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DEPARTMENT OF ELECTRONICS AND COMMUNICATION ENGINEERING

EC T35 CIRCUIT THEORY NOTES

II YEAR/ III SEM

EC T35 CIRCUIT THEORY

COURSE OBJECTIVE

- *To understand the need for various theorems to solve complicated Electrical circuits*
- *To explore the use of Resonant circuits and tuned circuits in the field of communication*
- *To analyze the transient behavior of Electrical circuits*
- *To identify the ways and means to solve magnetically coupled circuits*
- *To understand the use of network topology in circuit solving*

UNIT- I

DC Circuit Analysis: Sources-Transformation and manipulation, Network theorems - Superposition theorem, Thevenin's theorem, Norton's theorem, Reciprocity theorem, Millman's theorem, Compensation theorem, Maximum power transfer theorem and Tellegen's theorem – Application to DC circuit analysis.

UNIT- II

AC Circuit Analysis: Series circuits - RC, RL and RLC circuits and Parallel circuits –RLC circuits - Sinusoidal steady state response - Mesh and Nodal analysis - Analysis of circuits using Superposition, Thevenin's, Norton's and Maximum power transfer theorems.

Resonance - Series resonance - Parallel resonance - Variation of impedance with frequency - Variation in current through and voltage across L and C with frequency – Bandwidth – Q factor - Selectivity.

UNIT- III

Transient Analysis: Natural response-Forced response - Transient response of RC, RL and RLC circuits to excitation by DC and exponential sources - Complete response of RC, RL and RLC Circuits to sinusoidal excitation-Transient analysis by Laplace Transformation Technique.

UNIT- IV

Magnetically Coupled Circuits: Self inductance - Mutual inductance - Dot rule - Coefficient of coupling - Analysis of multi winding coupled circuits - Series, Parallel connection of coupled inductors - Single tuned and double tuned coupled circuits.

UNIT -V

Network Topology: Network terminology - Graph of a network - Incidence and reduced incidence matrices – Trees –Cutsets - Fundamental cutsets - Cutset matrix – Tiesets – Link currents and Tieset schedules -Twig voltages and Cutset schedules, Duality and dual networks.

1. William H. Hayt, Jr. Jack E. Kemmerly and Steven M. Durbin, ―Engineering Circuit Analysis‖, McGraw Hill Science Engineering, 8th Edition, 2013.

2. Joseph Edminister and Mahmood Nahvi, ―Electric Circuits‖, Schaum's Outline Series, Fourth Edition, Tata McGraw Hill Publishing Company, New Delhi, 2003.

Reference Books:

1. David A. Bell, ―Electric Circuits‖, Sixth Edition, PHI Learning, New Delhi, 2003.
2. P. Ramesh Babu, ―Circuits and Networks‖, Scitech Publications, First Edition 2010, Chennai.

Web References:

1. www.circuit_magic.com
2. www.learnabout_electronics.org

UNIT- I DC Circuit Analysis

1. Circuit : Acircuit is a closed conducting path through which an electric current either flows or is intended to flow.

2. Parameters. The various elements of an electric circuit are called its parameters like resistance, inductance and capacitance. These parameters may be ***lumped or distributed*.**

3. Liner Circuit. A linear circuit is one whose parameters are constant *i.e.* they do not change with voltage or current.

4. Non-linearCircuit. It is that circuit whose parameters change with voltage or current.

5. BilateralCircuit.A bilateral circuit is one whose properties or characteristics are the same In either direction. The usual transmission line is bilateral ,because it can be made to perform its function equally well in either direction.

6. UnilateralCircuit. It is that circuit whose properties or characteristics change with the direction of its operation. A diode rectifier is a unilateral circuit, because it cannot perform rectification in both directions.

7. ElectricNetwork. A combination of various electric elements, connected in any manner whatsoever, is called an electric network.

8. PassiveNetwork is one which contains no source of e.m.f. in it.

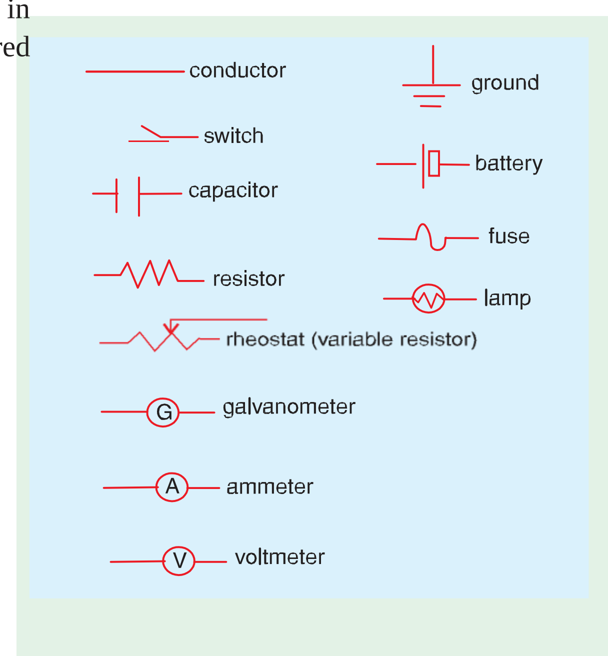
9. ActiveNetwork is one which contains one or more than one source of e.m.f.

10. Node is a junction in a circuit where two or more circuit elements are connected together.

11. Branch is that part of a network which lies between two junctions.

12. Loop. It is a close path in a circuit in which no element or node is encountered more than once.

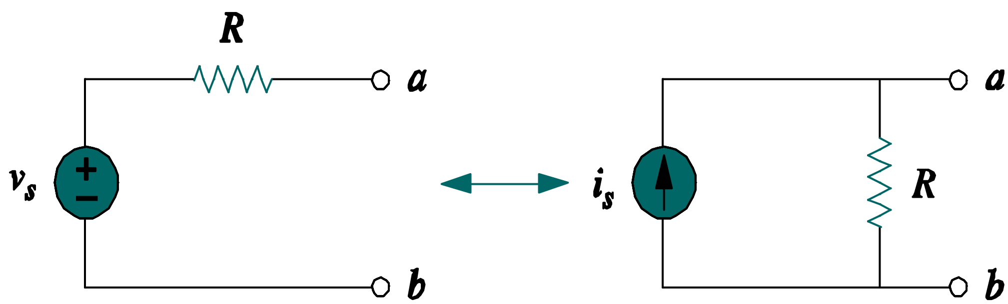
13. Mesh. It is a loop that contains No other loop within it.



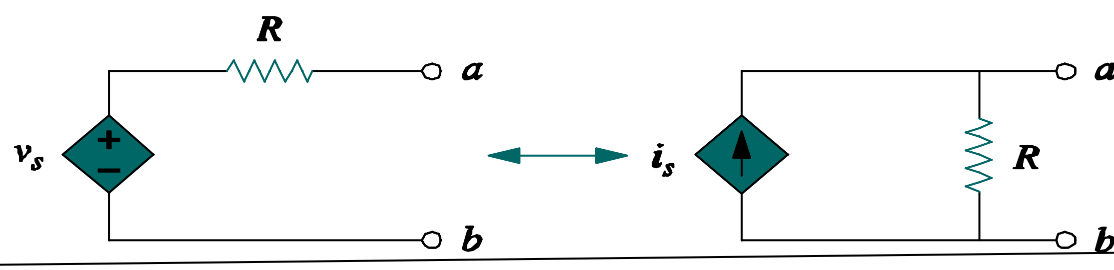
Sources-Transformation and manipulation

- A source transformation is the process of replacing a voltage source V_s in series with a resistor R by a current source i_s in parallel with a resistor R, or vice versa.

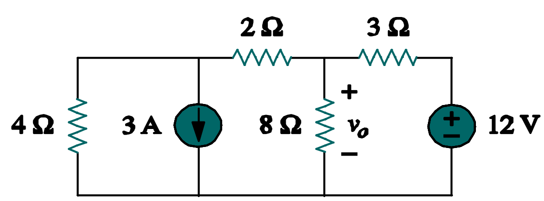
  $V_s = i_s R$ or $i_s = V_s/R$

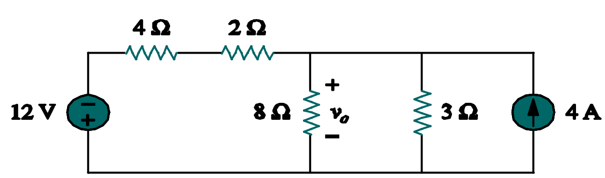


- It also applies to dependent sources:

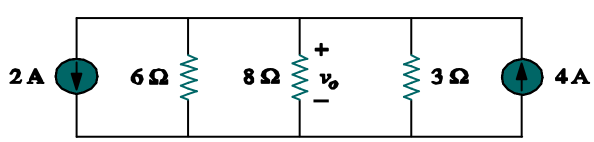


1. Example, find out Vo

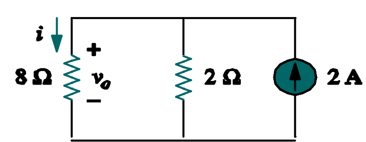




(a)

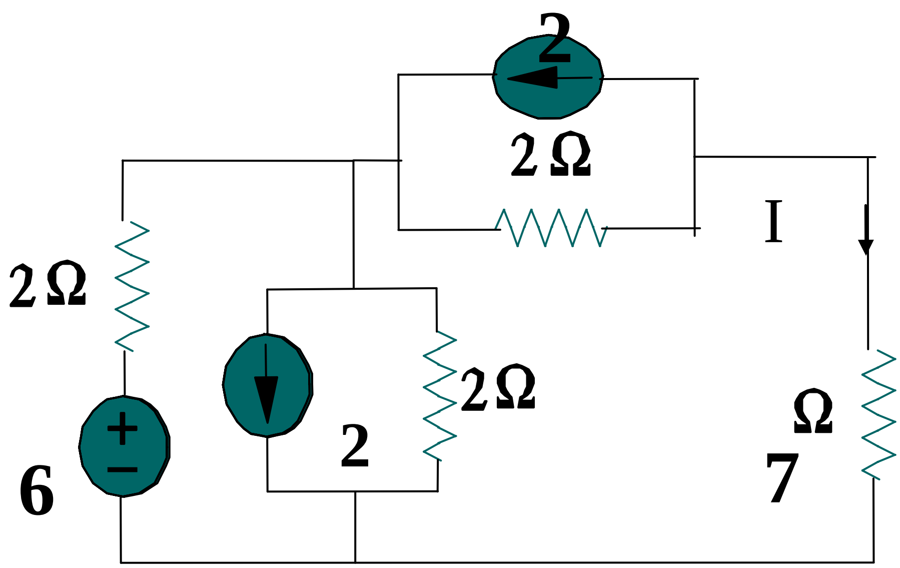


(b)



(c)

2. **find out I (use source transformation)**

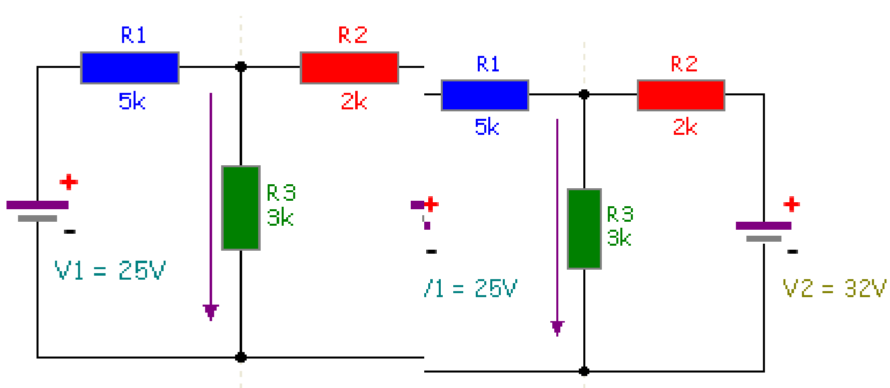


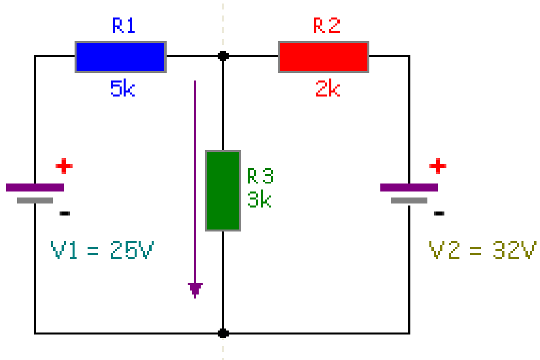
I =0.25 A

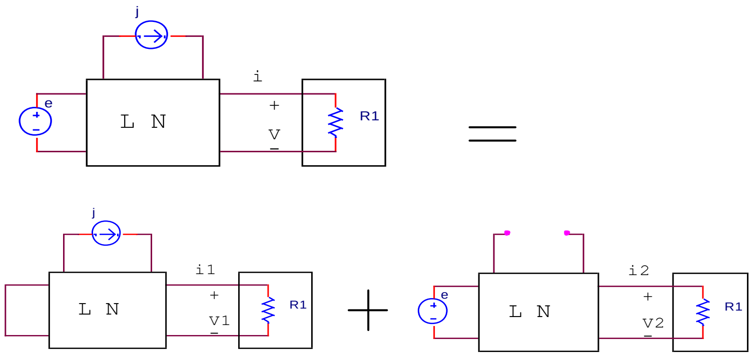
THE SUPERPOSITION THEOREM

In an electrical network made up from linear resistances and containing *more than one sourceof emf*, the resultant current flowing in any branch is the algebraic sum of the currents that would flow in that branch if the effects of each emf were considered separately all other emfs being suppressed and replaced by their respective internal resistances(normally this is a short circuit).

> *"The total power delivered to a resistive element must be determined using the total current through or the total voltage across the element and cannot be determined by a simple sum of the power levels established by each source."*





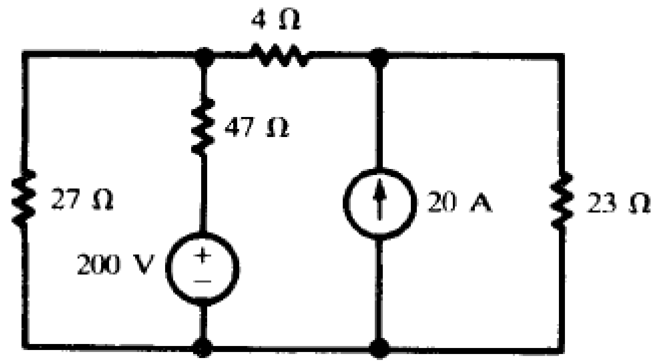


$V = V1 + V2; \quad i = i1 + i2$

Advantages

- Used to find the solution to networks with two or more sources that are not in series or parallel.
- The current through, or voltage across, an element in a network is equal to the algebraic sum of the currents or voltages produced independently by each source.
- Since the effect of each source will be determined independently, the number of networks to be analyzed will equal the number of sources.
- Linearity is the property of an element describing a linear relationship between cause and effect.
- A linear circuit is one whose output is linearly (or directly proportional) to its input.

1. Solve the circuit shown below by super position principle.



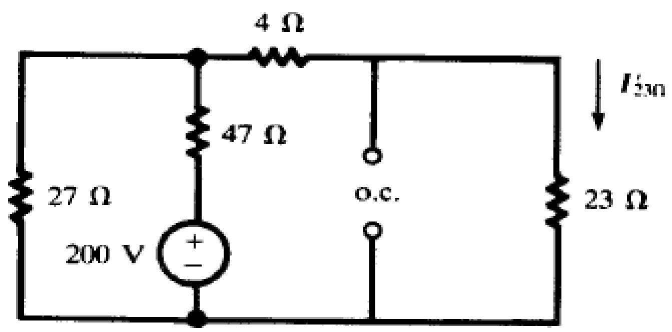
1. Find the total current i_T and Req in the circuit when 200v source alone acting.

2. Calculate the i_T and Req in the circuit when 20A source alone acting.

3. Determine the total current through 23Ω in the circuit.

4. Compute the current through 4 Ω resistor in the circuit.

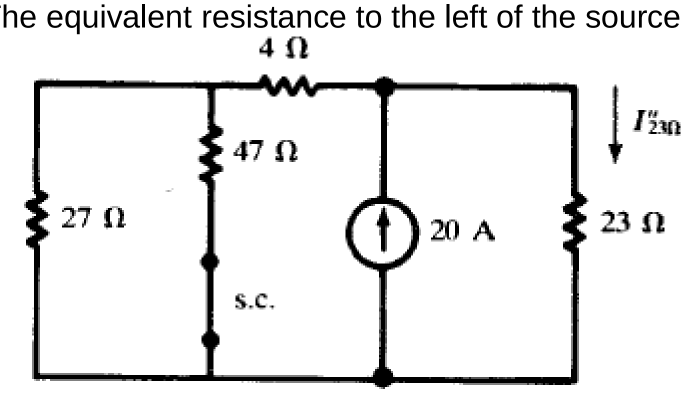
Solution: With the 200-V source acting alone, the 20-A current source is replaced by an open circuit is shown in figure (a)



(a)

$$R_{eq} = 47 + \frac{(27)(4+23)}{54} = 60.5\ \Omega$$
$$I_T = \frac{200}{60.5} = 3.31\ \text{A}$$
$$I'_{23\Omega} = \left(\frac{27}{54}\right)(3.31) = 1.65\ \text{A}$$

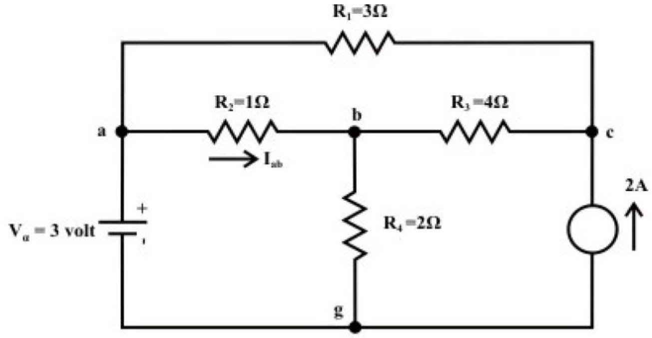
When the 20-A source acts alone, the 200-V source is replaced by a short circuit, Fig.(b). The equivalent resistance to the left of the source is



(b)

$$R_{eq} = 4 + \frac{(27)(47)}{74} = 21.15\ \Omega$$
$$I''_{23\Omega} = \left(\frac{21.15}{21.15+23}\right)(20) = 9.58\ \text{A}$$

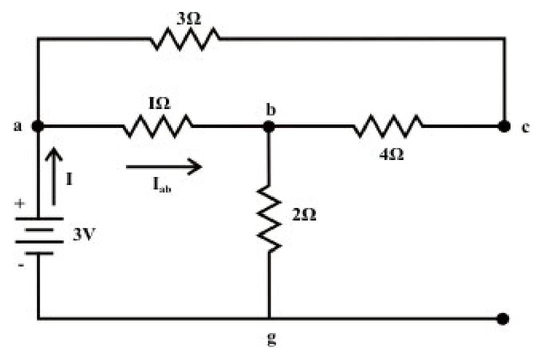
2. Consider the network shown in fig and solve by super position theorem.



1.Calculate V_{cg} using superposition theorem.
2.Calculate I_{ab} using superposition theorem.
3.Determine the total current flow when voltage source alone acting
4.Find the current through R_4 resistor.

Solution:

First consider the voltage source that acts only in the circuit and the current source is replaced by its internal resistance and it is shown below.



