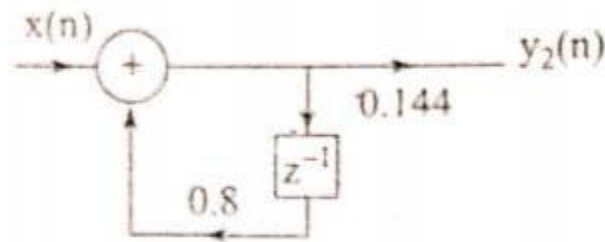
The system function of the difference equation is

$$\begin{aligned} H(z) &= \frac{0.7 - 0.252z^{-2}}{1 + 0.1z^{-1} - 0.72z^{-2}} \\ &= 0.35 + \frac{0.35 - 0.035z^{-1}}{1 + 0.1z^{-1} - 0.72z^{-2}} \\ &= 0.35 + \frac{0.296}{1 + 0.9z^{-1}} + \frac{0.144}{1 - 0.8z^{-1}} \\ &= c + H_1(z) + H_2(z) \end{aligned}$$

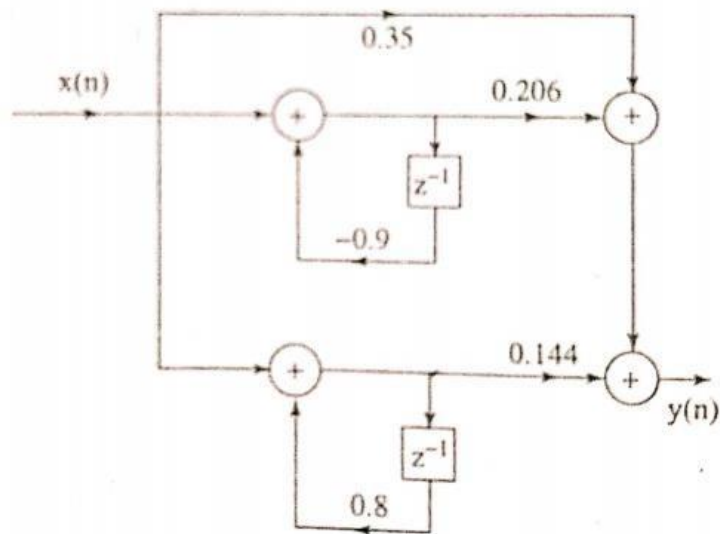
$H_1(z)$ can be realized in direct form II as



$H_2(z)$ can be realized in direct form II as



Now the realization of $H(z)$ is shown in fig.



5. Find the inverse Z- transform of $H(z) = \frac{1}{2}z^{-2} + 2z^{-1} + 1$. (May 2017)

Given

$$H(z) = \frac{1}{2}z^{-2} + 2z^{-1} + 1$$

The above equation can be simplified as

$$H(z) = \frac{z^{-2} + 4z^{-1} + 2}{2}$$

$$H(z) = z^{-2} \left(\frac{1 + 4z + 2z^2}{2} \right)$$

$$H(z) = \frac{2z^2 + 4z + 1}{2z^2}$$

$$H(z) = \frac{z^2 + 2z + 0.5}{z^2}$$

Multiply by z on both sides denominator, we get

$$\frac{H(z)}{z} = \frac{z^2 + 2z + 0.5}{z^3}$$

Apply partial fraction,

$$\frac{H(z)}{z} = \frac{z^2 + 2z + 0.5}{z^3} = \frac{A}{z^3} + \frac{B}{z^2} + \frac{C}{z} \quad \text{Eq. (1)}$$

$$A = \frac{z^2 + 2z + 0.5}{z^3}, \text{ where } z = 0$$

$$\mathbf{A = 0.5}$$

$$B = \frac{d}{dz}(z^2 + 2z + 0.5), \text{ where } z = 0$$

$$B = 2z + 2, \text{ where } z = 0$$

$$\mathbf{B = 2}$$

$$C = \frac{d}{dz}(2z + 2), \text{ where } z = 0$$

$$C = 2, \text{ where } z = 0$$

$$\mathbf{C = 2}$$

Eq. (1) becomes,

$$\begin{aligned} \frac{H(z)}{z} &= \frac{z^2 + 2z + 0.5}{z^3} = \frac{0.5}{z^3} + \frac{2}{z^2} + \frac{2}{z} \\ H(z) &= z \left(\frac{0.5}{z^3} + \frac{2}{z^2} + \frac{2}{z} \right) \\ H(z) &= \frac{0.5}{z^2} + \frac{2}{z} + 2 \end{aligned}$$

Apply Inverse Z- Transform,

$$\begin{aligned} Z^{-1}(H(z)) &= Z^{-1} \left(\frac{0.5}{z^2} + \frac{2}{z} + 2 \right) \\ h(n) &= 0.5nu(n) + 2u(n) + 2\delta(n) \end{aligned}$$

6. Obtain the cascade and parallel form realizations for the following systems. (May 2017)

$$y(n) - \frac{1}{4}y(n-1) - \frac{1}{8}y(n-2) = x(n) + 3x(n-1) + 2x(n-2)$$

Cascade Realization:

Given

$$y(n) - \frac{1}{4}y(n-1) - \frac{1}{8}y(n-2) = x(n) + 3x(n-1) + 2x(n-2)$$

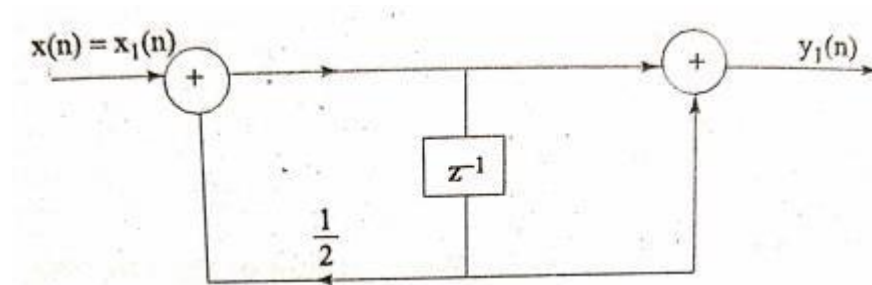
Taking z – transform on both sides, we get

$$\begin{aligned} Y(z) - \frac{1}{4}z^{-1}Y(z) - \frac{1}{8}z^{-2}Y(z) &= X(z) + 3z^{-1}X(z) + 2z^{-2}X(z) \\ \left[1 - \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2} \right] Y(z) &= X(z) \left[1 + 3z^{-1} + 2z^{-2} \right] \\ \frac{Y(z)}{X(z)} &= \frac{1 + 3z^{-1} + 2z^{-2}}{1 - \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}} \\ &= \left(\frac{(1 + z^{-1})(1 + 2z^{-1})}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + \frac{1}{4}z^{-1}\right)} \right) \\ &= H_1(z)H_2(z) \end{aligned}$$

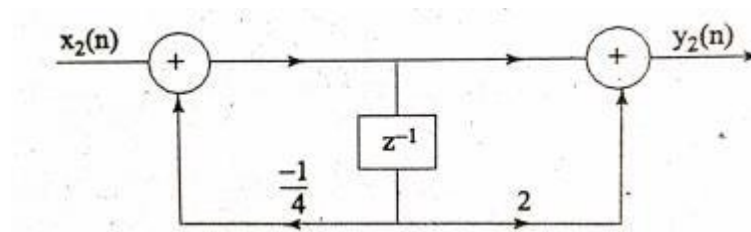
Where,

$$H_1(z) = \frac{1 + z^{-1}}{1 - \frac{1}{2}z^{-1}} \quad \text{and} \quad H_2(z) = \frac{1 + 2z^{-1}}{1 + \frac{1}{4}z^{-1}}$$

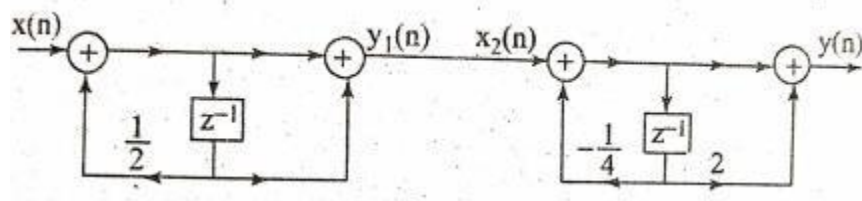
The realization of $H_1(z)$ is shown in fig.,



The realization of $H_2(z)$ is shown in fig.,



Now the cascade realization of $H(z)$ can be obtained by cascading $H_1(z)$ and $H_2(z)$



Parallel Realization:

Given

$$y(n) - \frac{1}{4}y(n-1) - \frac{1}{8}y(n-2) = x(n) + 3x(n-1) + 2x(n-2)$$

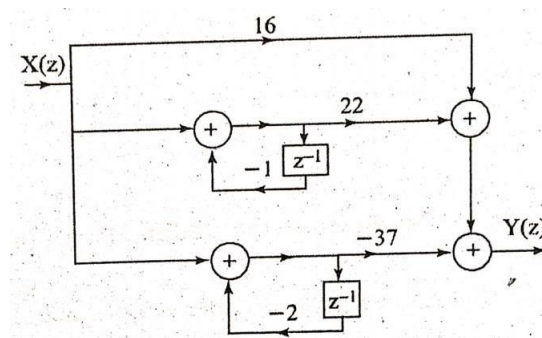
Taking z -transform on both sides, we get

$$\begin{aligned} Y(z) - \frac{1}{4}z^{-1}Y(z) - \frac{1}{8}z^{-2}Y(z) &= X(z) + 3z^{-1}X(z) + 2z^{-2}X(z) \\ \left[1 - \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2} \right] Y(z) &= X(z) \left[1 + 3z^{-1} + 2z^{-2} \right] \\ Y(z) \left[\frac{1}{1 - \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}} \right] &= X(z) \left[\frac{1 + 3z^{-1} + 2z^{-2}}{1 - \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}} \right] \end{aligned}$$

Now we express the system in partial fraction form as follows

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} = 16 + \frac{-15 + 7z^{-1}}{1 + 3z^{-1} + 2z^{-2}} \\ &= 16 + \frac{-15 + 7z^{-1}}{(1 + z^{-1})(1 + 2z^{-1})} \\ &= 16 + \frac{22}{1 + z^{-1}} - \frac{37}{1 + 2z^{-1}} \end{aligned}$$

The realization of above equation is,



7. Obtain the Direct form I, Direct form 2, Cascade form & parallel form realization of the LTI system by the equation. (Dec 2017)

$$y(n) = -0.1y(n-1) + 0.2y(n-2) + 3x(n) + 3.6x(n-1) + 0.6x(n-2)$$

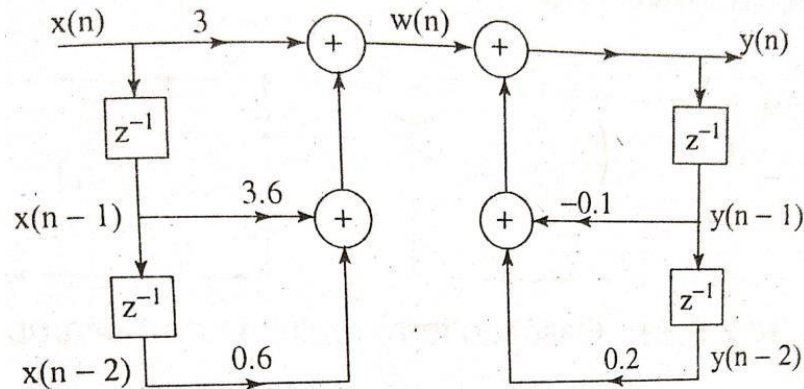
Direct form 1:

$$3x(n) + 3.6x(n-1) + 0.6x(n-2) = w(n)$$

Let

$$y(n) = -0.1y(n-1) + 0.2y(n-2) + w(n)$$

By inspection, the direct form 1 realization is shown in fig.



Direct form 2:

From the given difference equation we have

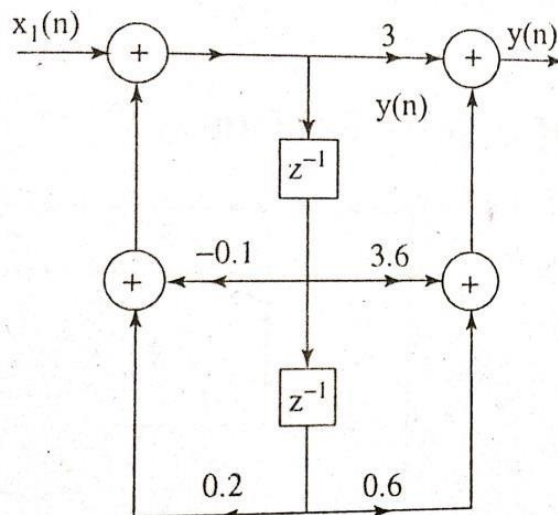
$$Y(z) = -0.1z^{-1}Y(z) + 0.2z^{-2}Y(z) + 3X(z) + 3.6z^{-1}X(z) + 0.6z^{-2}X(z)$$

$$Y(z) + 0.1z^{-1}Y(z) - 0.2z^{-2}Y(z) = 3X(z) + 3.6z^{-1}X(z) + 0.6z^{-2}X(z)$$

$$Y(z) \left[1 + 0.1z^{-1} - 0.2z^{-2} \right] = X(z) \left[3 + 3.6z^{-1} + 0.6z^{-2} \right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{3 + 3.6z^{-1} + 0.6z^{-2}}{1 + 0.1z^{-1} - 0.2z^{-2}}$$

The above system function can be realized in direct form 2 as shown in fig.



Cascade form:

We have

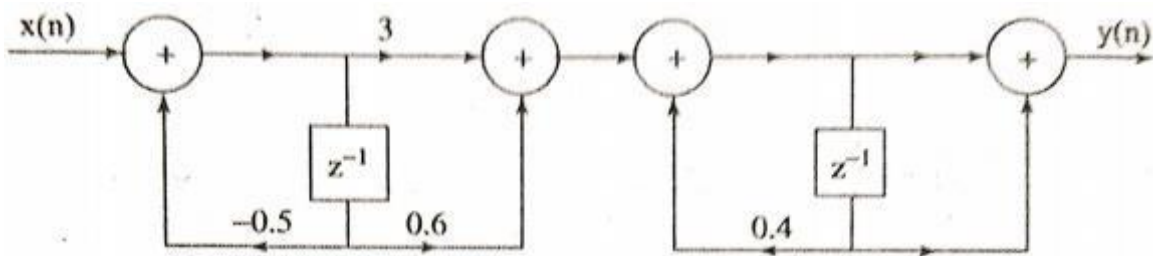
$$H(z) = \frac{Y(z)}{X(z)} = \frac{[3 + 3.6z^{-1} + 0.6z^{-2}]}{[1 + 0.1z^{-1} - 0.2z^{-2}]}$$

$$= \frac{(3 + 0.6z^{-1})(1 + z^{-1})}{(1 + 0.5z^{-1})(1 - 0.4z^{-1})}$$

Let

$$H_1(z) = \frac{3 + 0.6z^{-1}}{1 + 0.5z^{-1}} \text{ and}$$

$$H_2(z) = \frac{1 + 0.6z^{-1}}{1 - 0.4z^{-1}}$$

Now we realize $H_1(z)$ and $H_2(z)$ and cascade both to get realization of $H(z)$ **Parallel form:**

$$H(z) = \frac{Y(z)}{X(z)} = \frac{[3 + 3.6z^{-1} + 0.6z^{-2}]}{[1 + 0.1z^{-1} - 0.2z^{-2}]}$$

$$= -3 + \frac{7}{1 - 0.4z^{-1}} - \frac{1}{1 + 0.5z^{-1}}$$

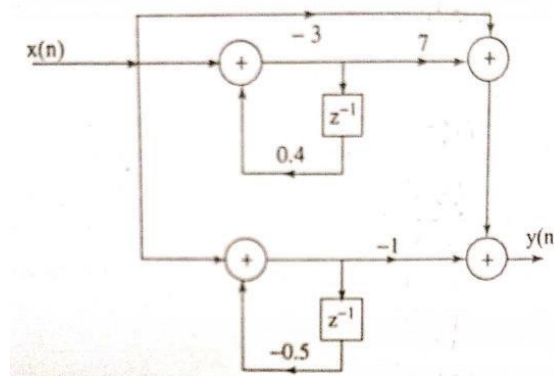
$$= c + H_1(z) + H_2(z)$$

$$\begin{array}{r} -3 \\ -0.2z^{-2} + 0.1z^{-1} + 1 \overline{) 0.6z^{-2} + 3.6z^{-1} + 3} \\ \underline{0.6z^{-2} - 0.3z^{-1} - 3} \\ 3.9z^{-1} + 6 \end{array}$$

$$\rightarrow H(z) = -3 + \frac{3.9z^{-1} + 6}{1 + 0.1z^{-1} - 0.2z^{-2}}$$

$$= -3 + \frac{A}{1 - 0.4z^{-1}} + \frac{B}{1 + 0.5z^{-1}}$$

where $A = 7, B = -1$

Now we realize $H(z)$ in parallel form as shown in fig.

8. Explain the properties of Z-transform and discrete time fourier transform. (Dec 2017, Dec 2019)

Properties of Z-transform:

(i) Linearity property:

$$\text{Let } x(n) = a_1x_1(n) + a_2x_2(n)$$

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x(n)z^{-n} \\ &= \sum_{n=-\infty}^{\infty} (a_1x_1(n) + a_2x_2(n))z^{-n} \\ &= a_1 \sum_{n=-\infty}^{\infty} x_1(n)z^{-n} + a_2 \sum_{n=-\infty}^{\infty} x_2(n)z^{-n} \\ &= a_1X_1(z) + a_2X_2(z) \end{aligned}$$

$$\text{ROC: At least } \text{ROC}_1 \cap \text{ROC}_2$$

(ii) Time shifting property:

$$\text{Let } y(n) = x(n - k).$$

$$\begin{aligned} Y(z) &= \sum_{n=-\infty}^{\infty} y(n)z^{-n} \\ &= \sum_{n=-\infty}^{\infty} (x(n - k))z^{-n} \\ \text{Let } m &= n - k \\ &= \sum_{m=-\infty}^{\infty} x(m)z^{-(m+k)} = \sum_{m=-\infty}^{\infty} x(m)z^{-m}z^{-k} \\ &= z^{-k} \left[\sum_{m=-\infty}^{\infty} x(m)z^{-m} \right] = z^{-k}X(z) \end{aligned}$$

(iii) Scaling in the z-domain:

$$\text{Let } y(n) = a^n x(n)$$

$$\begin{aligned} Y(z) &= \sum_{n=-\infty}^{\infty} y(n)z^{-n} \\ &= \sum_{n=-\infty}^{\infty} a^n x(n)z^{-n} \\ &= \sum_{n=-\infty}^{\infty} x(n)(a^{-1}z)^{-n} \\ &= X(a^{-1}z) \end{aligned}$$

$$\text{ROC: } r_1 < |a^{-1}z| = \frac{|z|}{|a|} < r_2 \equiv |a|r_1 < |z| < |a|r_2$$

(iv) Convolution property:

$$y(n) = x(n) \cdot h(n)$$

$$= \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

The z-transform of the above equation is:

$$Y(z) = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x(k)h(n-k)z^{-n}$$

Adjusting the summation and replacing z^{-n} by $z^{-(n-k)}z^{-k}$

$$\sum_{k=-\infty}^{\infty} x(k) \sum_{n=-\infty}^{\infty} h(n-k)z^{-(n-k)}z^{-k}$$

$$\text{But } \sum_{n=-\infty}^{\infty} h(n-k)z^{-(n-k)} = h(z)$$

Placing $h(z)$ in the prior equation

$$Y(z) = h(z) \sum_{k=-\infty}^{\infty} x(k)z^{-k}$$

$$= h(z) \cdot x(z). \text{ Hence, proved.}$$

(v) Time reversal property:

From the definition, the z-transform of $x(-n)$ is:

$$z[x(-n)] = \sum_{n=-\infty}^{\infty} x(-n)z^{-n}$$

$$\text{Put } -n=m$$

Incorporating the corresponding change in limits:

$$z[x(-n)] = \sum_{m=-\infty}^{\infty} x(m)z^m = \sum_{m=-\infty}^{\infty} x(m)z^{-m \cdot (-1)}$$

$$= \sum_{m=-\infty}^{\infty} x(m)(z^{-1})^{-m} = X[z^{-1}]$$

Thus ROC always is $r_1 < |z| < r_2$

In this case, ROC is

$$1/r_2 < |z| < 1/r_1$$

Properties of Discrete Time Fourier Transform:**(i) Linearity property:**

$$\mathcal{F}_{DT}\{x_1(n) + x_2(n)\} = \mathcal{F}_{DT}\{x_1(n)\} + \mathcal{F}_{DT}\{x_2(n)\} = X_1(e^{j\omega}) + X_2(e^{j\omega})$$

(ii) Time shifting property:

If $\mathcal{F}_{DT}\{x(n)\} = X(e^{j\omega})$, then:

$$\mathcal{F}_{DT}\{x(n - n_0)\} = e^{-j\omega n_0} X(e^{j\omega})$$

Proof:

$$\sum_{n=-\infty}^{\infty} x(n - n_0) e^{-j\omega n} = \sum_{m=-\infty}^{\infty} x(m) e^{-j\omega(m+n_0)} = e^{-j\omega n_0} \sum_{m=-\infty}^{\infty} x(m) e^{-j\omega m}$$

(iii) Time reversal property:

If $\mathcal{F}_{DT}\{x(n)\} = X(e^{j\omega})$, then:

$$\mathcal{F}_{DT}\{x(-n)\} = X(e^{-j\omega})$$

(iv) Convolution property:

If $\mathcal{F}_{DT}\{x(n)\} = X(e^{j\omega})$ and $\mathcal{F}_{DT}\{y(n)\} = Y(e^{j\omega})$, then:

$$G(e^{j\omega}) = \mathcal{F}_{DT}\{x(n) * y(n)\} = X(e^{j\omega}) Y(e^{j\omega})$$

Proof:

$$\begin{aligned} G(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x(m) y(n-m) e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x(m) \sum_{m=-\infty}^{\infty} y(n-m) e^{-j\omega n} \\ &= \sum_{m=-\infty}^{\infty} x(m) \sum_{r=-\infty}^{\infty} y(r) e^{-j\omega(m+r)} = \sum_{m=-\infty}^{\infty} x(m) e^{-j\omega m} \sum_{r=-\infty}^{\infty} y(r) e^{-j\omega r} \end{aligned}$$

(v) Correlation property:

If $\mathcal{F}_{DT}\{x(n)\} = X(e^{j\omega})$ and $\mathcal{F}_{DT}\{y(n)\} = Y(e^{j\omega})$, then:

$$\begin{aligned} \mathcal{F}_{DT}\{x(n) \star y(n)\} &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x(m) y(m-n) e^{-j\omega n} = \sum_{m=-\infty}^{\infty} x(m) \sum_{n=-\infty}^{\infty} y(m-n) e^{-j\omega n} \\ &= \sum_{m=-\infty}^{\infty} x(m) \sum_{r=-\infty}^{\infty} y(r) e^{-j\omega(m-r)} = \sum_{m=-\infty}^{\infty} x(m) e^{-j\omega m} \sum_{r=-\infty}^{\infty} y(r) e^{-j\omega(-r)} \\ &= X(e^{j\omega}) Y(e^{-j\omega}) \end{aligned}$$

9. Determine the DFT of the sequence $x(n) = \frac{1}{4}$, for $0 \leq n \leq 2$. (May 2018)
 $= 0$, otherwise

Given $L = 3$. For $N = 4$, the periodic extension of $x(n)$ shown in fig. can be obtained by adding one zero (i.e., $N - L$ zeros)

We have,

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \quad k = 0, 1, 2, \dots, N-1$$

From the given sequence,

$$x(0) = 1/4; x(1) = 1/4; x(2) = 1/4; x(3) = 0$$

For $N = 4$

$$X(k) = \sum_{n=0}^3 x(n)e^{-j2\pi kn/N} \quad k = 0, 1, 2, 3$$

For $k = 0$

$$\begin{aligned} X(0) &= \sum_{n=0}^3 x(n) = x(0) + x(1) + x(2) + x(3) \\ &= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + 0 = \frac{3}{4} \end{aligned}$$

For $k = 1$

$$\begin{aligned} X(1) &= \sum_{n=0}^3 x(n)e^{-j2\pi n/4} \\ &= x(0) + x(1)e^{-j\pi/2} + x(2)e^{-j\pi} + x(3)e^{-j3\pi/2} \\ &= \frac{1}{4} + \frac{1}{4} \left(\cos \frac{\pi}{2} - j \sin \frac{\pi}{2} \right) + \frac{1}{4} (\cos \pi - j \sin \pi) + 0 \\ &= \frac{1}{4} (1 - j - 1) = -\frac{j}{4} \end{aligned}$$

For $k = 2$

$$\begin{aligned} X(2) &= \sum_{n=0}^3 x(n)e^{-j\pi n} \\ &= x(0) + x(1)e^{-j\pi} + x(2)e^{-j2\pi} + x(3)e^{-j3\pi} \\ &= \frac{1}{4} + \frac{1}{4} (\cos \pi - j \sin \pi) + \frac{1}{4} (\cos 2\pi - j \sin 2\pi) + 0 \\ &= \frac{1}{4} (1 - 1 + 1) = \frac{1}{4} \end{aligned}$$

For $k = 3$

$$\begin{aligned} X(3) &= \sum_{n=0}^3 x(n)e^{-j3\pi n/4} \\ &= x(0) + x(1)e^{-j3\pi/4} + x(2)e^{-j3\pi/2} + x(3)e^{-j9\pi/4} \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{4} + \frac{1}{4} \left(\cos \frac{3\pi}{2} - j \sin \frac{3\pi}{2} \right) + \frac{1}{4} (\cos 3\pi - j \sin 3\pi) + 0 \\
 &= \frac{1}{4} (1 + j - 1) = \frac{j}{4}
 \end{aligned}$$

Therefore, $x(k) = \left\{ \frac{3}{4}, \frac{-j}{4}, \frac{1}{4}, \frac{j}{4} \right\}$

10. Find the z-transform of $x(n) = \left(\frac{1}{2} \right)^n - 5u(n-2) + 8u(n-5)$. (May 2018)

Given

$$\begin{aligned}
 x(n) &= \left(\frac{1}{2} \right)^n - 5u(n-2) + 8u(n-5) \\
 Z[x(n)] &= Z \left[\left(\frac{1}{2} \right)^n - 5u(n-2) + 8u(n-5) \right]
 \end{aligned}$$

We know that

$$\begin{aligned}
 Z[x(n)] &= \sum_{n=-\infty}^{\infty} x(n)z^{-n} \\
 X(z) &= \sum_{n=0}^{\infty} \left[\left(\frac{1}{2} \right)^n - 5u(n-2) + 8u(n-5) \right] z^{-n} \\
 X(z) &= \sum_{n=0}^{\infty} \left[\left(\frac{1}{2} \right)^n \right] z^{-n} - 5 \sum_{n=0}^{\infty} [u(n-2)] z^{-n} + 8 \sum_{n=0}^{\infty} [u(n-5)] z^{-n} \\
 X(z) &= \frac{z}{z - \frac{1}{2}} - 5z^{-2} \frac{z}{z-1} + 8z^{-5} \frac{z}{z-1} \quad (\text{• Timeshifting property}) \\
 X(z) &= \frac{z}{z - \frac{1}{2}} - 5 \frac{z^{-1}}{z-1} + 8 \frac{z^{-4}}{z-1}
 \end{aligned}$$

11. (a) Find the z-transform for the signal $x(n) = \cos \omega(n)$. (Dec 2018)