Calculate the current flowing through the 'a-b' branch

$$R_{eq} = [(R_{ac} + R_{cb}) \| \; R_{ab}] + R_{bg} = \frac{7}{8} + 2 = \frac{23}{8}\Omega$$

$$I = \frac{3}{\frac{23}{8}} A = 1.043A$$

Now current through a to b, is given by

$$I_{ab} = \frac{7}{8} \times \frac{24}{23} = 0.913A$$

$$I_{acb} = 1.043 - 0.913 = 0.13A$$

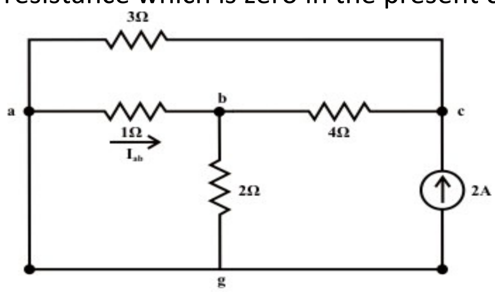
Voltage across c-g terminal :

$$V_{cg} = V_{bg} + V_{cb}$$

$$= 2 \times 1.043 + 4 \times 0.13 = 2.61 volts$$

Current source only (retain one source at a time):

Now consider the current source only acting and the voltage source is replaced by its internal resistance which is zero in the present case. The circuit diagram is shown below



Current in the following branches:

$$3\Omega\, resistor = \frac{(14/3)\times 2}{(14/3)+3} = 1.217A; \qquad 4\Omega\, resistor = 2 - 1.217 = 0.783A$$

$$1\Omega\ resistor = \left(\frac{2}{3}\right)\times 0.783 = 0.522\,A\ (b\ to\ a)$$

Voltage across 3Ω resistor (c & g terminals) $V_{cg} = 1.217\times 3 = 3.651\,volts$

The total current flowing through 1Ω resistor (due to the both sources) from a to b = 0.913 (due to voltage source only; current flowing from 'a' to 'b') – 0.522 (due to current source only; current flowing from 'b'to 'a') $= 0.391A$.

Total voltage across the current source $V_{cg} = 2.61\,volt$ (due to voltage source ; 'c' is higher potential than 'g') + 3.651 $volt$ (due to current source only; 'c' is higher potential than 'g') $= 6.26\,volt$.

<u>Thevenin's and Norton's Theorems</u>

- That if we are only interested in current, voltage and power delivered by a linear portion of a circuit, we can replace that portion (potentially a large complex network,) by an *equivalent circuit containing only an independent source and a single resistor*. The response will be unchanged in the rest of the original circuit.

- ***Thevenin's Theorem says that the independent source is a voltage source and we should place it in series with the resistor. The theorem also tells us how to calculate the value of the voltage source, V_s, and the value of the resistance, R_t, called the Thevenin Resistance.***

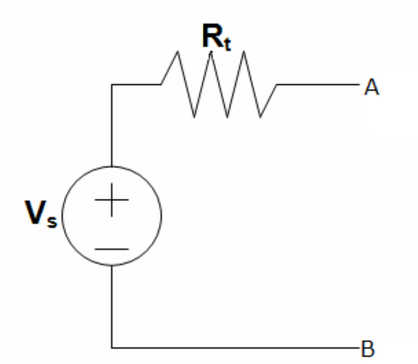
- ***Norton's Theorem says that the independent source is a current source and we should place it in parallel with the resistance. The theorem also tell us how to calculate the value of the current source, I_s, and the value of the resistance, R_t, called the Thevenin Resistance.***

- Of course, by source transformations, we can always switch from the "Thevenin" equivalent circuit to the "Norton" equivalent circuit.

To find the Thevenin Equivalent Network

1. **First you must identify the network to find the equivalent of.**
   You can rearrange any circuit in the form of two networks connected by two resistance-less conductors, labeled terminals A and B. (Note: If either network contains a dependant source, its control variable must be in the same network.)

   If one of the networks is linear it can be replaced by this Thevenin equivalent network:

   The only thing left to do is find the values of R_t and V_s.

2. **To find V_s:**
   Define a voltage, v_{oc}, as the open circuit voltage which would appear across the terminals A and B (of the original network) if there was an open circuit between A and B. This voltage is V_s.

3. **To find R_t:**
   There are three different cases that will require different methods to find R_t:

   a. If there are *only* independent sources in the network, then "kill" them.
      $R_t = R_{eq}$

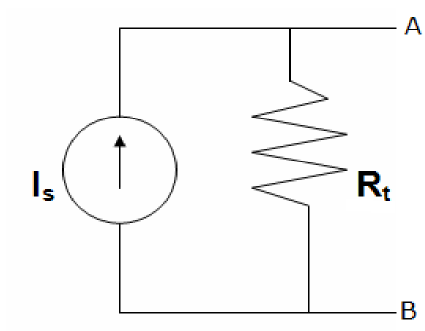
   b. If there are dependant sources *and* independent sources in the network, find both v_{oc} and i_{sc}.
      $R_t = v_{oc} / i_{sc}$.

   c. If there are *only* dependent sources apply a 1A current source at the terminals A and B. Calculate the resulting voltage, v, across this current source.
      $R_t = v / 1A$
      (Alternatively you can apply a 1V voltage source and measure resulting current, i, through it. $R_t = 1V / i$)

To find the Norton Equivalent Network

1. **First you must identify the network to find the equivalent of.**
   You can rearrange any circuit in the form of two networks connected by two resistance-less conductors, labeled terminals A and B. (Note: If either network contains a dependant source, its control variable must be in the same network.)

   If one of the networks is linear it can be replaced by this Norton equivalent network:

   The only thing left to do is find the values of R_t and I_s.

2. **To find I_s:**
   Define a current, i_{sc}, as the short circuit current which would be the current that would flow from terminal A to B (of the original network) if A and B were short circuited. This current is I_s.

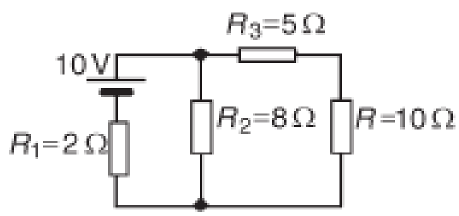
3. **To find R_t:**
   There are three different cases that will require different methods to find R_t:

a. If there are *only* independent sources in the network then "kill" them.
$R_t = R_{eq}$

b. If there are dependant sources *and* independent sources in the network, find both v_{oc} and i_{sc}.
$R_t = v_{oc} / i_{sc}$.

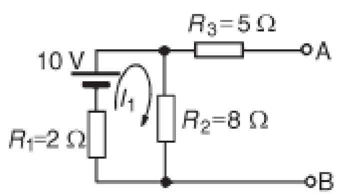
c. If there are *only* dependent sources apply a 1A current source at the terminals A and B. Calculate the resulting voltage, v, across this current source.
$R_t = v / 1A$
(Alternatively you can apply a 1V voltage source and measure resulting current, i, through it. $R_t = 1V / i$)

1. Solve the circuit shown below by thevenin's theorem.



1. Calculate current through 10Ω resistor by thevenin's theorem.
2. Find the Req after voltage source is removed.
3. Determine the voltage across 10Ω resistor.
4. Obtain the thevenin's equivalent circuit.

Solution: The 10 Ω resistance is removed from the circuit as shown in Figure

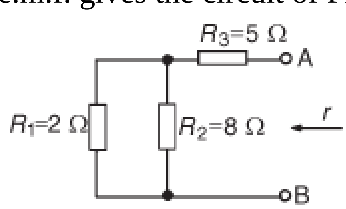


No current flowing in the 5 Ω resistor and current I_1 is

$$I_1 = \frac{10}{R_1 + R_2} = \frac{10}{2+8} = 1 \text{ A}$$

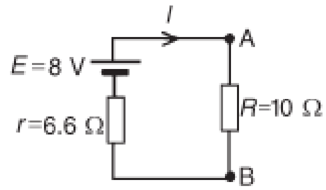
$$\text{P.d. across } R_2 = I_1 R_2 = 1 \times 8 = 8 \text{ V}$$

Removing the source of e.m.f. gives the circuit of Figure



$$\text{Resistance, } r = R_3 + \frac{R_1 R_2}{R_1 + R_2} = 5 + \frac{2 \times 8}{2 + 8}$$
$$= 5 + 1.6 = 6.6\ \Omega$$

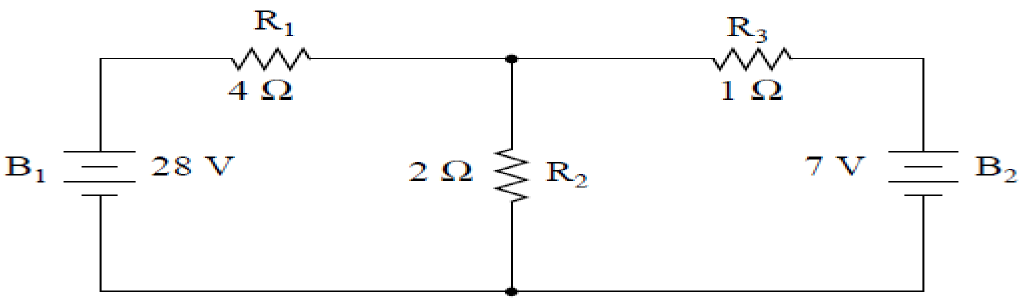
The equivalent Thevenin's circuit is shown in Figure



$$\text{Current } I = \frac{E}{R + r} = \frac{8}{10 + 6.6} = \frac{8}{16.6} = 0.482\ \text{A}$$

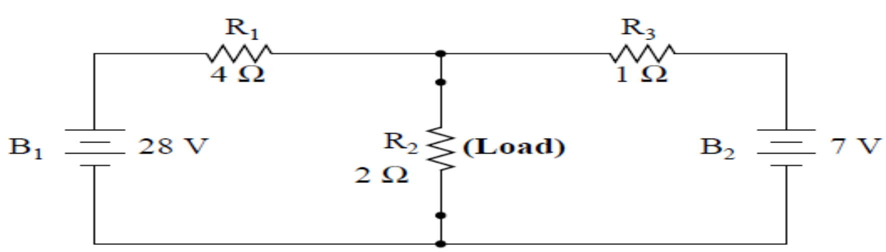
Hence the current flowing in the 10 Ω resistor of Figure is 0.482 **A.**

2. Determine the voltage across 2Ω resistor by thevenin's theorem.

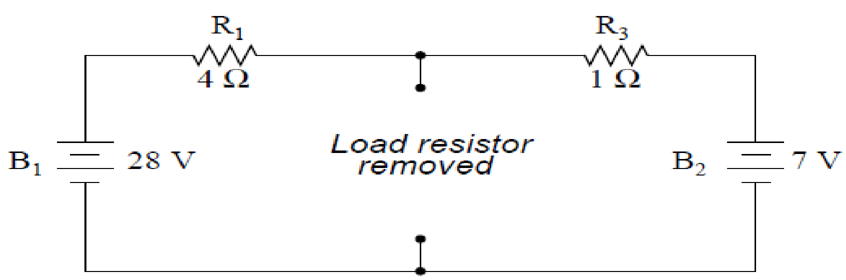


Solution:

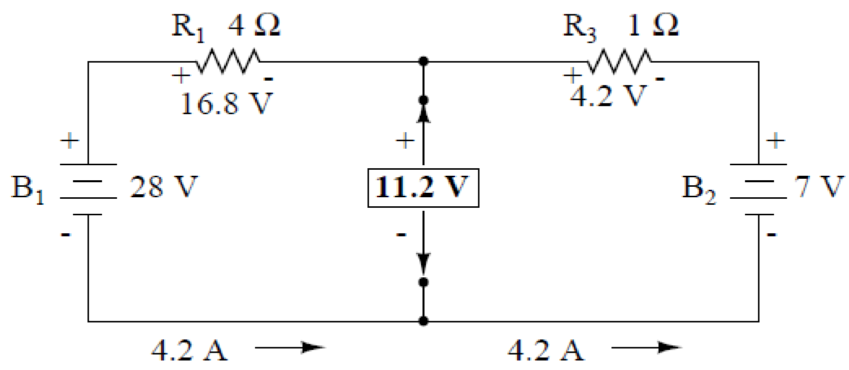
Step:1- Decide to designate R2 as the "load" resistor in this circuit.



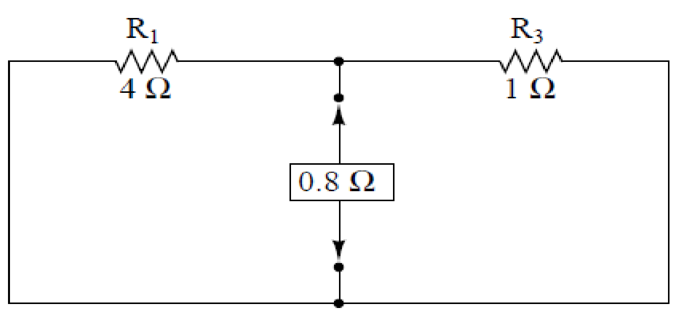
Step:2-Remove the load resistor



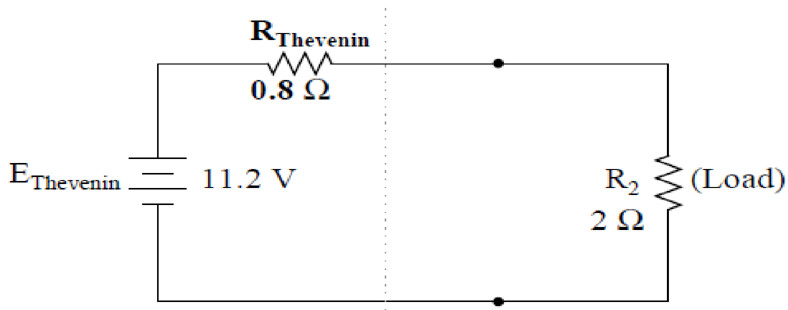
Step:3-Find the voltage across load resistor by applying the rules of series circuits, Ohm's Law, and Kirchhoff 's Voltage Law:(consider as V_{th})



Step:4-Find the equivalent resistance across load resistor:(consider as R_{th})

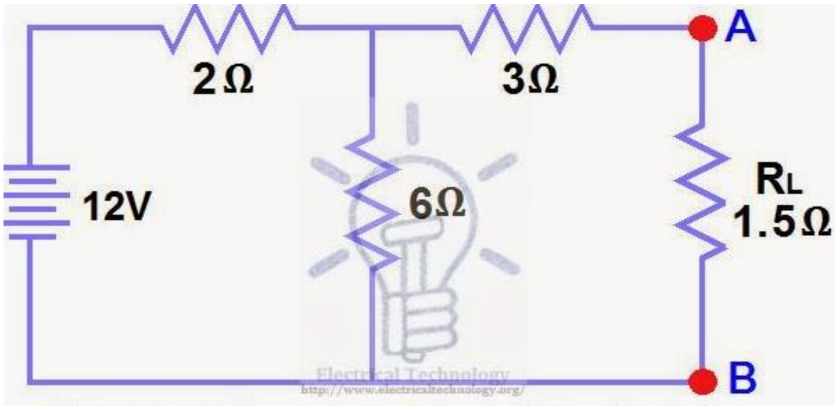


Step:5-Finally draw the Thevenin Equivalent circuit



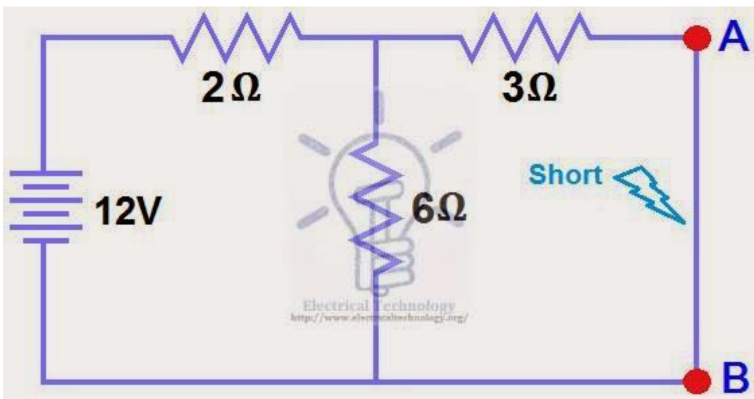
Voltage across 2 Ω resistor $V_L = \frac{11.2X\ 2}{0.8+2} = 8V$

3. Find R_N, I_N, the current flowing through and Load Voltage across the load resistor in fig (1) by using Norton's Theorem.



Step 1.

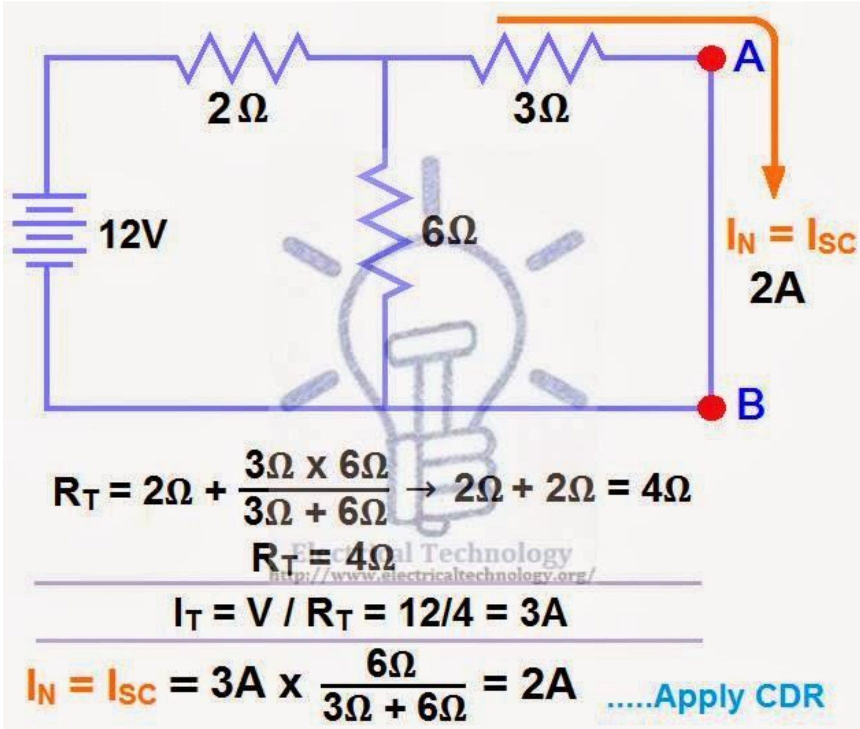
Short the 1.5Ω load resistor



Step 2.

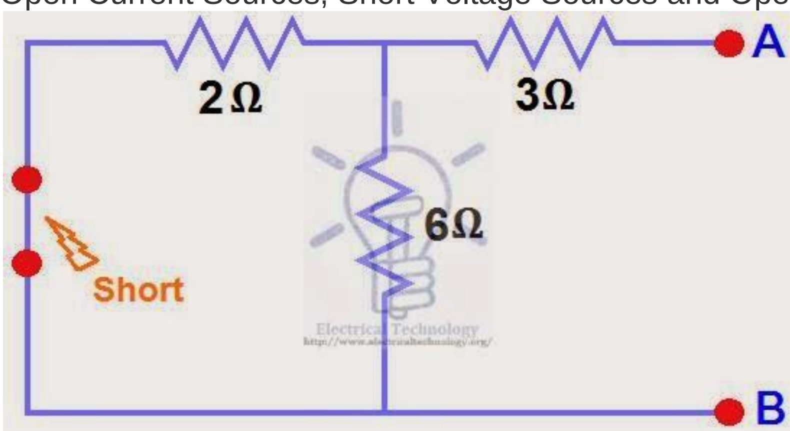
Calculate / measure the Short Circuit Current. This is the Norton Current (I_N).
We have shorted the AB terminals to determine the Norton current, $I_{N.}$ The 6Ω and 3Ω are then in parallel and this parallel combination of 6Ω and 3Ω are then in series with 2Ω.
So the Total Resistance of the circuit to the Source is:-
2Ω + (6Ω || 3Ω) ….. (|| = in parallel with).

$R_T = 2\Omega + [(3\Omega \times 6\Omega) / (3\Omega + 6\Omega)] \rightarrow I_T = 2\Omega + 2\Omega = 4\Omega.$
$R_T = 4\Omega$
$I_T = V / R_T$
$I_T = 12V / 4\Omega$
$I_T = 3A..$
Now we have to find $I_{SC} = I_N$… Apply CDR… (Current Divider Rule)…
$I_{SC} = I_N = 3A \times [(6\Omega / (3\Omega + 6\Omega)] = 2A.$
$I_{SC} = I_N = 2A.$



Step 3.

Open Current Sources, Short Voltage Sources and Open Load Resistor.



Step 4.