Given

$$x(n) = \cos \omega n u(n)$$

$$\begin{aligned}
 X(z) &= \sum_{n=-\infty}^{\infty} x(n)z^{-n} \\
 &= \sum_{n=-\infty}^{\infty} (\cos \omega n) u(n) z^{-n} = \sum_{n=0}^{\infty} (\cos \omega n) z^{-n} \\
 &= \sum_{n=0}^{\infty} \left[\frac{e^{j\omega n} + e^{-j\omega n}}{2} \right] z^{-n} \quad \cos \theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \\
 &= \frac{1}{2} \sum_{n=0}^{\infty} [e^{j\omega n} + e^{-j\omega n}] z^{-n}
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \left[\sum_{n=0}^{\infty} e^{j\omega n} z^{-n} + \sum_{n=0}^{\infty} e^{-j\omega n} z^{-n} \right] \\
&= \frac{1}{2} \left[\sum_{n=0}^{\infty} (e^{j\omega} z^{-1})^n + \sum_{n=0}^{\infty} (e^{-j\omega} z^{-1})^n \right] \\
X(z) &= \frac{1}{2} \left[\frac{1}{1 - e^{j\omega} z^{-1}} + \frac{1}{1 - e^{-j\omega} z^{-1}} \right] \\
&= \frac{1}{2} \left[\frac{1 - e^{-j\omega} z^{-1} + 1 - e^{j\omega} z^{-1}}{(1 - e^{j\omega} z^{-1})(1 - e^{-j\omega} z^{-1})} \right] \\
X(z) &= \frac{1}{2} \left[\frac{2 - 2(\cos \omega) z^{-1}}{(1 - e^{j\omega} z^{-1})(1 - e^{-j\omega} z^{-1}) + z^{-2}} \right] \\
X(z) &= \frac{1}{2} \left[\frac{2 - 2(\cos \omega) z^{-1}}{1 - 2(\cos \omega) z^{-1} + z^{-2}} \right] \\
X(z) &= \left[\frac{1 - (\cos \omega) z^{-1}}{1 - 2(\cos \omega) z^{-1} + z^{-2}} \right]
\end{aligned}$$

(b) Obtain the DTFT of $x(n) = a^n u(n)$. (Dec 2018)

Given

$$x(n) = a^n u(n)$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$F[a^n u(n)] = \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

$$= 1 + ae^{-j\omega} + a^2 e^{-j2\omega} + \dots \infty$$

$$= \frac{1}{1 - ae^{-j\omega}}$$

$$F[a^n u(n)] = \frac{1}{1 - ae^{-j\omega}}$$

12. Obtain the direct form I and direct form II realization for the system. (Dec 2018)

$$y(n) - \frac{5}{6}y(n-1] + \frac{1}{6}y(n-2) = x(n) + 2x(n-1)$$

Direct Form I:

Given

$$y(n) - \frac{5}{6}y(n-1] + \frac{1}{6}y(n-2) = x(n) + 2x(n-1)$$

Taking z-transform on both sides assuming zero initial condition, we get

$$Y(z) - \frac{5}{6}z^{-1}Y(z) + \frac{1}{6}z^{-2}Y(z) = X(z) + 2z^{-1}X(z)$$

Let

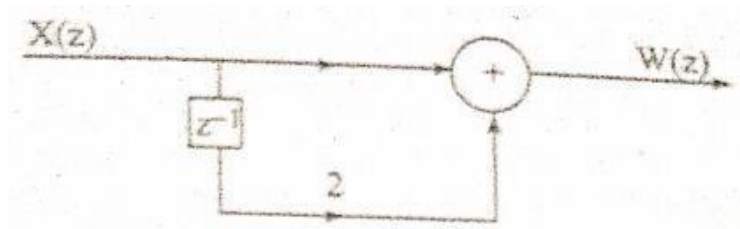
$$X(z) + 2z^{-1}X(z) = W(z) \quad \text{Eq. (1)}$$

Then

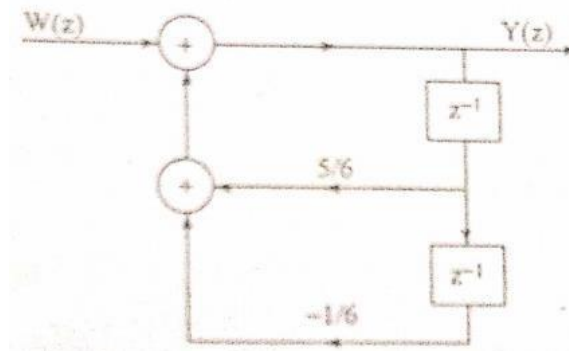
$$Y(z) - \frac{5}{6}z^{-1}Y(z) + \frac{1}{6}z^{-2}Y(z) = W(z) \quad \text{Eq. (2)}$$

$$Y(z) = \frac{5}{6}z^{-1}Y(z) - \frac{1}{6}z^{-2}Y(z) + W(z)$$

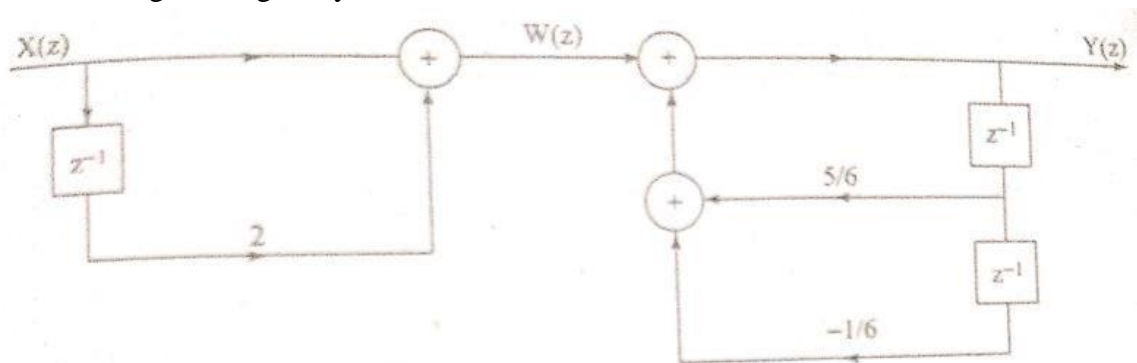
Realizing Eq. (1), we get



Similarly realization of Eq. (2) is shown in Fig.



Combining both figures yields



Direct Form II:

Given

$$y(n) - \frac{5}{6}y(n-1] + \frac{1}{6}y(n-2) = x(n) + 2x(n-1)$$

Taking z-transform on both sides assuming zero initial condition, we get

$$\begin{aligned} Y(z) - \frac{5}{6}z^{-1}Y(z) + \frac{1}{6}z^{-2}Y(z) &= X(z) + 2z^{-1}X(z) \\ Y(z) \left[1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2} \right] &= X(z) [1 + 2z^{-1}] \\ \Rightarrow \frac{Y(z)}{X(z)} &= \frac{1 + 2z^{-1}}{1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}} \end{aligned}$$

Let

$$\frac{Y(z)}{X(z)} = \frac{Y(z)}{W(z)} * \frac{W(z)}{X(z)}$$

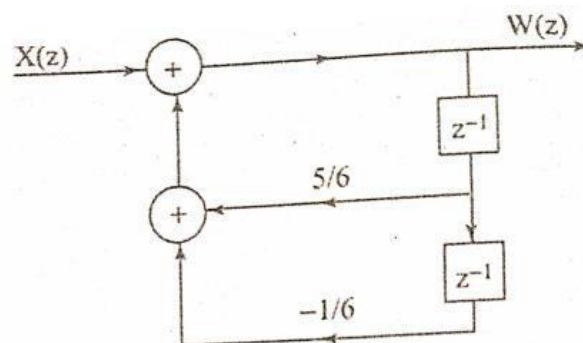
Where

$$\begin{aligned} \frac{W(z)}{X(z)} &= \frac{1}{1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}} \\ W(z) &= X(z) + \frac{5}{6}z^{-1}W(z) - \frac{1}{6}z^{-2}W(z) \end{aligned} \quad \text{Eq. (1)}$$

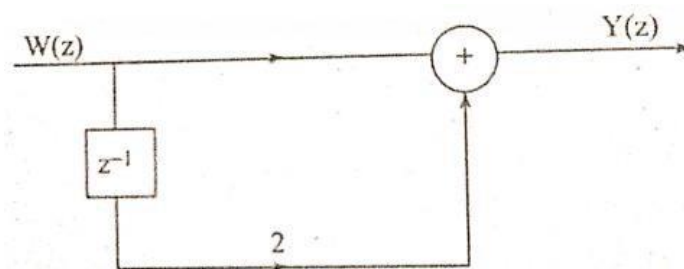
$$\frac{Y(z)}{W(z)} = 1 + 2z^{-1}$$

$$Y(z) = W(z) + 2z^{-1}W(z) \quad \text{Eq. (2)}$$

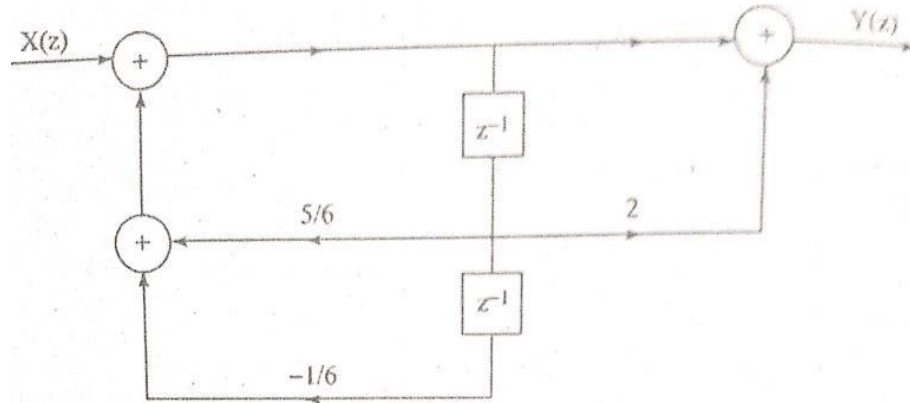
Realization of Eq. (1) is shown in Fig.



Realization of Eq. (2) is shown in Fig.



Combining Figures we get the direct form II Realization



13. (a) Obtain the direct form 1 for the following systems. (Dec 2019)

$$y(n] = 0.5y(n - 1) + 0.45y(n - 2) + x(n) + 987x(n - 1)$$

Direct Form I:

Given

$$y(n] = 0.5y(n - 1) + 0.45y(n - 2) + x(n) + 987x(n - 1)$$

Taking z-transform on both sides assuming zero initial condition, we get

$$Y(z) = 0.5z^{-1}Y(z) + 0.45z^{-2}Y(z) + X(z) + 987z^{-1}X(z)$$

$$Y(z) - 0.5z^{-1}Y(z) - 0.45z^{-2}Y(z) = X(z) + 987z^{-1}X(z)$$

Let

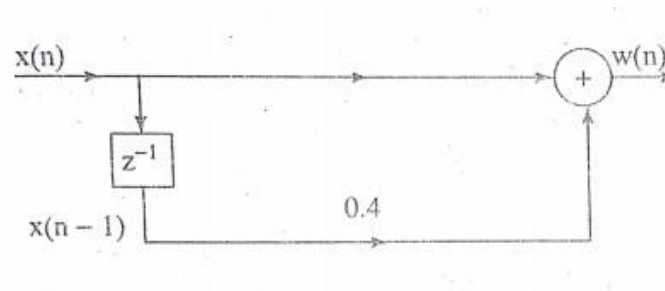
$$X(z) + 987z^{-1}X(z) = W(z) \quad \text{Eq. (1)}$$

Then

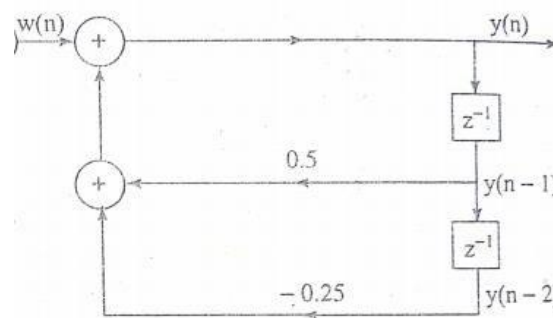
$$Y(z) - 0.5z^{-1}Y(z) - 0.45z^{-2}Y(z) = W(z) \quad \text{Eq. (2)}$$

$$Y(z) = 0.5z^{-1}Y(z) + 0.45z^{-2}Y(z) + W(z)$$

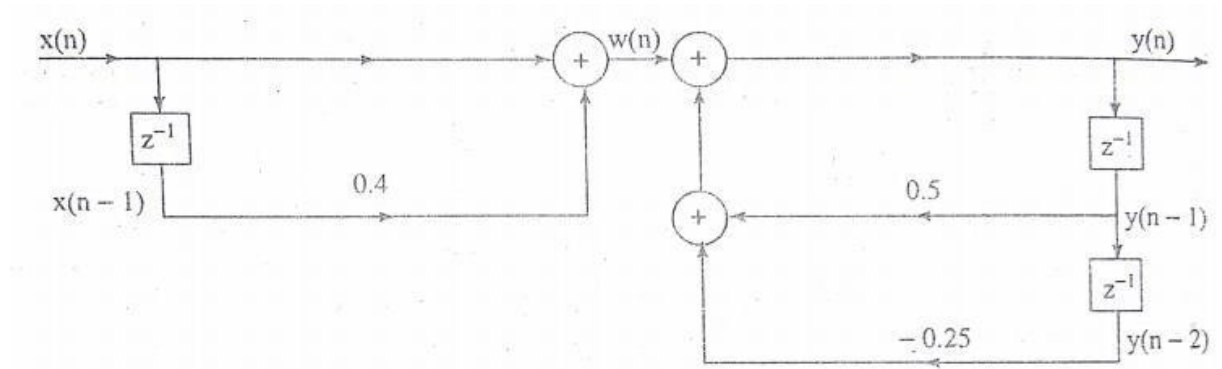
Realizing Eq. (1), we get



Similarly realization of Eq. (2) is shown in Fig.



Combining both figures yields



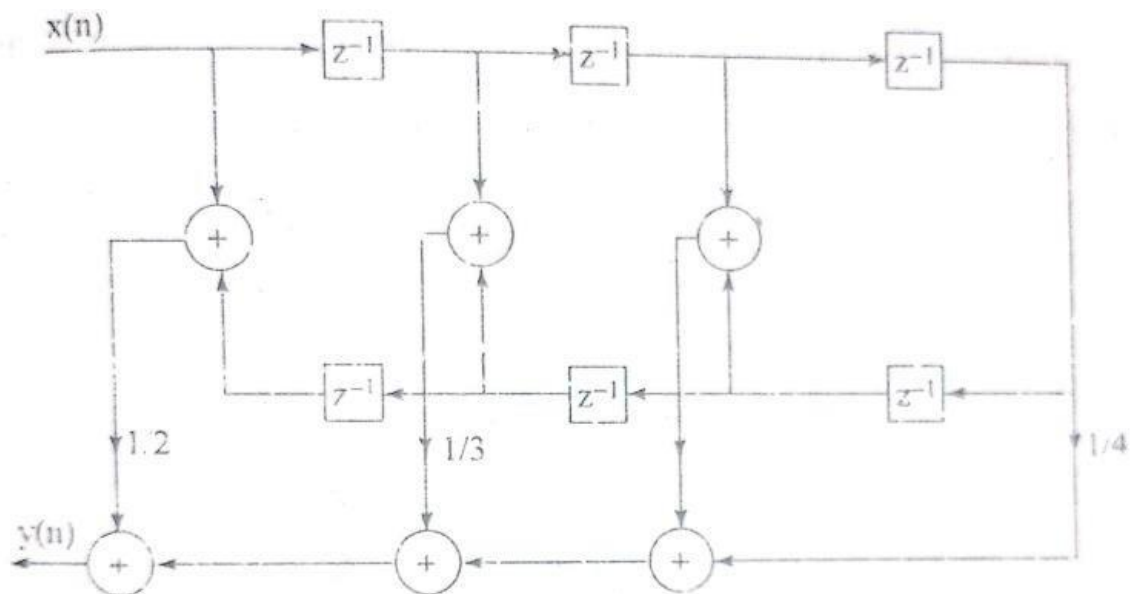
(b) Obtain the linear phase realization of the system function. (Dec 2019)

$$H(z) = \frac{1}{2} + \frac{1}{3}z^{-1} + z^{-2} + \frac{1}{4}z^{-3} + z^{-4} + \frac{1}{3}z^{-5} + \frac{1}{2}z^{-6}$$

By inspection we find that the system $H(z)$ is that of a linear phase FIR filter and

$$h(n) = h(N-1-n)$$

Therefore, we can realize the system function as shown in Fig.



BM T63 – BIOMEDICAL SIGNAL PROCESSING**UNIT-III: DFT AND FFT****PART- A****1. Define Discrete Fourier Transform with example. (May 2016, May 2017, May 2019)**

Discrete Fourier Transform (DFT) is a powerful computation tool which allows us to evaluate the Fourier Transform $X(e^{j\omega})$ on a digital computer or specially designed hardware. Unlike DTFT, which is defined for finite and infinite sequences, DFT defined only for sequences of finite length. Since $X(e^{j\omega})$ is continuous and periodic, DFT is obtained by sampling one period of the Fourier transform at a finite number of frequency points.

Example: (a) Implemented in many signal processing algorithms
(b) Perform linear filtering operations in frequency domain.

2. What is known as radix 2 algorithms? (May 2016)

The FFT algorithm is most efficient in calculating N -point DFT. If the number of output points N can be expressed as a power of 2 that is, $N = 2^M$, where M is an integer, then this algorithm is known as radix-2 FFT algorithm.

3. What is zero padding? What are its uses? (Nov 2016, Dec 2017)

Let the sequence $x(n)$ has a length L . If we want to find the N -point DFT ($N > L$) of the sequence $x(n)$, we have to add $(N - L)$ zeros to the sequence $x(n)$. This is known as zero padding. The uses of padding a sequence with zeros are

- (a) We can get “better display” of the frequency spectrum
- (b) With zero padding, the DFT can be used in linear filtering

4. Differentiate between DIT and DIF algorithm. (Nov 2016)

- (i) For DIT, the input is bit reversal while the output is in natural order, whereas for DIF, the input is in natural order while the output is bit reversal.
- (ii) The DIF butterfly is slightly different from the DIT butterfly, the difference being that the complex multiplication takes place after the add – subtract operation in DIF.

5. What are the applications of FFT algorithms? (May 2017, May 2019)

The applications of FFT algorithm include

- (i) Linear filtering

- (ii) Correlation
- (iii) Spectrum analysis
- (iv) Human speech and hearing use signals with this type of encoding
- (v) Modulation and demodulation of complex data symbols using orthogonal frequency division multiplexing (OFDM)

6. Compute the IDFT of $Y(k) = \{1, 0, 1, 0\}$. (Dec 2017)

$$\begin{aligned}
 y(n) &= \frac{1}{N} \sum_{k=0}^{N-1} Y(k) e^{j2\pi kn/N} \quad n = 0, 1, \dots, N-1 \\
 &= \frac{1}{4} \sum_{k=0}^3 Y(k) e^{j\pi kn/2} \quad n = 0, 1, 2, 3 \\
 y(0) &= \frac{1}{4} \left[\sum_{k=0}^3 Y(k) \right] = \frac{1}{4} [1 + 0 + 1 + 0] = \frac{1}{2} \\
 y(1) &= \frac{1}{4} \left[\sum_{k=0}^3 Y(k) e^{j\pi k/2} \right] = \frac{1}{4} [1 + 0 + 1(-1) + 0] = 0 \\
 y(2) &= \frac{1}{4} \left[\sum_{k=0}^3 Y(k) e^{j\pi k} \right] = \frac{1}{4} [1 + 0 + 1 + 0] = \frac{1}{2} \\
 y(3) &= \frac{1}{4} \left[\sum_{k=0}^3 Y(k) e^{j3\pi k/2} \right] = \frac{1}{4} [1 + 0 - 1 + 0] = 0 \\
 y(n) &= [0.5, 0, 0.5, 0]
 \end{aligned}$$

7. Determine the 4 point DFT for a 3 point sequence $x(n) = \{1, 2, 1\}$. (May 2018)

$$\begin{aligned}
 x(k) &= \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \quad k = 0, 1, \dots, N-1 \\
 &= \sum_{n=0}^3 x(n) e^{-j\pi kn/2} \quad n = 0, 1, 2, 3 \\
 x(0) &= 1; x(1) = 2; x(2) = 1; x(3) = 0 \\
 X(0) &= \left[\sum_{n=0}^3 x(n) \right] = [1 + 2 + 1 + 0] = 4 \\
 X(1) &= \left[\sum_{n=0}^3 x(n) e^{-j\pi n/2} \right] = [1 + 2(-j) + 1(-1) + 0] = -2j \\
 X(2) &= \left[\sum_{n=0}^3 x(n) e^{-j\pi n} \right] = [1 + 2(-1) + 1 + 0] = 0
 \end{aligned}$$

$$x(3) = \left[\sum_{n=0}^3 x(n) e^{-jn3\pi/2} \right] = [1 + 2(j) + 1(-1) + 0] = 2j$$

$$x(k) = [4, -2j, 0, 2j]$$

8. Write the relationship of the DFT to other transforms. (May 2018)

(i) Relation between DFT and z-transform:

- Sampling the z transform at $\omega = \frac{2\pi}{N}k$, we can obtain the DFT.
- The z transform can be obtained from the DFT.

$$\begin{aligned} X(z) &= \sum_{n=0}^{N-1} x(n) z^{-n} = \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn} \right] z^{-n} \\ &= \frac{1}{N} \sum_{k=0}^{N-1} X(k) \left[\sum_{n=0}^{N-1} W_N^{-kn} z^{-n} \right] = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \frac{1 - W_N^{-kN} z^{-N}}{1 - W_N^{-k} z^{-1}} \\ &= \frac{1 - z^{-N}}{N} \sum_{k=0}^{N-1} \frac{X(k)}{1 - W_N^{-k} z^{-1}} \end{aligned}$$

(ii) Relation between DFT and DTFT:

Given a length- N signal $x(n)$, n -point DFT is

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} = \sum_{n=0}^{N-1} x[n] e^{-j \frac{2\pi}{N} nk}, \quad 0 \leq k \leq N-1$$

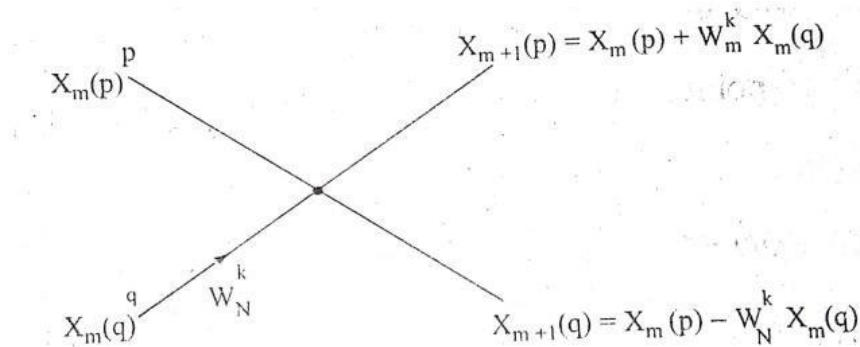
Its DTFT is

$$X(e^{j\Omega}) = \sum_{n=0}^{N-1} x[n] e^{-j\Omega n} \quad -\infty \leq \Omega \leq \infty$$

DFT is the DTFT sampled at N equally spaced frequencies between 0 and 2π

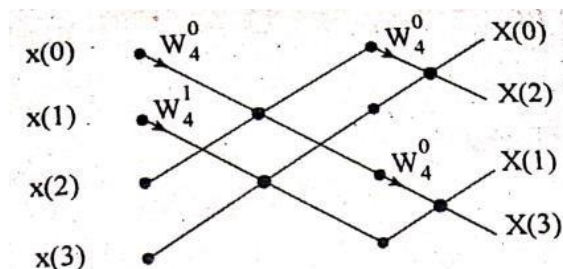
$$X[k] = X(e^{j\Omega}) \Big|_{\Omega = k \frac{2\pi}{N}}, \quad 0 \leq k \leq N-1$$

9. Draw the basic butterfly diagram for DIT – FFT algorithm using radix 2. (Dec 2018)

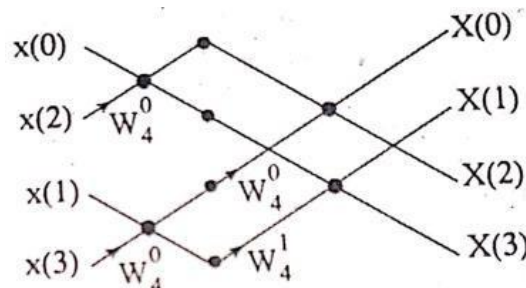


10. Draw the Radix -2 , 4 point butterfly diagram. (Dec 2019)

Radix 2 DIT:



Radix 2 DIF:



11. Define the twiddle factor of FFT. (Dec 2019)

- A twiddle factor, in fast Fourier transform (FFT) algorithms, is any of the trigonometric constant coefficients that are multiplied by the data in the course of the algorithm.
- Twiddle factors are a set of values that is used to speed up DFT and IDFT calculations

Twiddle factors are mathematically represented as

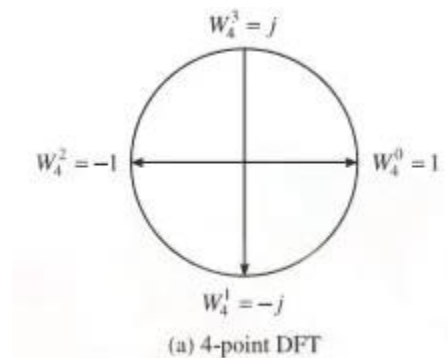
$$W_N = e^{-j2\pi/N}$$

Rewriting the equations for calculating DFT and IDFT using twiddle factors we get:

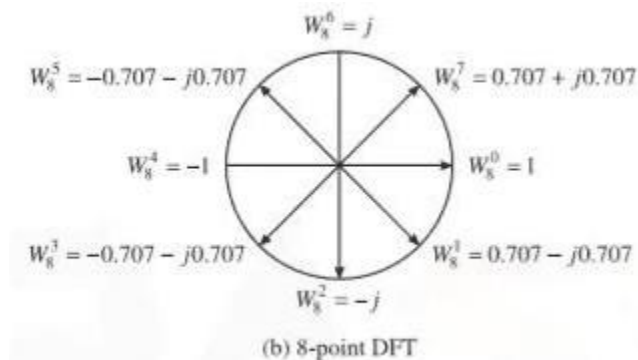
$$\text{DFT: } \sum_{n=0}^{N-1} x(n) W_N^{nk}$$

$$\text{IDFT: } \frac{1}{N} \sum_{n=0}^{N-1} x(k) W_N^{-nk}$$

4 Point:



8 Point:



PART – B

- Determine output response $y(n)$ if $h(n) = \{2,2,2\}$ and $x(n) = \{1,2,3,1\}$ by using linear and circular convolution. (May 2016)**

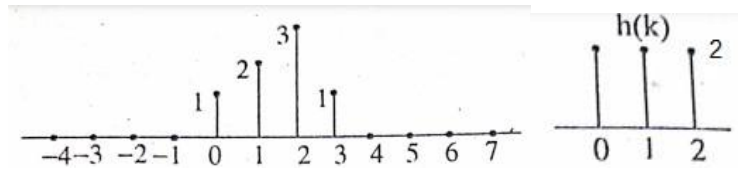
(i) Linear Convolution

Given, $x(n) = \{1,2,3,1\}$, $h(n) = \{2,2,2\}$

We know

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

The number of samples in linear convolution is $L + M - 1 = 4 + 3 - 1 = 6$



$$y(0) = \sum_{k=-\infty}^{\infty} x(k)h(-k) = 2$$

$$y(1) = \sum_{k=-\infty}^{\infty} x(k)h(1-k) = 2 + 4 = 6$$

$$y(2) = \sum_{k=-\infty}^{\infty} x(k)h(2-k) = 2 + 4 + 6 = 12$$

$$y(3) = \sum_{k=-\infty}^{\infty} x(k)h(3-k) = 4 + 6 + 2 = 12$$

$$y(4) = \sum_{k=-\infty}^{\infty} x(k)h(4-k) = 6 + 2 = 8$$

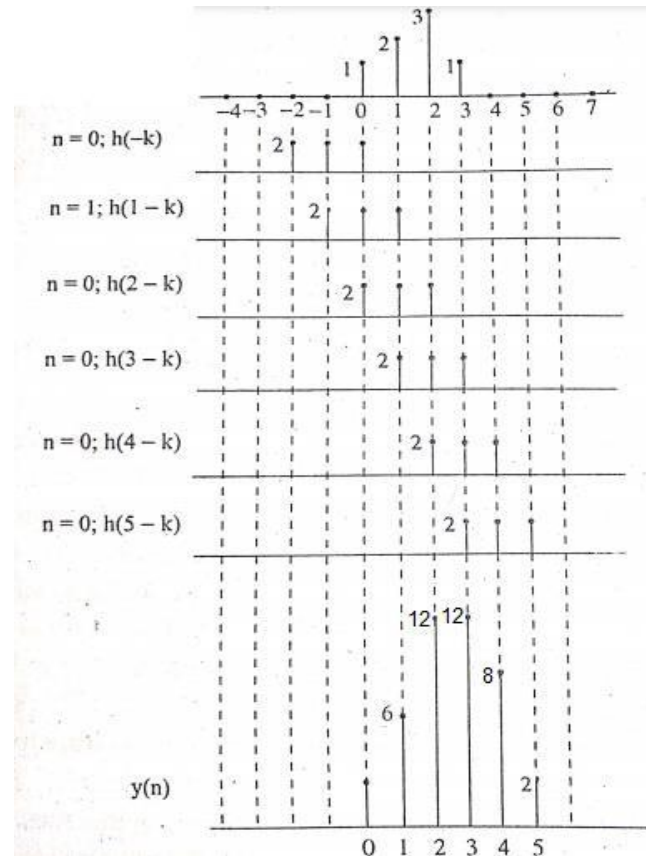
$$y(5) = \sum_{k=-\infty}^{\infty} x(k)h(5-k) = 2$$

$$y(n) = \{2, 6, 12, 12, 8, 2\}$$

(ii) Circular Convolution:

$$\text{Given, } x(n) = \{1, 2, 3, 1\},$$

$$h(n) = \{2, 2, 2, 0\}$$



Using matrix approach we can write $h(n)$ as $N \times N$ matrix form and $x(n)$ as column matrix.

$$\begin{bmatrix} 2 & 0 & 2 & 2 \\ 2 & 2 & 0 & 2 \\ 2 & 2 & 2 & 0 \\ 0 & 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2+0+6+2=10 \\ 2+4+0+2=8 \\ 2+4+6+0=12 \\ 0+4+6+2=12 \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \\ 12 \\ 12 \end{bmatrix}$$

$$y(n) = x(n) \otimes h(n) = \{10, 8, 12, 12\}$$

(iii) Circular Convolution with Zero padding:

To get the result of linear convolution with circular convolution we have to add appropriate number of zeros to both sequences.

$$x(n) = \{1, 2, 3, 1, 0, 0\}$$

$$h(n) = \{2, 2, 2, 0, 0, 0\}$$

$$\begin{bmatrix} 2 & 0 & 0 & 0 & 2 & 2 \\ 2 & 2 & 0 & 0 & 0 & 2 \\ 2 & 2 & 2 & 0 & 0 & 0 \\ 0 & 2 & 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 & 2 & 0 \\ 0 & 0 & 0 & 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2+0+0+0+0+0=2 \\ 2+4+0+0+0+0=6 \\ 2+4+6+0+0+0=12 \\ 0+4+6+2+0+0=12 \\ 0+0+6+2+0+0=8 \\ 0+0+0+2+0+0=2 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \\ 12 \\ 12 \\ 8 \\ 2 \end{bmatrix}$$

$$y(n) = \{2, 6, 12, 12, 8, 2\}$$

2. Write short notes on FFT algorithm with butterfly structure as example. (May 2016)

The FFT algorithm is most efficient in calculating N -point DFT. If the number of output points N can be expressed as a power of 2 that is, $N = 2^M$, where M is an integer, then this algorithm is known as radix-2 FFT algorithm.