Calculate /measure the Open Circuit Resistance. This is the Norton Resistance (R_N)

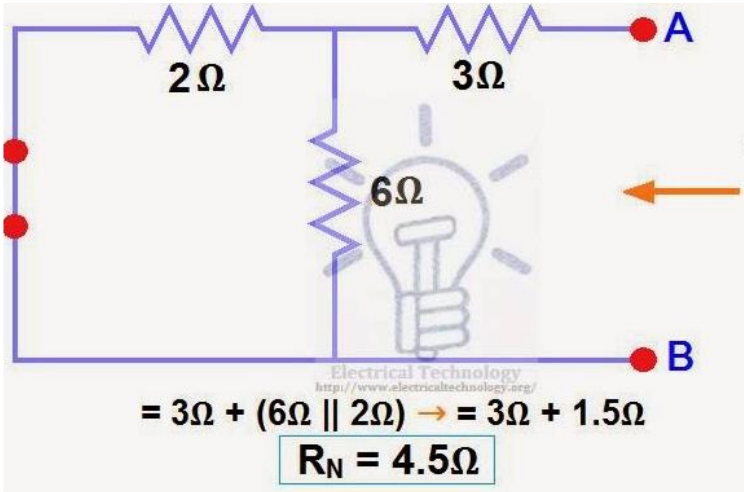
We have Reduced the 12V DC source to zero is equivalent to replace it with a short in step (3), as shown in figure (4) We can see that 3Ω resistor is in series with a parallel combination of 6Ω resistor and 2Ω resistor. i.e.:

3Ω + (6Ω || 2Ω) ….. (|| = in parallel with)

R_N = 3Ω + [(6Ω x 2Ω) / (6Ω + 2Ω)]

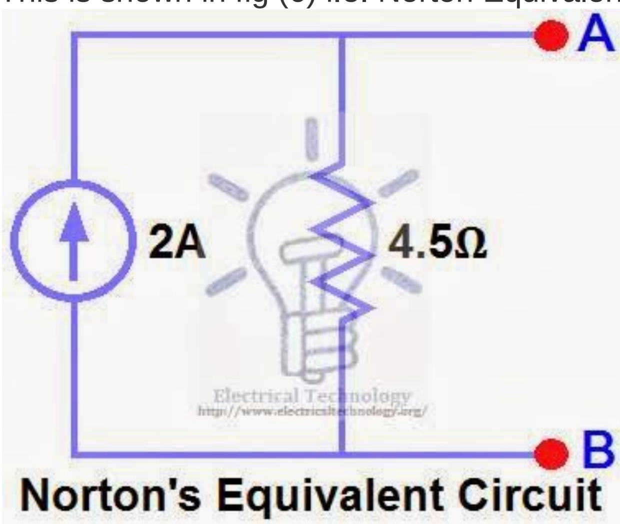
R_N = 3Ω + 1.5Ω

R_N = 4.5Ω



Step 5.

Connect the R_N in Parallel with Current Source $I_N$and re-connect the load resistor. This is shown in fig (6) i.e. Norton Equivalent circuit with load resistor.



Step 6.

Now apply the last step i.e. calculate the load current through and Load voltage across load resistor by Ohm's Law as shown in fig 7.

Load Current through Load Resistor…

$I_L = I_N \times [R_N / (R_N + R_L)]$

$= 2A \times (4.5\Omega / 4.5\Omega + 1.5k\Omega) \rightarrow = 1.5A$

I_L = 1. 5A

And

Load Voltage across Load Resistor…

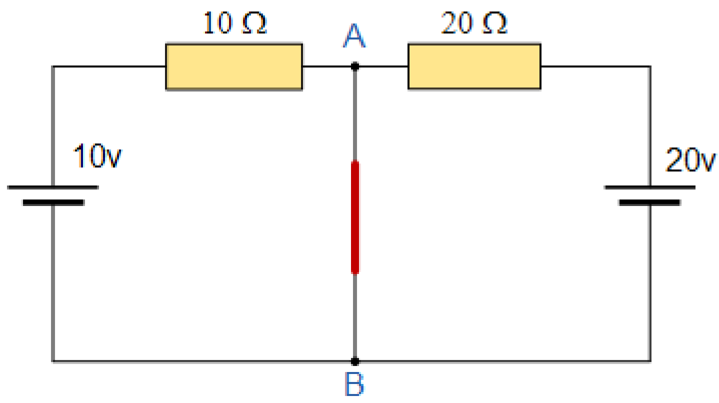
$V_L = I_L \times R_L$

$V_L = 1.5A \times 1.5\Omega$

V_L= 2.25V

---

4. **Find the Norton's Equivalent of the above circuit we firstly have to remove the centre 40Ω load resistor and short out the terminals A and B to give us the following circuit.**

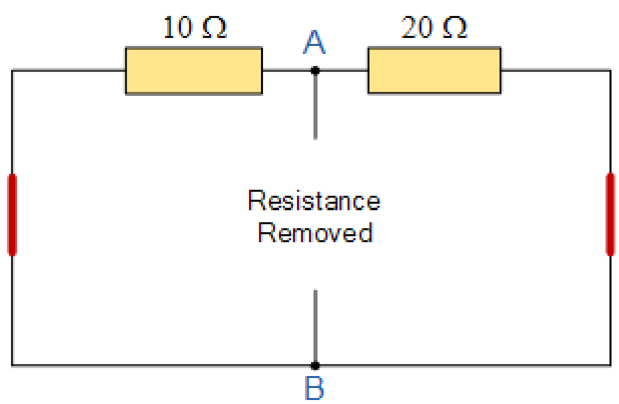


with A-B Shorted Out

$$I_1 = \frac{10v}{10\Omega} = 1amp, \quad I_2 = \frac{20v}{20\Omega} = 1amp$$

$$\text{therefore,} \quad I_{short\text{-}circuit} = I_1 + I_2 = 2amps$$

If we short-out the two voltage sources and open circuit terminals A and B, the two resistors are now effectively connected together in parallel. The value of the internal resistor Rs is found by calculating the total resistance at the terminals A and B giving us the following circuit.



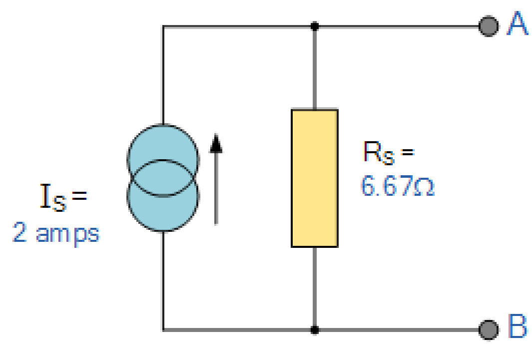
Find the Equivalent Resistance (Rs)

$$10\Omega \text{ Resistor in Parallel with the } 20\Omega \text{ Resistor}$$

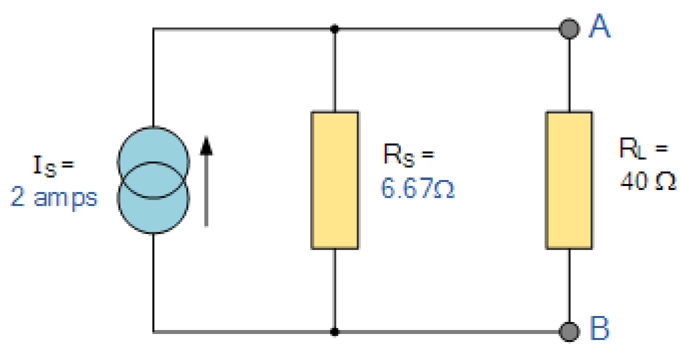
$$R_T = \frac{R_1 \times R_2}{R_1 + R_2} = \frac{20 \times 10}{20 + 10} = 6.67\Omega$$

Having found both the short circuit current, Is and equivalent internal resistance, Rs this then gives us the following Nortons equivalent circuit.

Nortons equivalent circuit.



Ok, so far so good, but we now have to solve with the original 40Ω load resistor connected across terminals A and B as shown below.



Again, the two resistors are connected in parallel across the terminals A and B which gives us a total resistance of:

$$R_T = \frac{R_1 \times R_2}{R_1 + R_2} = \frac{6.67 \times 40}{6.67 + 40} = 5.72\Omega$$

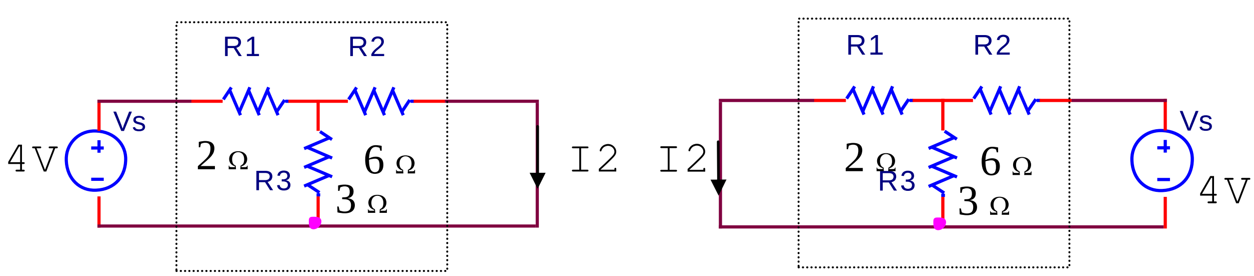
The voltage across the terminals A and B with the load resistor connected is given as:

$$V_{A\text{-}B} = I \times R = 2 \times 5.72 = 11.44v$$

Then the current flowing in the 40Ω load resistor can be found as:

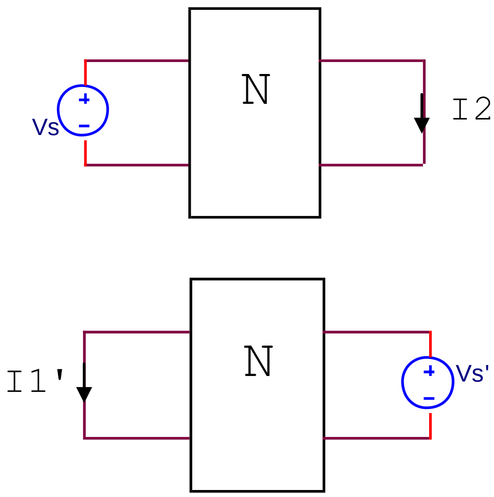
$$I = \frac{V}{R} = \frac{11.44}{40} = 0.286 \text{ amps}$$

RECIPROCITY THEOREM



$I_2 = \frac{1}{3}A \; I_2 = \frac{1}{3}A$

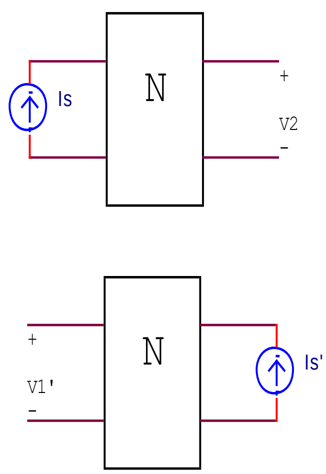
- **Case 1** The current in any branch of a network, due to a single voltage source E anywhere else in the network, will equal the current through the branch in which the source was originally located if the source is placed in the branch in which the current I was originally measured.



$$if \quad Vs = Vs' \quad then \quad I1' = I2$$

$$actually \quad exists: \quad \frac{I1'}{Vs'} = \frac{I2}{Vs}$$

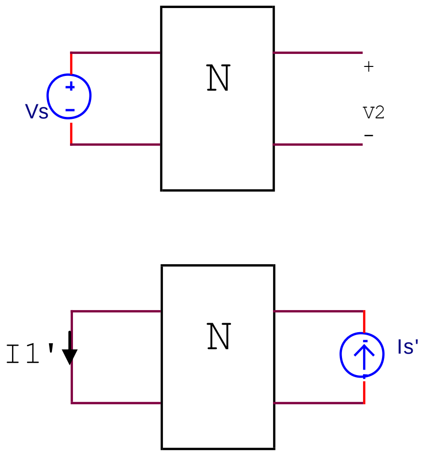
Case 2 :



$$if \quad Is = Is' \quad then \quad V1' = V2$$

$$actually \quad exists: \quad \frac{V1'}{Is'} = \frac{V2}{Is}$$

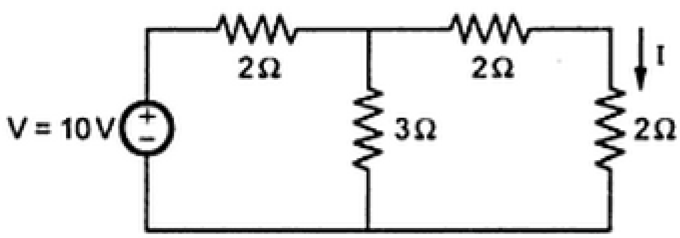
Case 3 :



$$if \quad Vs = Is' \quad then \quad I1' = V2$$

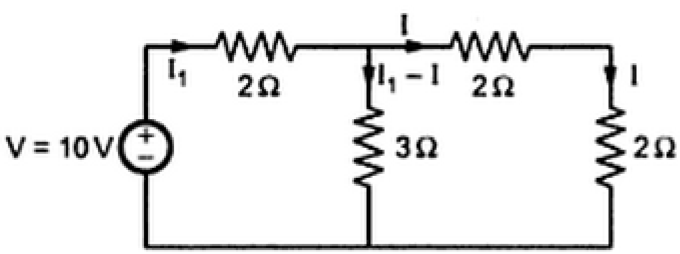
$$actually \quad exists: \quad \frac{I1'}{Is'} = \frac{V2}{Vs}$$

Verify reciprocity theorem for the voltage V and current I for the network shown in figure.



Solution

The various branch currents are shown as



Applying KVL to the two loops,

$$-2\,I_1 - 3\,(I_1 - I) + 10 = 0$$

$$\therefore \qquad 5\,I_1 - 3\,I = 10$$

$$-2\,I - 2I + 3\,(I_1 - I) = 0$$

$$\therefore \qquad 3\,I_1 - 7\,I = 0$$

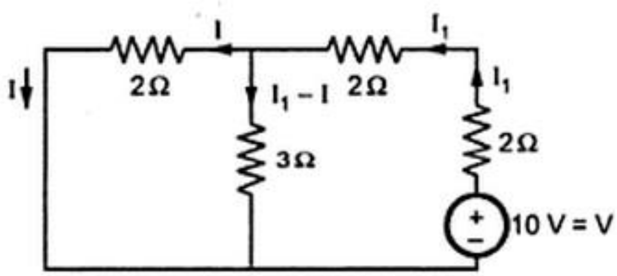
$$D = \begin{vmatrix} 5 & -3 \\ 3 & -7 \end{vmatrix} = -35 + 9 = -26$$

$$D_2 = \begin{vmatrix} 5 & 10 \\ 3 & 0 \end{vmatrix} = -30$$

$$I = \frac{D_2}{D} = \frac{-30}{-26} = 1.1538$$

$$\frac{V}{I} = \frac{10}{1.1538} = \mathbf{8.67\,\Omega}$$

Now interchange the positions of V and I



Applying KVL to the two loops,

$$-3\,(I_1 - I) + 2\,I = 0$$

$$\therefore \quad -3\,I_1 + 5\,I = 0$$

$$+2\,I_1 + 2\,I_1 - 10 + 3\,(I_1 - I) = 0$$

$$\therefore \quad 7\,I_1 - 3\,I = 10$$

$$D = \begin{vmatrix} -3 & 5 \\ 7 & -3 \end{vmatrix} = -26$$

$$D_2 = \begin{vmatrix} -3 & 0 \\ 7 & 10 \end{vmatrix} = -30$$

$$I = \frac{D_2}{D} = \frac{-30}{-26} = 1.1538$$

$$\frac{V}{I} = \frac{10}{1.1538}$$

$$= \mathbf{8.67\,\Omega}$$

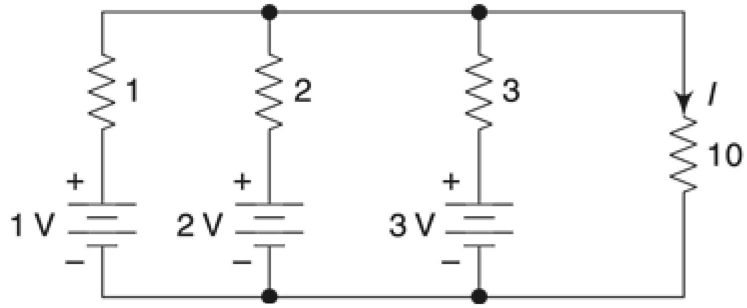
In both the cases the ratio of V/I is same and hence reciprocity theorem is verified.

MILLMAN'S THEOREM,

- Any number of parallel voltage sources can be reduced to one.
- This permits finding the current through or voltage across R_L without having to apply a method such as mesh analysis, nodal analysis, superposition and so on.
- Convert all voltage sources to current sources.
- Combine parallel current sources.

- Convert the resulting current source to a voltage source and the desired single-source network is obtained.

Find the load current using Millman's theorem. All values are in ohm.



Solution

Here, E_1 =1 V, E_2 = 2 V, E_3 = 3 V

Z_1 = 1 Ω, Z_2 = 2 Ω, Z_3 = 3 Ω

$\therefore Y_1 = 1\ ℧,\ Y_2 = 0.5\ ℧,\ Y_3 = \frac{1}{3}\ ℧$

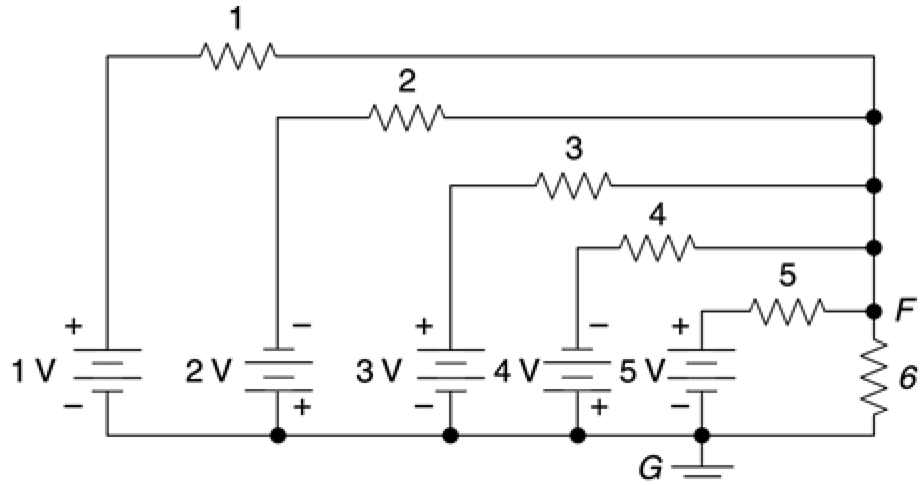
By Millman's theorem, the equivalent **circuit** is shown.

$$\therefore E = \frac{\sum_{i=1}^{3} E_i Y_i}{\sum_{i=1}^{3} Y_i} = \frac{1 \times 1 + 2 \times 0.5 + 3 \times \frac{1}{3}}{1 + 0.5 + \frac{1}{3}} = \frac{3}{\frac{11}{6}} = \frac{18}{11}\ \text{V}$$

and $$Z = \frac{1}{\sum_{i=1}^{3} Y_i} = \frac{6}{11}\ \Omega$$

$$\therefore I = \frac{E}{Z + 10} = \frac{\frac{18}{11}}{\frac{6}{11} + 10} = \frac{18}{116} = \frac{9}{58}\ \text{A}$$

Obtain the potential of node *F* with respect to node *G* in the circuit of the figure. All values are in ohm.