DIT Algorithm:

Decimation-in-time algorithm is used to calculate the DFT of a N point sequence. The idea is to break the N -point sequence into two sequences, the DFTs of which can be obtained to give the DFT of the original N -point sequence. Initially the N -point

sequence is divided into two $\frac{N}{2}$ -point sequences $x_e(n)$ and $x_o(n)$, which have even

and odd members of $x(n)$ respectively. The $\frac{N}{2}$ -point DFTs of these two sequences are

evaluated and combined to give N -point DFT. Similarly the $\frac{N}{2}$ -point DFTs can be

expressed as a combination of $\frac{N}{4}$ -point DFTs. This process is continued until we are

left with 2-point DFT. This algorithm is called decimation in time because the sequence $x(n)$ is often split into smaller subsequences.

DIF Algorithm:

In this output sequence $X(k)$ is divided into smaller and smaller subsequences, that is why the name decimation in frequency. Initially the input sequence $x(n)$ is divided into

two sequences $x_1(n)$ and $x_2(n)$ consisting of first $\frac{N}{2}$ -samples of $x(n)$ and the last $\frac{N}{2}$ -

samples of $x(n)$ respectively. Then, we find the $\frac{N}{2}$ -point sequences $f(n)$ and $g(n)$ as

$$f(n) = x_1(n) + x_2(n) \text{ and } g(n) = [x_1(n) - x_2(n)]W_N^n$$

The $\frac{N}{2}$ DFT as a combination of two $\frac{N}{2}$ -point DFTs. This process is continued until we are left with 2 point DFT.

Differences between DIT and DIF algorithm:

- (i) For DIT, the input is bit reversal while the output is in natural order, whereas for DIF, the input is in natural order while the output is bit reversal.
- (ii) The DIF butterfly is slightly different from the DIT butterfly, the difference being that the complex multiplication takes place after the add – subtract operation in DIF.

Basic operation of DIT and DIF algorithm:

The basic operation of DIT algorithm is the so called butterfly in which two inputs $X_m(p)$ and $X_m(q)$ are combined to give the outputs $X_{m+1}(p)$ and $X_{m+1}(q)$ via. the operation

$$X_{m+1}(p) = X_m(p) + W_N^k X_m(q)$$

$$X_{m+1}(q) = X_m(p) - W_N^k X_m(q)$$

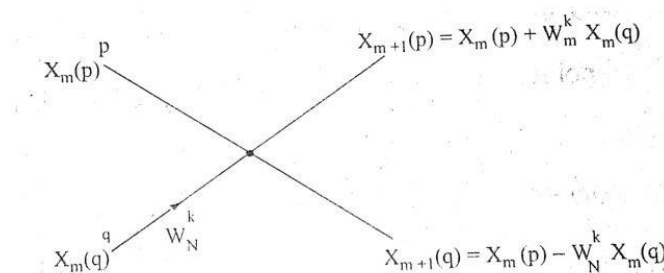
Similarly for DIF algorithm,

$$X_{m+1}(p) = X_m(p) + X_m(q)$$

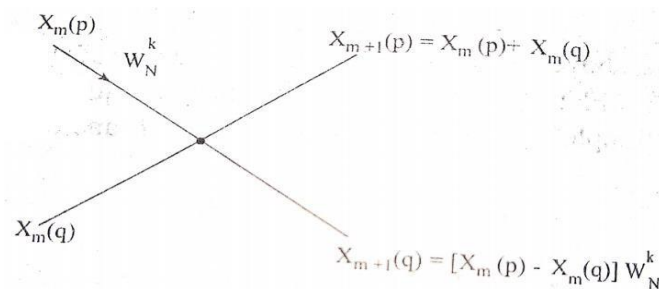
$$X_{m+1}(q) = [X_m(p) - X_m(q)]W_N^k$$

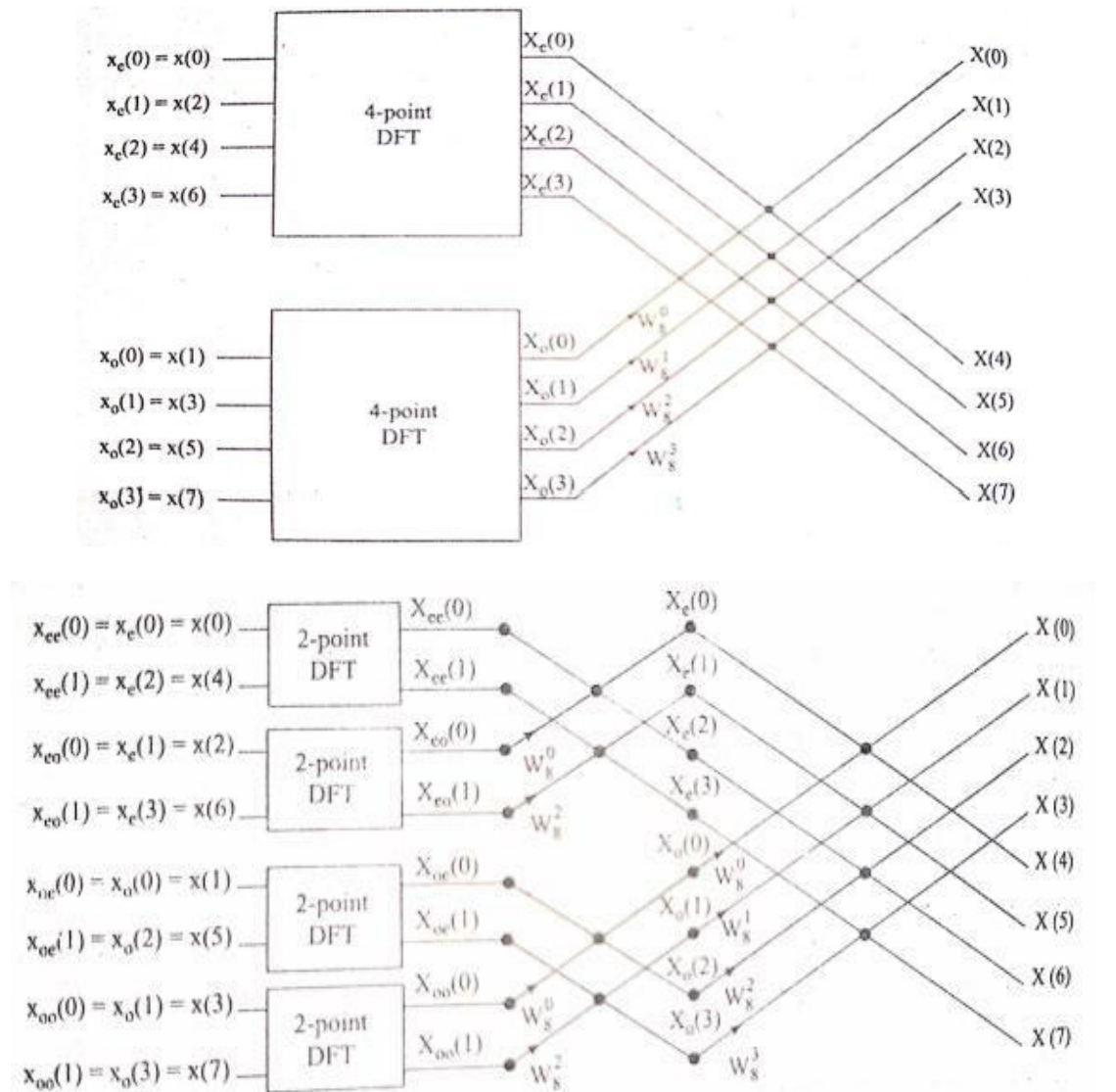
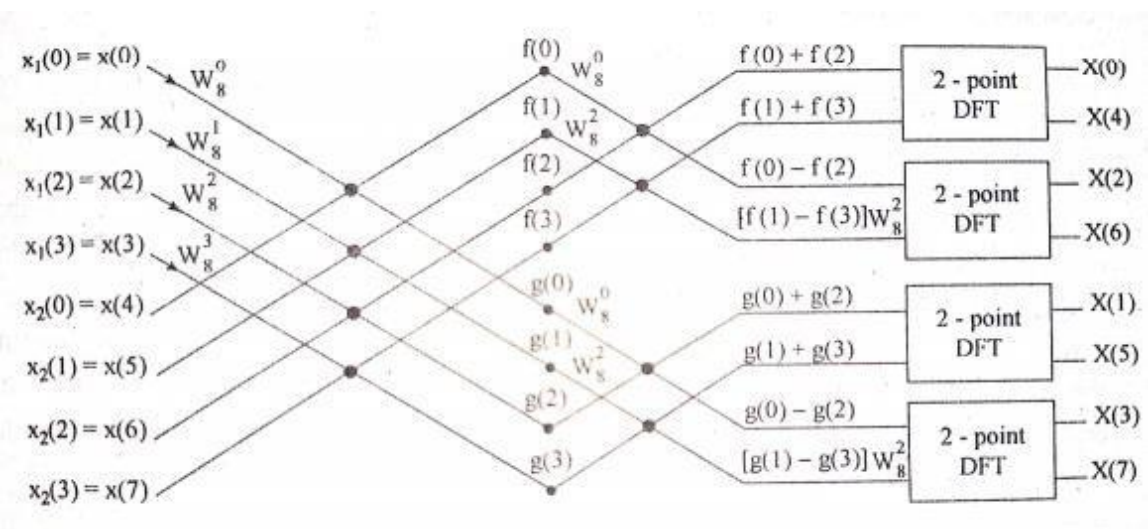
where W_N^k is twiddle factor.

Basic butterfly diagram of DIT algorithm is



Basic butterfly diagram of DIF algorithm is



DIT Radix 2 diagram:**DIF Radix 2 diagram:**

3. Determine the 8 point DFT of the signal $x(n) = \{1,1,1,1,0,0,0,0\}$ and sketch its magnitude and phase. (Nov 2016)

Given

$$x(n) = \{1,1,1,1,0,0,0,0\}$$

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}; \quad k = 0,1,\dots,N-1$$

$$= \sum_{n=0}^{N-1} x(n)e^{-j\pi kn/4}; \quad k = 0,1,2,3,4,5,6,7$$

$$X(0) = \sum_{n=0}^7 x(n) = x(0) + x(1) + x(2) + x(3) + x(4) + x(5) + x(6) + x(7)$$

$$= 1 + 1 + 1 + 1 + 0 + 0 + 0 + 0 = 4$$

$$X(1) = \sum_{n=0}^7 x(n)e^{-j\pi n/4}$$

$$= x(0) + x(1)e^{-j\pi/4} + x(2)e^{-j\pi/2} + x(3)e^{-j3\pi/4} + x(4)e^{-j\pi} + x(5)e^{-j5\pi/4} + x(6)e^{-j3\pi/2} + x(7)e^{-j7\pi/4}$$

$$= 1 + 0.707 - j0.707 - j + (-0.707 - j0.707)$$

$$= 1 - j2.414$$

Since $x(4) = x(5) = x(6) = x(7) = 0$, we consider first four terms only

$$X(2) = \sum_{n=0}^7 x(n)e^{-j\pi n/2}$$

$$= x(0) + x(1)e^{-j\pi/2} + x(2)e^{-j\pi} + x(3)e^{-j3\pi/2}$$

$$= 1 + 1(-j) + 1(-1) + 1(j)$$

$$= 0$$

$$X(3) = \sum_{n=0}^7 x(n)e^{-j3\pi n/4}$$

$$= x(0) + x(1)e^{-j3\pi/4} + x(2)e^{-j3\pi/2} + x(3)e^{-j9\pi/4}$$

$$= 1 + (-0.707 - j0.707) + 1(j) + 1(0.707 - j0.707)$$

$$= 1 - j0.414$$

$$X(4) = \sum_{n=0}^7 x(n)e^{-j\pi n}$$

$$= x(0) + x(1)e^{-j\pi} + x(2)e^{-j2\pi} + x(3)e^{-j3\pi}$$

$$= 1 - 1 + 1 - 1$$

$$= 0$$

$$\begin{aligned}
 X(5) &= \sum_{n=0}^7 x(n)e^{-j5\pi n/4} \\
 &= x(0) + x(1)e^{-j5\pi/4} + x(2)e^{-j5\pi/2} + x(3)e^{-j15\pi/4} \\
 &= 1 + 1(-0.707 + j0.707) + 1(-j) + 1(0.707 + j0.707) \\
 &= 1 + j0.414
 \end{aligned}$$

$$\begin{aligned}
 X(6) &= \sum_{n=0}^7 x(n)e^{-j3\pi n/2} \\
 &= x(0) + x(1)e^{-j3\pi/2} + x(2)e^{-j3\pi} + x(3)e^{-j9\pi/2} \\
 &= 1 + (j) + 1(-1) + 1(j) \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 X(7) &= \sum_{n=0}^7 x(n)e^{-j7\pi n/4} \\
 &= x(0) + x(1)e^{-j7\pi/4} + x(2)e^{-j7\pi/2} + x(3)e^{-j21\pi/4} \\
 &= 1 + 1(0.707 + j0.707) + 1(j) + 1(-0.707 + j0.707) \\
 &= 1 + j2.414
 \end{aligned}$$

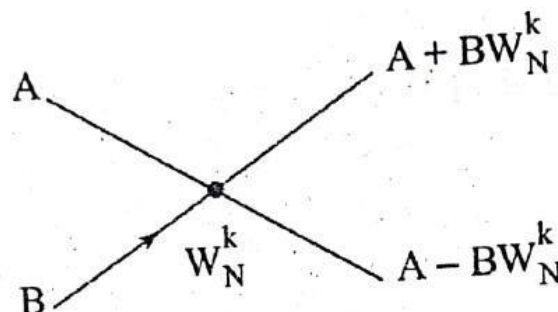
$$X(k) = \{4, 1 - j2.414, 0, 1 - j0.414, 0, 1 + j0.414, 0, 1 + j2.414\}.$$

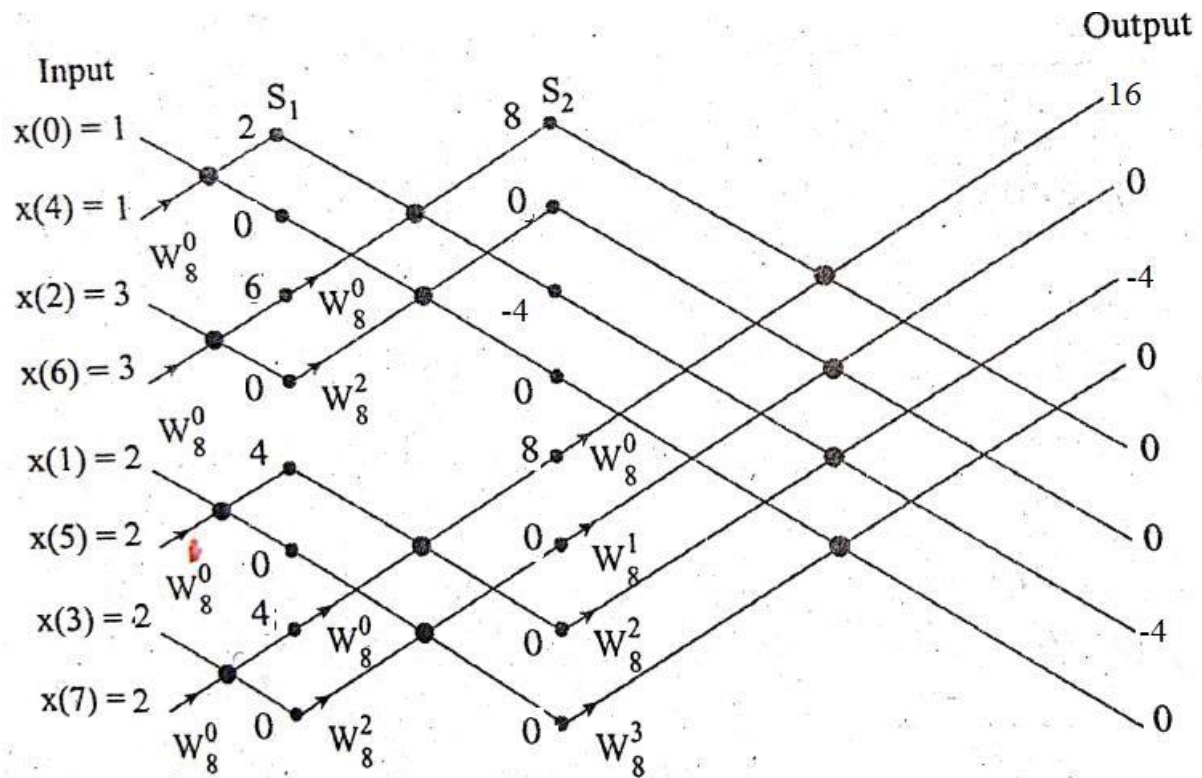
4. Find the DFT of a sequence $x(n) = \{1, 2, 3, 2, 1, 2, 3, 2\}$ using DIT algorithm. (Nov 2016)

The twiddle factors associated with the flow graph are

$$\begin{aligned}
 W_8^0 &= 1 \\
 W_8^1 &= (e^{-j2\pi/8})^1 = e^{-j\pi/4} = 0.707 - j0.707 \\
 W_8^2 &= (e^{-j2\pi/8})^2 = e^{-j\pi/2} = -j \\
 W_8^3 &= (e^{-j2\pi/8})^3 = e^{-j3\pi/4} = -0.707 - j0.707
 \end{aligned}$$

The basic operation is





Input		Output of Stage 1	Output of Sate 2	Output
$x(0)$	1	$1+1 = 2$	$2+6 = 8$	$8+8 = 16$
$x(4)$	1	$1-1 = 0$	$0+(0-j) = 0$	$0+0(0.707-j0.707)=0$
$x(2)$	3	$3+3 = 6$	$2-6 = -4$	-4
$x(6)$	3	$3-3 = 0$	$0-0(-j) = 0$	$0+0(-0.707-j0.707) = 0$
$x(1)$	2	$2+2 = 4$	$4+4 = 8$	$8-8 = 0$
$x(5)$	2	$2-2 = 0$	$0+0(-j)=0$	$0-0(0.707-j0.707) = 0$
$x(3)$	2	$2+2 = 4$	$4-4 = 0$	-4
$x(7)$	2	$2-2 = 0$	$0-0(-j) = 0$	$0+0(-0.707-j0.707) = 0$

$$X(k) = \{16, 0, -4, 0, 0, 0, -4, 0\}$$

5. Determine output response $y(n)$ if $h(n) = \{1, 1, 1\}$ and $x(n) = \{1, 2, 3, 1\}$ by using linear and circular convolution. (May 2017, May 2019)

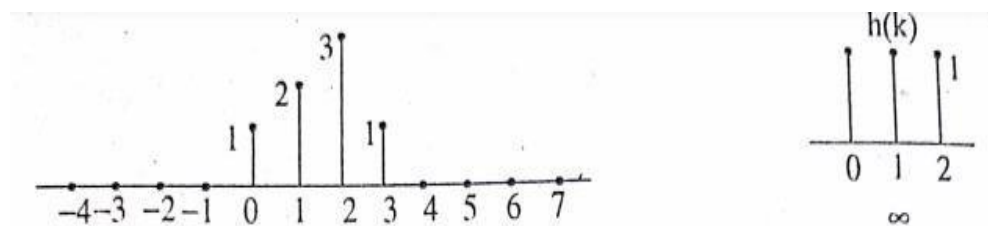
(i) Linear Convolution

Given, $x(n) = \{1, 2, 3, 1\}$, $h(n) = \{1, 1, 1\}$

We know

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

The number of samples in linear convolution is $L + M - 1 = 4 + 3 - 1 = 6$



$$y(0) = \sum_{k=-\infty}^{\infty} x(k)h(-k) = 1$$

$$y(1) = \sum_{k=-\infty}^{\infty} x(k)h(1-k) = 1 + 2 = 3$$

$$y(2) = \sum_{k=-\infty}^{\infty} x(k)h(2-k) = 1 + 2 + 3 = 6$$

$$y(3) = \sum_{k=-\infty}^{\infty} x(k)h(3-k) = 2 + 3 + 1 = 6$$

$$y(4) = \sum_{k=-\infty}^{\infty} x(k)h(4-k) = 3 + 1 = 4$$

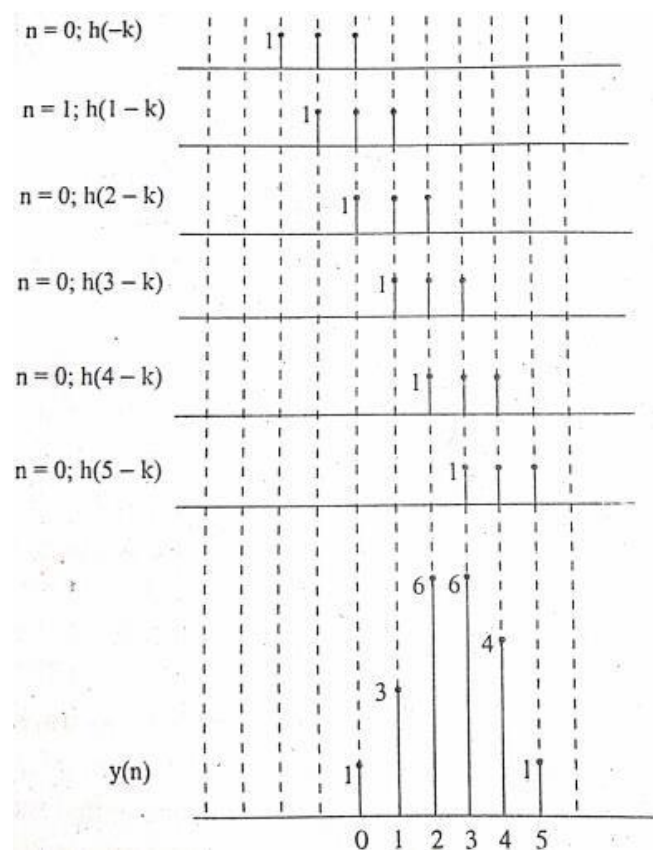
$$y(5) = \sum_{k=-\infty}^{\infty} x(k)h(5-k) = 1$$

$$y(n) = \{1, 3, 6, 6, 4, 1\}$$

(ii) Circular Convolution:

Given, $x(n) = \{1, 2, 3, 1\}$,

$h(n) = \{1, 1, 1, 0\}$



Using matrix approach we can write $h(n)$ as $N \times N$ matrix form and $x(n)$ as column matrix.