Solution

By Millman's theorem, equivalent voltage is,

$$V = \frac{\sum_{i=1}^{5} E_i Y_i}{\sum_{i=1}^{5} Y_i} = \frac{1\times 1 - 2\times 1/2 + 3\times 1/3 - 4\times 1/4 + 5\times 1/5}{1 + 1/2 + 1/3 + 1/4 + 1/5} = \frac{60}{137}\ \text{V}$$

Equivalent impedance,

$$Z = \frac{1}{\sum_{i=1}^{5} Y_i} = \frac{1}{1 + 1/2 + 1/3 + 1/4 + 1/5} = \frac{60}{137}\ \Omega$$

Therefore, the current through the 6Ω resistance is,

$$I = \frac{V}{Z+6} = \frac{60/137}{60/137+6} = \frac{60}{882}\text{A}$$

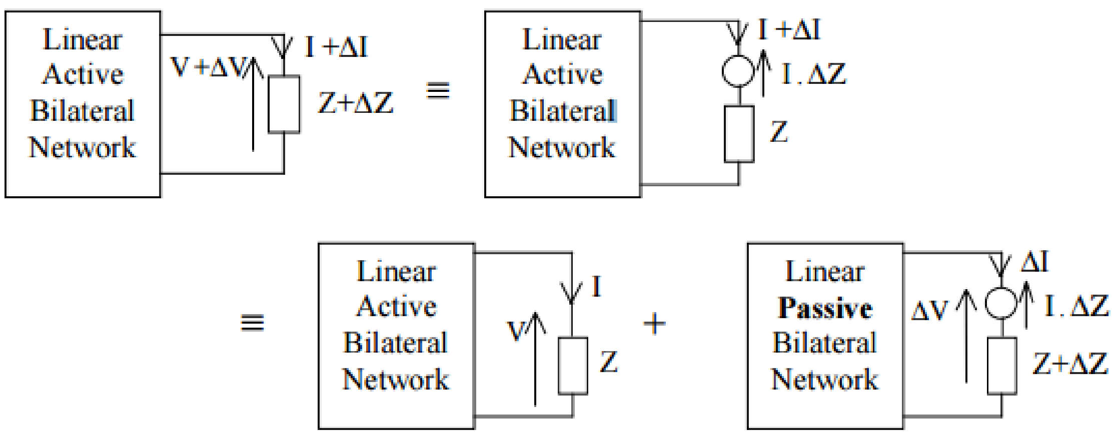
Hence, the voltage between the points *F* and *G* is,

$$V_{FG} = 6\times I = 6\times \frac{60}{882} = \frac{60}{147}\quad \text{Volt}$$

COMPENSATION THEOREM,

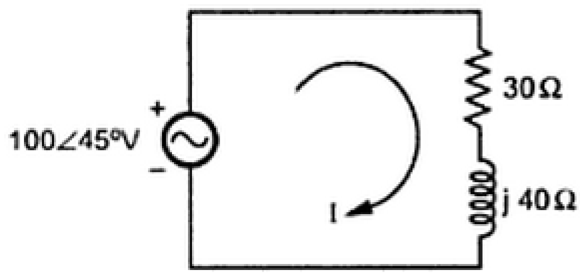
In any linear network consisting of linear and bilateral impedances and active sources, if the impedance Z of the branch carrying current I increases by dI, then the increment of voltage or current in each branch of the network is that voltage or current that would be produced by an opposing voltage source of value V_C (= I.dZ) introduced in the altered branch after replacing original sources by their internal impedances.

In many circuits, after the circuit is analysed, it is realised that only a small change need to be made to a component to get a desired result. In such a case we would normally have to recalculate. The compensation theorem allows us to compensate properly for such changes without sacrificing accuracy. In any linear bilateral active network, if any branch carrying a current I has its impedance Z changed by an amount ΔZ, the resulting changes that occur in the other branches are the same as those which would have been caused by the injection of a voltage source of (-) I . ΔZ in the modified branch



Consider the voltage drop across the modified branch. V +ΔV = (Z+ΔZ)(I+ΔI) = Z . I + ΔZ . I + (Z + ΔZ) . ΔI from the original network, V = Z . I ∴ ΔV = ΔZ . I + (Z + ΔZ) . ΔI Since the value I is already known from the earlier analysis, and the change required in the impedance, ΔZ , is also known, I .ΔZ is a known fixed value of voltage and may thus be represented by a source of emf I. ΔZ .

Calculate the change in current in the network shown in figure using compensation theorem when the reactance has changed to j35Ω.



Solution:

Applying KVL, we get,

$$I = \frac{100\angle 45^\circ}{30 + j40} = \frac{100\angle 45^\circ}{50\angle 53.13^\circ} = 2\angle -8.13^\circ \text{ A}$$

$$I = (1.9798 - j\,0.2828) \text{ A}$$

Now the reactance has changed to j 35. Hence the current in network will also change to I′. The change in the reactance is given by,

$$\delta Z = j40 - j35 = j5\,\Omega$$

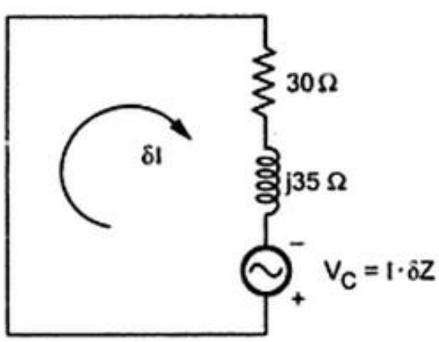


Fig. 3.59 (a)

Now the reactance is decreased. Modifying the network by replacing voltage source by short circuit and introducing compensation source $V_C = I \cdot \delta Z$ in the branch altered as shown in the Fig. 3.59 (a)

The compensation source is given by,

$$V_C = I \cdot \delta Z$$

$$= (2\angle -8.13^\circ)(j5) = (2\angle -8.13^\circ)(5\angle 90^\circ)$$

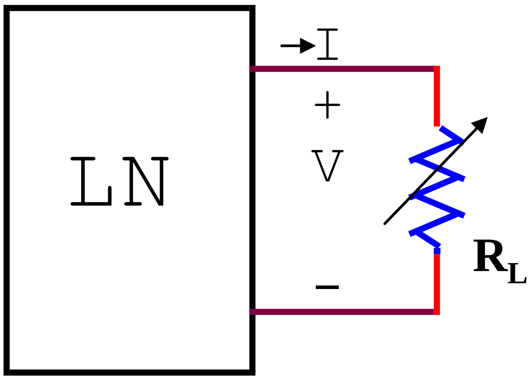
$$\therefore \quad V_C = 10\angle 81.87^\circ \text{ V}$$

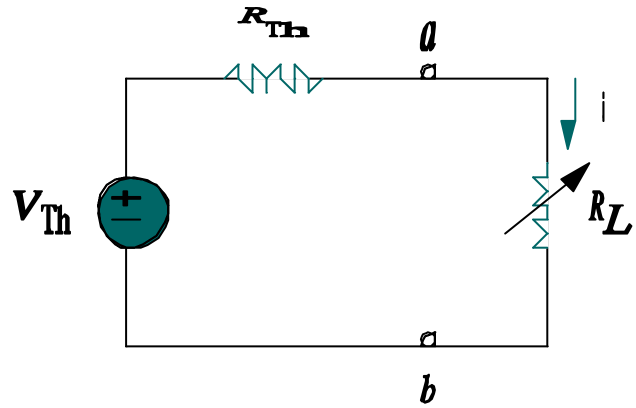
Thus, change in current is given by,

$$\delta I = \frac{V_C}{30 + j35} = \frac{10\angle 81.87^\circ}{46.0977\angle 49.4^\circ} = \mathbf{0.2169\angle 32.47^\circ \text{ A}}$$

MAXIMUM POWER TRANSFER THEOREM

Replacing the original network by its Thevenin equivalent, then the power delivered to the load is





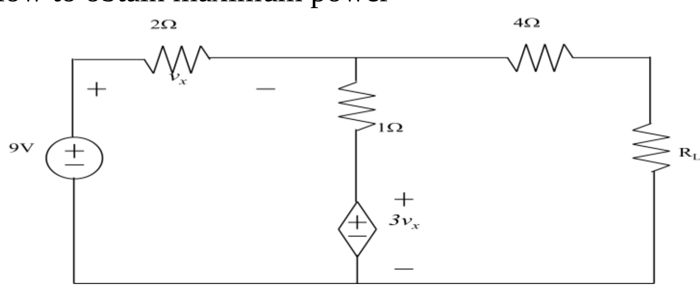
Power delivered to the load as a function of R_L

$$\frac{dp}{dR_L} = T_{Th}^2 \left[\frac{(R_{Th} - R_L)}{(RTh + RL)^3} \right] = 0$$

$$so \quad yields \quad R_L = R_{Th} \quad and \quad p = \frac{V_{Th}^2}{4R_{Th}}$$

The variable resistor in the circuit in Fig. shown below is adjusted for maximum power transfer to R_0.

Solve the circuit given below to obtain maximum power



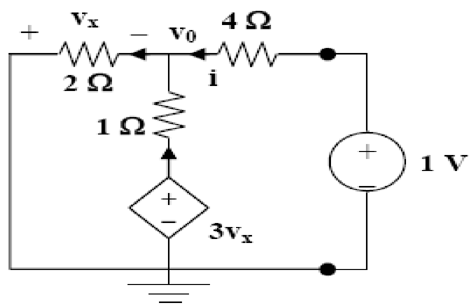
1.Find the value of R_L for maximum power transfer in the circuit.

2. Calculate the $R_{th.}$

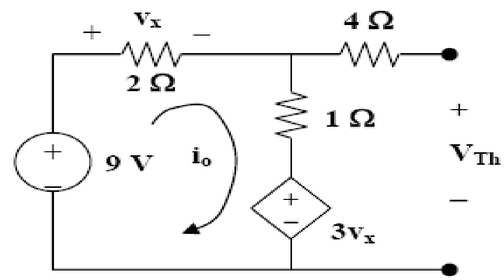
3.Calculate the $V_{th.}$

4. Find the maximum power.

Solution:



(a)



(b)

Applying KCL at the top node gives

$$\frac{1-v_o}{4}+\frac{3v_x-v_o}{1}=\frac{v_o}{2}$$

But $v_x = -v_o$. Hence

$$\frac{1-v_o}{4}-4v_o=\frac{v_o}{2} \longrightarrow v_o = 1/(19)$$

$$i=\frac{1-v_o}{4}=\frac{1-\frac{1}{19}}{4}=\frac{9}{38}$$

$R_{Th} = 1/i \ = 38/(9) = 4.222\Omega$

To find V_{Th}, consider the circuit in Fig. (b),

$$-9 + 2i_o + i_o + 3v_x = 0$$

But $v_x = 2i_o$. Hence,

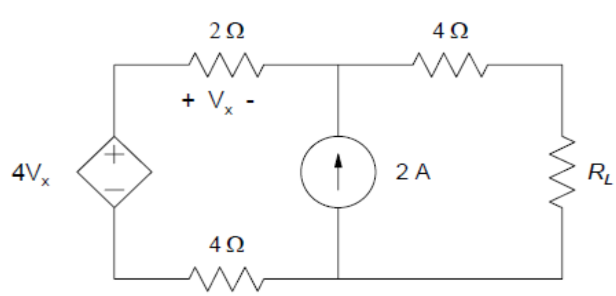
$$9 = 3i_o + 6i_o = 9i_o \longrightarrow i_o = 1A$$

$$V_{Th} = 9 - 2i_o = 7V$$

$R_L = R_{Th} =$ $\underline{\mathbf{4.222\Omega}}$

$$P_{max} = \frac{v_{Th}^2}{4R_L}=\frac{49}{4(4.222)}=\underline{\mathbf{2.901W}}$$

Solve the circuit given below to obtain maximum power



1. Find R so that maximum power is transferred to the resistance R.

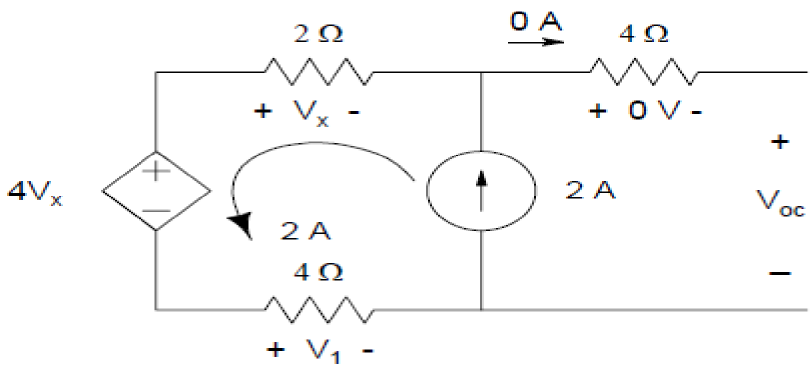
2. Calculate the $R_{th.}$

3.Calculate the $V_{th.}$

4. Find the maximum power.

Solution:

In order to find V_{TH}, open circuit voltage V_{oc} is found



KVL around the outer loop:

$$-4V_x + V_x + V_{OC} - V_1 = 0$$

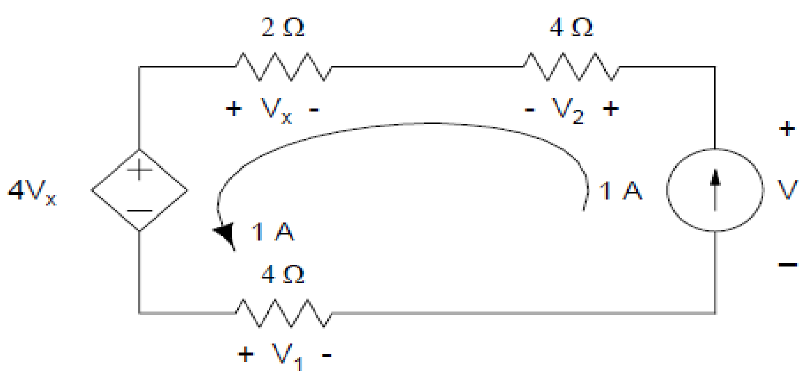
$$V_{OC} = 3V_x + V_1$$

where

$$V_1 = 2 \times 4 = 8V$$

$$V_x = -2 \times 2 = -4V$$

$$\therefore V_{OC} = -12_x + 8 = -4V$$



KVL around the loop:

$$-4V_x + V_x - V_2 + V - V_1 = 0$$

$$V = 3V_x + V_2 + V_1$$

where

$$V_x = -1 \times 2 = -2V$$

$$V_1 = 4 \times 1 = 4V$$

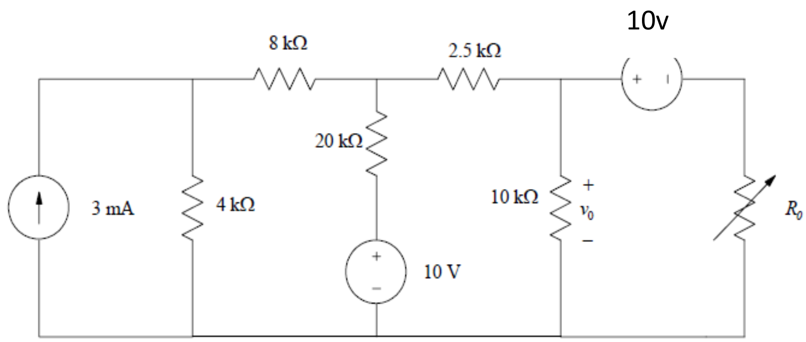
$$V_2 = 4 \times 1 = 4V$$

$$\therefore V = -6 + 4 + 4 = 2V$$

$$R_{TH} = \frac{V}{1} = 2\Omega$$

When $R_L = 2\Omega$ it absorbs maximum power.

$$P_{max} = \frac{(-4)^2}{4 \times 2} = \frac{16}{8} = 2W$$



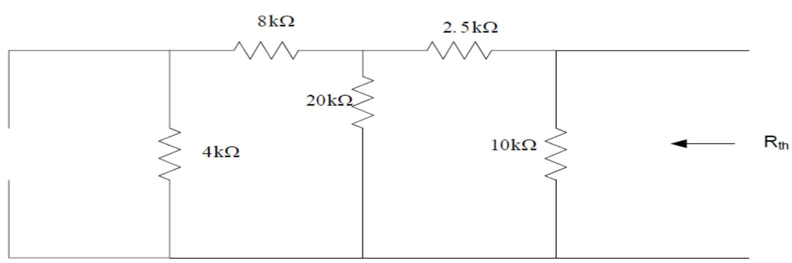
1.Find the value of R_o for maximum power transfer in the circuit.

2. Calculate the $R_{th.}$

3.Calculate the $V_{th.}$

4. Find the maximum power transfer to Ro.

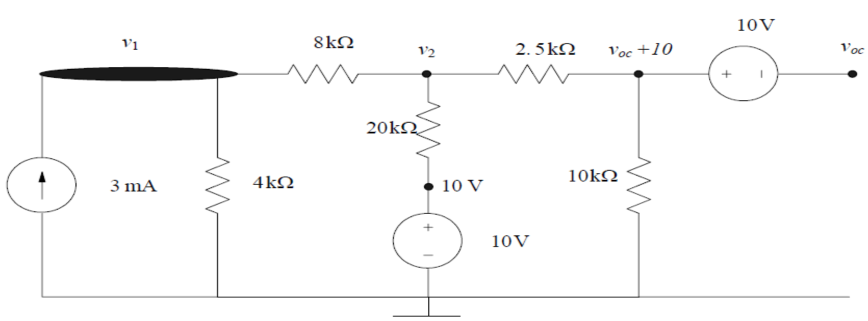
Solution:



$$R_{th} = ((8k + 4k)//20k + 2.5k)//10k$$

$$R_{th} = (7.5k + 2.5k)//10k$$

$$\boxed{R_{th} = 5k\Omega}$$



KCL at v_1 :

$$\left(\frac{1}{4k}+\frac{1}{8k}\right)v_1-\frac{1}{8k}v_2=3m$$

multiply both sides by 8k yields:

$$3v_1-v_2=24$$

$$v_1=8+\frac{1}{3}v_2............(1)$$

KCL at v_2:

$$-\frac{1}{8k}v_1+\left(\frac{1}{8k}+\frac{1}{20k}+\frac{1}{2.5k}\right)v_2-\frac{1}{20k}10-\frac{1}{2.5k}(v_{oc}+10)=0$$

multiply both sides by 40k yields:

$$-5v_1+23v_2-16v_{oc}=180............(2)$$

KCL at v_3:

$$-\frac{1}{2.5k}v_2+\left(\frac{1}{2.5k}+\frac{1}{10k}\right)(v_{oc}+10)=0$$

multiply both sides by 10:

$$-4v_2+5v_{oc}=-50$$

$$v_{oc}=-10+\frac{4}{5}v_2..............(3)$$

Subst. Eqs. (1) and (3) into (2) gives:

$$-5\left(8+\frac{1}{3}v_2\right)+23v_2-16\left(-10+\frac{4}{5}v_2\right)=180$$

$$\left(-\frac{5}{3}+23-\frac{64}{5}\right)v_2=180+40-160=60$$

$$(-25+345-192)v_2=900$$

$$128v_2=900$$

$$v_2=7.03125V$$

$$v_{oc}=-10+0.8(7.03125)=-4.375V$$

When $R_0=R_{th}=5k\Omega$ it absorbs maximum power.

$$P_{max}=\frac{V_{oc}^{\ 2}}{4R_{th}}=\frac{(-4.375)^2}{4(5k)}=0.957mW$$

Tellegen‘s theorem

- If there are b branches in a lumped circuit, and the voltage u_k, current i_k of each branch apply passive sign convention, then we have

$$\sum_{k=1}^{b}u_k i_k=0$$

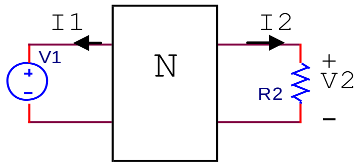
- If two lumped circuits and have the same topological graph with b branches, and the voltage, current of each branch apply passive sign convention, then we have not only

$$\sum_{k=1}^{b}u_k i_k=0 \quad \sum_{k=1}^{b}\hat{u}_k \hat{i}_k=0$$

$$but\ also \quad \sum_{k=1}^{b}\hat{u}_k i_k=0 \quad \sum_{k=1}^{b}u_k \hat{i}_k=0$$

Example:

N *is a network including resistors only. When* $R_2 = 2\Omega, V_1 = 6V$, *We can get* $I_1 = -2A$, $V_2 = 2V$; *When* $R_2' = 4\Omega, V_1' = 10V$, *We can get* $I_1' = -3A$, *find out* V_2' *then.*



According to the Tellegen Theorem

$$V_1 I_1' + V_2 I_2' + \sum_{k=3}^{b} V_k I_k' = 0\,;\, V_1' I_1 + V_2' I_2 + \sum_{k=3}^{b} V_k' I_k = 0$$

and $V_k I_k' = R I_k I_k' = R I_k' I_k = V_k' I_k$

$$\therefore \quad \sum_{k=3}^{b} V_k I_k' = \sum_{k=3}^{b} V_k' I_k$$

$$\therefore \quad V_1 I_1' + V_2 I_2' = V_1' I_1 + V_2' I_2$$

$$6 \times (-3) + 2 \times \frac{V_2'}{4} = 10 \times (-2) + V_2' \times \frac{2}{2}$$

$$\therefore \quad V_2' = 4V$$

UNIT- II

AC Circuit Analysis: Series circuits - RC, RL and RLC circuits and Parallel circuits –RLC circuits - Sinusoidal steady state response - Mesh and Nodal analysis - Analysis of circuits using Superposition, Thevenin‘s, Norton‘s and Maximum power transfer theorems.

Alternating Current Circuits

Review of rms values. rms values are root-mean-square values of quantities (such as voltage and current) that vary periodically with time. In AC circuits voltage and current vary *sinusoidally* with time:

$$v = V\sin(\omega t), \qquad i = I\sin(\omega t - \phi)$$

where V and I are the voltage and current amplitudes, respectively. ω is the angular frequency ($\omega = 2\pi f$, where f is the frequency) and ϕ is a phase constant that we will discuss later. The rms values of voltage and current are defined to be

$$V_{\text{rms}} = \sqrt{\overline{V^2\sin^2(\omega t)}}, \qquad I_{\text{rms}} = \sqrt{\overline{I^2\sin^2(\omega t - \phi)}}$$

Where the overbar indicates the average value of the function over one cycle. Since the average value of $\sin^2\theta$ over one cycle is ½, we get

$$V_{\text{rms}} = \frac{V}{\sqrt{2}} \quad \text{and} \quad I_{\text{rms}} = \frac{I}{\sqrt{2}}$$

Note that these formulas are valid *only* if the voltage varies <u>sinusoidally</u> with time.

What we will study in this chapter is what happens to the current and power in an AC series circuit if a resistor, a capacitor and an inductor are present in the circuit.

Resistors and Resistance

If *just a resistor* of resistance R is connected across an AC generator the generator is said to have a *purely resistive load*. The phase constant ϕ is zero and we write

$$v = V\sin(\omega t), \qquad i = I\sin(\omega t) \quad \text{and} \quad V_{\text{rms}} = I_{\text{rms}}R.$$

Since the angle for *v* and *i* is the same, the instantaneous voltage and current are said to be *in phase*. Note that

$$\frac{V_{\text{rms}}}{I_{\text{rms}}} = R$$

is a constant independent of the frequency f of the AC generator. We assume that the resistor maintains its resistance regardless of how fast or slow the generator's armature is turning. *R*, of course, is measured in ohms.

For a purely resistive load the average power delivered to the circuit by the generator is given by

$$\overline{P} = I_{\text{rms}}V_{\text{rms}} \;\; \text{or} \;\; \overline{P} = I_{\text{rms}}^2 R$$

which are analogous to the familiar formulas for DC circuits. *P*, as usual, is measured in watts.

Capacitors and Capacitive Reactance

Now let us connect *just a capacitor* of capacitance *C* across an AC generator. In this case the generator is said to have a *purely capacitive load*. The phase constant ϕ is $-\frac{\pi}{2}$ and we write

$$v = V\sin(\omega t), \qquad i = I\sin\left(\omega t + \frac{\pi}{2}\right) \;\; \text{and} \quad V_{\text{rms}} = I_{\text{rms}}X_{\text{C}}$$

where X_C is called the *capacitive reactance*. Capacitive reactance, like resistance, is measured in ohms.

Since the angle for the instantaneous current is *greater* than the angle for the instantaneous voltage by π/2 radians or 90°, the current is said to *lead the voltage by 90°* or *lead the voltage by a quarter cycle.* (Remember that a full cycle is 360° - a "complete trip" around a circle.) We can also say that *the voltage* ***lags*** *the current by 90° or lags the current by a quarter cycle.*

For a capacitive load the ratio $\frac{V_{\text{rms}}}{I_{\text{rms}}}$ is *not a constant independent of the frequency of the generator*. It can be shown that in fact

$$\frac{V_{\text{rms}}}{I_{\text{rms}}} = \frac{1}{2\pi fC} \quad \text{so that} \quad X_{\text{C}} = \frac{1}{2\pi fC}.$$

Units check: $\frac{1}{\text{Hz}\cdot\text{F}} = \frac{1}{\frac{1}{s}\cdot\frac{\text{C}}{\text{V}}} = \frac{1}{\frac{\text{C}}{s}\cdot\frac{1}{\text{V}}} = \frac{\text{V}}{\text{A}} = ohms$. See Figure 23.2 on page 714 of your text.

For a purely capacitive load the average power delivered to the circuit by the generator is *zero*. The reason for this is that the instantaneous voltage and current in the circuit are exactly 90° out of phase. Over one cycle the generator delivers as much power to the capacitor as it gets back from the capacitor. (Remember that over a generator cycle the capacitor will *charge* then *discharge*.)

Example

Two stripped wires from the end of a lamp cord are soldered to the terminals of a 200 μF capacitor. The lamp cord, which has a standard electric plug on the other end, is then plugged into a 120 V, 60 Hz AC outlet.

(*Do not* try this at home.)

a. Find the reactance of the capacitor.

$$X_{\text{C}} = \frac{1}{2\pi fC}; \quad X_{\text{C}} = \frac{1}{2\pi(60)(2\times10^{-4})}; \quad \boxed{X_{\text{C}} = 13.3 \text{ ohms}}$$

b. Find the rms current drawn from the wall outlet.

$$V_{\text{rms}} = I_{\text{rms}} X_{\text{C}}; \quad I_{\text{rms}} = \frac{V_{\text{rms}}}{X_{\text{C}}}; \quad I_{\text{rms}} = \frac{120 \text{ V}}{13.3\ \Omega}; \quad \boxed{I_{\text{rms}} = 9.02 \text{ A}}$$

Inductors and Inductive Reactance

Now let us connect *just an inductor* of inductance *L* across an AC generator. In this case the generator is said to have a *purely inductive load*. The phase constant ϕ is $+\frac{\pi}{2}$ and we write

$$v = V\sin(\omega t), \qquad i = I\sin\left(\omega t - \frac{\pi}{2}\right) \text{ and } \quad V_{\text{rms}} = I_{\text{rms}} X_{\text{L}}$$

where X_L is called the *inductive reactance*. Inductive reactance, like resistance, is measured in ohms.

Since the angle for the instantaneous current is *smaller* than the angle for the instantaneous voltage by $\pi/2$ radians or 90°, the current is said to *lag the voltage by 90°* or *lag the voltage by a quarter cycle.* (Alternatively one can say that *thevoltage leads the current by 90°* or *the voltage leads the current by a quarter cycle*.)

For an inductive load the ratio $\frac{V_{\text{rms}}}{I_{\text{rms}}}$ is *not a constant independent of the frequency of the generator.* It can be shown that in fact

$$\frac{V_{\text{rms}}}{I_{\text{rms}}} = 2\pi f L \quad \text{so that} \quad X_{\text{L}} = 2\pi f L.$$

Units check: $\text{Hz} \cdot \text{H} = \frac{1}{\text{s}} \cdot \frac{\text{V}}{\frac{\text{A}}{\text{s}}} = \frac{\text{V}}{\text{A}} = ohms$. See Figure 23.6 on page 716 of your text.

For a purely inductive load the average power delivered to the circuit by the generator is *zero*. The reason for this is that the instantaneous voltage and current in the circuit are exactly 90° out of phase. Over one cycle the generator delivers as much power to the inductor as it gets back from the inductor. (Remember that over a generator cycle the induced emf in the inductor will reverse direction.)

Example

Two stripped wires from the end of a lamp cord are soldered to the terminals of a 200 mH inductor. The lamp cord, which has a standard electric plug on the other end, is then plugged into a 120 V, 60 Hz AC outlet.

(*Do not* try this at home.)

a. Find the reactance of the inductor.

$$X_L = 2\pi f L;\quad X_L = 2\pi(60)(0.200);\quad \boxed{X_L = 75.4 \text{ ohms}}$$

b. Find the rms current drawn from the wall outlet.

$$V_{rms} = I_{rms} X_L;\quad I_{rms} = \frac{V_{rms}}{X_L};\quad I_{rms} = \frac{120\text{ V}}{75.4\ \Omega};\quad \boxed{I_{rms} = 1.59\text{ A}}$$

RCL Series Circuits

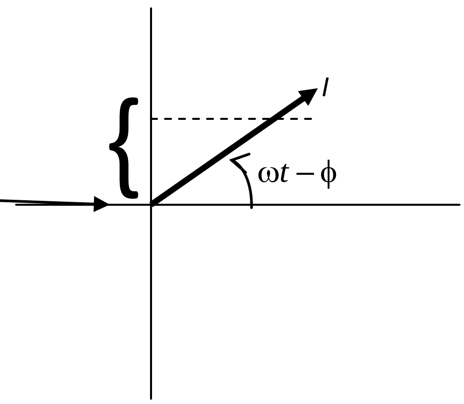
An RCL series circuit consists of a resistor, a capacitor, an inductor and an AC generator connected in series. See the figure.

The mathematical analysis of this circuit requires the solution of a differential equation. However, there is a way to solve the circuit using a geometrical device that is analogous to a vector. This device is called a *phasor* (or *rotor*). A phasor is a vector whose tail sits at the origin of an xy-coordinate system. The phasor *rotates counterclockwise* about the origin with angular frequency ω (the angular frequency of the AC generator). The phasor represents either voltage or current, and its y-component is the instantaneous value of the quantity it represents.

$$V\sin(\omega t)$$

We will assume that at any instant the current through each circuit element is given by

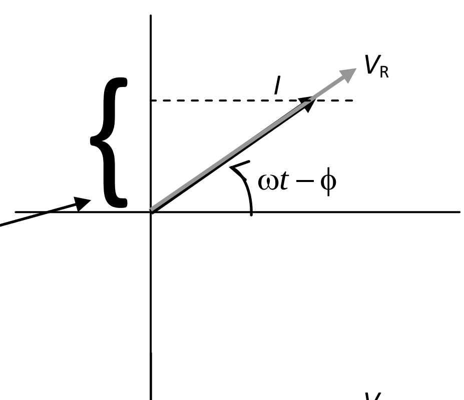
$$i = I\sin(\omega t - \phi).$$



The current phasor has length I and makes an angle of ωt - ϕ with respect to the x-axis. At any instant its y-component equals the current in the circuit.

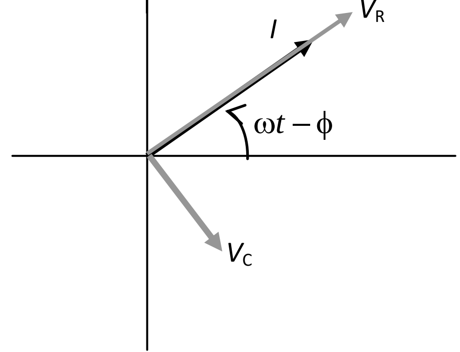
Now consider the voltage phasor of the resistor. The instantaneous voltage across the resistor is just

$$iR = IR\sin(\omega t - \phi) \text{ or } v_R = V_R\sin(\omega t - \phi)$$



The length of the resistor's voltage phasor is the voltage amplitude V_R. At any instant the angle it makes with the x-axis is ωt - ϕ. The y-component of this phasor is then

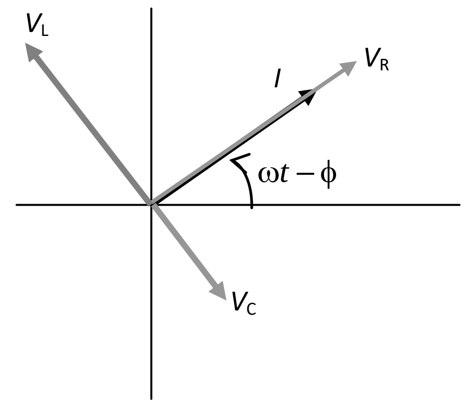
$$V_R\sin(\omega t - \phi),$$



which is the instantaneous voltage across the resistor. Note that the current and voltage across the resistor are *in phase*. Hence the voltage phasor for the resistor lies on top the current phasor.

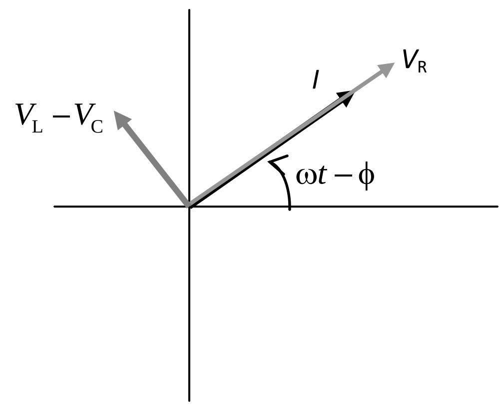
Now consider the voltage phasor for the *capacitor*. Here it is *critically important* to remember the phase relationship between the current and voltage for a capacitor. Does the current lead or lag the voltage in a capacitor? By how many degrees? The current *leads* the voltage by 90°. Since the phasors rotate counterclockwise, the voltage phasor for the capacitor must lie 90°*clockwise* from the current phasor.

Now consider the voltage phasor for the *inductor*. It is *critically important* to remember the phase relationship between the current and voltage for an inductor. Does the current lead or lag the voltage in an inductor? By how many degrees? The current *lags* the voltage by 90°. Since the phasors rotate counterclockwise, the voltage phasor for the inductor must lie 90°*counterclockwise* from the current phasor.



Note that the voltage phasors for the inductor and the capacitor lie along the same line. (We have arbitrarily assumed that V_L is larger than V_c.)Using the rules of vector addition we may combine them to obtain the next diagram.

By Kirchhoff's loop rule the voltage drops across the capacitor, resistor and inductor must, at any instant, equal the voltage rise across the generator. This will be satisfied if *the vector sum of the* $V_L - V_C$ *and the* V_R *phasors matches the voltage phasor of the generator.* See the last diagram below.



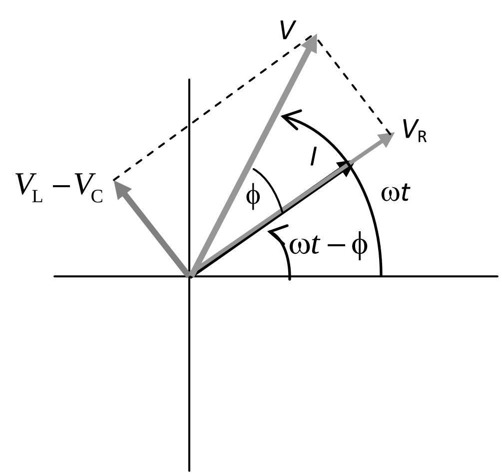
From the last diagram we obtain some very important relationships. In particular, note that

$$V^2 = (V_L - V_C)^2 + V_R^2 \quad \text{or} \quad V = \sqrt{(V_L - V_C)^2 + V_R^2}$$

since $V_L = IX_L$, $V_C = IX_C$ and $V_R = IR$ we can write

$$V = \sqrt{(IX_L - IX_C)^2 + (IR)^2} \quad \text{or} \quad V = I\sqrt{(X_L - X_C)^2 + R^2}$$

$$\text{or} \quad V = IZ \quad \text{where} \quad \boxed{Z = \sqrt{(X_L - X_C)^2 + R^2}}$$



Z is called the *impedance* of the circuit and is measured in ohms. Note that we have dropped the "rms" subscripts for the voltage and the current in the $V = IX$ formulas above because the formulas are also valid if we replace each rms value with its corresponding amplitude (the square root of 2 cancels from both sides of each equation).

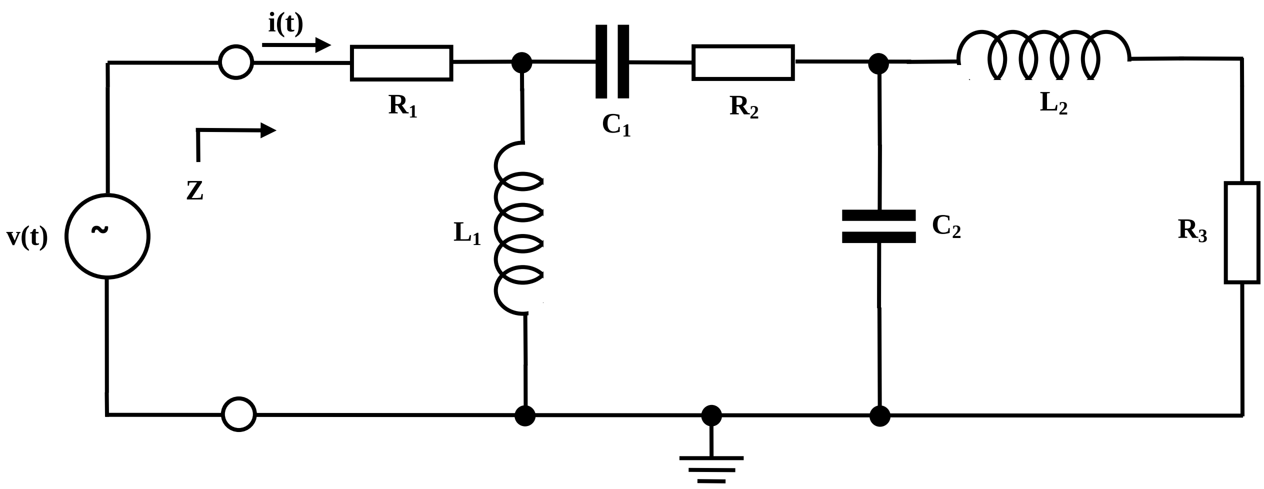
We can now find a formula for the phase ϕ of the current. From the right triangle with sides V, V_R and V_L - V_C in the diagram above we have

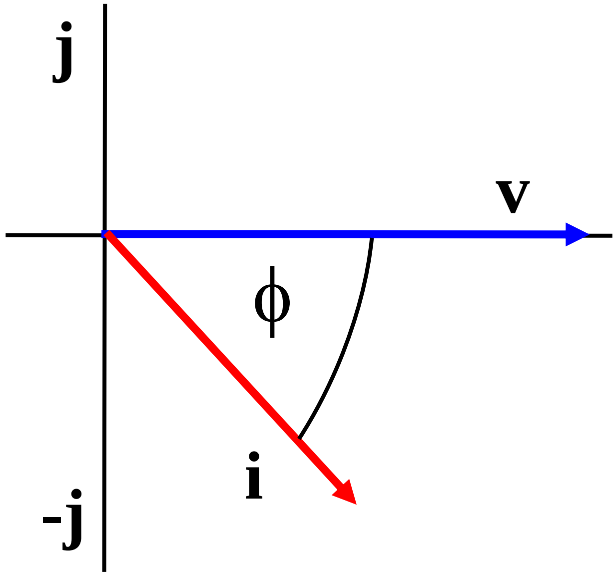
$$\tan\phi = \frac{V_L - V_C}{V_R} = \frac{IX_L - IX_C}{IR} \quad \text{so that} \quad \boxed{\tan\phi = \frac{X_L - X_C}{R}}$$

Power in AC Circuits II

15.1 Power Dissipation in an AC circuit

In general, an ac circuit will contain a combination of resistive and reactive components and the reactive elements may be either inductive or capacitive as shown in Fig.1 below. This means that at different points in the circuit the current and voltage relationships will vary depending on the elements involved. From the point of view of a voltage source driving such a circuit, the overall network will have an impedance, which has a magnitude and phase and a current will flow into the circuit which also possesses a corresponding magnitude and phase as shown below.





$$Z = \frac{v(t)}{i(t)}$$

$$i(t) = \frac{v(t)}{Z} = \frac{|v|\angle 0}{|Z|\angle \phi}$$

$$i(t) = \frac{|v|\angle 0}{|Z|}\angle -\phi$$

Fig. 1 The Phase Relationship Associated with an AC Circuit having Reactance

A plot of the voltage, which is taken as the reference zero angle, and the current with the instantaneous power is shown in Fig. 2 below. The current is seen to lag behind the voltage by an angle ϕ. Note that, unlike the case for resistive and purely reactive circuits, the instantaneous power profile is not symmetrical. It can be seen in this example that the power profile is positive for longer than it is negative and also that it reaches a higher positive peak than negative peak. This means that more power is delivered to the network in each cycle of the sinusoidal source than is returned to the source. Therefore there is a net transfer of power from the source to the circuit and this power is dissipated in the resistive components of the network.