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+0+3+1=5 \\ 1+2+0+1=4 \\ 1+2+3+0=6 \\ 0+2+3+1=6 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 6 \\ 6 \end{bmatrix}$$

$$y(n) = x(n) \otimes h(n) = \{5, 4, 6, 6\}$$

(iii) Circular Convolution with Zero padding:

To get the result of linear convolution with circular convolution we have to add appropriate number of zeros to both sequences.

$$x(n) = \{1, 2, 3, 1, 0, 0\}$$

$$h(n) = \{1, 1, 1, 0, 0, 0\}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+0+0+0+0+0=1 \\ 1+2+0+0+0+0=3 \\ 1+2+3+0+0+0=6 \\ 0+2+3+1+0+0=6 \\ 0+0+3+1+0+0=4 \\ 0+0+0+1+0+0=1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 6 \\ 6 \\ 4 \\ 1 \end{bmatrix}$$

$$y(n) = \{1, 3, 6, 6, 4, 1\}$$

6. Find the DFT of a sequence $x(n) = \{1, 2, 3, 2, 1, 2, 3, 2\}$ using DIF algorithm. (May 2017)

The twiddle factors associated with the flow graph are

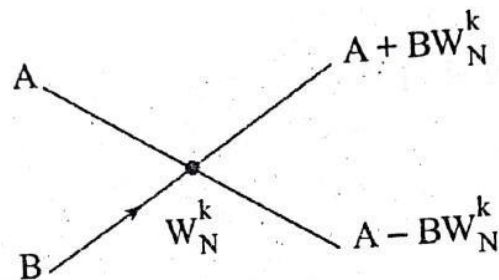
$$W_8^0 = 1$$

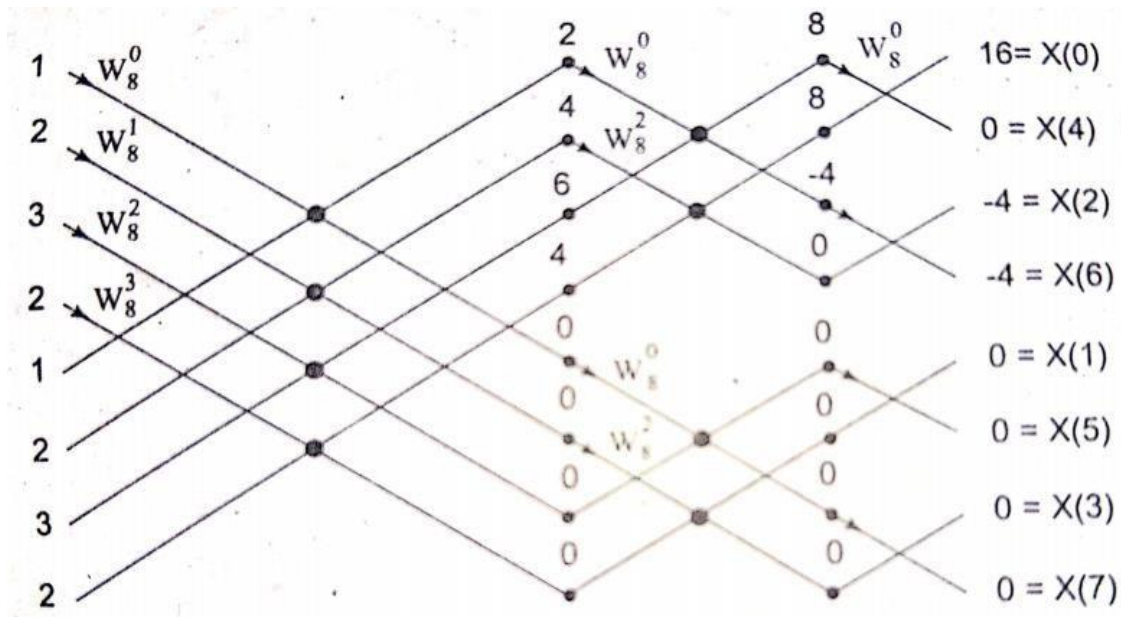
$$W_8^1 = (e^{-j2\pi/8})^1 = e^{-j\pi/4} = 0.707 - j0.707$$

$$W_8^2 = (e^{-j2\pi/8})^2 = e^{-j\pi/2} = -j$$

$$W_8^3 = (e^{-j2\pi/8})^3 = e^{-j3\pi/4} = -0.707 - j0.707$$

The basic operation is





Input		Output of Stage 1	Output of Sate 2	Output	
$x(0)$	1	$1+1 = 2$	$2+6 = 8$	$8+8 = 16$	$x(0)$
$x(1)$	2	$2+2 = 4$	$4+4 = 8$	$8-8 = 0$	$x(4)$
$x(2)$	3	$3+3 = 6$	$2-6 = -4$	-4	$x(2)$
$x(3)$	2	$2+2 = 4$	$4-4 = 0$	-4	$x(6)$
$x(4)$	1	$(1-1)(1) = 0$	$0+0 = 0$	$0+0 = 0$	$x(1)$
$x(5)$	2	$(2-2)(0.707-j0.707) = 0$	$0+0 = 0$	$0+0 = 0$	$x(5)$
$x(6)$	3	$(3-3)(-j) = 0$	$0(1)+0 = 0$	$0+0 = 0$	$x(3)$
$x(7)$	2	$(2-2)(-0.707-j0.707) = 0$	$0(-j)+0 = 0$	$0+0 = 0$	$x(7)$

$$X(k) = \{16, 0, -4, 0, 0, 0, -4, 0\}$$

7. Find DFT for $x(n) = \{1, 1, 2, 0, 1, 2, 0, 1\}$ using FFT DIT butterfly algorithm. (Dec 2017)

The twiddle factors associated with the flow graph are

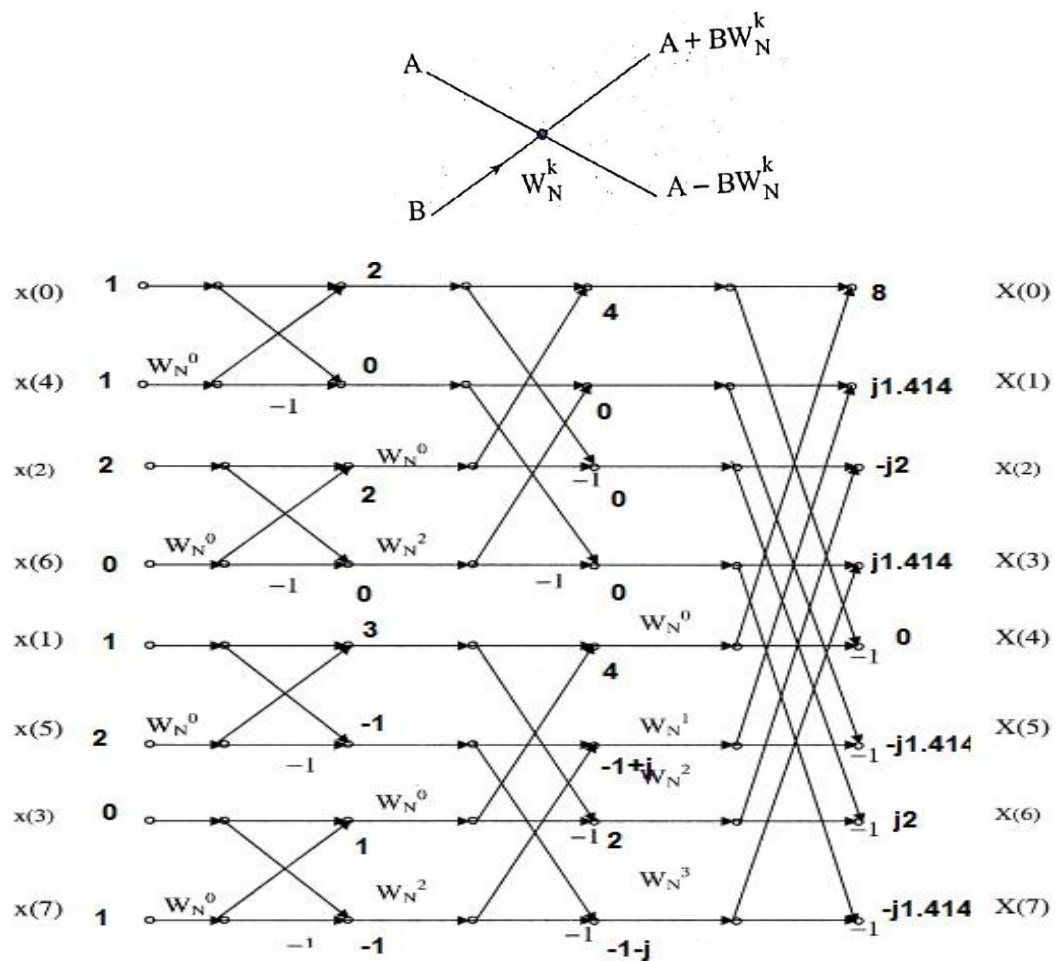
$$W_8^0 = 1$$

$$W_8^1 = (e^{-j2\pi/8})^1 = e^{-j\pi/4} = 0.707 - j0.707$$

$$W_8^2 = (e^{-j2\pi/8})^2 = e^{-j\pi/2} = -j$$

$$W_8^3 = (e^{-j2\pi/8})^3 = e^{-j3\pi/4} = -0.707 - j0.707$$

The basic operation is



Input	Output of Stage 1	Output of Sate 2	Output
x(0)	1	$1+1 = 2$	$2+2 = 4$
x(4)	1	$1-1 = 0$	$0+(-1+j)(0.707-j0.707) = j1.414$
x(2)	2	$2+0 = 2$	$2-2 = 0$
x(6)	0	$2-0 = 0$	$0+(-1-j)(-0.707-j0.707) = j1.414$
x(1)	1	$1+2 = 3$	$3+1 = 4$
x(5)	2	$1-2 = -1$	$-1-1(-j) = -1+j$
x(3)	0	$0+1 = 1$	$3-1 = 2$
x(7)	1	$0-1 = -1$	$-1-(-1)(-j) = -1-j$

$$X(k) = \{8, j1.414, -j2, j1.414, 0, -j1.414, j2, -j1.414\}$$

8. A 8 point sequence is given by $x(n) = 1$ for $-3 \leq n \leq 3$ and $x(n) = 0$ otherwise. Compute a 8 point DFT of $x(n)$ by butterfly using DIF method. (Dec 2017)

From the given sequence $x(n) = \{1, 1, 1, 1, 1, 1, 1, 0\}$

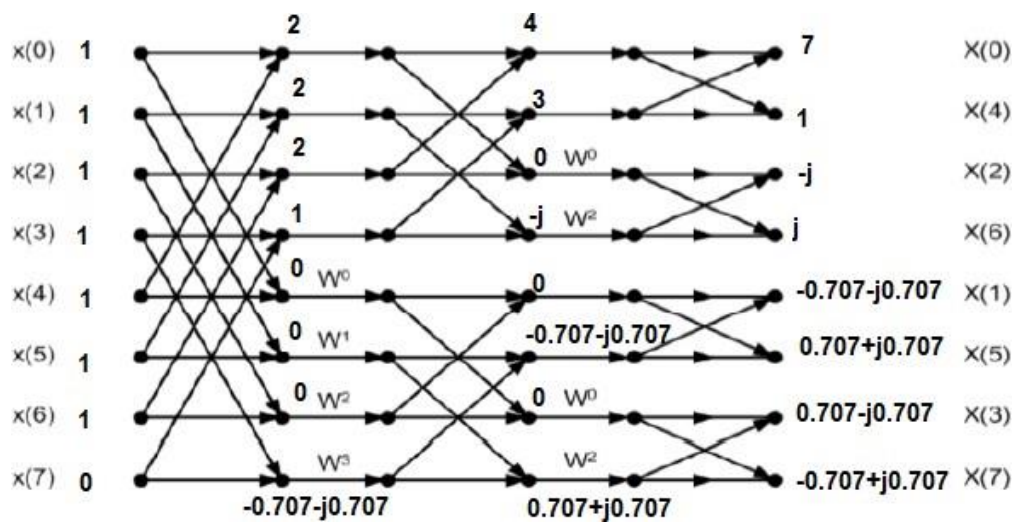
The twiddle factors associated with the flow graph are

$$W_8^0 = 1$$

$$W_8^1 = (e^{-j2\pi/8})^1 = e^{-j\pi/4} = 0.707 - j0.707$$

$$W_8^2 = (e^{-j2\pi/8})^2 = e^{-j\pi/2} = -j$$

$$W_8^3 = (e^{-j2\pi/8})^3 = e^{-j3\pi/4} = -0.707 - j0.707$$



Input		Output of Stage 1	Output of Sate 2	Output	
$x(0)$	1	$1+1 = 2$	$2+2 = 4$	$4+3 = 7$	$x(0)$
$x(1)$	1	$1+1 = 2$	$2+1 = 3$	$4-3 = 1$	$x(4)$
$x(2)$	1	$1+1 = 2$	$2-2 = 0$	$-j$	$x(2)$
$x(3)$	1	$1+0 = 1$	$2-1(-j) = -j$	$0+(-1)(-j) = j$	$x(6)$
$x(4)$	1	$(1-1)(1) = 0$	$0+0(-j) = 0$	$-0.707-j0.707$	$x(1)$
$x(5)$	1	$(1-1)(0.707-j0.707) = 0$	$0+(-0.707-j0.707) = -0.707-j0.707$	$0.707+j0.707$	$x(5)$
$x(6)$	1	$(1-1)(-j) = 0$	$0+0 = 0$	$0.707-j0.707$	$x(3)$
$x(7)$	0	$(1-0)(-0.707-j0.707) = -0.707-j0.707$	$0(-j)+(-1)(-0.707-j0.707) = 0.707+j0.707$	$-0.707+j0.707$	$x(7)$

$$X(k) = \{7, -0.707 - j0.707, -j, 0.707 - j0.707, 1, 0.707 + j0.707, j, -0.707 + j0.707\}$$

9. Given $x(n) = \{0, 1, 2, 3, 4, 5, 6, 7\}$ **find** $x(k)$ **using DIT FFT algorithm.** (May 2018)

The twiddle factors associated with the flow graph are

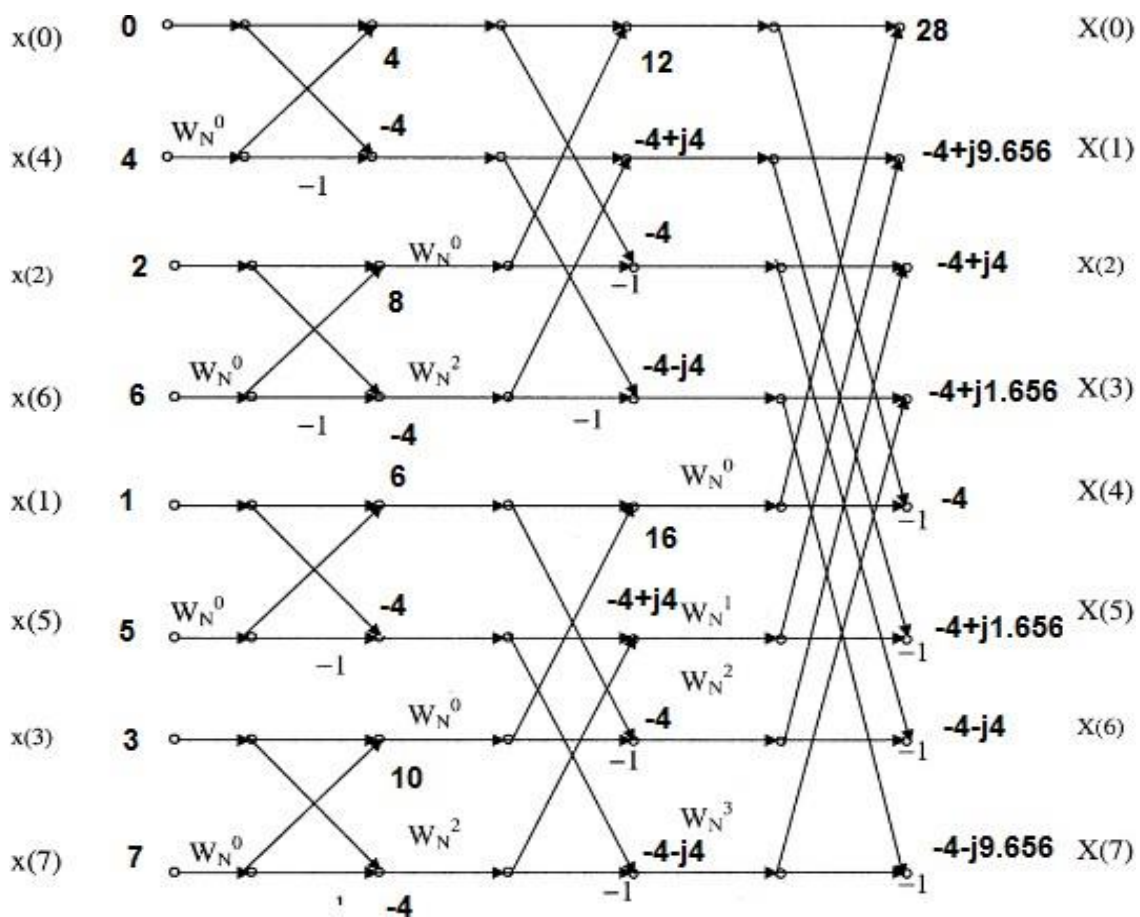
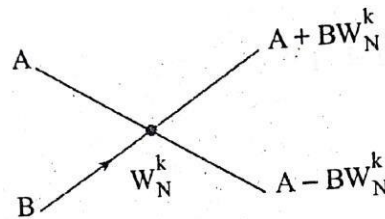
$$W_8^0 = 1$$

$$W_8^1 = (e^{-j2\pi/8})^1 = e^{-j\pi/4} = 0.707 - j0.707$$

$$W_8^2 = (e^{-j2\pi/8})^2 = e^{-j\pi/2} = -j$$

$$W_8^3 = (e^{-j2\pi/8})^3 = e^{-j3\pi/4} = -0.707 - j0.707$$

The basic operation is



Input		Output of Stage 1	Output of Sate 2	Output
$x(0)$	0	$0+4 = 4$	$4+8 = 12$	$12+16 = 28$
$x(4)$	4	$0-4 = -4$	$-4+(-4)(-j) = -4+j4$	$-4+j9.656$
$x(2)$	2	$2+6 = 8$	$4-8 = -4$	$-4+(-4)(-j) = -4+j4$
$x(6)$	6	$2-6 = -4$	$-4+(-1)(j4) = -4-j4$	$-4+j1.656$
$x(1)$	1	$1+5 = 6$	$6+10 = 16$	$12-16 = -4$
$x(5)$	5	$1-5 = -4$	$-4+(-4)(-j) = -4+j4$	$-4-j1.656$
$x(3)$	3	$3+7 = 10$	$6-10 = -4$	$-4+(-1)(j4) = -4-j4$
$x(7)$	7	$3-7 = -4$	$-4+(-1)(j4) = -4-j4$	$-4-j9.656$
$X(k) = \{28, -4 + j9.656, -4 + j4, -4 + j1.656, -4, -4 + j1.656, -4 - j4, -4 - j9.656\}$				

10. Find the inverse DFT of $X(k) = \{1, 2, 3, 4\}$. (May 2018)

Given $X(k) = \{1, 2, 3, 4\}$

$$x(0) = 1; x(1) = 2; x(3) = 3; x(4) = 4$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N} \quad ; n = 0, 1, \dots, N-1$$

For $N = 4$

$$x(n) = \frac{1}{4} \sum_{k=0}^3 X(k) e^{j\pi kn/2} ; n = 0, 1, 2, 3$$

$$\begin{aligned} x(0) &= \frac{1}{4} \sum_{k=0}^3 X(k) = \frac{1}{4} [x(0) + x(1) + x(2) + x(3)] \\ &= \frac{1}{4} [1 + 2 + 3 + 4] = \frac{10}{4} = \frac{5}{2} \end{aligned}$$

$$\begin{aligned} x(1) &= \frac{1}{4} \sum_{k=0}^3 X(k) e^{j\pi k/2} = \frac{1}{4} [x(0) + x(1)e^{j\pi/2} + x(2)e^{j\pi} + x(3)e^{j3\pi/2}] \\ &= \frac{1}{4} [1 + 2(j) + 3(-1) + 4(-j)] = \frac{1 + j2 - 3 - j4}{4} \\ &= \frac{-2 - j2}{4} \end{aligned}$$

$$\begin{aligned}
 x(2) &= \frac{1}{4} \sum_{k=0}^3 X(k) e^{j\pi k} = \frac{1}{4} [x(0) + x(1)e^{j\pi} + x(2)e^{j2\pi} + x(3)e^{j3\pi}] \\
 &= \frac{1}{4} [1 + 2(-1) + 3(1) + 4(-1)] = \frac{1 - 2 + 3 - 4}{4} \\
 &= \frac{-2}{4} = \frac{-1}{2} \\
 x(3) &= \frac{1}{4} \sum_{k=0}^3 X(k) e^{j3\pi k/2} = \frac{1}{4} [x(0) + x(1)e^{j3\pi/2} + x(2)e^{j3\pi} + x(3)e^{j9\pi/2}] \\
 &= \frac{1}{4} [1 + 2(-j) + 3(-1) + 4(j)] = \frac{1 - j2 - 3 + j4}{4} \\
 &= \frac{-2 + j2}{4} \\
 x(n) &= \left\{ \frac{5}{2}, \frac{-2 - j2}{4}, \frac{-1}{2}, \frac{-2 + j2}{4} \right\}
 \end{aligned}$$

11. Find the circular convolution for the sequence $x_1(n) = \{1, 2, 3, 4, 5\}$ and

$x_2(n) = \{1, 2, 2, 1, 0\}$. (Dec 2018)

(i) Circular Convolution:

Given, $x_1(n) = \{1, 2, 3, 4, 5\}$, $x_2(n) = \{1, 2, 2, 1, 0\}$

Using matrix approach we can write $x_2(n)$ as $N \times N$ matrix form and $x_1(n)$ as column matrix.

$$\begin{bmatrix} 1 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 1 & 2 \\ 2 & 2 & 1 & 0 & 1 \\ 1 & 2 & 2 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 1+0+3+8+10=22 \\ 2+2+0+4+10=18 \\ 2+4+3+0+5=14 \\ 1+4+6+4+0=15 \\ 0+2+6+8+5=21 \end{bmatrix} = \begin{bmatrix} 22 \\ 18 \\ 14 \\ 15 \\ 21 \end{bmatrix}$$

$$y(n) = x(n) \otimes h(n) = \{22, 18, 14, 15, 21\}$$

(ii) Circular Convolution with Zero padding:

To get the result of linear convolution with circular convolution we have to add appropriate number of zeros to both sequences.

No. of sequences = $L+M-1 = 5+5-1 = 9$

$$x_1(n) = \{1, 2, 3, 4, 5, 0, 0, 0, 0\}$$

$$x_2(n) = \{1, 2, 2, 1, 0, 0, 0, 0, 0\}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+0+0+0+0+0+0+0+0=1 \\ 2+2+0+0+0+0+0+0+0=4 \\ 2+4+3+0+0+0+0+0+0=9 \\ 1+4+6+4+0+0+0+0+0=15 \\ 0+2+6+8+5+0+0+0+0=21 \\ 0+0+3+8+10+0+0+0+0=21 \\ 0+0+0+4+10+0+0+0+0=14 \\ 0+0+0+0+5+0+0+0+0=5 \\ 0+0+0+0+0+0+0+0+0=0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 9 \\ 15 \\ 21 \\ 21 \\ 14 \\ 5 \\ 0 \end{bmatrix}$$

$$y(n) = \{1, 4, 9, 15, 21, 21, 14, 5, 0\}$$

12. Find the DFT of the sequence $x(n) = \{1, 2, 3, 4, 4, 3, 2, 1\}$ using DIT – FFT algorithm.
(Dec 2018)

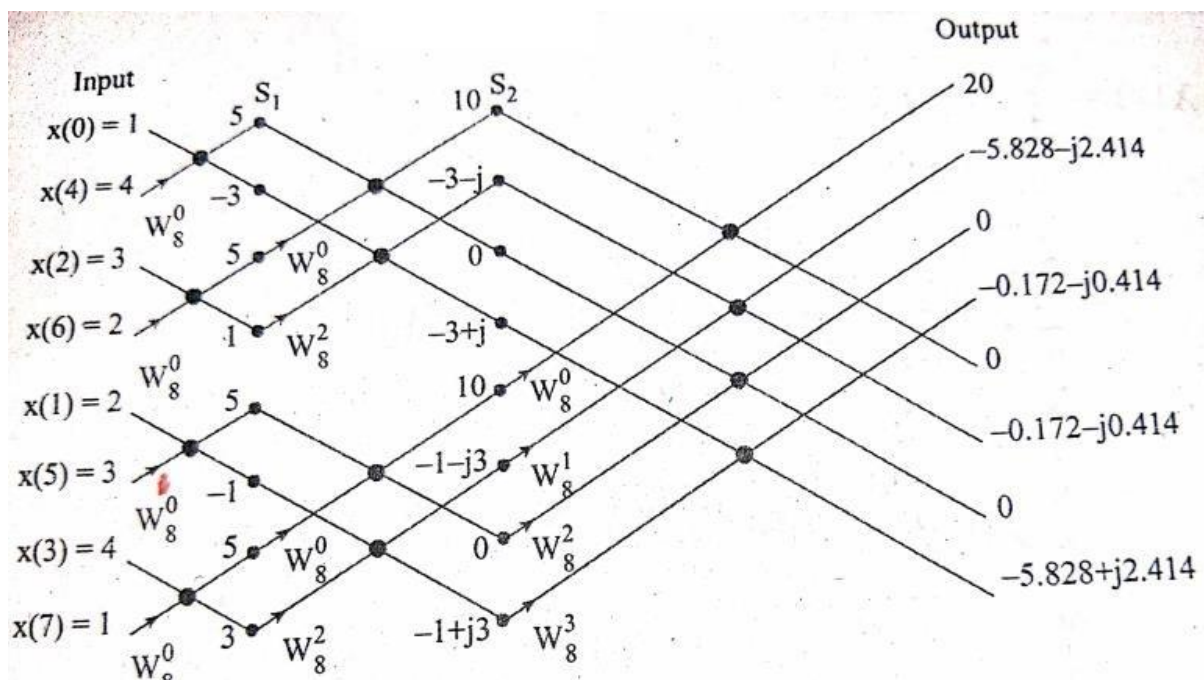
The twiddle factors associated with the flow graph are

$$W_8^0 = 1$$

$$W_8^1 = (e^{-j2\pi/8})^1 = e^{-j\pi/4} = 0.707 - j0.707$$

$$W_8^2 = (e^{-j2\pi/8})^2 = e^{-j\pi/2} = -j$$

$$W_8^3 = (e^{-j2\pi/8})^3 = e^{-j3\pi/4} = -0.707 - j0.707$$



Input		Output of Stage 1	Output of Sate 2	Output
$x(0)$	1	$1+4 = 5$	$5+5 = 10$	$10+10 = 20$
$x(4)$	4	$1-4 = -3$	$-3+(1)(-j) = -3-j$	$-3-j+(0.707-j0.707)(-1-3j) = -5.828-j2.414$
$x(2)$	3	$3+2 = 5$	$5-5 = 0$	0
$x(6)$	2	$3-2 = 1$	$-3-(-j)1 = -3+j$	$(-3+j)+(-1+3j)(-0.707-j0.707) = -0.172-j0.414$
$x(1)$	2	$2+3 = 5$	$5+5 = 10$	$10-10 = 0$
$x(5)$	3	$2-3 = -1$	$-1+(3)(-j) = -1-j3$	$3-j-(0.707-j0.707)(-1-3j) = -0.172+j0.414$
$x(3)$	4	$4+1 = 5$	$5-5 = 0$	0
$x(7)$	1	$4-1 = 3$	$-1-(-j)3 = -1+j3$	$(-3+j)-(-1+3j)(-0.707-j0.707) = -5.828+j0.414$

$$X(k) = \{20, -5.828 - j2.414, 0, -0.172 - j1.414, 0, -0.172 + j0.414, 0, -5.828 + j2.414\}$$

13. Find the DFT of a sequence $x(n) = \{1, 2, 3, 2, 1, 2, 3, 2\}$ using DIT algorithm. (May 2019)

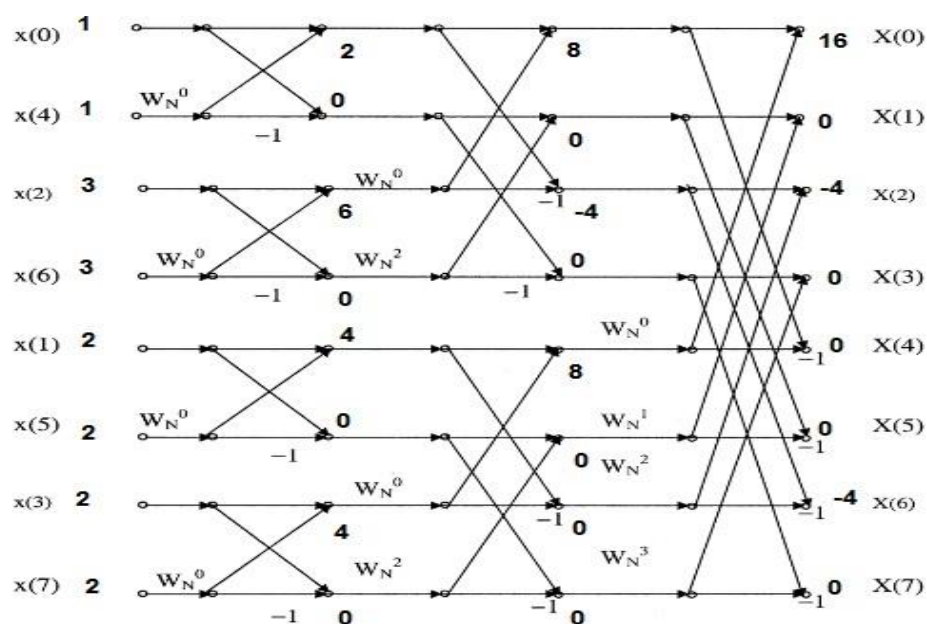
The twiddle factors associated with the flow graph are

$$W_8^0 = 1$$

$$W_8^1 = (e^{-j2\pi/8})^1 = e^{-j\pi/4} = 0.707 - j0.707$$

$$W_8^2 = (e^{-j2\pi/8})^2 = e^{-j\pi/2} = -j$$

$$W_8^3 = (e^{-j2\pi/8})^3 = e^{-j3\pi/4} = -0.707 - j0.707$$



Input		Output of Stage 1	Output of Sate 2		Output
$x(0)$	1	$1+1 = 2$	$2+6 = 8$	16	$x(0)$
$x(4)$	1	$1-1 = 0$	$0+0(-j) = 0$	0	$x(1)$
$x(2)$	3	$3+3 = 6$	$2-6 = -4$	-4	$x(2)$
$x(6)$	3	$3-3 = 0$	$0+0(-j) = 0$	0	$x(3)$
$x(1)$	2	$2+2 = 4$	$4+4 = 8$	0	$x(4)$
$x(5)$	2	$2-2 = 0$	$0+0(-j) = 0$	0	$x(5)$
$x(3)$	2	$2+2 = 4$	$4-4 = 0$	-4	$x(6)$
$x(7)$	2	$2-2 = 0$	$0+0(-j) = 0$	0	$x(7)$

$$X(k) = \{16, 0, -4, 0, 0, 0, -4, 0\}$$

14. Determine the eight point DFT of the sequences $x(n) = \{0, 0, 1, 1, 1, 0, 0, 0\}$ using DIT FFT algorithm. (Dec 2019)

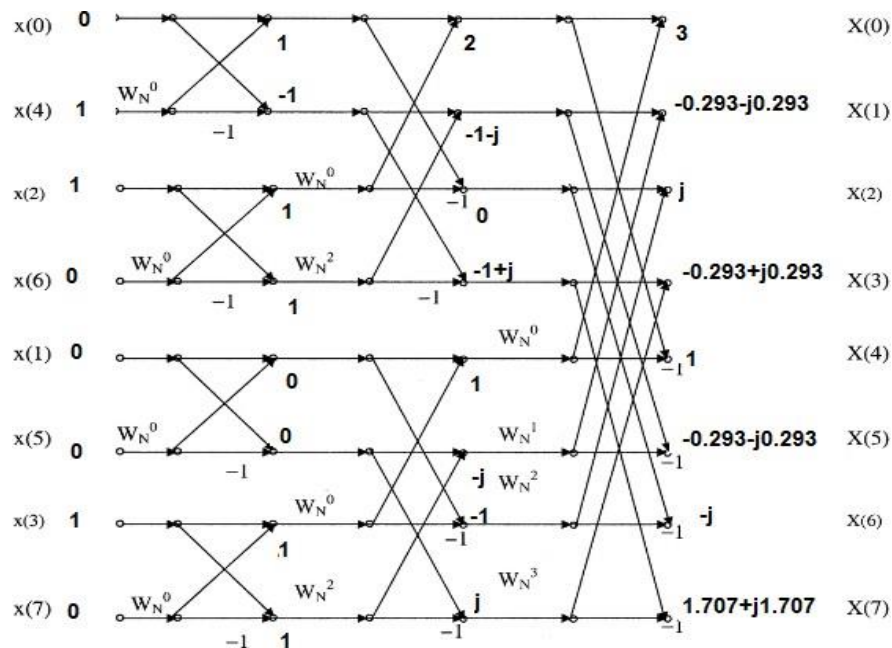
The twiddle factors associated with the flow graph are

$$W_8^0 = 1$$

$$W_8^1 = (e^{-j2\pi/8})^1 = e^{-j\pi/4} = 0.707 - j0.707$$

$$W_8^2 = (e^{-j2\pi/8})^2 = e^{-j\pi/2} = -j$$

$$W_8^3 = (e^{-j2\pi/8})^3 = e^{-j3\pi/4} = -0.707 - j0.707$$



Input		Output of Stage 1	Output of Sate 2	Output	
$x(0)$	0	$0+1 = 1$	$1+1 = 2$	$1+2 = 3$	$x(0)$
$x(4)$	1	$0-1 = -1$	$-1-j$	$(-1-j)+(-j)(-0.707+j0.707)$ $= -0.293-j0.293$	$x(1)$
$x(2)$	1	$1+0 = 1$	0	j	$x(2)$
$x(6)$	0	$1-1 = 0$	$-1+j$	$(-1+j)+(j)(-0.707-j0.707)$ $= -0.293+j0.293$	$x(3)$
$x(1)$	0	0	1	$2-1 = 1$	$x(4)$
$x(5)$	0	0	$-j$	$(-1-j)-(-j)(0.707-j0.707)$ $= -0.293-j0.293$	$x(5)$
$x(3)$	1	$1+0 = 1$	-1	$-j$	$x(6)$
$x(7)$	0	$1-0 = 1$	j	$(-1+j)-(j)(-0.707-j0.707)$ $= -1.707+j1.707$	$x(7)$

$$X(k) = \{3, -0.293 - j0.293, j, -0.293 + 0.293, 1, -0.293 - j0.293, -j, -1.707 + j1.707\}$$

15. Perform the linear convolution for the given sequences $x(n) = \{1, -1, 1\}$ and $h(n) = \{2, 2, 1\}$ using DFT method. (Dec 2019)

The length of both sequences $L = M = 3$. Therefore we add $M - 1 = 2$ zero samples to the sequences $x(n)$ and $L - 1 = 2$ zero samples to the sequences to $h(n)$ so that the length of the both sequences is $L + M - 1 = 5$. The length of the DFT that we use is $N = 5$

Now the sequences are

$$x(n) = \{1, -1, 1, 0, 0\}$$

$$h(n) = \{2, 2, 1, 0, 0\}$$

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi nk / N}$$

$$X(k) = \sum_{n=0}^4 x(n)e^{-j2\pi nk / 5}$$

$$X(0) = \sum_{n=0}^4 x(n) = x(0) + x(1) + x(2) + x(3) + x(4) = 1 - 1 + 1 + 0 + 0 = 1$$

$$X(1) = \sum_{n=0}^4 x(n)e^{-j2\pi n / 5}$$

$$X(1) = x(0) + x(1)e^{-j2\pi / 5} + x(2)e^{-j4\pi / 5} (\because x(3) = x(4) = 0)$$

$$X(1) = 1 - 1 \left(\cos \frac{2\pi}{5} - j \sin \frac{2\pi}{5} \right) + 1 \left(\cos \frac{4\pi}{5} - j \sin \frac{4\pi}{5} \right)$$

$$X(1) = 1 - 1(0.309 - j0.951) + 1(-0.809 - j0.587) = -0.118 + j0.364$$

$$X(2) = \sum_{n=0}^4 x(n)e^{-j4n\pi/5}$$

$$X(2) = x(0) + x(1)e^{-j4\pi/5} + x(2)e^{-j8\pi/5} + x(3)e^{-j12\pi/5} + x(4)e^{-j16\pi/5}$$

$$X(2) = 1 - 1 \left[\cos \frac{4\pi}{5} - j \sin \frac{4\pi}{5} \right] + 1 \left[\cos \frac{8\pi}{5} - j \sin \frac{8\pi}{5} \right] + 1 \left[\cos \frac{12\pi}{5} - j \sin \frac{12\pi}{5} \right] + 1 \left[\cos \frac{16\pi}{5} - j \sin \frac{16\pi}{5} \right]$$

$$X(2) = 1 - 1(-0.809 - j0.587) + 1(0.309 + j0.951) = 2.118 + j1.538$$

$$X(3) = \sum_{n=0}^4 x(n)e^{-j6n\pi/5}$$

$$X(3) = x(0) + x(1)e^{-j6\pi/5} + x(2)e^{-j12\pi/5} + x(3)e^{-j18\pi/5} + x(4)e^{-j24\pi/5}$$

$$X(3) = 1 - 1 \left[\cos \frac{6\pi}{5} - j \sin \frac{6\pi}{5} \right] + 1 \left[\cos \frac{12\pi}{5} - j \sin \frac{12\pi}{5} \right] + 1 \left[\cos \frac{18\pi}{5} - j \sin \frac{18\pi}{5} \right] + 1 \left[\cos \frac{24\pi}{5} - j \sin \frac{24\pi}{5} \right]$$

$$X(3) = 1 - 1(-0.809 + j0.587) + 1(0.309 - j0.951) = 2.118 - j1.538$$

$$X(4) = \sum_{n=0}^4 x(n)e^{-j8n\pi/5}$$

$$X(4) = x(0) + x(1)e^{-j8\pi/5} + x(2)e^{-j16\pi/5} + x(3)e^{-j24\pi/5} + x(4)e^{-j32\pi/5}$$

$$X(4) = 1 - 1 \left[\cos \frac{8\pi}{5} - j \sin \frac{8\pi}{5} \right] + 1 \left[\cos \frac{16\pi}{5} - j \sin \frac{16\pi}{5} \right] + 1 \left[\cos \frac{24\pi}{5} - j \sin \frac{24\pi}{5} \right] + 1 \left[\cos \frac{32\pi}{5} - j \sin \frac{32\pi}{5} \right]$$

$$X(4) = 1 - 1(0.309 + j0.951) + 1(-0.809 + j0.589) = -0.118 - j0.364$$

$$H(k) = \sum_{n=0}^{N-1} h(n)e^{-j2\pi nk/N}$$

$$H(k) = \sum_{n=0}^4 h(n)e^{-j2\pi nk/5}$$

$$H(0) = \sum_{n=0}^4 h(n) = x(0) + x(1) + x(2) + x(3) + x(4) = 2 + 2 + 1 + 0 + 0 = 5$$

$$H(1) = 2 + 2(0.309 - j0.951) + 1(-0.809 - j0.587) = 1.809 - j2.489$$

$$H(2) = 2 + 2(-0.809 - j0.587) + 1(0.309 + j0.951) = 0.691 - j0.223$$

$$H(3) = 2 + 2(-0.809 + j0.587) + 1(0.309 - j0.951) = 0.691 + j0.223$$

$$H(4) = 2 + 2(0.309 + j0.951) + 1(-0.809 + j0.589) = 1.809 + j2.489$$

As per convolution;

$$Y(k) = X(k)H(k)$$

$$Y(0) = X(0)H(0) = (1)(5) = 5$$

$$Y(1) = X(1)H(1) = (-0.118 + j0.364)(0.809 - j2.489) = 0.692 + j0.952$$

$$Y(2) = X(2)H(2) = (2.118 + j1.538)(0.691 - j0.223) = 1.806 + j0.591$$

$$Y(3) = X(3)H(3) = (2.118 - j1.538)(0.691 + j0.223) = 1.806 - j0.591$$

$$Y(4) = X(4)H(4) = (-0.118 - j0.364)(0.809 + j2.489) = 0.692 - j0.952$$

Therefore

$$Y(k) = \{5, 0.692 + j0.952, 1.806 + j0.591, 1.806 - j0.591, 0.692 - j0.952\}$$

BM T63 – BIOMEDICAL SIGNAL PROCESSING**UNIT-IV: DESIGN OF DIGITAL FILTERS****PART- A****1. What is the use of choosing a window? (May 2016)**

One possible way of finding an FIR filter that approximates $H(e^{j\omega})$ would be to truncate the infinite series at $n = \pm \left(\frac{N-1}{2} \right)$. Abrupt truncation of the series will lead to

oscillations in the pass band and stop band. These oscillations can be reduced through the use of less abrupt truncation of the Fourier series. This can be achieved by multiplying infinite impulse response by a finite weighing sequence $w(n)$, called a window.

2. What is known as warping effect? (May 2016, Nov 2016, May 2018)

The relation between the analog and digital frequencies in bilinear transformation is given by

$$\Omega = \frac{2}{T} \tan \frac{\omega}{2}$$

For smaller values of ω there exist linear relationship between ω and Ω . But for large values of ω the relationship is non-linear. This non linearity introduces distortion in the frequency axis. This is known as warping effect. This effect compresses the magnitude and phase response at high frequencies.

3. Write the equation of Rectangular and Hanning window. (Nov 2016)**Rectangular Window:**

The rectangular window sequence is given by

$$w_R(n) = 1 \text{ for } -(N-1)/2 \leq n \leq (N-1)/2$$

$$= 0 \text{ otherwise}$$

Hanning Window:

The hanning window sequence is given by

$$W_{Hn}(e^{j\omega}) = 0.5 + 0.5 \cos(2\pi n / N - 1) \text{ for } -(N-1)/2 \leq n \leq (N-1)/2$$

$$= 0 \text{ otherwise}$$

4. Write the equation of Triangular and Hamming window. (May 2017, May 2019)

Triangular or Bartlett Window:

The N – point triangular window is given by

$$w_T(n) = 1 - \frac{2|n|}{N-1} \text{ for } -(N-1)/2 \leq n \leq (N-1)/2$$

Hamming Window:

The hamming window sequence is given by

$$W_H(e^{j\omega}) = 0.54 + 0.46 \cos(2\pi n / N - 1) \text{ for } -(N-1)/2 \leq n \leq (N-1)/2$$

$$= 0 \text{ otherwise}$$

5. Mention the steps for designing a Butterworth filter. (May 2017)

- (i) Find the given specifications; find the order of the filter N .
- (ii) Round off it to the next higher integer.
- (iii) Find the transfer function $H(s)$ for $\Omega_C = 1$ rad/sec for the value of N .
- (iv) Calculate the value of cutoff frequency Ω_C .
- (v) Find the transfer function $H_a(s)$ for the above value of Ω_C by substituting
 $s \rightarrow \frac{s}{\Omega_C}$ in $H(s)$.

6. What are the properties of Chebyshev filter? (Dec 2017)

- (i) The magnitude response of the chebyshev filter exhibits ripple either in pass-band or in stop-band according to type.
- (ii) The poles of the chebyshev filter lie on an ellipse.

7. Distinguish between analog filter and digital filter. (Dec 2017)

Analog Filter	Digital Filter
1. Analog filter processes analog inputs and generates analog outputs.	1. A digital filter processes and generates digital data.
2. Analog filters are constructed from active or passive electronic components.	2. A digital filter consists of elements like adder, multiplier and delay unit.
3. Analog filter is described by a differential equation.	3. Digital filter is described by a difference equation.

4. The frequency response of an analog filter can be modified by changing the components.	4. The frequency response of an digital filter can be changed by changing the filter coefficients.
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8. Differentiate between FIR and IIR with respect to their characteristics. (May 2018)

FIR Filter	IIR Filter
1. These filters can be easily designed to have perfectly linear phase..	1. These filters do not have linear phase.
2. FIR filters can be realized recursively and non-recursively.	2. IIR filters are easily realized recursively.
3. Greater flexibility to control the shape of their magnitude response.	3. Less flexibility, usually limited to specific kind of filters..
4. Errors due to round off noise are less severe in FIR filters, mainly because feedback is not used	4. The round off noise in IIR filters are More.

9. Define hamming window. (Dec 2018)

Hamming Window:

The hamming window sequence is given by

$$W_H(e^{j\omega}) = 0.54 + 0.46 \cos(2\pi n / N - 1) \text{ for } -(N-1)/2 \leq n \leq (N-1)/2$$

$$= 0 \text{ Otherwise}$$

The main lobe width is equal to $\frac{8\pi}{N}$ and the peak side lobe level is -41dB. The low pass FIR filter design will have first side lobe peak of -53dB.

10. What is meant by frequency prewarping? (Dec 2018, May 2019)

The effect of the non-linear compression at high frequencies can be compensated. When the desired magnitude response is piece-wise constant over frequency, this compression can be compensated by introducing a suitable pre-scaling, or pre-warping the critical frequencies by using the formula.

$$\Omega = \frac{2}{T} \tan \frac{\omega}{2}$$

11. Define Gibb's phenomenon. (Dec 2019)

One possible way of finding an FIR filter that approximates $H(e^{j\omega})$ would be to truncate the infinite Fourier series at $n = \pm \left(\frac{N-1}{2} \right)$. Abrupt truncation of the series

will lead to oscillation both in pass-band and in stop-band. This phenomenon is known as Gibbs phenomenon.

12. Give any two properties of Butterworth and Chebyshev filter. (Dec 2019)**Properties of Butterworth filter:**

- (i) The magnitude response of the butterworth filter decreases monotonically as the frequency Ω increases from 0 to ∞
- (ii) The poles of the butterworth filter lie on a circle.

Properties of Chebyshev filter:

- (i) The magnitude response of the chebyshev filter exhibits ripple either in pass-band or in stop-band according to type.
- (ii) The poles of chebyshev filter lie on an ellipse.

PART – B**1. Explain in detail about frequency sampling method. (May 2016)**

- The frequency sampling method is use to design recursive and non-recursive FIR filters for both standard frequency selective filters and with arbitrary frequency response.
- The main idea of the frequency sampling design method is that a desired frequency response can be approximated by sampling it at N evenly spaced points and then obtaining N-point filter response.
- A continuous frequency response is then calculated as an interpolation of the sampled frequency response.
- The approximation error would then be exactly zero at the sampling frequencies and would be finite in frequencies between them.
- The smoother the frequency response being approximated, the smaller will be the error of interpolation between the sample points.
- There are two distinct types of Non-Recursive Frequency Sampling method of FIR filter design, depending on where the initial frequency sample occurred.
- The type 1 designs have the initial point at $\omega=0$, whereas the type 2 designs have the initial point at $f = 1/2N$ or $\omega = \pi/N$.

Procedure for Type-1 Design:

- (i) Choose the desired frequency response $H_d(\omega)$
- (ii) Sample $H_d(\omega)$ at N -points taking $\omega = \omega_K = 2\pi k / N$, where $K = 0, 1, 2, 3, \dots, (N-1)$, generate the sequence $H(k)$. To obtain a good approximation of the desired frequency response, a sufficiently large number of the frequency samples should be taken. $H(k) = H_d(\omega) \big|_{\omega=2\pi k/N}$ for $k = 0, 1, \dots, (N-1)$.
- (iii) The N -point inverse DFT of the sequence $H(k)$ gives the impulse response of the filter $h(n)$. For practical realization of the filter, samples of impulse response should be real. This can happen if all the complex terms appear in conjugate pairs.
 - Desired filter coefficients

$$h(n) = \text{InvDFT}\{H(k)\} = \frac{1}{N} \sum_{k=0}^{N-1} H(k) e^{j2\pi kn/N}$$

- For linear phase filters, with positive symmetrical impulse response,

$$h(n) = \frac{1}{N} \left\{ h(0) + 2 \sum_{k=1}^{UL} \text{Re} \left[H(k) e^{j2\pi kn/N} \right] \right\}$$

$$\text{When, } N \text{ is odd } UL = \frac{N-1}{2} \text{ and when } N \text{ is even } UL = \frac{N}{2} - 1$$

- (iv) Take z-transform of the impulse response $h(n)$ to get the filter transfer function,

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$$

Procedure for Type-2 Design:

(Same steps as above except step 2)

- (ii) Sample $H_d(\omega)$ at N -points by $\omega = \omega_k = \frac{2\pi}{2N} (2k+1)$ where $k = 0, 1, 2, 3, \dots, (N-1)$ generate the sequence $H(k)$

$$H(k) = H_d(\omega) \big|_{\omega=2\pi(k+1)/2N} \text{ for } k = 0, 1, \dots, (N-1)$$

Type 2 frequency samples give additional flexibility in the design method to specify the desired frequency response at a second possible set of frequencies.

Advantage:

- Unlike the window method, this technique can be used for any given magnitude response.
- This method is useful for the design of non-prototype filters where the desired magnitude response can take any irregular shape.
- Major advantage of Frequency sampling method lies in the efficient frequency sampling structure, which is obtained when most of the frequency samples are zero.

Disadvantage:

- One disadvantage with this method is that the frequency response obtained by interpolation is equal to the desired frequency response only at the sampled points. At the other points, there will be a finite error present.
2. Describe in detail about design of Butterworth filters with an example. (May 2016)

The steps to design an analog Butterworth filter.

- Find the given specifications; find the order of the filter N .
- Round off it to the next higher integer.
- Find the transfer function $H(s)$ for $\Omega_C = 1$ rad/sec for the value of N .
- Calculate the value of cutoff frequency Ω_C .
- Find the transfer function $H_a(s)$ for the above value of Ω_C by substituting $s \rightarrow \frac{s}{\Omega_C}$ in $H(s)$.

Example:

Design an analog butterworth filter that has a -2 dB passband attenuation at a frequency of 20 rad/sec. and atleast -10 dB stopband attenuation at 30 rad/sec.

Solution:

Given $\alpha_p = 2dB$; $\Omega_p = 20rad / sec$

$\alpha_s = 10dB$; $\Omega_s = 30rad / sec$

- To find the order of the filter N :

$$N \geq \frac{\log \sqrt{\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}}}{\log \frac{\Omega_s}{\Omega_p}}$$

$$N \geq \frac{\log \sqrt{\frac{10 - 1}{10^{0.2} - 1}}}{\log \frac{30}{20}}$$

$$N \geq 3.37$$

- Round off N to the next higher integer, $N \geq 3.37$, We get $N = 4$

The normalized low pass butterworth filter for $N = 4$ can be found from the table as

N	Denominator of $H(s)$
1	$s + 1$
2	$s^2 + \sqrt{2}s + 1$
3	$(s + 1)(s^2 + s + 1)$
4	$(s^2 + 0.76537s + 1)(s^2 + 1.8477s + 1)$
5	$(s + 1)(s^2 + 0.61803s + 1)(s^2 + 1.61803s + 1)$
6	$(s^2 + 1.931855s + 1)(s^2 + \sqrt{2}s + 1)(s^2 + 0.51764s + 1)$
7	$(s + 1)(s^2 + 1.80194s + 1)(s^2 + 1.247s + 1)(s^2 + 0.445s + 1)$

- (iii) Find the transfer function $H(s)$ for $\Omega_C = 1$ rad/sec for the value of N .

$$H(s) = \frac{1}{(s^2 + 0.76537s + 1)(s^2 + 1.8477s + 1)}$$

- (iv) Calculate the value of cutoff frequency Ω_C .

$$\Omega_C = \frac{\Omega_P}{(10^{0.1\alpha_P} - 1)^{1/2N}} = \frac{20}{(10^{0.2} - 1)^{1/8}} = 21.3868$$

- (v) Find the transfer function $H(s)$ for $\Omega_C = 1$ rad/sec for the value of N .

The transfer function for $\Omega_C = 21.3868$ can be obtained by substituting,

$$s \rightarrow \frac{s}{21.3868} \text{ in } H(s)$$

$$H(s) = \frac{1}{\left(\frac{s}{21.3868}\right)^2 + 0.76537\left(\frac{s}{21.3868}\right) + 1} \times \frac{1}{\left(\frac{s}{21.3868}\right)^2 + 1.8477\left(\frac{s}{21.3868}\right) + 1}$$

$$H(s) = \frac{0.20921 \times 10^6}{(s^2 + 16.3686s + 457.394)(s^2 + 39.5176s + 457.394)}$$

3.Design a third order Butterworth digital filter using impulse invariant technique.**Assume sampling period T = 1sec. (Nov 2016)**

From the table, for N = 3, the transfer function of a normalized Butterworth filter is given by

N	Denominator of $H(s)$
1	$s + 1$
2	$s^2 + \sqrt{2}s + 1$
3	$(s + 1)(s^2 + s + 1)$
4	$(s^2 + 0.76537s + 1)(s^2 + 1.8477s + 1)$
5	$(s + 1)(s^2 + 0.61803s + 1)(s^2 + 1.61803s + 1)$
6	$(s^2 + 1.931855s + 1)(s^2 + \sqrt{2}s + 1)(s^2 + 0.51764s + 1)$
7	$(s + 1)(s^2 + 1.80194s + 1)(s^2 + 1.247s + 1)(s^2 + 0.445s + 1)$

$$H(s) = \frac{1}{(s + 1)(s^2 + s + 1)}$$

$$H(s) = \frac{A}{s + 1} + \frac{B}{s + 0.5 + j0.866} + \frac{C}{s + 0.5 - j0.866}$$

$$A = (s + 1) \frac{1}{(s + 1)(s^2 + s + 1)} \text{ for } s = -1$$

$$A = \frac{1}{(-1)^2 - 1 + 1} = 1$$

$$B = (s + 0.5 + j0.866) \frac{1}{(s + 1)(s + 0.5 + j0.866)(s + 0.5 - j0.866)}; \text{ for } s = -0.5 - j0.866$$

$$B = \frac{1}{(-0.5 - j0.866 + 1)(-j0.866 - j0.866)}$$

$$= \frac{1}{-j1.732(0.5 - j0.866)} = \frac{1}{-j0.866 - 1.5}$$

$$= \frac{-1.5 + j0.866}{3} = -0.5 + j0.288$$

$$C = B^* = -0.5 - j0.288$$

Hence

$$H(s) = \frac{1}{s + 1} + \frac{-0.5 + 0.288j}{s + 0.5 + j0.866} + \frac{-0.5 - 0.288j}{s + 0.5 - j0.866}$$

$$= \frac{1}{s - (-1)} + \frac{-0.5 + 0.288j}{s - (-0.5 - j0.866)} + \frac{-0.5 - 0.288j}{s - (-0.5 + j0.866)}$$

In impulse invariant technique

$$\text{If } H(s) = \sum_{k=1}^N \frac{c_K}{s - p_K}, \text{ then } H(z) = \sum_{k=1}^N \frac{c_K}{1 - e^{p_K T} z^{-1}}$$

Therefore,

$$\begin{aligned} H(z) &= \frac{1}{1 - e^{-1} z^{-1}} + \frac{-0.5 + 0.288j}{1 - e^{-0.5} e^{-j0.866} z^{-1}} + \frac{-0.5 - 0.288j}{1 - e^{-0.5} e^{j0.866} z^{-1}} \\ &= \frac{1}{1 - 0.368z^{-1}} + \frac{-1 + 0.66z^{-1}}{1 - 0.786z^{-1} + 0.368z^{-2}} \end{aligned}$$

4. Apply bilinear transformation to $H(s) = \frac{2}{(s+1)(s+2)}$ with $T=1$ sec and find $H(z)$.

(Nov 2016, May 2019)

Solution:

$$\text{Given } H(s) = \frac{2}{(s+1)(s+2)}$$

$$\text{Substitute } s = \frac{2 \left[\frac{1 - z^{-1}}{1 + z^{-1}} \right]}{T} \text{ in } H(s) \text{ to get } H(z)$$

$$\begin{aligned} H(z) &= H(s); \text{ for } s = \frac{2 \left(\frac{1 - z^{-1}}{1 + z^{-1}} \right)}{T} \\ &= \frac{2}{(s+1)(s+2)} \text{ for } s = \frac{2 \left(\frac{1 - z^{-1}}{1 + z^{-1}} \right)}{T} \end{aligned}$$

Given $T = 1$ sec

$$\begin{aligned} H(z) &= \frac{2 \left(\frac{1 - z^{-1}}{1 + z^{-1}} \right)}{\left\{ 2 \left(\frac{1 - z^{-1}}{1 + z^{-1}} \right) + 1 \right\} \left\{ 2 \left(\frac{1 - z^{-1}}{1 + z^{-1}} \right) + 2 \right\}} \\ &= \frac{2(1 + z^{-1})^2}{(3 - z^{-1})(4)} \\ &= \frac{(1 + z^{-1})^2}{6 - 2z^{-1}} = \frac{0.166(1 + z^{-1})^2}{(1 - 0.33z^{-1})} \end{aligned}$$

5. Using hamming window design a FIR filter approximating the ideal frequency response

$$H_d(e^{j\omega}) = e^{-j\alpha\omega} \quad \text{for } |\omega| \leq \pi/6$$

$$= 0 \quad \text{for } \pi/6 < |\omega| \leq \pi$$

Determine the filter coefficients for $N = 13$. (May 2017)

Solution:

We know

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi/6}^{\pi/6} e^{-j\alpha\omega} e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi/6}^{\pi/6} e^{-j(\alpha-n)\omega} d\omega$$

$$= \frac{1}{-2\pi(\alpha-n)j} e^{-j(\alpha-n)\omega} \quad \text{for } \frac{\pi}{6} < \omega < \frac{\pi}{6}$$

$$= \frac{e^{j(n-\alpha)\pi/6} - e^{-j(n-\alpha)\pi/6}}{2\pi j(n-\alpha)}$$

$$= \frac{\sin \frac{\pi}{6} (n-\alpha)}{\pi (n-\alpha)}$$

For the frequency response we can find that the filter coefficients are symmetrical about $\alpha = \frac{N-1}{2} = 6$, satisfying $h(n) = h(N-1-n)$.