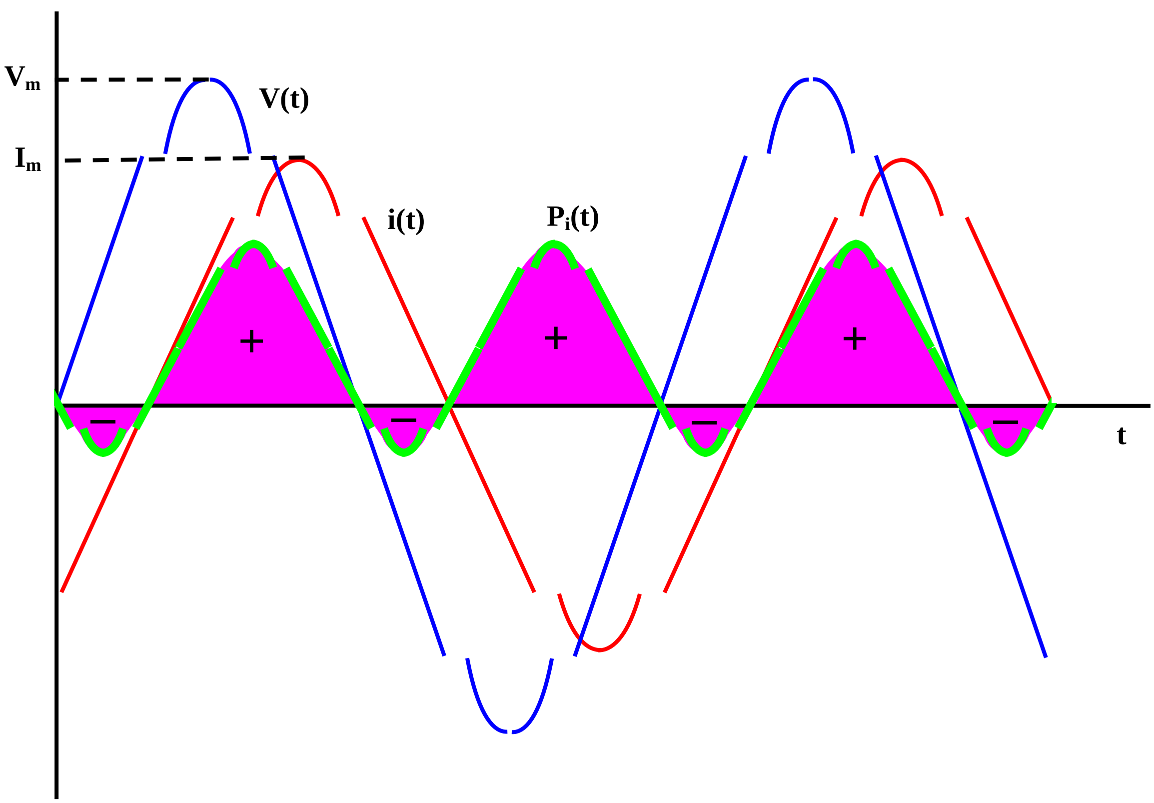


Fig. 2 Waveforms Showing Power Relations in an AC Circuit having Reactance

Instantaneous Power:

The instantaneous power can be found as before as the product of the voltage and current as continuous functions of time:

If

$$v(t) = V_m \, \text{Sin}\, \omega t \quad \text{and} \quad i(t) = I_m \, \text{Sin}(\omega t - \phi)$$

Then

$$P_i = V_m I_m \text{Sin}\, \omega t . \text{Sin}(\omega t - \phi)$$

Average Power:

$$P_{AVE} = \frac{1}{T}\int_0^T P_i dt = \frac{1}{T}\int_0^T v(t) i(t) dt$$

Using the trigonometric expansion

Using the trigonometric expansion

gives:

But

So that:

The factors Cos and Sin ϕ are constants for a given circuit where there is a given phase shift between the supply voltage and the current drawn by the circuit so that:

$$P_{AVE} = \frac{V_m I_m}{2T} Cos\phi \int_0^T dt - \frac{V_m I_m}{2T} Cos\phi \int_0^T Cos 2\omega t.dt$$

$$- \frac{V_m I_m}{2T} Sin\phi \int_0^T Sin 2\omega t.dt$$

$$P_{AVE} = \frac{V_m I_m}{2T} Cos\phi \left| t \right|_0^T - \frac{V_m I_m}{2T} Cos\phi \frac{1}{2\omega} \left| Sin 2\omega t \right|_0^T$$

$$+ \frac{V_m I_m}{2T} Sin\phi \frac{1}{2\omega} \left| Cos 2\omega t \right|_0^T$$

$$P_{AVE} = \frac{V_m I_m}{2T} Cos\phi (T - 0) - \frac{V_m I_m}{4\omega T} Cos\phi (Sin 4\pi - Sin 0)$$

$$+ \frac{V_m I_m}{4\omega T} Sin\phi (Cos 4\pi - Cos 0)$$

The last two terms in this expression have a value of zero as before so that finally:

$$P_{AVE} = \frac{V_m I_m}{2} Cos\phi = \frac{V_m}{\sqrt{2}} \text{ x } \frac{I_m}{\sqrt{2}} Cos\phi$$

$$P_{AVE} = V_{RMS} I_{RMS} Cos\phi$$

The term Cos ϕ is referred to as the *Power Factor* of the circuit. This is a property of the ac network and is determined by the phase angle of the network impedance.

$$\text{Power Factor} = \cos\phi$$

The Power Factor varies between a value of 0 and 1.

$$\phi = 0^{o} \Rightarrow \cos\phi = 1 \quad P_{AVE} = V_{RMS} I_{RMS} \qquad \text{purely resistive circuit}$$

$$\phi = 90^{o} \Rightarrow \cos\phi = 0 \quad P_{AVE} = 0 \qquad \text{purely reactive circuit}$$

The average power calculated above is the actual power consumed from the power delivered to the network. This is dissipated by the resistive elements of the circuit. However, the source must be rated to generate and deliver the total power demanded by the circuit even though not all of this is consumed. The power dissipated is also referred to as Active Power and represents energy consumed.

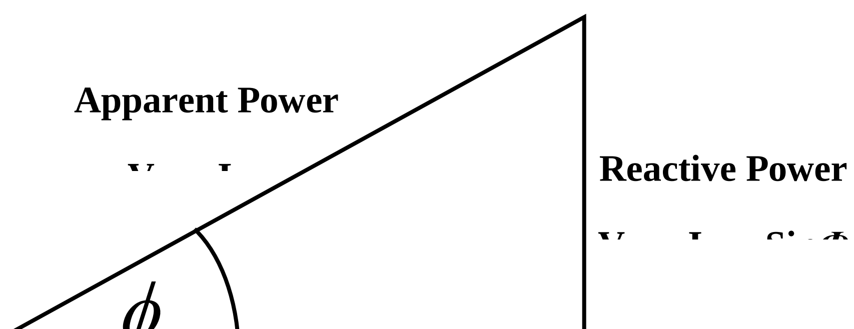
15.2 Complex Power

It has been seen from the previous waveform showing the instantaneous power that the positive excursion is greater than the negative excursion, so that there is a net transfer of power from the source to the load per cycle of the source voltage. The phase of the impedance of the network results in a phase angle between voltage and current which gives the Power Factor in the Average or Active Power drawn by the network. However, as with purely reactive circuits, there is also some power which is drawn from the source, stored temporarily in the reactive elements and then returned to the source in a later part of each cycle. This is referred to as the Reactive Power. In practice the source driving the network must be rated to handle and deliver both the active and reactive power, even though only the active power will be dissipated by the circuit. The vector sum of the Active and Reactive Power is referred to as the Apparent Power and gives the concept of Complex Power as illustrated in phasor form in Fig. 3 below.

$$\text{Apparent Power} = \text{Active Power} + j\,\text{Reactive Power}$$

$$\text{Apparent Power} = \text{Average Power} + j\,\text{Reactive Power}$$

Fig. 3 A Phasor Representation of Complex Power



$$\text{Apparent Power} = V_{RMS} I_{RMS} = \frac{V_m I_m}{2}$$

$$\text{Active or Average Power} = V_{RMS} I_{RMS} \cos\phi$$

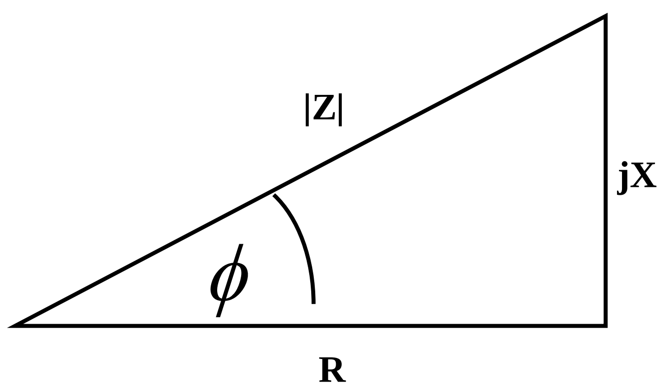
$$\text{Reactive Power} = j\, V_{RMS} I_{RMS} \sin\phi$$

and

$$V_{RMS}^2 I_{RMS}^2 = V_{RMS}^2 I_{RMS}^2 \cos^2\phi + V_{RMS}^2 I_{RMS}^2 \sin^2\phi$$

In order to avoid having to have a source which must be capable of providing much more power than is actually going to be consumed by a network, the aim is to minimise the amount of reactive power demanded of the source. Therefore the aim is to make the apparent power and the active power equal. This means making the power factor as close to unity as is possible.

Consider the network impedance shown in Fig. 4 below:



$$Z = R + jX$$

$$\sin\phi = \frac{X}{|Z|}$$

$$\cos\phi = \frac{R}{|Z|}$$

$$\text{Power Factor} = \cos\phi = \frac{R}{|Z|}$$

Fig. 4 Power Factor in Complex Power

where R is the overall equivalent resistance of the ac network as seen by the source. This may not actually be a resistive element but can represent work done by some piece of equipment or machine which is provided with electrical power and consumes energy.

Consider the circuit shown in Fig. 5 below.

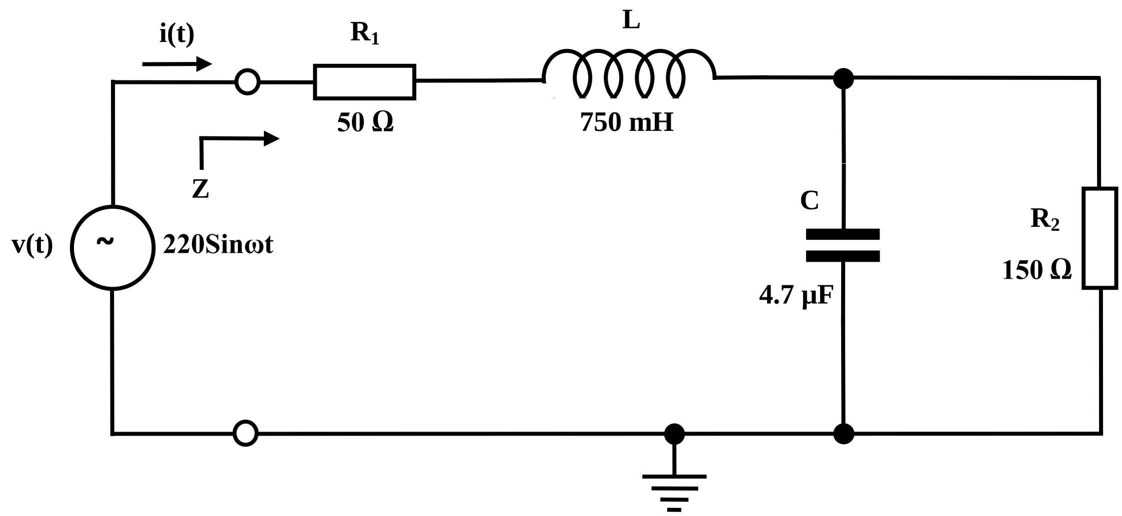


Fig. 5 An Example Circuit for AC Power Analysis

$$f = 50\ \text{Hz} \quad \textbf{and} \quad \omega = 2\pi f = 314\ \text{rad/s}$$

then

$$j\omega L = j \times 314 \times 750 \times 10^{-3} = j236\Omega$$

$$-j\frac{1}{\omega C} = -j\frac{1}{314 \times 4.7 \times 10^{-6}} = -j\frac{10^{6}}{1475.8} = -j678\Omega$$

Then:

$$Z = R_1 + j\omega L + \frac{-j\dfrac{R_2}{\omega C}}{R_2 - j\dfrac{1}{\omega C}}$$

$$Z = 50 + j236 + \frac{-j678x150}{150 - j678}$$

$$Z = 50 + j236 - j\frac{101700}{150 - j678}$$

Rationalising:

$$Z = 50 + j236 - j\frac{101700(150 + j678)}{(150 - j678)(150 + j678)}$$

$$Z = 50 + j236 + \frac{-j15.3x10^6 + 69x10^6}{150^2 + 678^2}$$

$$Z = 50 + j236 + \frac{-j15.3x10^6 + 69x10^6}{482184}$$

$$Z = 50 + j236 - j32 + 143$$

$$Z = 193 + j236 - j32 \quad \Omega$$

resistance **inductive reactance** **capacitive reactance**

So that overall $Z = 193 + j204 \quad \Omega$

The net impedance is more reactive than resistive and the reactance appears inductive.

$$\left|Z\right| = \sqrt{193^2 + 204^2} = 281\ \Omega$$

$$\angle\phi_Z = \mathrm{Tan}^{-1}\frac{204}{193} = \mathrm{Tan}^{-1}1.056 = 46.6^\circ$$

The current flowing into the circuit from the source can be found as:

$$i = \frac{V}{Z} = \frac{\left|V\right|\angle 0^\circ}{\left|Z\right|\angle\phi_Z} = \frac{220\angle 0^\circ}{281\angle 46.6^\circ} = 0.78\angle -46.6^\circ \quad A$$

The Power factor of the network is given as:

$$\text{Power Factor} = \mathrm{Cos}\phi = \mathrm{Cos}46.6^\circ = 0.687$$

The complex power can be evaluated as:

$$\text{Apparent Power} = V_{RMS}I_{RMS} = \frac{V_m I_m}{2} = \frac{220x0.78}{2} = 85.8W$$

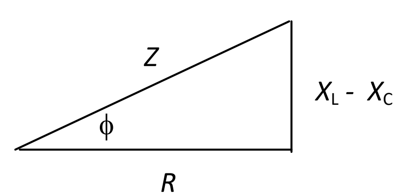
$$\text{Active Power} = V_{RMS}I_{RMS} = \frac{V_m I_m}{2}\mathrm{Cos}\phi = 85.8x0.687 = 58.9W$$

$$\text{Reactive Power} = j\,V_{RMS}I_{RMS}\mathrm{Sin}\phi = j\frac{V_m I_m}{2}\mathrm{Sin}\phi = j85.8x0.727 = j62.4W$$

Average Power

On average, only the resistance in the RCL series circuit consumes power. The average rate of power consumption is given by

$$\overline{P} = I_{rms}^2 R$$



The triangle at the right is useful to remember since one can quickly obtain the formulas that were derived above from it:

$$Z = \sqrt{\left(X_L - X_C\right)^2 + R^2} \quad \text{and} \quad \tan\phi = \frac{X_L - X_C}{R}$$

also note that $\dfrac{R}{Z} = \cos\phi$ so that $R = Z\cos\phi$ and $\overline{P} = I_{rms}^2 Z\cos\phi$ or $\overline{P} = I_{rms}\left(I_{rms}Z\right)\cos\phi$ from which we obtain

$$\boxed{\overline{P} = I_{rms}V_{rms}\cos\phi}$$

cosφ is called the *power factor* of the RCL circuit.

Using the formula $\tan\phi = \dfrac{X_L - X_C}{R}$ we make the following observations and definitions:

If $X_L > X_C$, $\phi > 0$ and the circuit is said to have *an inductive load*.
If $X_L < X_C$, $\phi < 0$ and the circuit is said to have *a capacitive load*.
If $X_L = X_C$, $\phi = 0$ and the circuit is said to have *a resistive load*.

Example

A series RCL circuit has a 75.0 Ω resistor, a 20.0 μF capacitor and a 55.0 mH inductor connected across an 800 volt rms AC generator operating at 128 Hz.

a. Is the load on the circuit inductive, capacitive or resistive? What is the phase angle φ?

$$X_L = 2\pi f L; \quad X_L = 2\pi(128)\left(5.5\times10^{-2}\right) = 44.2\ \Omega$$

$$X_C = \frac{1}{2\pi f C}; \quad X_C = \frac{1}{2\pi(128)\left(2.0\times10^{-5}\right)} = 61.2\ \Omega$$

Since $X_C > X_L$ the load is **capacitive**. The phase angle is

$$\phi = \arctan\left(\frac{X_L - X_C}{R}\right); \quad \phi = \arctan\left(\frac{44.2 - 61.2}{75.0}\right)$$

$$\phi = -0.223 \text{ rad} \quad \text{or} \quad \boxed{\phi = -12.8^\circ.}$$

b. What is the rms current in the circuit?

To answer this question we must determine the circuit's *impedance* Z then use $I_{rms} = V_{rms}/Z$:

$$Z = \sqrt{\left(X_L - X_C\right)^2 + R^2}; \quad Z = \sqrt{\left(44.2 - 61.2\right)^2 + \left(75.0\right)^2}$$

$$Z = 76.9\ \Omega. \quad I_{rms} = \frac{V_{rms}}{Z}; \quad I_{rms} = \frac{800\ \text{V}}{76.9\ \Omega}; \quad \boxed{I_{rms} = 10.4\ \text{A}}$$

c. Write the formula for the current in the circuit as a function of time.

$i = I\sin(\omega t - \phi)$ where I is the *current amplitude.*

$I = I_{\text{rms}}\sqrt{2}$: $I = 10.4(1.414)$; $I = 14.7$ A

$\omega = 2\pi f$: $\omega = 2\pi(128)$; $\omega = 804$ rad/s

$\boxed{i = 14.7\sin(804t + 0.223)}$ ← *Note the use of radians.*

(t in seconds and i in amperes.)

d. Find the rms voltage across each circuit element.

$V_{\text{Rrms}} = I_{\text{rms}}R$; $V_{\text{Rrms}} = (10.4\text{ A})(75.0\ \Omega)$; $\boxed{V_{\text{Rrms}} = 780\text{ V}}$

$V_{\text{Crms}} = I_{\text{rms}}X_{\text{C}}$; $V_{\text{Rrms}} = (10.4\text{ A})(61.2\ \Omega)$; $\boxed{V_{\text{Crms}} = 636\text{ V}}$

$V_{\text{Lrms}} = I_{\text{rms}}X_{\text{L}}$; $V_{\text{Rrms}} = (10.4\text{ A})(44.2\ \Omega)$; $\boxed{V_{\text{Lrms}} = 460\text{ V}}$

Question: Shouldn't these voltages add to 800 V?
Answer: No. One must take into account the *phase* of the voltage across each element. See part e.

e. Find the instantaneous voltage across each circuit element at t = 0 seconds.

$v_{\text{R}} = iR$; $v_{\text{R}} = (14.7)(75.0)\sin(0.223)$; $\boxed{v_{\text{R}} = 244\text{ V}}$

$v_{\text{C}} = i_{-\pi/2}X_{\text{C}}$; $v_{\text{C}} = (14.7)(61.2)\sin(0.223 - 1.57)$; $\boxed{v_{\text{C}} = -877\text{ V}}$

$v_{\text{L}} = i_{+\pi/2}X_{\text{L}}$; $v_{\text{L}} = (14.7)(44.2)\sin(0.223 + 1.57)$; $\boxed{v_{\text{L}} = 633\text{ V}}$

The voltage across the capacitor lags the current by 90°.

The voltage across the inductor leads the current by 90°.

Question: Why do these voltages add to *zero*?
Answer: Their sum is in agreement with Kirchhoff's loop rule; the voltage across the generator is $v = V\sin(\omega t)$ or $v = 800\sqrt{2}\sin(804t) = 0$ at $t = 0$ s.

f. Find the average power delivered to the circuit by the generator.

$\overline{P} = I_{\text{rms}}V_{\text{rms}}\cos(\phi)$; $\overline{P} = (10.4\text{ A})(800\text{ V})\cos(-0.223)$

$\boxed{\overline{P} = 8.11\text{ kW}}$

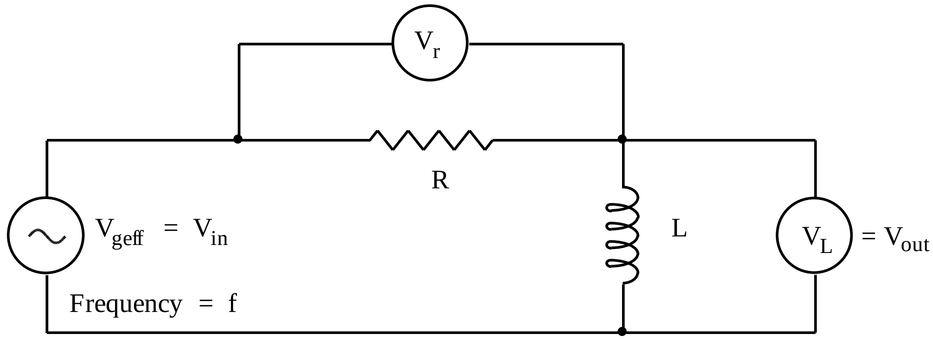
The Limiting Behavior of Capacitors and Inductors

Unlike a resistor, which has a constant resistance R independent of the ac frequency, capacitors and inductors have *reactances* that do depend on it.

The inductive reactance is given by

$$X_{\text{L}} = 2\pi fL$$

If f is large, so is X_{L}, and the inductor acts almost like an open circuit. If f is small, so is X_{L}, and the inductor acts almost like a short circuit.

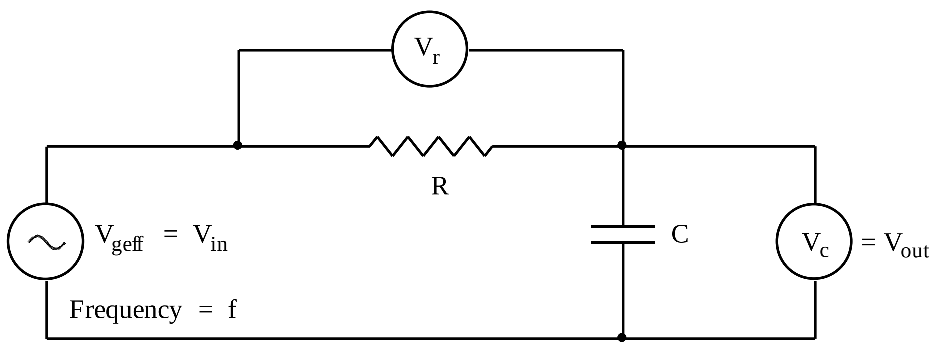


This circuit can be regarded as a high-pass filter. At very-high frequencies the inductor has a high reactance and acts almost like an open circuit. Thus, the current is low, the voltage drop in the resistor is low, and $V_{out} = V_{in}$. At very-low frequencies the inductor has a low reactance and acts like a short circuit. The output voltage is virtually zero. Hence, the circuit passes high-frequency AC voltages but stops low-frequency AC voltages.

The capacitive reactance is given by

$$X_C = \frac{1}{2\pi f C}$$

If f is large, X_C is small, and the capacitor acts almost like a short circuit. If f is small, X_C is large, and the capacitor acts almost like an open circuit.

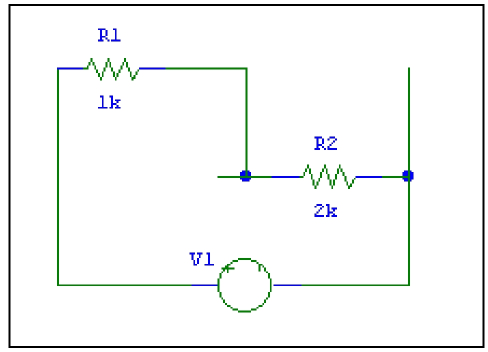


This circuit can be regarded as a low-pass filter. At very low frequencies the capacitor has a high reactance and is almost like an open circuit. Thus, the current is low, the voltage drop in the resistor is low, and $V_{out} = V_{in}$. At very high frequencies the capacitor has a low reactance and acts like a short circuit. The output voltage is virtually zero. Hence this circuit passes low-frequency AC voltages but stops high-frequency AC voltages.

Example

Suppose that an RC circuit (as shown in the last diagram above) is used in a crossover network in a 2-way stereo speaker. (A 2-way stereo speaker has a small speaker – a “tweeter” – for high frequencies and a large speaker – a “woofer” – for low frequencies. A *crossover network* in the speaker system directs low frequencies to the woofer and high frequencies to the tweeter). In the last diagram above V_{in} is the voltage supplied by the speaker output jacks of a stereo receiver; V_{out} is the voltage to be delivered to the woofer. If R is 30 ohms, find the capacitance C so that the amplitude of frequency 8,000 Hz is reduced to half its value at output.

$$V_{\text{out}} = IX_{\text{C}}; \quad I = \frac{V_{\text{in}}}{Z}; \quad V_{\text{out}} = \frac{V_{\text{in}} X_{\text{C}}}{\sqrt{X_{\text{C}}^2 + R^2}}; \quad \frac{V_{\text{out}}}{V_{\text{in}}} = \frac{X_{\text{C}}}{\sqrt{X_{\text{C}}^2 + R^2}}$$



$$\frac{1}{2} = \frac{X_{\text{C}}}{\sqrt{X_{\text{C}}^2 + R^2}}; \quad \frac{1}{2}\sqrt{X_{\text{C}}^2 + R^2} = X_{\text{C}} \quad \leftarrow \text{Square both sides.}$$

$$\frac{1}{4}\left(X_{\text{C}}^2 + R^2\right) = X_{\text{C}}^2; \quad \frac{1}{4}R^2 = \frac{3}{4}X_{\text{c}}^2 \Rightarrow X_{\text{C}}^2 = \frac{1}{3}R^2$$

$$X_{\text{C}}^2 = \frac{1}{3}(30)^2 = 300\ \Omega^2; \quad X_{\text{C}} = 17.3\ \Omega; \quad X_{\text{C}} = \frac{1}{2\pi fC}$$

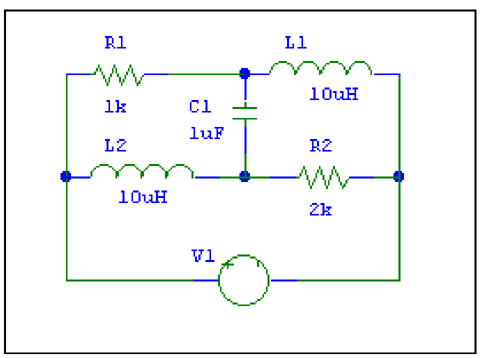
$$C = \frac{1}{2\pi f X_{\text{C}}}; \quad f = 8{,}000 \text{ Hz.} \quad \therefore C = \frac{1}{2\pi(8000)(17.3)}; \quad \boxed{C = 1.15\ \mu\text{F}}$$

Remark. The frequency whose amplitude is reduced to half by the crossover network is called the *crossover frequency*. In the above example 8,000 Hz is the crossover frequency.

Example

Estimate the impedance of the circuit shown at the left for a generator frequency of
a. 1,000,000 Hz
b. 0.001 Hz

a. For a high frequency the inductors act like *open* circuits and the capacitor acts like a *short* circuit, effectively producing the circuit shown in the diagram on the next page.



The impedance is now just the net resistance of the circuit.
Since the resistors are in series,

$$R = R_1 + R_2; \quad R = 1\text{k} + 2\text{k}; \quad \boxed{R = 3\text{ k}\Omega,} \quad Z = 3\text{ k}\Omega$$

b. For a low frequency the inductors act as *short* circuits and the capacitor acts as an *open* circuit, effectively producing the circuit shown in the diagram below.

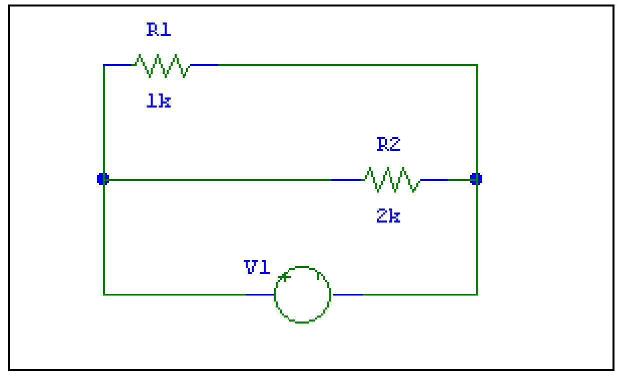
The impedance is now just the net resistance of the circuit. Since the resistors are in parallel,

$$\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2};\quad \frac{1}{R}=\frac{1}{1\text{k}}+\frac{1}{2\text{k}};\quad \frac{1}{R}=\frac{3}{2\text{k}};\quad R=0.67\ \text{k}\Omega$$

$$\boxed{Z=0.67\ \text{k}\Omega}$$

As the frequency of the AC generator is changed from very low values to very high values the impedance of the circuit will increase from the lower limit of 0.67 kΩ to the upper limit of 3 kΩ.

Note: The formula for impedance we found earlier, $Z=\sqrt{\left(X_{\text{L}}-X_{\text{C}}\right)^2+R^2}$, *does not* apply to the given circuit in this example because the circuit elements *are not* connected in series! The formulas for the reactances, however, always apply.



Electrical Resonance

For an RCL series circuit the current amplitude is given by

$$I=\frac{V}{Z}=\frac{V}{\sqrt{\left(X_{\text{L}}-X_{\text{C}}\right)^2+R^2}}$$

where V is the voltage amplitude. If V , R, C, and L are fixed and the frequency of the AC generator is variable, we can change the reactances of the inductor and capacitor by changing the frequency of the generator. As the frequency of the generator changes, so does the impedance Z of the circuit and the current amplitude I. If we look at the above formula we see that Z can be minimized (made as small as possible) by making the reactances X_{L} and X_{C} equal to one another. The current amplitude I will then be *maximized* (made as large as possible). If these conditions are met, *electrical resonance* is said to occur in the circuit. The RCL series circuit is said to be *at resonance*. For resonance,

$$X_{\text{L}}=X_{\text{C}}$$

$$2\pi fL=\frac{1}{2\pi fC}$$

$$\left(2\pi f\right)^2=\frac{1}{LC}$$

$$2\pi f=\frac{1}{\sqrt{LC}}$$

$$f=\boxed{\frac{1}{2\pi\sqrt{LC}}}$$

This value of f is called *the resonant frequency* of the RCL series circuit. At resonance the phase angle ϕ is *zero* and the circuit has a *resistive load*. The power factor $\cos\phi$ is 1 and *maximum power* is delivered to the circuit by the generator. At resonance the impedance Z equals the resistance R.

Example

An RCL series circuit is powered by an AC generator with rms voltage 200 V.
$R=20.0\ \Omega,\ \ C=5.00\ \mu\text{F},\ \ L=200\ \text{mH}.$

a. Find the resonant frequency of the circuit.

$$f = \frac{1}{2\pi\sqrt{LC}}; \; f = \frac{1}{2\pi\sqrt{(0.200\ \text{H})(5.00\times10^{-6}\ \text{F})}}; \; f = \frac{1}{2\pi\sqrt{1.00\times10^{-6}\ \frac{\text{V}}{\text{A/s}}\cdot\frac{\text{C}}{\text{V}}}}$$

$$f = \frac{1000}{\pi\sqrt{\frac{\text{V}}{\text{C/s}^2}\cdot\frac{\text{C}}{\text{V}}}} = \frac{1000}{\pi\cdot\text{s}}; \quad \boxed{f = 159\ \text{Hz}}$$

b. Find the rms current at resonance.

$$I_{\text{rms}} = \frac{V_{\text{rms}}}{Z}; \quad I_{\text{rms}} = \frac{200\ \text{V}}{20.0\ \Omega}; \quad \boxed{I_{\text{rms}} = 10.0\ \text{A}}$$

c. Find the average power delivered to the circuit at resonance.

$$\overline{P} = I_{\text{rms}}V_{\text{rms}}; \quad \overline{P} = (10.0\ \text{A})(200\ \text{V}); \quad \boxed{\overline{P} = 2.00\ \text{kW}}$$

SINUSOIDAL STEADY STATE ANALYSIS

Analyzing ac circuits usually requires three steps.

Steps to analyze AC Circuits:

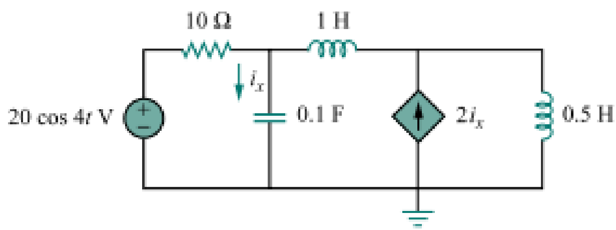
1. Transform the circuit to the phasor or frequency domain.
2. Solve the problem using circuit techniques (nodal analysis, mesh analysis, superposition, etc.).
3. Transform the resulting phasor to the time domain.

NODAL ANALYSIS

The basis of nodal analysis is Kirchhoff's current law. Since KCL is valid for phasors, as demonstrated previously, we can analyze ac circuits by nodal analysis. The following examples illustrate this.

Example:

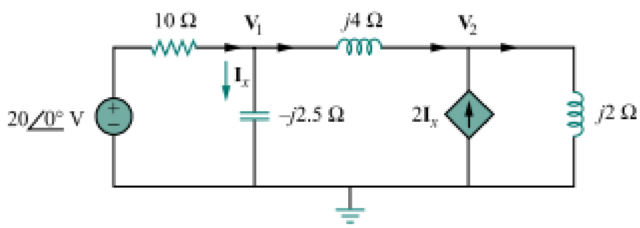
Find i_x in the circuit shown using nodal analysis.



1. Convert the circuit to the frequency domain and draw the equivalent circuit in the freq domain:

$$\begin{aligned} 20\cos 4t &\implies 20\angle 0^\circ, \qquad \omega = 4 \text{ rad/s} \\ 1\text{ H} &\implies j\omega L = j4 \\ 0.5\text{ H} &\implies j\omega L = j2 \\ 0.1\text{ F} &\implies \frac{1}{j\omega C} = -j2.5 \end{aligned}$$

Thus, the frequency-domain equivalent circuit is as shown



2. Now apply KCL at node 1

$$\frac{20 - \mathbf{V}_1}{10} = \frac{\mathbf{V}_1}{-j2.5} + \frac{\mathbf{V}_1 - \mathbf{V}_2}{j4}$$

$$(1 + j1.5)\mathbf{V}_1 + j2.5\mathbf{V}_2 = 20$$

At node 2

$$2\mathbf{I}_x + \frac{\mathbf{V}_1 - \mathbf{V}_2}{j4} = \frac{\mathbf{V}_2}{j2}$$

But $I_x = V_1 / -j2.5$. Substituting this gives

$$\frac{2\mathbf{V}_1}{-j2.5} + \frac{\mathbf{V}_1 - \mathbf{V}_2}{j4} = \frac{\mathbf{V}_2}{j2}$$

Simplifying we get:

$$11V_1 + 15V_2 = 0$$

We now have two equations in V_1 and V_2. We can solve this system of equations be substitution or using a matrix.

$$\begin{bmatrix} 1+j1.5 & j2.5 \\ 11 & 15 \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 0 \end{bmatrix}$$

$$\Delta = \begin{vmatrix} 1+j1.5 & j2.5 \\ 11 & 15 \end{vmatrix} = 15 - j5$$

$$\Delta_1 = \begin{vmatrix} 20 & j2.5 \\ 0 & 15 \end{vmatrix} = 300, \qquad \Delta_2 = \begin{vmatrix} 1+j1.5 & 20 \\ 11 & 0 \end{vmatrix} = -220$$

$$\mathbf{V}_1 = \frac{\Delta_1}{\Delta} = \frac{300}{15 - j5} = 18.97\angle 18.43^\circ \text{ V}$$

$$\mathbf{V}_2 = \frac{\Delta_2}{\Delta} = \frac{-220}{15 - j5} = 13.91\angle 198.3^\circ \text{ V}$$

The current I x is given by

$$\mathbf{I}_x = \frac{\mathbf{V}_1}{-j2.5} = \frac{18.97\angle 18.43^\circ}{2.5\angle -90^\circ} = 7.59\angle 108.4^\circ \text{ A}$$

Transforming this to the time domain,

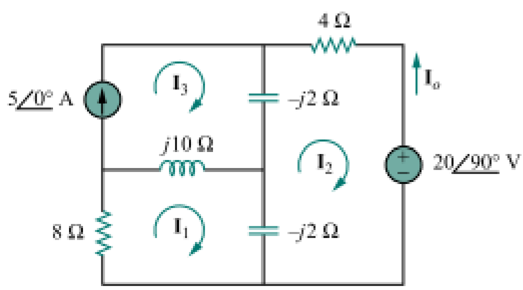
$$i_x = 7.59 \cos(4t + 108.4^\circ) \text{ A}$$

MESH ANALYSIS

Kirchhoff's voltage law (KVL) forms the basis of mesh analysis. The validity of KVL for ac circuits was shown previously and is illustrated in the following examples.

Example:

Determine current I_o in the circuit below using mesh analysis.



1. Apply KVL to mesh 1

Applying KVL to mesh 1, we obtain

$8I_1 + j10*(I_1-I_3) - j2*(I_1 - I_2) = 0$

$(8 + j10 - j2)\mathbf{I_1} - (-j2)\mathbf{I_2} - j10\mathbf{I_3} = 0$

2. Apply KVL to mesh 2

For mesh 2,

$(4 - j2 - j2)\mathbf{I}_2 - (-j2)\mathbf{I}_1 - (-j2)\mathbf{I}_3 + 20\ ,\ 90^\circ = 0$

3. Given that for mesh 3, $I_3 = 5$, use this system of equations to solve for I_1 and I_2.

$$(8 + j8)I_1 + j2I_2 = j50$$

$$j2I_1 + (4 - j4)I_2 = - j20 - j10$$

In matrix form as

$$\begin{bmatrix} 8+j8 & j2 \\ j2 & 4-j4 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} j50 \\ -j30 \end{bmatrix}$$

$$\Delta = \begin{vmatrix} 8+j8 & j2 \\ j2 & 4-j4 \end{vmatrix} = 32(1+j)(1-j) + 4 = 68$$

$$\Delta_2 = \begin{vmatrix} 8+j8 & j50 \\ j2 & -j30 \end{vmatrix} = 340 - j240 = 416.17\underline{/-35.22^\circ}$$

$$\mathbf{I}_2 = \frac{\Delta_2}{\Delta} = \frac{416.17\underline{/-35.22^\circ}}{68} = 6.12\underline{/-35.22^\circ}\text{ A}$$

Since $I_0 = - I_2$ then we know I_0.

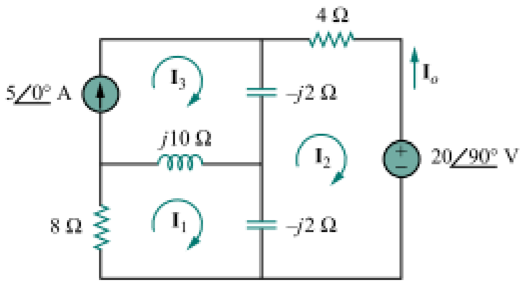
SUPERPOSITION THEOREM

Since ac circuits are linear, the superposition theorem applies to ac circuits the same way it applies to dc circuits. The theorem becomes important if the circuit has sources operating at different frequencies. In this case, since the impedances depend on frequency, we must have a different frequency-domain circuit for each frequency. The total response must be obtained by adding the individual responses in the time domain. It is incorrect to try to add the responses in the phasor or frequency domain.