Truncating $h_d(n)$ to $N=13$ samples we have

$$h(n) = h_d(n) \quad \text{for } 0 \leq n \leq 12$$

$$= 0 \quad \text{otherwise}$$

$$h(n) = \frac{\sin \frac{\pi}{6} (n - \alpha)}{\pi (n - \alpha)}$$

$$= \frac{\sin \frac{\pi}{6} (n - 6)}{\pi (n - 6)}$$

$$h(0) = h(13 - 1 - 0) = h(12) = 0$$

$$h(1) = h(11) = 0.0318$$

$$h(2) = h(10) = 0.0689$$

$$h(3) = h(9) = 0.106$$

$$h(4) = h(8) = 0.1378$$

$$h(5) = h(7) = 0.159$$

$$h(6) = \lim_{n \rightarrow 6} \frac{\sin \frac{\pi}{6} (n - 6)}{\pi (n - 6)} = 1/6 = 0.167$$

The hamming window sequence is given by

$$W_H(e^{j\omega}) = 0.54 + 0.46 \cos(2\pi n / N - 1) \text{ for } |n| \leq (N - 1) / 2$$

$$= 0$$

Otherwise

$$\Rightarrow W_H(e^{j\omega}) = 0.54 + 0.46 \cos \frac{n\pi}{6} \text{ for } |n| \leq 6$$

$$= 0$$

Otherwise

$$w_H(0) = 0.54 + 0.46 = 1$$

$$w_H(1) = w_H(-1) = 0.54 + 0.46 \cos \frac{\pi}{6} = 0.94$$

$$w_H(2) = w_H(-2) = 0.54 + 0.46 \cos \frac{\pi}{3} = 0.77$$

$$w_H(3) = w_H(-3) = 0.54 + 0.46 \cos \frac{\pi}{2} = 0.54$$

$$w_H(4) = w_H(-4) = 0.54 + 0.46 \cos \frac{2\pi}{3} = 0.31$$

$$w_H(5) = w_H(-5) = 0.54 + 0.46 \cos \frac{5\pi}{6} = 0.142$$

$$w_H(6) = w_H(-6) = 0.54 + 0.46 \cos \pi = 0.08$$

The causal window sequence can be obtained by shifting the above sequence right by six samples

i.e.,

$$w_H(6) = 1$$

$$w_H(5) = w_H(7) = 0.94; w_H(4) = w_H(8) = 0.77$$

$$w_H(3) = w_H(9) = 0.54; w_H(2) = w_H(10) = 0.31$$

$$w_H(1) = w_H(11) = 0.08; w_H(0) = w_H(12) = 0.08$$

The FIR filter coefficients are

$$h(n) = h_d(n)w_H(n) \quad \text{for } 0 \leq n \leq N-1$$

$$h_d(n) \text{ for } 0 \leq n \leq 12$$

$$h(0) = h(12) = h_d(0)w_H(0) = (0)(0.08) = 0$$

$$h(1) = h(11) = h_d(1)w_H(1) = (0.0318)(0.142) = 0.0045$$

$$h(2) = h(10) = h_d(2)w_H(2) = (0.0689)(0.31) = 0.02136$$

$$h(3) = h(9) = h_d(3)w_H(3) = (0.106)(0.54) = 0.05724$$

$$h(4) = h(8) = h_d(4)w_H(4) = (0.1378)(0.77) = 0.1061$$

$$h(5) = h(7) = h_d(5)w_H(5) = (0.159)(0.94) = 0.149$$

$$h(6) = h_d(6)w_H(6) = (0.167)(1) = 0.167$$

6. Using a rectangular window technique design a low pass filter with pass band gain of unity, cutoff frequency of 1000 Hz and working at a sampling frequency of 5kHz. The length of the impulse response should be 7. (May 2017)

Solution:

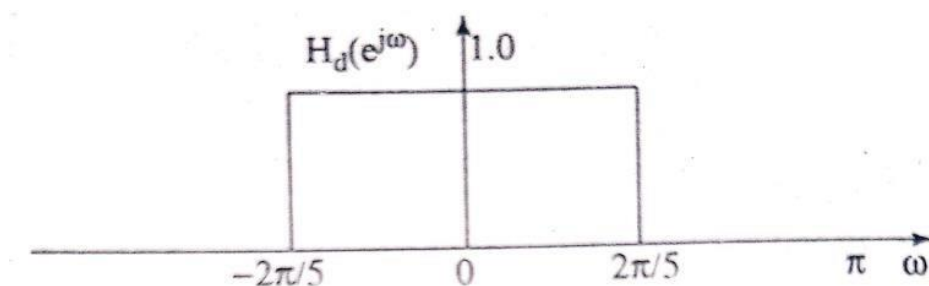
Given

$$f_c = 1000\text{Hz}$$

$$F = 5000\text{Hz}$$

The cutoff frequency $\omega_c = 2\pi fT$

$$= \frac{2\pi(1000)}{5000} = \frac{2\pi}{5}$$



The desired frequency response of the LPF is shown in fig.

The filter coefficients are given by

$$h_d(n) = \frac{1}{2\pi} \int_{-\frac{2\pi}{5}}^{\frac{2\pi}{5}} e^{j\omega n} d\omega$$

$$= \frac{\sin \frac{2\pi}{5} n}{\pi n}; -\infty \leq n \leq \infty$$

The rectangular window for $N = 7$ is given by

$$w_R(n) = 1 \quad \text{for } -3 \leq n \leq 3$$

$$= 0 \quad \text{otherwise}$$

$$\text{For } n = 0; \quad h(n) = \lim_{n \rightarrow 0} \frac{\sin \frac{2\pi}{5} n}{\pi n} = \frac{2}{5} = 0.4$$

$$\text{For } n = 1; \quad h(1) = h(-1) = \frac{\sin \frac{2\pi}{5}}{\pi} = 0.3027$$

$$\text{For } n = 2; \quad h(2) = h(-2) = \frac{\sin \frac{4\pi}{5}}{2\pi} = 0.0935$$

$$\text{For } n = 3; \quad h(3) = h(-3) = \frac{\sin \frac{6\pi}{5}}{3\pi} = -0.06236$$

The filter coefficients of realizable filter are

$$h(0) = h(6) = -0.06236; h(1) = h(5) = 0.0935; h(2) = h(4) = 0.0327, h(3) = 0.4$$

- 7. Prove that an FIR filter has linear phase if the unit sample response satisfies the condition $h(n) = h(N-1-n)$ also discuss symmetric and anti-symmetric cases of FIR filter when N is even. (Dec 2017)**

Case 1: Symmetric Impulse response for N Even

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n)e^{-j\omega n}$$

$$\begin{aligned}
&= \sum_{n=0}^{\frac{N-2}{2}} h(n) e^{-j\omega n} + \sum_{n=\frac{N}{2}}^{N-1} h(n) e^{-j\omega n} \\
&= \sum_{n=0}^{\frac{N-2}{2}} h(n) e^{-j\omega n} + \sum_{n=0}^{\frac{N-2}{2}} h(N-1-n) e^{-j\omega(N-1-n)}
\end{aligned}$$

We know, $h(n) = h(N-1-n)$

$$\begin{aligned}
H(e^{j\omega}) &= \sum_{n=0}^{\frac{N-2}{2}} h(n) e^{-j\omega n} + \sum_{n=0}^{\frac{N-2}{2}} h(n) e^{-j\omega(N-1-n)} \\
&= e^{-j\omega(N-1/2)} \left[\sum_{n=0}^{\frac{N-2}{2}} h(n) e^{j\omega \left[\frac{N-1}{2} - n \right]} + \sum_{n=0}^{\frac{N-2}{2}} h(n) e^{-j\omega \left[\frac{N-1}{2} - n \right]} \right] \\
&= e^{-j\omega(N-1/2)} \left[\sum_{n=0}^{\frac{N-2}{2}} 2h(n) \cos \omega \left(\frac{N-1}{2} - n \right) \right] \\
&= e^{-j\omega(N-1/2)} \left[\sum_{n=0}^{\frac{N}{2}} 2h(n) \left(\frac{N}{2} - n \right) \cos \left(n - \frac{1}{2} \right) \omega \right] \\
&= e^{-j\omega(N-1/2)} \left[\sum_{n=0}^{\frac{N}{2}} b(n) \cos \left(n - \frac{1}{2} \right) \omega \right]
\end{aligned}$$

Where

$$b(n) = 2h\left(\frac{N}{2} - n\right)$$

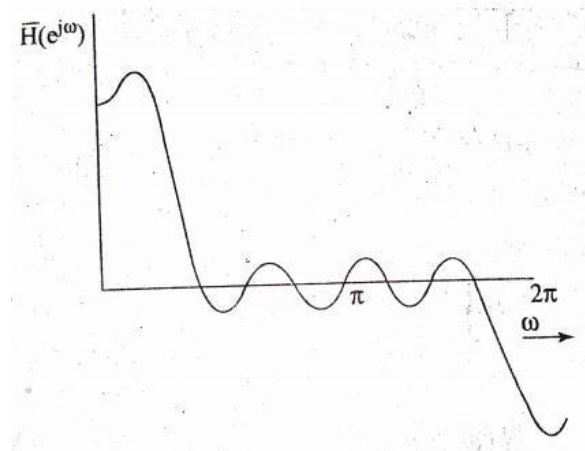
We can write the eqn. as

$$H(e^{j\omega}) = e^{-j\omega(N-1/2)} \overline{H}(e^{j\omega}) = \overline{H}(e^{j\omega}) e^{j\theta(\omega)}$$

Where

$$\begin{aligned}
\overline{H}(e^{j\omega}) &= \left[\sum_{n=0}^{\frac{N}{2}} b(n) \cos \left(n - \frac{1}{2} \right) \omega \right] \\
\theta(\omega) &= -\alpha\omega = -\left(\frac{N-1}{2} \right) \omega
\end{aligned}$$

The frequency response of linear phase filter with symmetric impulse response for N even is shown in fig.



Case 2: Antisymmetric Impulse response for N Even

$$\begin{aligned}
 H(e^{j\omega}) &= \sum_{n=0}^{N-1} h(n)e^{-j\omega n} \\
 &= \sum_{n=0}^{\frac{N-2}{2}} h(n)e^{-j\omega n} + \sum_{n=\frac{N}{2}}^{N-1} h(n)e^{-j\omega n} \\
 &= \sum_{n=0}^{\frac{N-2}{2}} h(n)e^{-j\omega n} + \sum_{n=0}^{\frac{N-2}{2}} h(N-1-n)e^{-j\omega(N-1-n)}
 \end{aligned}$$

We know, $h(n) = -h(N-1-n)$

$$\begin{aligned}
 H(e^{j\omega}) &= \sum_{n=0}^{\frac{N-2}{2}} h(n)e^{-j\omega n} - \sum_{n=0}^{\frac{N-2}{2}} h(n)e^{-j\omega(N-1-n)} \\
 &= e^{-j\omega(N-1)/2} \left[\sum_{n=0}^{\frac{N-2}{2}} h(n)e^{j\omega[N-1/2-n]} - \sum_{n=0}^{\frac{N-2}{2}} h(n)e^{-j\omega[N-1/2-n]} \right] \\
 &= e^{-j\omega(N-1)/2} e^{j\pi/2} \left[\sum_{n=0}^{\frac{N-2}{2}} h(n) \frac{\sin \omega(n - \frac{N-1}{2})}{\sin \omega \frac{1}{2}} \right] \\
 &= e^{-j\omega(N-1)/2} e^{j\pi/2} \left[\sum_{n=1}^{\frac{N}{2}} d(n) \sin \omega \left(n - \frac{1}{2} \right) \right]
 \end{aligned}$$

Where

$$d(n) = 2h\left(\frac{N-n}{2}\right)$$

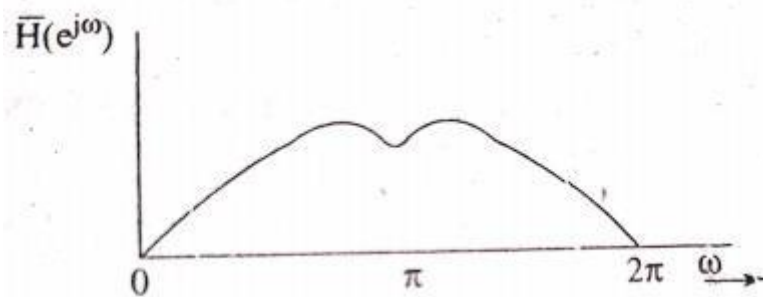
$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)} e^{j\frac{\pi}{2}} H(e^{j\omega})$$

$$H(e^{j\omega}) = H(e^{j\omega}) e^{j\theta(\omega)} = H(e^{j\omega}) e^{j\left(\frac{\pi}{2} - \alpha\omega\right)}$$

$$H(e^{j\omega}) = \left| \sum_{n=1}^{\frac{N}{2}} d(n) \sin \omega \left(n - \frac{1}{2} \right) \right|$$

$$\theta(\omega) = \frac{\pi}{2} - \alpha\omega$$

The frequency response of linear phase filter with antisymmetric impulse response for N even is shown in fig.



8. Determine H(z) for a butterworth filter satisfying the following constraints

$$\begin{aligned} 0.707 \leq |H(j\Omega)| \leq 1, & \quad 0 \leq \Omega \leq \frac{\pi}{2} \\ |H(j\Omega)| \leq 0.2 & \quad \frac{3\pi}{4} \leq \Omega \leq \pi \end{aligned} \quad \text{with } T = 1 \text{ sec, apply impulse invariant}$$

transformation. (Dec 2017)

Solution:

The relationship between analog and digital frequencies in Impulse invariant method is $\omega = \Omega T$

From the given data $T = 1 \text{ sec}$ i.e., $\omega = \Omega$

$$\Rightarrow \Omega_P = \omega_P; \Omega_S = \omega_S$$

We know $\lambda = 4.898$; $\varepsilon = 1$

The order of the filter

$$N \geq \frac{\log \frac{\lambda}{\varepsilon}}{\log \frac{\Omega_s}{\Omega_p}} = \frac{\log 4.898}{\log \frac{3\pi/4}{\pi/2}}$$

$$N \geq 3.924$$

i.e., $N = 4$

The transfer function of a fourth order normalized butterworth filter is

$$H(s) = \frac{1}{(s^2 + 0.76537s + 1)(s^2 + 1.8477s + 1)}$$

As $\varepsilon = 1$; $\Omega_p = \Omega_c = 0.5\pi = 1.57$

$$H_a(s) = H(s); s \rightarrow \frac{s}{1.57}$$

$$= \frac{(1.57)^4}{(s^2 + 1.202s + 2.465)(s^2 + 2.902s + 2.465)}$$

$H_a(s)$ in the partial fraction form is given by

$$H_a(s) = \frac{A}{(s + 1.45 + j0.6)} + \frac{A^*}{(s + 1.45 - j0.6)} + \frac{B}{(s + 0.6 + j1.45)} + \frac{B^*}{(s + 0.6 - j1.45)}$$

$$A = \cancel{(s + 1.45 + j0.6)} \frac{(1.57)^4}{(\cancel{s + 1.45 + j0.6})(s + 1.45 - j0.6)(s^2 + 1.202s + 2.465)}; s = -1.45 - j0.6$$

$$= \frac{(1.57)^4}{(-j0.6 - 0.6)[(-1.45 - j0.6)^2 + 1.202(-1.45 - j0.6) + 2.465]}$$

$$= \frac{(1.57)^4}{-j(1.2)[1.745 + 1.74j - 1.7429 - j0.7212 + 2.465]}$$

$$= \frac{(1.57)^4}{-j(1.2)[2.465 + j1.0188]}$$

$$= \frac{5.063}{1.0188 - j2.465} = \frac{5.063(1.0188 + j2.465)}{7.114} = 0.7253 + j1.754$$

$$B = \cancel{(s + 0.6 + j1.45)} \frac{(1.57)^4}{(\cancel{s + 0.6 + j1.45})(s + 0.6 - j1.45)(s^2 + 2.902s + 2.465)}; s = -0.6 - j1.45$$

$$\begin{aligned}
 &= \frac{(1.57)^4}{-j(2.9)[(-0.6 - j1.45)^2 + 2.902(-0.6 - j1.45) + 2.465]} \\
 &= \frac{(1.57)^4}{-j(2.9)[-1.745 + 1.74j - 1.7412 - j4.208 + 2.465]} \\
 &= \frac{2.095}{-j[-1.0187 - j2.468]} \\
 &= \frac{2.095}{-2.468 + j1.0187} = \frac{2.095(-2.468 - j1.0187)}{7.1287} = -0.7253 - 0.3j
 \end{aligned}$$

$$H_a(s) = \frac{0.7253 + j1.754}{s - (-1.45 - j0.6)} + \frac{0.7253 - j1.754}{s - (-1.45 + j0.6)} + \frac{-0.7253 - 0.3j}{s - (-0.6 - j1.45)} + \frac{-0.7253 + 0.3j}{s - (-0.6 + j1.45)}$$

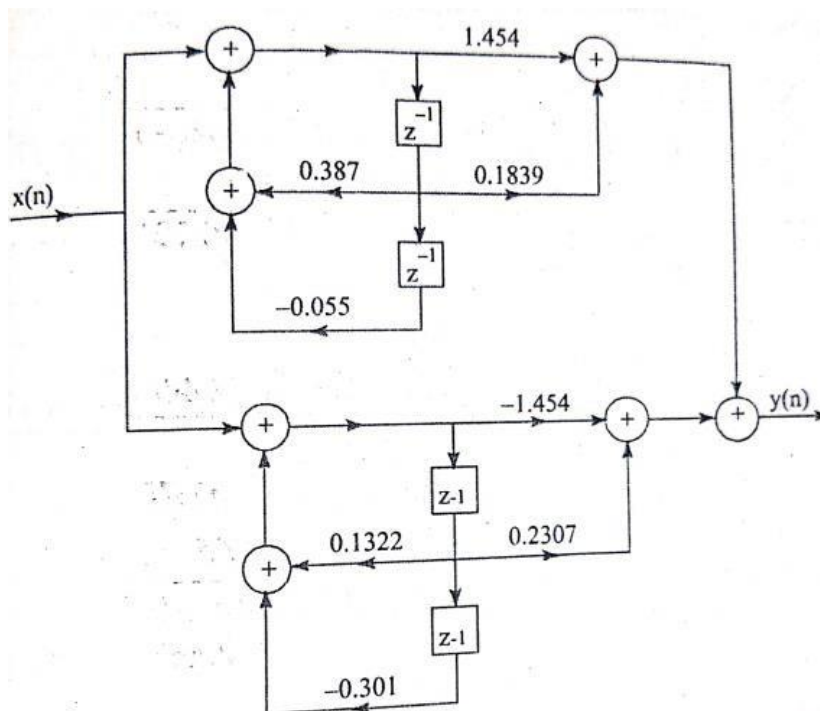
We know that $T = 1$ sec

$$H(z) = \sum_{k=1}^{\infty} \frac{c_k}{1 - e^{p_k} z^{-1}}$$

Therefore,

$$\begin{aligned}
 H_a(z) &= \frac{0.7253 + j1.754}{1 - e^{-1.45} e^{-j0.6} z^{-1}} + \frac{0.7253 - j1.754}{1 - e^{-1.45} e^{j0.6} z^{-1}} + \frac{-0.7253 - 0.3j}{1 - e^{-0.6} e^{-j1.45} z^{-1}} + \frac{-0.7253 + 0.3j}{1 - e^{-0.6} e^{j1.45} z^{-1}} \\
 &= \frac{1.454 + 0.1839z^{-1}}{1 - 0.387z^{-1} + 0.055z^{-2}} + \frac{-1.454 + 0.2307z^{-1}}{1 - 0.1322z^{-1} + 0.301z^{-2}}
 \end{aligned}$$

This can be realized using parallel form as shown in fig.



9. Design a 4th order digital butterworth low pass filter with cutoff 40Hz, T=1 sec and sampling frequency 200Hz to eliminate high frequency noise in ECG signal. (May 2018)

Solution:

Given $\omega_C = 2\pi \times 40 = 80\pi \text{ rad/sec}$

$$T = \frac{1}{f} = \frac{1}{200} = 5 \times 10^{-3} \text{ sec}$$

Prewarping the digital frequency we have

$$\Omega_c = \frac{2}{T} \tan \frac{\omega_C T}{2} = \frac{2}{5 \times 10^{-3}} \tan \left(\frac{80\pi \times 5 \times 10^{-3}}{2} \right) = 400 \tan \frac{1.256}{2} = 290 \text{ rad/sec}$$

Given, the order of the filter $N = 4$

The normalized low pass butterworth filter for $N = 4$ can be found from the table as

N	Denominator of $H(s)$
1	$s + 1$
2	$s^2 + \sqrt{2}s + 1$
3	$(s + 1)(s^2 + s + 1)$
4	$(s^2 + 0.76537s + 1)(s^2 + 1.8477s + 1)$
5	$(s + 1)(s^2 + 0.61803s + 1)(s^2 + 1.61803s + 1)$
6	$(s^2 + 1.931855s + 1)(s^2 + \sqrt{2}s + 1)(s^2 + 0.51764s + 1)$
7	$(s + 1)(s^2 + 1.80194s + 1)(s^2 + 1.247s + 1)(s^2 + 0.445s + 1)$

Find the transfer function $H(s)$ for $\Omega_C = 1 \text{ rad/sec}$ for the value of N .

$$H(s) = \frac{1}{(s^2 + 0.76537s + 1)(s^2 + 1.8477s + 1)}$$

The transfer function for $\Omega_C = 290 \text{ rad/sec}$ can be obtained by substituting,

$$s \rightarrow \frac{s}{290} \text{ in } H(s)$$

$$\begin{aligned}
 H(s) &= \frac{1}{\left(\frac{s}{290}\right)^2 + 0.76537\left(\frac{s}{290}\right) + 1} \\
 &\times \frac{1}{\left(\frac{s}{290}\right)^2 + 1.8477\left(\frac{s}{290}\right) + 1} \\
 &= \frac{290^4}{(s^2 + 221s + 84100)(s^2 + 535s + 84100)}
 \end{aligned}$$

$H_a(s)$ in the partial fraction form is given by

$$H(s) = \frac{290^4}{(s + 110.5 + j268.12)(s + 110.5 - j268.12)(s + 267.5 + j111.99)(s + 267.5 - j111.99)}$$

$$H_a(s) = \frac{A}{(s + 110.5 + j268.12)} + \frac{A^*}{(s + 110.5 - j268.12)} + \frac{B}{(s + 267.5 + j111.99)} + \frac{B^*}{(s + 267.5 - j111.99)}$$

$$\begin{aligned}
 A &= \cancel{(s + 110.5 + j268.12)} \frac{(290)^4}{(s + 110.5 + j268.12)(s + 110.5 - j268.12)(s^2 + 535s + 84100)} \\
 ; s &= -110.5 - j268.12
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{(290)^4}{(-110.5 - j268.12 + 110.5 - j268.12)(-110.5 - j268.12 + 267.5 + j111.99)} \\
 &\quad (-110.5 - j268.12 + 267.5 - j111.99) \\
 &= \frac{(290)^4}{(-j536.24)[(157 - j156.13)(157 - j380.11)]} \\
 &= -133.9 - j55.192
 \end{aligned}$$

$$A^* = -133.9 + j55.192$$

$$\begin{aligned}
 B &= \cancel{(s + 267.5 + j111.99)} \frac{(290)^4}{(s + 110.5 + j268.12)(s + 110.5 - j268.12)(s + 267.5 + j111.99)} \\
 &\quad (s + 267.5 - j111.99) \\
 ; s &= -267.5 - j111.99
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{(290)^4}{(-267.5 - j111.99 + 110.5 + j268.12)(-267.5 - j111.99 + 110.5 - j268.12)} \\
 &\quad (-267.5 - j111.99 + 267.5 - j111.99)
 \end{aligned}$$

$$\frac{(290)^4}{(-157 + j156.13)(157 - j380.11)(-j223.98)} \\ = 133.9 + j319.88$$

$$B^* = 133.9 - j319.88$$

$$H_a(s) = \frac{-133.9 - j55.192}{(s + 110.5 + j268.12)} + \frac{-133.9 + j55.192}{(s + 110.5 - j268.12)} + \frac{133.9 + j319.88}{(s + 267.5 + j111.99)} + \frac{133.9 - j319.88}{(s + 267.5 - j111.99)}$$

$$H_a(s) = \frac{-133.9 - j55.192}{(s - (-110.5 - j268.12))} + \frac{-133.9 + j55.192}{(s - (-110.5 + j268.12))} + \frac{133.9 + j319.88}{(s - (-267.5 - j111.99))} + \frac{133.9 - j319.88}{(s - (-267.5 + j111.99))}$$

We know that $T = 1$ sec

$$H(z) = \sum_{k=1}^N \frac{c_k}{1 - e^{p_k} z^{-1}}$$

Therefore,

$$H_a(z) = \frac{-133.9 - j55.192}{1 - e^{-110.5} e^{-j268.12} z^{-1}} + \frac{-133.9 + j55.192}{1 - e^{-110.5} e^{j268.12} z^{-1}} + \frac{133.9 + j319.88}{1 - e^{-267.5} e^{-j111.99} z^{-1}} + \frac{133.9 - j319.88}{1 - e^{-267.5} e^{j111.99} z^{-1}}$$

10. Explain the following terms briefly. (May 2018)

(a) Frequency sampling structure

Set the frequency response of an FIR filter at M equally-spaced points $\Omega_k = 2\pi k / M$, $k = 0, 1, 2, \dots, M-1$ and find the impulse response $h[n]$.

Frequency response and impulse response are related by

$$H(e^{j\Omega}) = \sum_{n=0}^{M-1} h[n] e^{-j\Omega n}$$

and it follows that

$$H(e^{j\Omega_k}) = H(e^{j2\pi k/M}) = \sum_{n=0}^{M-1} h[n] e^{-j2\pi kn/M}, \quad k = 0, 1, 2, \dots, M-1$$

This is the DFT of $h[n]$.

Since $H(e^{j2\pi k/M})$ is the DFT of $h[n]$, $h[n]$ must be the inverse DFT of $H(e^{j2\pi k/M})$.

$$h[n] = \frac{1}{M} \sum_{k=0}^{M-1} H(e^{j2\pi k/M}) e^{j2\pi kn/M}, \quad n = 0, 1, 2, \dots, M-1$$

The transfer function and impulse response are related by

$$H(z) = \sum_{n=0}^{M-1} h[n] z^{-n} = \frac{1}{M} \sum_{n=0}^{M-1} \sum_{k=0}^{M-1} H(e^{j2\pi k/M}) e^{j2\pi kn/M} z^{-n}.$$

Exchanging the order of summation,

$$H(z) = \frac{1}{M} \sum_{k=0}^{M-1} H(e^{j2\pi k/M}) \sum_{n=0}^{M-1} e^{j2\pi kn/M} z^{-n}$$

Using the formula for summing a geometric series,

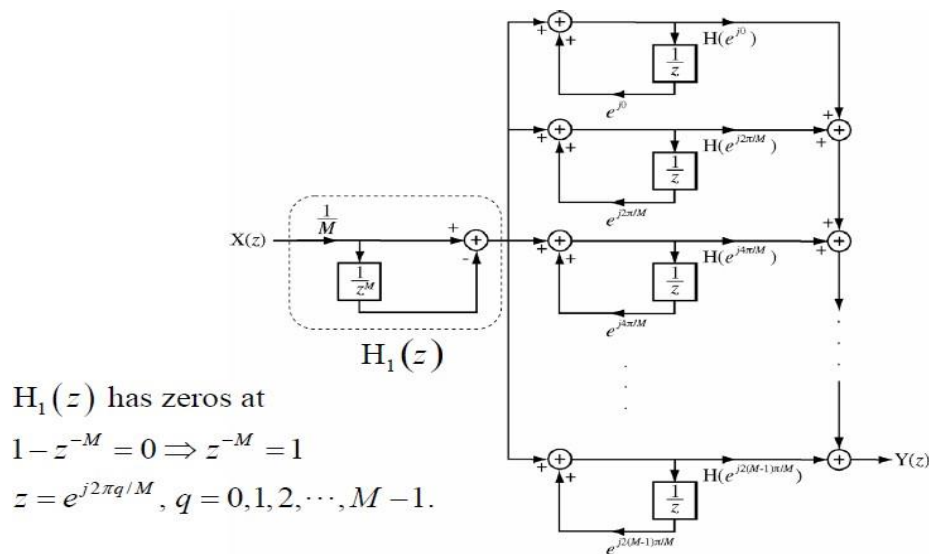
$$H(z) = \frac{1}{M} \sum_{k=0}^{M-1} H(e^{j2\pi k/M}) \frac{1 - z^{-M} e^{j2\pi k}}{1 - z^{-1} e^{j2\pi k/M}} = \frac{1 - z^{-M}}{M} \sum_{k=0}^{M-1} \frac{H(e^{j2\pi k/M})}{1 - z^{-1} e^{j2\pi k/M}}$$

We can think of this as the cascade of two filters, one with transfer function

$$H_1(z) = \frac{1 - z^{-M}}{M}$$

and one with transfer function

$$H_2(z) = \sum_{k=0}^{M-1} \frac{H(e^{j2\pi k/M})}{1 - z^{-1} e^{j2\pi k/M}}.$$

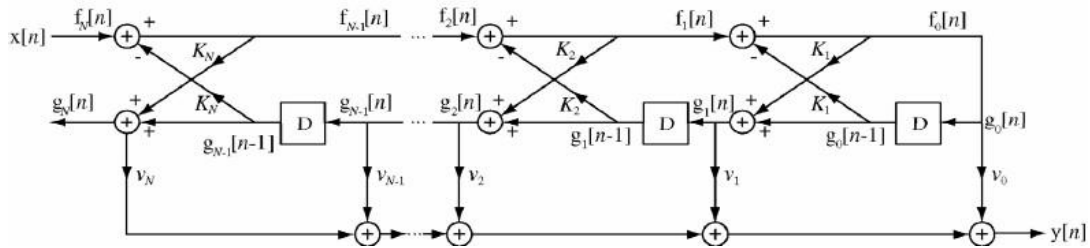


(b) Lattice Structure for IIR Filter:

Then the transfer function will be of the form

$$H(z) = \frac{\gamma_N(0) + \gamma_N(1)z^{-1} + \cdots + \gamma_N(N)z^{-N}}{1 + \alpha_N(1)z^{-1} + \alpha_N(2)z^{-2} + \cdots + \alpha_N(N)z^{-N}} = \frac{\Gamma_N(z)}{A_N(z)}$$

$$y[n] = \sum_{m=0}^N v_m g_m[n] \Rightarrow Y(z) = \sum_{m=0}^N v_m G_m(z)$$

**(c) Perturbation error:**

Perturbation, in mathematics, method for solving a problem by comparing it with a similar one for which the solution is known. Usually the solution found in this way is only approximate. Perturbation is used to find the roots of an algebraic equation that differs slightly from one for which the roots are known.

(d) Limit cycles:

A limit cycle oscillation is a periodic low-level oscillatory disturbance (useless signal) that may exist in an otherwise stable filter. It creeps into the system due to the nonlinearities that arise from the inherent quantization in the system.

11. Design an ideal HPF with a frequency response (Dec 2018)

$$H_d(e^{j\omega}) = 1 \quad \text{for } \pi/4 \leq |\omega| \leq \pi$$

$$= 0 \quad \text{for } 0 \leq |\omega| \leq \pi/4$$

Find the values of h(n) for N = 11. Find H(z) using hanning window function.

Solution:

We know

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega$$

$$\begin{aligned}
&= \frac{1}{2\pi} \left[\int_{-\pi}^{-\pi/4} e^{j\omega n} d\omega + \int_{\pi/4}^{\pi} e^{j\omega n} d\omega \right] \\
&= \frac{1}{2\pi j n} \left[e^{j\omega n} \right]_{-\pi/4}^{\pi/4} + e^{j\omega n} \left[\frac{\pi}{4} \right] \\
&= \frac{1}{\pi n (2j)} \left[e^{-j\pi n/4} - e^{-j\pi n} + e^{j\pi n} - e^{j\pi n/4} \right] \\
&= \frac{1}{\pi n} \left[\sin \pi n - \sin \frac{\pi}{4} n \right]
\end{aligned}$$

Truncating $h_d(n)$ to 11 samples we have,

$$\begin{aligned}
h(n) &= h_d(n) \quad \text{for } |n| \leq 5 \\
&= 0 \quad \text{Otherwise}
\end{aligned}$$

For $n = 0$

$$h(0) = \lim_{n \rightarrow 0} \frac{\sin \pi n}{\pi n} - \lim_{n \rightarrow 0} \frac{\sin \frac{\pi}{4} n}{\pi n} = \left(1 - \frac{1}{4} \right) = 0.75$$

For $n = 1$

$$\begin{aligned}
h(1) &= h(-1) = \frac{\sin \pi - \sin \frac{\pi}{4}}{\pi} = -0.225 \\
h(2) &= h(-2) = \frac{\sin 2\pi - \sin \frac{\pi}{2}}{2\pi} = -0.159 \\
h(3) &= h(-3) = \frac{\sin 3\pi - \sin \frac{3\pi}{4}}{4\pi} = -0.075 \\
h(4) &= h(-4) = \frac{\sin 4\pi - \sin \pi}{5\pi} = 0 \\
h(5) &= h(-5) = \frac{\sin 5\pi - \sin \frac{5\pi}{4}}{5\pi} = 0.045
\end{aligned}$$

Hanning window:

The hanning window sequence is given by

$$\begin{aligned}
W_{Hn}(e^{j\omega}) &= 0.5 + 0.5 \cos(2\pi n / N - 1) \quad \text{for } -(N-1)/2 \leq n \leq (N-1)/2 \\
&= 0 \quad \text{Otherwise}
\end{aligned}$$

For $N = 11$

$$w_{Hn}(n) = 0.5 + 0.5 \cos \frac{\pi n}{5} \quad \text{for } -5 \leq n \leq 5$$

$$= 0 \quad \text{Otherwise}$$

$$w_{Hn}(0) = 0.5 + 0.5 = 1$$

$$w_{Hn}(1) = w_{Hn}(-1) = 0.5 + 0.5 \cos \frac{\pi}{5} = 0.9045$$

$$w_{Hn}(2) = w_{Hn}(-2) = 0.5 + 0.5 \cos \frac{2\pi}{5} = 0.655$$

$$w_{Hn}(3) = w_{Hn}(-3) = 0.5 + 0.5 \cos \frac{3\pi}{5} = 0.345$$

$$w_{Hn}(4) = w_{Hn}(-4) = 0.5 + 0.5 \cos \frac{4\pi}{5} = 0.0945$$

$$w_{Hn}(5) = w_{Hn}(-5) = 0.5 + 0.5 \cos \pi = 0$$

The filter coefficients using hanning window are

$$h(n) = h_d(n)w_{Hn}(n) \quad \text{for } -5 \leq n \leq 5$$

$$= 0 \quad \text{Otherwise}$$

$$h(0) = h_d(0)w_{Hn}(0) = (1)(0.75) = 0.75$$

$$h(-1) = h(1) = h_d(1)w_{Hn}(1) = (-0.225)(0.905) = -0.204$$

$$h(-2) = h(2) = h_d(2)w_{Hn}(2) = (-0.159)(0.655) = -0.104$$

$$h(-3) = h(3) = h_d(3)w_{Hn}(3) = (-0.075)(0.345) = -0.026$$

$$h(-4) = h(4) = h_d(4)w_{Hn}(4) = (0)(0.0945) = 0$$

$$h(-5) = h(5) = h_d(5)w_{Hn}(5) = (0.045)(0) = 0$$

The transfer function of the filter is given by

$$H(z) = h(0) + \sum_{n=1}^5 h(n) [z^{-n} + z^n]$$

$$= 0.75 - 0.204(z + z^{-1}) - 0.104(z^2 + z^{-2}) - 0.026(z^3 + z^{-3})$$

The transfer function of the realizable filter is

$$H(z) = z^{-5} H(z)$$

$$H(z) = -0.026z^{-2} - 0.104z^{-3} - 0.204z^{-4} + 0.75z^{-5} - 0.204z^{-6} - 0.104z^{-7} - 0.026z^{-8}$$

12. Design a digital butterworth IIR filter for the following constraints using bilinear transformation method. (Dec 2018, Dec 2019)

$$0.8 \leq |He^{j\omega}| \leq 1, \quad 0 \leq \Omega \leq 0.2\pi$$

$$|He^{j\omega}| \leq 0.2 \quad 0.6\pi \leq \Omega \leq \pi$$

Solution:

Given $\frac{1}{\sqrt{1+\varepsilon^2}} = 0.8$ for which $\varepsilon = 0.75$, $\frac{1}{\sqrt{1+\lambda^2}} = 0.2$ from which $\lambda = 4.899$.

$$\omega_s = 0.6\pi \text{ rad}; \omega_p = 0.2\pi \text{ rad}$$

$$\frac{\omega_s}{\omega_p} = \frac{\Omega_s T}{\Omega_p T} = \frac{\Omega_s}{\Omega_p} = \frac{0.6\pi}{0.2\pi} = 3$$

$$N = \frac{\log \lambda / \varepsilon}{\log 1/k} = \frac{\log \frac{4.899}{0.75}}{\log 3} = 1.71$$

Approximating to nearest higher values we have $N = 2$

For $N = 2$ the transfer function of normalized butterworth filter is

N	Denominator of $H(s)$
1	$s + 1$
2	$s^2 + \sqrt{2}s + 1$
3	$(s + 1)(s^2 + s + 1)$
4	$(s^2 + 0.76537s + 1)(s^2 + 1.8477s + 1)$
5	$(s + 1)(s^2 + 0.61803s + 1)(s^2 + 1.61803s + 1)$
6	$(s^2 + 1.931855s + 1)(s^2 + \sqrt{2}s + 1)(s^2 + 0.51764s + 1)$
7	$(s + 1)(s^2 + 1.80194s + 1)(s^2 + 1.247s + 1)(s^2 + 0.445s + 1)$

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

$$\Omega_c = \frac{\Omega_p}{(\varepsilon)^{1/N}} = \frac{0.2\pi}{(0.75)^{1/2}} = 0.231\pi$$

$$H_a(s) = H(s); \text{ for } s \rightarrow s/0.231\pi$$

$$= \frac{0.5266}{s^2 + 1.03s + 0.5266}$$

$$\begin{aligned}
&= \frac{0.516j}{s + 0.51 + j0.51} - \frac{0.516j}{s + 0.51 - j0.51} \\
&= \frac{0.516j}{s - (-0.51 - j0.51)} - \frac{0.516j}{s - (-0.51 + j0.51)} \\
H(z) &= \frac{0.516j}{1 - e^{-0.51T} e^{-j0.51T} z^{-1}} - \frac{0.516j}{1 - e^{-0.51T} e^{j0.51T} z^{-1}} \quad (T = 1\text{sec}) \\
H(z) &= \frac{0.3019z^{-1}}{1 - 1.048z^{-1} + 0.36z^{-2}}
\end{aligned}$$

13. Explain the principle and procedure for designing FIR filter using rectangular window. (May 2019)

The design steps for FIR filter using window techniques are:

- (i) The specifications required for FIR filter design are-
 - a) Type of Filter: Low Pass, High Pass, band-pass and band-stop filters.
 - b) Order N (or length M) of the filter, $N = M - 1$
 - c) Desired Frequency Response

$$H_d(e^{j\omega}) = Ce^{-j\alpha\omega}, \text{ where } \alpha = M-12$$

d) Cut-off frequency: ω_c for LPF and HPF or ω_{c1} and ω_{c2} for band-pass and band-stop filters. If Cut-off frequency is F_c Hz and Sampling frequency is F_s Hz, then $\omega_c = 2\pi F_c F_s$

- (ii) Desired Impulse Response $h_d(n)$ is computed using above specifications by Inverse DTFT of $H_d(e^{j\omega})$

$$\therefore h_d(n) = \frac{1}{2\pi} \int_a^b H_d(\omega) e^{j\omega n} d\omega,$$

where the limits of integration are the Cut-off frequency depending on the type of filter.

- (iii) Desired Impulse Response $h_d(n)$ is of infinite duration. To make it of finite duration, it is multiplied with a suitable window function

$$W(n) = \begin{cases} A & 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

\therefore Finite Impulse Response $h(n) = h_d(n) \times W(n)$

For Rectangular Window:

The rectangular window sequence is given by

$$w_R(n) = 1 \text{ for } -(N-1)/2 \leq n \leq (N-1)/2$$

$$= 0 \text{ otherwise}$$

Use symmetric conditions $h(n)=h(N-1-n)$ when centre of Impulse response is at α and $h(n)=h(-n)$ when centre of Impulse response is at origin. So only half the number of Impulse response are calculated.

(iv) Transfer Function of the FIR Filter is obtained by taking Z Transform of Finite Impulse Response $h(n)$.

\therefore Transfer Function of Filter

$$H(z) = \sum_{n=0}^{N-1} h(n) \cdot z^{-n}$$

14. Design a butterworth LPF satisfying the following specifications $f_p = 0.10$, $\alpha_p = 0.5\text{dB}$, $f_s = 0.15$, $\alpha_s = 15\text{dB}$, $F = 1\text{Hz}$. (Dec 2019)

Given $\alpha_p = 0.5\text{dB}$; $f_p = 0.10\text{rad/sec}$

$\alpha_s = 15\text{dB}$; $f_s = 0.15\text{rad/sec}$

(i) To find the order of the filter N :

$$N \geq \frac{\log \sqrt{\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}}}{\log \frac{f_s}{f_p}} = \frac{\log \sqrt{\frac{10^{0.1 \times 15} - 1}{10^{0.1 \times 0.5} - 1}}}{\log \frac{0.15}{0.10}}$$

$$N \geq \frac{\log \sqrt{\frac{31.62 - 1}{1.122 - 1}}}{\log \frac{0.15}{0.10}}$$

$$N \geq \frac{\log \sqrt{\frac{30.62}{0.122}}}{\log \frac{0.15}{0.10}}$$

$$N \geq 1.715$$

(ii) Round off N to the next higher integer, $N \geq 1.715$, We get $N = 2$

The normalized low pass butterworth filter for $N = 2$ can be found from the table as

N	Denominator of $H(s)$
1	$s + 1$
2	$s^2 + \sqrt{2}s + 1$
3	$(s + 1)(s^2 + s + 1)$
4	$(s^2 + 0.76537s + 1)(s^2 + 1.8477s + 1)$
5	$(s + 1)(s^2 + 0.61803s + 1)(s^2 + 1.61803s + 1)$
6	$(s^2 + 1.931855s + 1)(s^2 + \sqrt{2}s + 1)(s^2 + 0.51764s + 1)$
7	$(s + 1)(s^2 + 1.80194s + 1)(s^2 + 1.247s + 1)(s^2 + 0.445s + 1)$

- (iii) Find the transfer function $H(s)$ for the value of $N = 2$.

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

- (iv) Calculate the value of cutoff frequency Ω_c .

$$\Omega_c = \frac{\Omega_p}{(10^{0.1\alpha_p} - 1)^{1/2N}} = \frac{0.10}{(10^{0.1 \times 0.5} - 1)^{1/4}} = 0.169$$

- (v) Find the transfer function $H(s)$ for $\Omega_c = 1$ rad/sec for the value of N .

The transfer function for $\Omega_c = 0.169$ can be obtained by substituting,

$$s \rightarrow \frac{s}{0.169} \text{ in } H(s)$$

$$, H(s) = \frac{1}{\left(\frac{s}{0.169}\right)^2 + \sqrt{2} \times \left(\frac{s}{0.169}\right) + 1}$$

$$H(s) = \frac{0.028}{(s^2 + 0.238s + 0.028)}$$

$$H(s) = \frac{0.028}{(s + 0.119 + j0.117)(s + 0.119 - j0.117)}$$

$$\frac{0.028}{(s + 0.119 + j0.117)(s + 0.119 - j0.117)} = \frac{A}{(s + 0.119 + j0.117)} + \frac{A^*}{(s + 0.119 - j0.117)}$$

$$\frac{A}{(s + 0.119 + j0.117)} = \frac{0.028}{(s + 0.119 + j0.117)(s + 0.119 - j0.117)}; s = -0.119 - j0.117$$

$$A = \frac{0.028}{-0.119 - j0.117 + 0.119 - j0.117} = \frac{0.028}{-j0.234} = j0.1196$$

$$A = j0.1196$$

$$A^* = -j0.1196$$

$$H(s) = \frac{j0.1196}{s - (-0.119 - j0.117)} + \frac{-j0.1196}{s - (-0.119 + j0.117)}$$

We know that $T = 1$ sec

$$H(z) = \sum_{k=1}^{\infty} \frac{c_k}{1 - e^{p_k} z^{-1}}$$

Therefore,

$$H_a(z) = \frac{j0.1196}{1 - e^{-0.119} e^{-j0.117} z^{-1}} + \frac{-j0.1196}{1 - e^{-0.119} e^{j0.117} z^{-1}}$$

BM T63 – BIOMEDICAL SIGNAL PROCESSING**UNIT-V: BIOMEDICAL APPLICATIONS****PART- A****1. What are the data reduction techniques? (May 2016)**

- Data reduction is a process that reduced the volume of original data and represents it in a much smaller volume.
- Data reduction includes 3 techniques namely dimensionality reduction, numerosity reduction and data compression.

2. Define EEG Signals. (May 2016)

- The signals obtained from the electrical activity of brain are referred to as EEG signals.
- These signals are generally categorized as delta, theta, alpha, beta and gamma based on signal frequencies ranges from 0.1 Hz to more than 100 Hz.

3. Define systole and diastole. (Nov 2016)

- Systole and diastole are two phases of the cardiac cycle.
- Systole occurs when the heart contracts to pump blood out whereas diastole occurs when the heart relaxes after contraction.

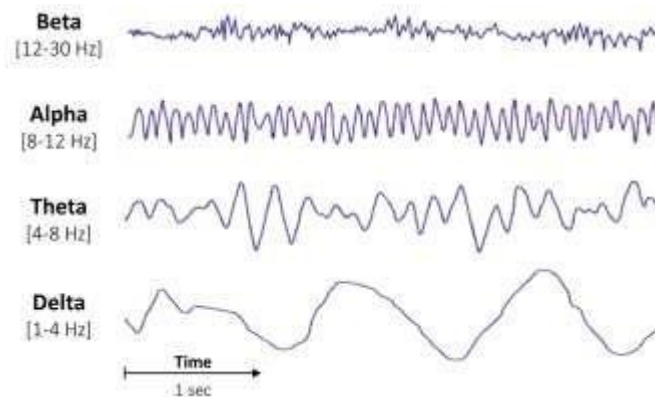
4. Write about R-R interval? (Nov 2016, Dec 2018)

- The RR interval is the time between QRS complexes.
- The instantaneous heart rate can be calculated from the time between any two QRS complexes.

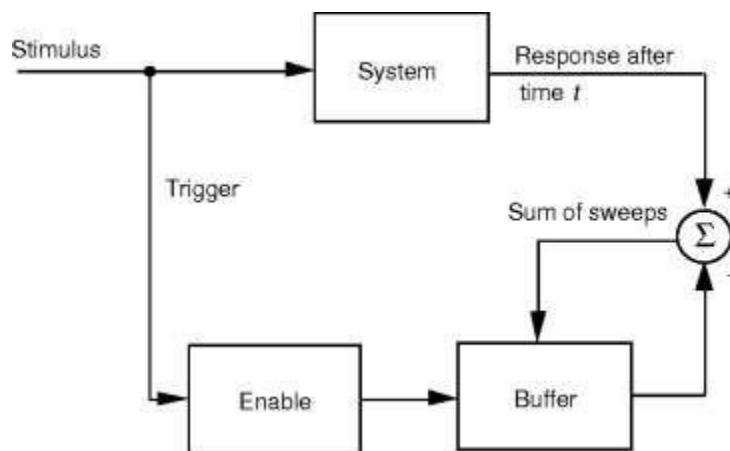
5. What are the various artifacts encountered in ECG analysis. (May 2017, May 2019)

- Loose lead artifact.
- Wandering baseline artifact.
- Muscle tremor artifact.
- CPR compression artifact.
- Neuromodulation artifact.
- Echo distortion artifact.

6. Draw the various EEG waveforms (May 2017, May 2019)



7. Draw the block diagram of typical signal averaging. (Dec 2017)



8. What are all the ranges occur in ECG intervals? (Dec 2017)

- **RR interval:** 0.6-1.2 seconds.
- **P wave:** 80 milliseconds.
- **PR interval:** 120-200 milliseconds.
- **PR segment:** 50-120 milliseconds.
- **QRS complex:** 80-100 milliseconds.
- **ST segment:** 80-120 milliseconds.
- **T wave:** 160 milliseconds.

9. What is EEG signals? Give its characteristics. (May 2018)

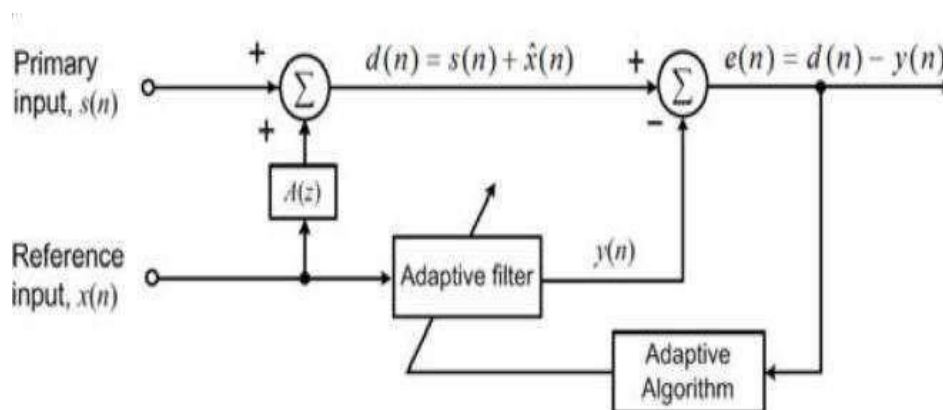
- A complex signal that represents the electrical activity of the brain is termed as EEG signals.
- Like other signals, the EEG can also be split into a series of sinusoids.

- Most signal-processed EEG parameters are based on power spectral analysis, which represents the amplitude of sinusoids as a function of frequency.

10. State the need for frequency information in signal analysis. (May 2018)

- Frequency-domain analysis is an important tool in signal processing applications.
- Frequency-domain analysis shows how the signal's energy is distributed over a range of frequencies.
- It provides the information on the phase shift that must be applied to each frequency component in order to recover the original time signal with a combination of all the individual frequency components.

11. Draw the block diagram of high frequency noise cancellation in electro surgery. (Dec 2018)



12. Define Notch filter. (Dec 2019)

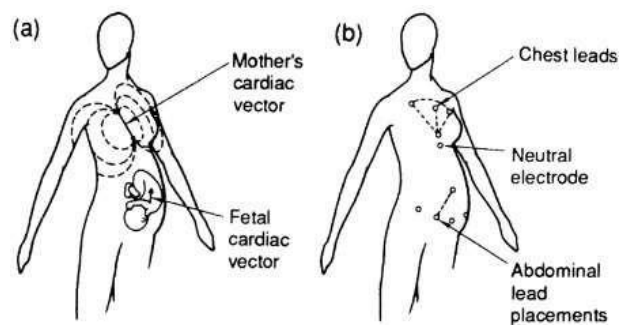
- A notch filter is a type of band-stop filter, which is a filter that attenuates frequencies within a specific range while passing all other frequencies unaltered.
- Notch filters has a very narrow range of frequencies.
- Notch filters have a high Q factor which is the ratio of center frequency to bandwidth.

13. How will you estimate R-R interval? (Dec 2019)

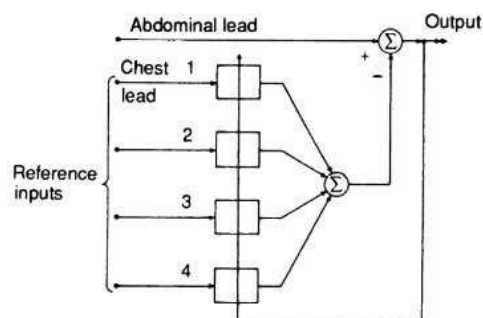
- The RR interval is the time between QRS complexes.
- It can be estimated by counting the number of QRS complexes in a 10 second period and multiplying that by 6.
- R-R interval = (QRS complexes for 10 sec period) * 6

PART – B**1. Explain in detail about the Maternal ECG cancellation in fetal electrocardiography. (May 2016, May 2017, Dec 2018)**

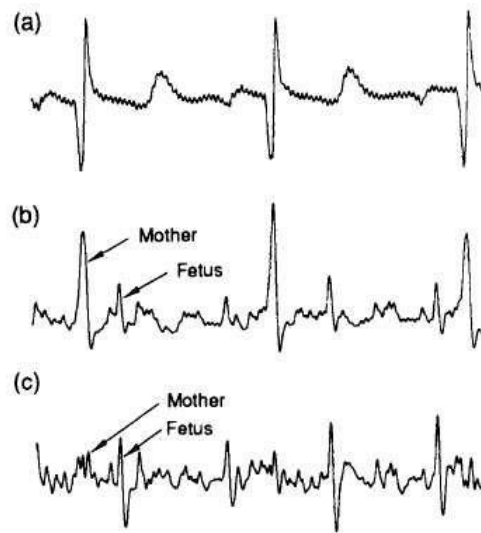
- In practice, fetal heart rate and number of fetuses are detected by recording abdominal electrocardiograms during labor and delivery.
- However, abdominal electrocardiograms are contaminated with background noise due to muscular activity and fetal motion.
- The detection of the fetal heartbeat is further obscured by the mother's heartbeat which is about twice as strong.
- The adaptive filter has been used to eliminate the background noise and enhance the fetal ECG.
- The input to the adaptive canceler, consisting of maternal and fetal heartbeats, is recorded from the mother's abdomen.
- In addition, four electrodes located at the mother's chest are used to record the mother's ECG. Signals from these ECG electrodes are employed as reference inputs to the ANC on the basis of the LMS algorithm.
- The following figure shows the cardiac field vectors of mother and fetus, and placement of leads.



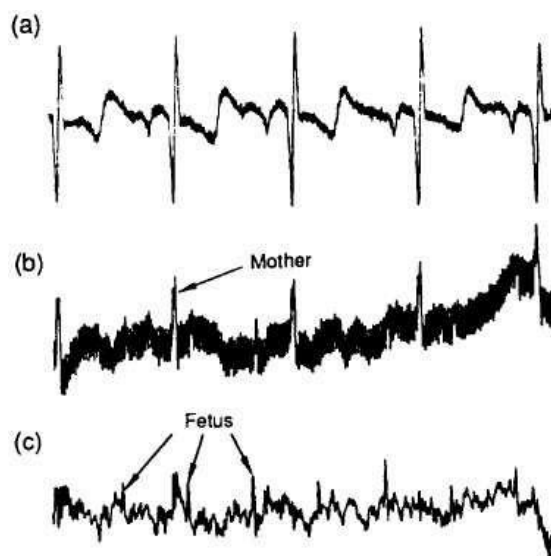
- The below figure shows the block diagram of the multiple-reference noise canceller used in this study.



- The following figure shows one of the reference inputs, the main input signal, and the adaptive noise canceling output.



- It is obvious from the above figure that the adaptive noise canceler output shows the fetal ECG by attenuating the mother's ECG present in the input signal.
- In this, each channel had 32 weights. The sampling frequency is 256 Hz.
- The main and multi-reference input signals are filtered before being presented to the adaptive noise canceler for elimination of frequencies above 35 Hz and below 3 Hz.
- It also checks the performance of the ANC by using different pre-filters (bandwidth 0.3-75 Hz) and a different sampling frequency (512 Hz).
- The below figure shows the result of wideband fetal ECG enhancement.



2. Describe in detail about Time and Frequency domain methods. (May 2016, Dec 2017, Dec 2019)

Propose a technique to detect alpha activity in EEG signal both in Time domain and Fourier domain. (May 2018)

- EEG signal is considered as a realization of a random process (RP).
- Random processes are characterized by probability distributions and their moments or correlation function and their Fourier spectra.
- EEG signal can be characterized by a number of average characteristics of the signal such as covariance /correlation (time domain) and spectra (frequency domain).
- The covariance $\gamma(t1, t2)$ between any 2 time samples $x(t1)$ and $x(t2)$ of a random signal is given by the following expectation:

$$\gamma(t1, t2) = \text{Cov}[x(t1), x(t2)] = E[(x(t1) - E(x(t1)))(x(t2) - E(x(t2)))]$$

- Estimating the covariance between any 2 signals, $x(t1)$ and $x(t2)$, requires averaging over a number of realizations of an ensemble.
- Another way to estimate the covariance of a stationary and ergodic signal is by computing time-average of the product of the signal and a replica of itself shifted by a certain time, τ , along the time axis.
- The time-average, $R_{xx}(\tau_k)$ is called the auto-correlation function and is given by

$$R_{xx}(\tau_k) = \langle x(t1), x(t1 + \tau_k) \rangle$$

- Assuming that every sample function or realization is representative of the whole signal being analyzed, it can be shown that for stationary and ergodic processes, the time-average $R_{xx}(\tau)$ for one realization $x(t)$ is an estimate of the ensemble

$$\gamma_{xx} = E[x(t) x(t + \tau)]$$

assuming that the signal $x(t)$ is of zero mean.

For the value $\tau = 0$,

$$\gamma_{xx}(0) = \lim_{n \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x^2(t) dt$$

which is the signal's average power or variance σ^2 . An important feature of the auto-correlation function is that its Fourier transform is

$$\varphi_{xx}(f) = \int_{-\infty}^{\infty} \gamma_{xx}(\tau) e^{-j2\pi f\tau} d\tau$$

$\varphi_{xx}(f)$ is called the power spectrum of the process $x(t)$. For the discrete case $x(n)$, the power spectral density (PSD) is defined as discrete-time Fourier transform (DTFT) of the auto-correlation sequence. Thus,

$$P_{xx}(f) = T \sum_{n=-\infty}^{\infty} \gamma_{xx}(n) e^{-j2\pi fnT}$$

where the PSD is assumed to be bandlimited to $\pm \frac{1}{2T}$ Hz and is periodic in frequency with period $\frac{1}{T}$ Hz.

3. What are the methods used for QRS complex detection? Explain PAN and TOMPKINS algorithm with block diagram. (Nov 2016, May 2019)

TECHNIQUES OR ALGORITHM USED FOR QRS DETECTION:

1. Template matching techniques

- Concept - Similarity between patterns related to human cognitive process
- Two methods - require a certain pattern and then compared against it for similarity and dissimilarity.

a) The template cross correlation method:

- Concept - Correlation
- Signals correlate when wave shapes are similar
- Correlation coefficient is higher when the similarity is high.
- Algorithm begins by computing cross correlation function and finding out the point where the similarity is maximum.
- Presence of similarity itself determines the occurrence of QRS complex.

b) The Template Subtraction Method:

- The incoming signal is compared with the reference signal and then the difference between them is noted.
- Algorithm begins by saving the incoming signal and then they are compared with the reference template and their difference is noted.
- When the difference is close to 0, the incoming signal is aligned with the reference template and the number of 0's determine their time duration.

- Advantage - Less complex (only summation and no product)
- Disadvantage - SNR is high only when the difference is 0 but at other points there are artifacts. So this process requires preprocessing before application.

2. Differentiation based QRS detection technique

- Since QRS complex is a high frequency complex, differentiation method magnifies the high frequency components and attenuates the low frequency waves such as P, T and U.
- This also implies two different techniques. The first method gives an estimation on the width of the QRS complex.
- The second method yields slope, amplitude and width of the QRS complex.

a) Simple high speed QRS width detection algorithm:

- Steps involved in this method are:
 - ❖ ECG waveform is BP filtered to remove low frequency components and high frequency noise.
 - ❖ Then the waveform is differentiated and rectified.
 - ❖ Then it is once again differentiated and we get second differentiation and rectification.
 - ❖ The first and second derivative is added and then smoothing of the derivatives is carried out.
 - ❖ The smoothed derivatives are scaled and then summed.
 - ❖ The final step is thresholding to obtain a rectangular pulse.
- Advantage - simple to implement and produce a pulse that is proportional to QRS width.
- Disadvantage - sensitive to high frequency noise.

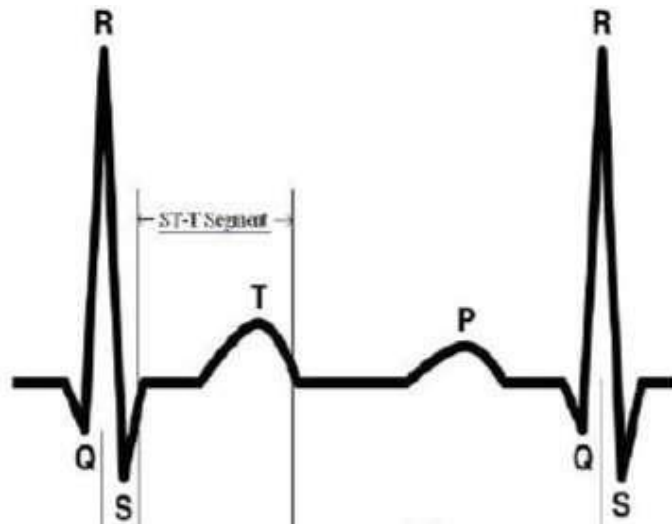
b) High Speed QRS detection algorithm:

- Steps followed in this method are:
 - ❖ LPF - Muscle noise, Power line frequency (50 / 60 Hz)
 - ❖ HPF - Base line wander, T wave noise
 - ❖ Differentiation- To obtain info on the QRS wave
 - ❖ Squaring – non-linear process to make all points in the signal positive

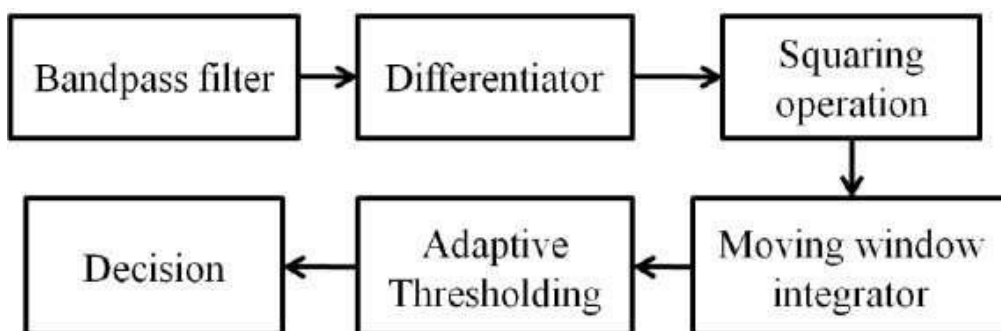
- ❖ Time averaged - window integration method is used to refine our output signal.
- Integer coefficient digital filters are used.

PAN and TOMPKINS algorithm:

- The below figure shows a complete ECG wave and ST-T segment.
- In order to detect ST-T segment, first peak of the R location is found.



- There are several methods for detecting QRS complex and one such algorithm is Pan& Tompkins method.
- This method is based on analysis of the slope, amplitude and width of QRS complexes.
- The algorithm includes a series of filters and methods that perform **low-pass, high-pass, derivative, squaring, integration, adaptive thresholding and search procedures.**
- The following figure illustrates the steps of the algorithm in the schematic form.



- In the first step the algorithm passes the signal through a low pass and a high pass filter in order to reduce the influence of the muscle noise, the power line interference, the baseline wander and the T-wave interference.
- After filtering, the signal is differentiated to provide the QRS slope information using the following formula:

$$y = \frac{1}{8} [2x(n) + x(n-1) - x(n-3) - 2x(n-4)]$$

- Then the signal is squared point by point making all data point positive and emphasizing the higher frequencies.
- After squaring, the algorithm performs sliding window integration in order to obtain waveform feature information.
- A temporal location of the QRS is marked from the rising edge of the integrated waveform.
- In the last step, two thresholds are adjusted.
- The higher of the two thresholds identifies peaks of the signal.
- The lower threshold is used when no peak has been detected by the higher threshold in a certain time interval.
- In this case the algorithm has to search back in time for a lost peak.
- When a new peak is identified (as a local maximum – change of direction within a predefined time interval), then this peak is classified as a signal peak if it exceeds the high threshold (or the low threshold if we search back in time for a lost peak) or as a noise peak otherwise.
- In order to detect a QRS complex the integration waveform and the filtered signals are investigated and different values for the above thresholds are used.
- To be identified as a QRS complex, a peak must be recognized as a QRS in both integration and filtered waveform.

4. Describe in detail about the Arrhythmia analysis monitoring. (Nov 2016, Dec 2019)

- An arrhythmia is a problem with the rate or rhythm of your heartbeat.
- The most common type of arrhythmia is atrial fibrillation, which causes an irregular and fast heart-beat.
- Many factors can affect your heart's rhythm, such as having had a heart attack, smoking, congenital heart defects, and stress.

- Some substances or medicines may also cause arrhythmias.
- Symptoms of arrhythmias include
 - ❖ Fast or slow heart beat
 - ❖ Skipping beats
 - ❖ Lightheadedness or dizziness
 - ❖ Chest pain
 - ❖ Shortness of breath
 - ❖ Sweating
- A person can measure their heart rate using their pulse. This is a point at which they can feel the heartbeat through the skin. The best locations on the body for this are:
 - ❖ the wrists
 - ❖ the insides of the elbows
 - ❖ the side of the neck
 - ❖ the top of the foot
- Automated arrhythmia analysis is the detection of arrhythmias through the use of a computer.
- Examples of medical devices that use automated arrhythmia detection include implantable defibrillators, public access automated external defibrillators, atrial defibrillators, and more.
- **Types of Arrhythmias:**
 - ❖ Bradycardia - Decreased heart beat
 - ❖ Tachycardia - Increased heart beat
 - ❖ Asystole - Heart block or Profound Bradycardia
 - ❖ Ventricular fibrillation - rapid and erratic electrical impulses
 - ❖ Skipped beat - Skipping of an entire cycle
 - ❖ Premature ventricular contraction - premature beats + compensatory pause
 - ❖ R on T phenomenon - R wave reoccurrence on T wave
 - ❖ Bigeminy - Normal beat + premature beat

- ❖ Trigeminy - Normal beat + 2 premature beats
- ❖ Interpolated PVC - premature beats minus compensatory pause

5. Describe in detail about ECG data compression techniques. (May 2017)

1) The AZTEC Technique:

- The amplitude zone – time epoch coding (AZTEC) algorithm developed for preprocessing real-time ECG's for rhythm analysis.
- It has become a popular data reduction algorithm for ECG monitors and databases with an achieved compression ratio of 10: 1.
- However, the reconstructed signal demonstrates significant discontinuities and distortion.
- In particular, most of the signal distortion occurs in the reconstruction of the P and T waves due to their slow varying slopes.
- The AZTEC algorithm converts raw ECG sample points into plateaus and slopes.
- The AZTEC plateaus (horizontal lines) are produced by utilizing the zero-order interpolation (ZOI).
- The stored values for each plateau are the amplitude value of the line and its length (the number of samples with which the line can be interpolated within aperture ϵ).
- The production of an AZTEC slope starts when the number of samples needed to form a plateau is less than three.
- The slope is saved whenever a plateau of three samples or more can be formed.
- The stored values for the slope are the duration (number of samples of the slope) and the final elevation (amplitude of last sample point).
- Signal reconstruction is achieved by expanding the AZTEC plateaus and slopes into a discrete sequence of data points.
- Even though the AZTEC provides a high data reduction ratio, the fidelity of the reconstructed signal is not acceptable to the cardiologist because of the discontinuity (step-like quantization) that occurs in the reconstructed ECG waveform.
- A significant reduction of such discontinuities is usually achieved by utilizing a smoothing parabolic filter.
- The disadvantage of utilizing the smoothing process is the introduction of amplitude distortion to the ECG waveform.

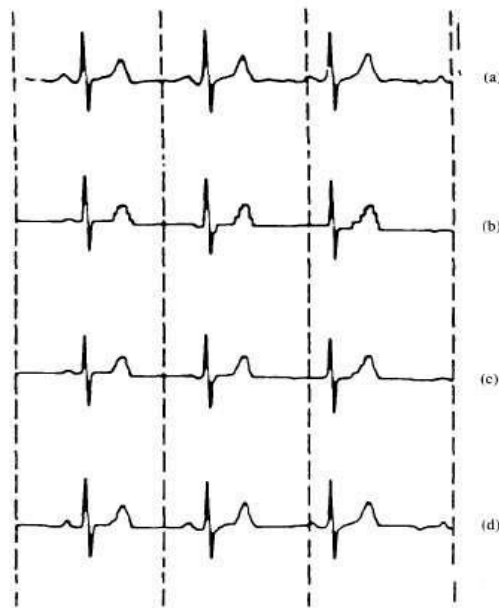
2) The Turning Point Technique:

- The turning point (TP) data reduction algorithm developed for the purpose of reducing the sampling frequency of an ECG signal from 200 to 100 Hz without diminishing the elevation of large amplitude QRS's.
- The algorithm processes three data points at a time; a reference point (X_1) and two consecutive data points (X_1 and X_2). Either X_1 or X_2 is to be retained.
- This depends on which point preserves the slope of the original three points.

- The TP algorithm produces a fixed compression ratio of 2:1 whereby the reconstructed signal resembles the original signal with some distortion.
- A disadvantage of the TP method is that the saved points do not represent equally spaced time intervals.

3) The CORTES Scheme:

- The coordinate reduction time encoding system (CORTES) algorithm is a hybrid of the AZTEC and TP algorithms.
- CORTES applies the TP algorithm to the high frequency regions (QRS complexes), whereas it applies the AZTEC algorithm to the isoelectric regions of the ECG signal.
- The AZTEC and TP algorithms are applied in parallel to the incoming sampled ECG data.
- Whenever an AZTEC line is produced, a decision based on the length of the line is used to determine whether the AZTEC data or the TP data is to be saved.
- If the line is longer than an empirically determined threshold, the AZTEC line is saved, otherwise the TP data are saved.



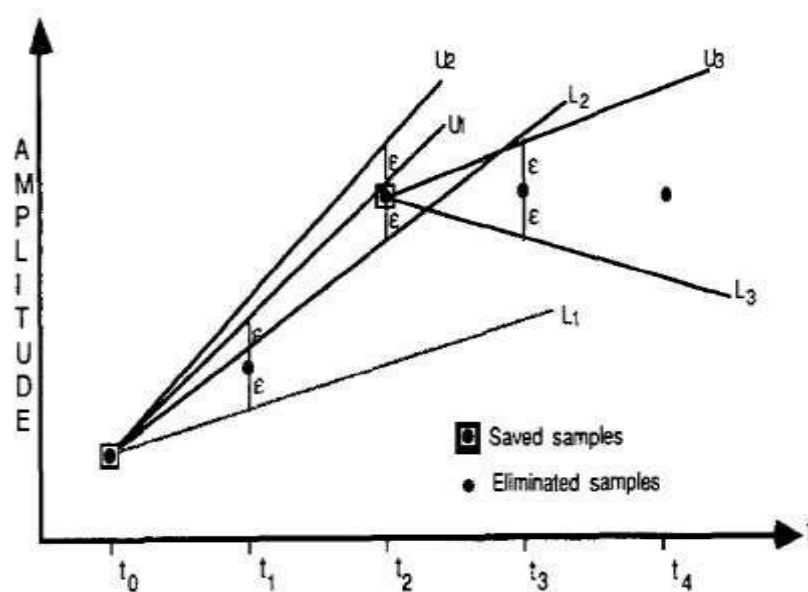
- Only AZTEC plateaus (lines) are generated; no slopes are produced.
- The CORTES signal reconstruction is achieved by expanding the AZTEC plateaus into discrete data points and interpolating between each pair of the TP data.
- Parabolic smoothing is applied to AZTEC portions of the reconstructed CORTES signal to reduce distortion.
- The above figure shows the effect of the AZTEC (b), filtered AZTEC (c) and CORTES (d) algorithms on the ECG (a).

4) Fan and SAPA Techniques:

Fan and scan-along polygonal approximation (SAPA) algorithms, developed for ECG data compression, are based on the first-order interpolation with two degrees of freedom (FOI-2DF) technique.

A) The Fan Algorithm:

- In essence, the Fan is a method of implementing the FOI-2DF without requiring the storage of all the actual data points between the last transmitted point and the present point during program execution.



- An illustration of the Fan method is shown in above figure.
- The Fan algorithm starts by accepting the first data point as a non-redundant (permanent) point (t_0) and functions as the origin.
- Two slopes (U_1, L_1) are drawn between the originating point and the next sample plus a specified threshold ($\pm\epsilon$).
- One upper slope (U_1) passes through a point greater than the second sample point value by a tolerance (ϵ), while the other lower slope (L_1) passes through a point less than the second sample point value by an ϵ .
- If the third sample point (t_2) falls within the area bounded by the two slopes, then new slopes (U_2, L_2) are calculated between the originating point and an ϵ greater and an ϵ lower than the third sample point.
- These new slopes (U_2, L_2) are compared to the previously stored slopes (U_1, L_1) and the most converging (restrictive) slopes are retained (U_1, L_2).
- The process is repeated whereby future sample values are compared with the values of the most convergent slopes.

- Whenever a sample value falls outside the area bounded by the converging slopes, the sample immediately preceding this sample point is saved as the next permanent sample.
- This permanent sample point also becomes the new originating point and the algorithm repeats.
- The sketch of the slopes drawn from the originating sample to future samples form a set of radial lines similar to a “fan,” giving this algorithm its name.
- Upon signal reconstruction, the retained (permanent) samples are connected with straight lines.
- The Fan method guarantees the error between the line, joining any two permanent sample points, and any actual (redundant) sample along the line is less than or equal to the magnitude of the preset error tolerance (ϵ).

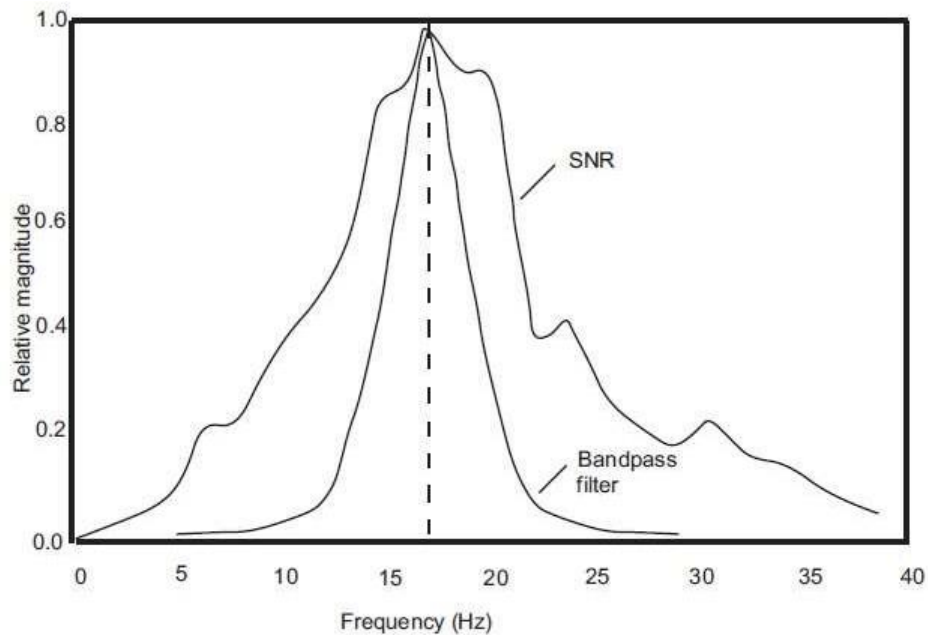
B) SAPA-2 Algorithm:

- The SAPA-2 algorithm, one of the three SAPA algorithms shows the best results.
- The theoretical bases of this algorithm is that the deviation between the straight lines (approximated signal) and the original signal is never more than the preset error tolerance (ϵ).
- The only difference between the Fan and SAPA-2 algorithms is that, in addition to the two slopes calculated in the Fan algorithm, SAPA-2 calculates a third slope between the originating sample point and the actual future sample point (called center slope).
- Whenever the center slope value does not fall within the two converging slopes boundary, the immediately preceding sample point is considered as a permanent sample.
- In other words, the SAPA-2 algorithm uses the center slope criterion, for verifying whether the sample is permanent or redundant, instead of the actual sample value criterion as is the case in the Fan algorithm.

6. Explain in detail about following ECG QRS detection technique (a) Bandpass filtering techniques (b) Differentiation techniques. (Dec 2017, Dec 2018)

BANDPASS FILTERING TECHNIQUES

- From the power spectral analysis of the various signal components in the ECG signal, a filter can be designed which effectively selects the QRS complex from the ECG.
- Figure shows a plot of the signal-to-noise ratio (SNR) as a function of frequency.
- It represents the plots of the signal-to-noise ratio (SNR) of the QRS complex referenced to all other signal noise based on heart beats.
- The optimal bandpass filter for a cardio-tachometer maximizes the SNR.



- The study of the power spectra of the ECG signal, QRS complex, and other noises also revealed that a maximum SNR value is obtained for a bandpass filter with a center frequency of 17 Hz and a Q of 3.

Filter responses for different values of Q

- The value of Q of the bandpass filter centered at $f_c = 17$ Hz determines how well the signal of interest is passed without being attenuated.
- It is also necessary to increase the SNR of the signal of interest; that is, the QRS complex. The Q of the filter is calculated as

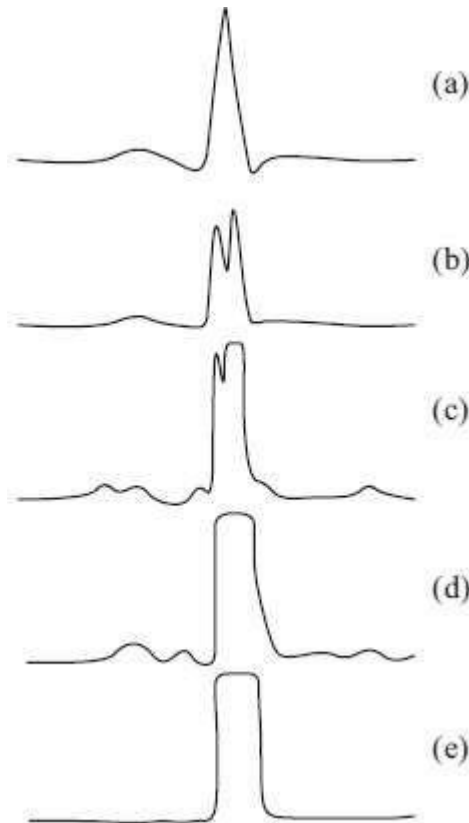
$$Q = \frac{f_c}{BW}$$

- A value of Q that is too high will result in a very oscillatory response. The ripples must die down within 200 ms.
- This is necessary so that the ripples from one QRS complex do not interfere with the ripples from the next one.
- With a center frequency of 17 Hz, the maximal permissible Q was found to be 5.
- For a bandpass filter with $f_c = 17$ Hz, a Q value of 5 was found to maximize the SNR.

DIFFERENTIATION TECHNIQUES

- Differentiation forms the basis of many QRS detection algorithms.
- Since it is basically a high-pass filter, the derivative amplifies the higher frequencies characteristic of the QRS complex while attenuating the lower frequencies of the P and T waves.

- An algorithm based on first and second derivatives was developed and modified for use in high-speed analysis of recorded ECGs.
- Subsequently an algorithm to compare noise sensitivity among certain types of QRS detection algorithms were implemented.
- Figure shows the various signal stages in the QRS detection algorithm based on differentiation. (a) Original ECG. (b) Smoothed and rectified first derivative. (c) Smoothed and rectified second derivative. (d) Smoothed sum of (b) and (c). (e) Square pulse output for each QRS complex.



- The absolute values of the first and second derivative are calculated from the ECG signal

$$y_0(nT) = |x(nT) - x(nT - 2T)|$$

$$y_1(nT) = |x(nT) - 2x(nT - 2T) + x(nT - 4T)|$$

- These two data buffers, $y_0(nT)$ and $y_1(nT)$, are scaled and then summed

$$y_2(nT) = 1.3y_0(nT) + 1.1y_1(nT)$$

- The data buffer $y_2(nT)$ is now scanned until a certain threshold is met or exceeded

$$y_2(iT) \geq 1.0$$

- In addition to detecting the QRS complex, this algorithm has the advantage that it produces a pulse which is proportional in width to the complex.
- However, a disadvantage is that it is particularly sensitive to higher-frequency noise. Once this condition is met for a data point in $y_2(iT)$, the next eight points are compared to the threshold.
- If six or more of these eight points meet or exceed the threshold, then the segment might be part of the QRS complex.

7. Write short notes on interference associated with biomedical signals and summarize various filtering techniques. (May 2018)

POWERLINE INTERFERENCE

- Electromagnetic fields caused by a power line represent a common noise source in ECG, as well as to any other bioelectrical signal recorded from the body surface.
- Such noise is characterized by 50 or 60Hz sinusoidal interference, possibly accompanied by a number of harmonics.
- It is necessary to remove the power line interference from ECG signal as it completely superimposes the low frequency ECG waves like P wave and T wave.
- The figure shows an ECG signal typically affected by a power line interference of 50 /60 Hz.



FILTERING TECHNIQUES

- Different filtering techniques has been proposed for Cancellation of Power line Interference from ECG signal
 - ❖ IIR Notch filter
 - ❖ FIR filtering
 - ❖ Adaptive filtering

IIR Notch Filter:

- Mostly researchers use digital infinite impulse response to remove the effects of power line interference and baseline wander from ECG signal.
- Notch filter can be used to remove the stationary power line interference which means do not vary with amplitude, frequency and phase over time.

- For a second order notch filter, bandwidth (Δf), notch frequency (f_o), and quality factor (referred as Q factor) are related by

$$Q = f_o / \Delta f$$

FIR Filtering

- FIR is chosen since it is simple and stable.
- FIR filters are particularly useful for applications where exact linear phase response is required.
- This FIR filter uses different window techniques in order to make ease of design and simplicity of programming.
- The windows aims to meet the proposed filter requirements.

Adaptive Filtering:

- The word adaptive means to adjust with the other environment by having same response as the system.
- The adaptive filter reduces the mean square error between the primary input (ECG signal) and the reference input (Noise ECG signal).
- The power line interference 50/60Hz from ECG signal such as harmonics and high frequency noise can be removed by adaptive filtering.
- These filters can able to remove the noise and extract the information and separate two or more combined signals.

8. Discuss in detail the parametric model in automated analysis of EEG. (May 2019)

Parametric Model:

- The FFT does not provide good frequency resolution and thus may render the spectral responses of two or more signals with short length and low SNR indistinguishable.
- The limitations of the FFT method may be overcome by use of parametric analysis methods.
- Parametric modeling of biomedical signals involves choosing an appropriate model for the signal and estimating the model parameters.
- These parameters can then be used to calculate the power spectrum of the signal, to classify the signal, or to perform data compression and pattern recognition.
- To model biomedical signals accurately the model parameters must be fewer in number than the samples of the related signal.
- Among the prominent parametric modeling methods, autoregressive modeling of signals shows outstanding performance when the signal has very sharp peaks.
- The autoregressive moving average (ARMA) model can be used for modeling signals with sharp frequency peaks and valleys.

Autoregressive Moving Average (ARMA) Method:

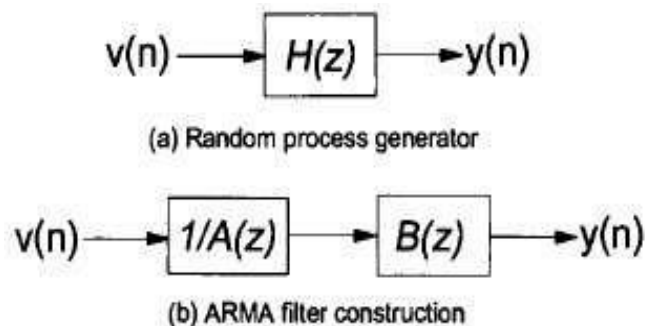
- Biomedical signals may not be appropriately modeled using autoregressive (AR) methods.
- For example, signals that are heavily corrupted by background noise are not adequately modeled by AR but are well suited to analysis by the ARMA method.
- The AR method is appropriate for signals with frequency responses that have sharp peaks, and MA works better on deep valleys.
- The ARMA method has been described as a pole-zero method.
- The output sequence $y(n)$ can be modeled by assuming an input driving sequence $v(n)$ as follows,

$$y = - \sum_{m=1}^p ay(n-m) + \sum_{m=0}^q bv(n-m)$$

where a represents the AR coefficient of the AR process at the m^{th} stage,

b represents the MA coefficient of the MA process at the m^{th} stage,

p represents the AR model order and q represents the MA model order.



- The above figure shows the generation of the ARMA process $y(n)$ by means of a causal, linear, time-invariant system.
- The input to the system $v(n)$ is a Gaussian process.
- The objective here is to separate the driving force $v(n)$ from any background noise.
- The transfer function of the ARMA process, $H(z)$, can be given in terms of the transfer function of the AR and MA methods,

$$H(z) = \frac{B(z)}{A(z)}$$

where $B(z)$ represents the transfer function of the MA part of $H(z)$ and $A(z)$ represents the transfer function of the AR part of $H(z)$.

- It is evident that the ARMA method calculates the AR and MA coefficients from non-linear equations.