Why? Because the exponential factor $e^{j\omega t}$ is implicit in sinusoidal analysis, and that factor would change for every angular frequency ω. It would therefore not make sense to add responses at different frequencies in the phasor domain. Thus, when a circuit has sources operating at different frequencies, one must add the responses due to the individual frequencies in the time domain.

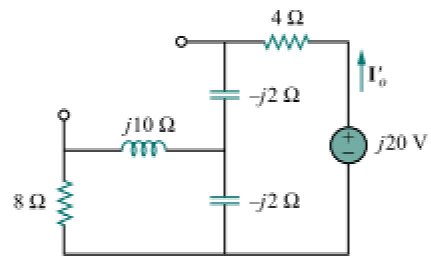
Example:

Use superposition to find I_0 in the circuit below:



Let $I_o = I_o' + I_o''$

where I_o' and I_o'' are due to the voltage and current sources, respectively.



To find I_o', recall that we open circuit current sources and short circuit voltage sources. Open circuiting the current source gives the circuit at right. If we let Z be the parallel combination of − j2 and 8 + j10, then

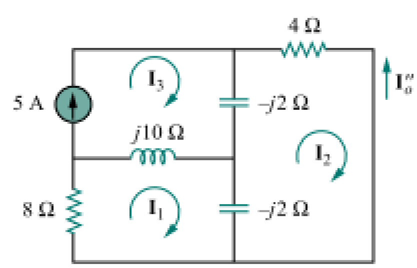
$$\mathbf{Z} = \frac{-j2(8+j10)}{-2j+8+j10} = 0.25 - j2.25$$

And the current is

$$\mathbf{I}_o' = \frac{j20}{4 - j2 + \mathbf{Z}} = \frac{j20}{4.25 - j4.25}$$

$$\mathbf{I}_o' = -2.353 + j2.353$$

To find I_o'', use circuit at right. For mesh 1:



$(8 + j8)I_1 - j10I_3 + j2I_2 = 0$

For mesh 2,

$(4 - j4)I_2 + j2I_1 + j2I_3 = 0$

For mesh 3,

$I_3 = 5$

Substituting

$(4 - j4)I_2 + j2I_1 + j10 = 0$

Expressing I_1 in terms of I_2 gives

$I_1 = (2 + j2)I_2 - 5$

Substituting, we get

$(8 + j8)[(2 + j2)I_2 - 5] - j50 + j2I_2 = 0$

Solving for I_2:

$$\mathbf{I}_2 = \frac{90 - j40}{34} = 2.647 - j1.176$$

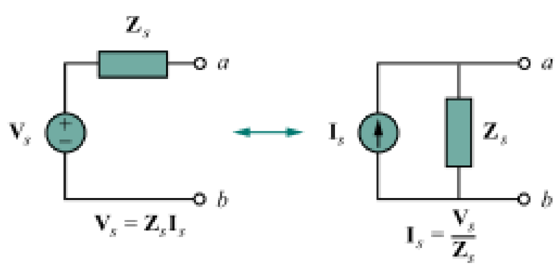
The total current is then the sum of these two currents:

$$\mathbf{I}_o = \mathbf{I}'_o + \mathbf{I}''_o = -5 + j3.529 = 6.12\angle 144.78^\circ \text{ A}$$

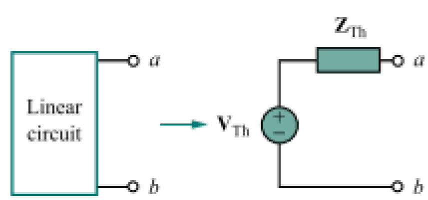
SOURCE TRANSFORMATION

Source transformation in the frequency domain involves transforming a voltage source in series with an impedance to a current source in parallel with an impedance, or vice versa. As we go from one source type to another, we must keep the following relationship in mind:

$$\mathbf{V}_s = \mathbf{Z}_s\mathbf{I}_s \quad \Longleftrightarrow \quad \mathbf{I}_s = \frac{\mathbf{V}_s}{\mathbf{Z}_s}$$



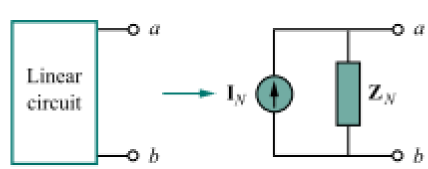
THEVENIN AND NORTON EQUIVALENT CIRCUITS



Thevenin's and Norton's theorems are applied to ac circuits in the same way as they are to dc circuits. The only additional effort arises from the need to manipulate complex numbers. The frequency-domain version of a Thevenin equivalent circuit is depicted in (a), where a linear circuit is replaced by a voltage source in series with an impedance. The Norton equivalent circuit is illustrated in (b), where a linear circuit is replaced by a current source in parallel with an impedance. Keep in mind that the two equivalent circuits are related as

$$\mathbf{V}_{Th} = \mathbf{Z}_N\mathbf{I}_N, \qquad \mathbf{Z}_{Th} = \mathbf{Z}_N$$

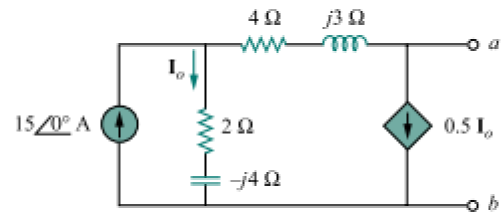
just as in source transformation. V_{Th} is the open-circuit voltage while I_N is the short-circuit current.



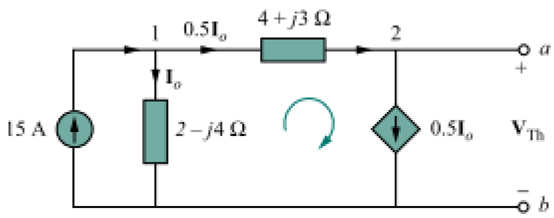
If the circuit has sources operating at different frequencies, the Thevenin or Norton equivalent circuit must be determined at each frequency. This leads to entirely different equivalent circuits, one for each frequency, not one equivalent circuit with equivalent sources and equivalent impedances.

Example:

Find the Thevenin equivalent of the circuit as seen from terminals a-b.



1. Redraw the circuit by combining series impedances:



2. To find V_{Th}, we apply KCL at node 1 to find I_0. Then apply KVL to the right hand loop.

$$15 = I_o + 0.5I_o \Rightarrow I_o = 10\text{ A}$$

Applying KVL to the loop, we obtain

$$-I_o(2 - j4) + 0.5I_o(4 + j3) + V_{Th} = 0$$

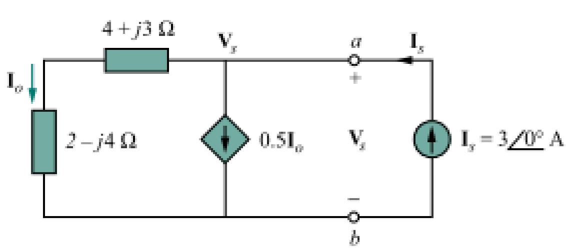
or

$$V_{Th} = 10(2 - j4) - 5(4 + j3) = -j55$$

Thus, the Thevenin voltage is

$$V_{Th} = 55, -90^\circ\text{V}$$

3. To find Z_{th}, remove the independent source and connect an arbitrary fixed current source (In this case 3A since it makes the math easy) to terminals a and b and redraw the circuit:



4. Now apply KCL at the node and KVL to the outer loop. Find Zth as the ratio of the Voltage to the Current.

At the node, KCL gives

$3 = I_o + 0.5 I_o \Rightarrow I_o = 2$ A

Applying KVL to the outer loop gives

$V_s = I_o (4 + j3 + 2 - j4) = 2(6 - j)$

The Thevenin impedance is

$$\mathbf{Z}_{Th} = \frac{\mathbf{V}_s}{\mathbf{I}_s} = \frac{2(6-j)}{3} = 4 - j0.6667\ \Omega$$

Resonance

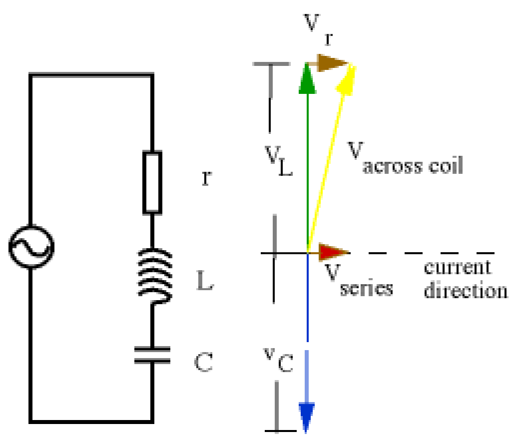
the expression for the series impedance goes to infinity at high frequency because of the presence of the inductor, which produces a large emf if the current varies rapidly. Similarly it is large at very low frequencies because of the capacitor, which has a long time in each half cycle in which to charge up. As we saw in the plot of $Z_{series}\omega$ above, there is a minimum value of the series impedance, when the voltages across capacitor and inductor are equal and opposite, ie $v_L(t) = - v_C(t)$ so $V_L(t) = V_C$, so

$\omega L = 1/\omega C$ so the frequency at which this occurs is

$$\omega_o = \frac{1}{\sqrt{LC}}$$

$$f_o = \frac{1}{2\pi}\frac{1}{\sqrt{LC}}$$

where ω_o and f_o are the angular and cyclic frequencies of resonance, respectively. At resonance, series impedance is a minimum, so the voltage for a given current is a minimum (or the current for a given voltage is a maximum).



This phenomenon gives the answer to our teaser question at the beginning. In an RLC series circuit in which the inductor has relatively low internal resistance r, it is possible to have a large voltage across the the inductor, an almost equally large voltage across capacitor but, as the two are nearly 180° degrees out of phase, their voltages almost cancel, giving a total series voltage that is quite small. This is one way to produce a large voltage oscillation with only a small voltage source. In the circuit diagram at right, the coil corresponds to both the inducance L and the resistance r, which is why they are drawn inside a box representing the physical component, the coil. Why are they in series? Because the current flows through the coil and thus passes through both the inductance of the coil and its resistance.

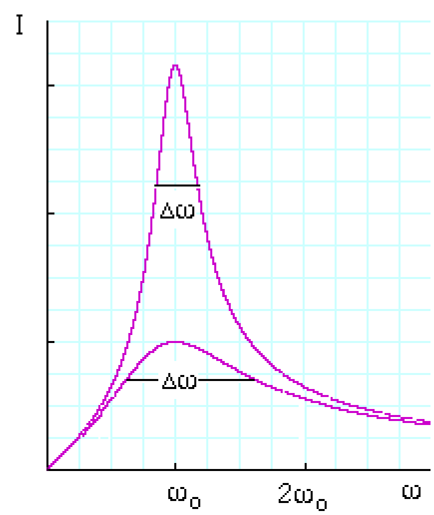
You get a big voltage in the circuit for only a small voltage input from the power source. You are not, of course, getting something for nothing. The energy stored in the large oscillations is gradually supplied by the AC source when you turn on, and it is then exchanged between capacitor and inductor in each cycle.

Bandwidth and Q factor

At resonance, the voltages across the capacitor and the pure inductance cancel out, so the series impedance takes its minimum value: $Z_o = R$. Thus, if we keep the voltage constant, the current is a maximum at resonance. The current goes to zero at low frequency, because X_C becomes infinite (the capacitor is open circuit for DC). The current also goes to zero at high frequency because X_L increases with ω (the inductor opposes rapid changes in the current). The graph shows I(ω) for circuit with a large resistor (lower curve) and for one with a small resistor (upper curve). A circuit with low R, for a given L and C, has a sharp resonance. Increasing the resistance makes the resonance less sharp. The former circuit is more selective: it produces high currents only for a narrow **bandwidth**, ie a small range of ω or f. The circuit with higher R responds to a wider range of frequencies and so has a larger bandwidth. The bandwidth Δω (indicated by the horiztontal bars on the curves) is defined as the difference between the two frequencies ω_+ and ω_- at which the circuit converts power at half the maximum rate.

Now the electrical power converted to heat in this circuit is I^2R, so the maximum power is converted at resonance, $\omega = \omega_o$. The circuit converts power at half this rate when the current is $I_o/\sqrt{2}$. The **Q value** is defined as the ratio

$$Q = \omega_o/\Delta\omega.$$



Complex impedance

You have perhaps been looking at these phasor diagrams, noticing that they are all two-dimensional, and thinking that we could simply use the complex plane. Good idea! But not original: indeed, that is the most common way to analyse such circuits.

The only difference from the presentation here is to consider cosusoids, rather than sinusoids. In the animations above, we used sin waves so that the vertical projection of the phasors would correspond to the height on the v(t) graphs. In complex algebra, we use cos waves and take their projections on the (horizontal) real axis. The phasor diagrams have now become diagrams of complex numbers, but otherwise look exactly the same. They still rotate at ωt, but in the complex plane. The resistor has a real impedance R, the inductor's reactance is a positive imaginary impedance

$$X_L = j\omega L$$

and the capacitor has a negative imaginary impedance

$$X_C = -j.1/\omega C = 1/j\omega C.$$

Consequently, using bold face for complex quantities, we may write:

$$\mathbf{Z}_{series} = (R^2 + (j\omega L + 1/j\omega C)^2)^{1/2}$$

and so on. The algebra is relatively simple. The magnitude of any complex quantity gives the magnitude of the quantity it represents, the phase angle its phase angle. Its real component is the component in phase with the reference phase, and the imaginary component is the component that is 90° ahead.

BANDWIDTH

At a certain frequency the power dissipated by the resistor is half of the maximum power which as mentioned occurs at $\omega_0 = \frac{1}{\sqrt{LC}}$. The half power occurs at the frequencies for which the amplitude of the voltage across the resistor becomes equal to $\frac{1}{\sqrt{2}}$ of the maximum.

$$P_{1/2} = \frac{1}{4}\frac{V_{max}^2}{R}$$

Figure 3 shows in graphical form the various frequencies of interest.

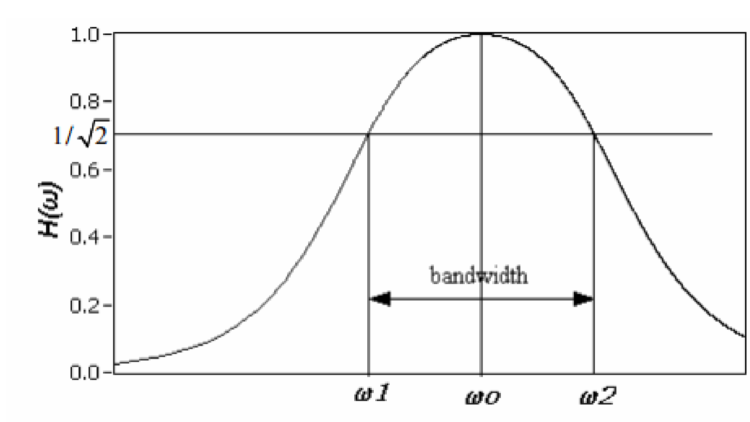


Figure 3

Therefore, the ½ power occurs at the frequencies for which

$$\frac{1}{\sqrt{2}} = \frac{\omega RC}{\sqrt{\left(1-\omega^2 LC\right)^2 + \left(\omega RC\right)^2}} \tag{1.8}$$

The bandwidth is the difference between the half power frequencies

$$Bandwidth = B = \omega_2 - \omega_1 \tag{1.11}$$

By multiplying Equation (1.9) with Equation (1.10) we can show that ω_0 is the geometric mean of ω_1 and ω_2.

$$\omega_0 = \sqrt{\omega_1 \omega_2}$$

As we see from the plot on Figure 2 the bandwidth increases with increasing R. Equivalently the sharpness of the resonance increases with decreasing R. For a fixed L and C, a decrease in R corresponds to a narrower resonance and thus a higher selectivity regarding the frequency range that can be passed by the circuit. As we increase R, the frequency range over which the dissipative characteristics dominate the behavior of the circuit increases. In order to quantify this behavior we define a parameter called the Quality Factor Q which is related to the sharpness of the peak and it is given by

$$Q = 2\pi \frac{\text{maximum energy stored}}{\text{total energy lost per cycle at resonance}} = 2\pi \frac{E_S}{E_D}$$

which represents the ratio of the energy stored to the energy dissipated in a circuit. The energy stored in the circuit is

$$E_S = \frac{1}{2}LI^2 + \frac{1}{2}CVc^2$$

For $Vc = A\sin(\omega t)$ the current flowing in the circuit is $I = C\frac{dVc}{dt} = \omega CA\cos(\omega t)$. The total energy stored in the reactive elements is

$$E_S = \frac{1}{2}L\omega^2 C^2 A^2 \cos^2(\omega t) + \frac{1}{2}CA^2 \sin^2(\omega t)$$

At the resonance frequency where $\omega = \omega_0 = \frac{1}{\sqrt{LC}}$ the energy stored in the circuit becomes

$$E_S = \frac{1}{2}CA^2$$

The energy dissipated per period is equal to the average resistive power dissipated times the oscillation period.

$$E_D = R\langle I^2 \rangle \frac{2\pi}{\omega_0} = R\left(\frac{\omega_0^2 C^2 A^2}{2}\right)\frac{2\pi}{\omega_0} = 2\pi\left(\frac{1}{2}\frac{RC}{\omega_0 L}A^2\right)$$

And so the ratio Q becomes

$$Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 RC}$$

- The quality factor increases with decreasing R.
- The bandwidth decreases with decreasing R.

Problems

A series RLC circuit with $L = 160\text{ mH}$, $C = 100\,\mu\text{F}$, and $R = 40.0\,\Omega$ is connected to a sinusoidal voltage $V(t) = (40.0\text{ V})\sin\omega t$, with $\omega = 200\text{ rad/s}$.

(a) What is the impedance of the circuit?

(b) Let the current at any instant in the circuit be $I(t) = I_0 \sin(\omega t - \phi)$. Find I_0.

(c) What is the phase ϕ?

Solution:

(a) The impedance of a series *RLC* circuit is given by

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

where

$$X_L = \omega L$$

and

$$X_C = \frac{1}{\omega C}$$

are the inductive reactance and the capacitive reactance, respectively. Since the general expression of the voltage source is $V(t) = V_0 \sin(\omega t)$, where V_0 is the maximum output voltage and ω is the angular frequency, we have $V_0 = 40$ V and $\omega = 200$ rad/s. Thus, the impedance Z becomes

$$Z = \sqrt{(40.0\ \Omega)^2 + \left((200\text{ rad/s})(0.160\text{ H}) - \frac{1}{(200\text{ rad/s})(100\times10^{-6}\text{ F})}\right)^2}$$
$$= 43.9\,\Omega$$

(b) With $V_0 = 40.0\,\text{V}$, the amplitude of the current is given by

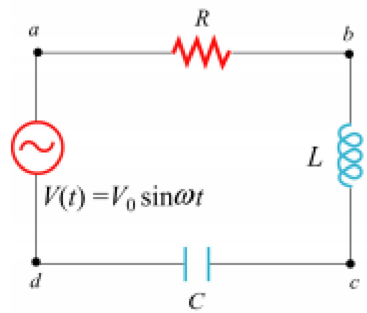
$$I_0 = \frac{V_0}{Z} = \frac{40.0\,\text{V}}{43.9\,\Omega} = 0.911\,\text{A}$$

(c) The phase between the current and the voltage is determined by

$$\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right) = \tan^{-1}\left(\frac{\omega L - \dfrac{1}{\omega C}}{R}\right)$$

$$= \tan^{-1}\left(\frac{(200\text{ rad/s})(0.160\text{ H}) - \dfrac{1}{(200\text{ rad/s})(100\times10^{-6}\text{ F})}}{40.0\ \Omega}\right) = -24.2°$$

Suppose an AC generator with $V(t)=(150\text{ V})\sin(100t)$ is connected to a series RLC circuit with $R=40.0\ \Omega$, $L=80.0$ mH, and $C=50.0\ \mu$F



(a) Calculate V_{R0}, V_{L0} and V_{C0}, the maximum of the voltage drops across each circuit element.

(b) Calculate the maximum potential difference across the inductor and the capacitor between points *b* and *d* shown in Figure

Solutions:

(a) The inductive reactance, capacitive reactance and the impedance of the circuit are given by

(a) The inductive reactance, capacitive reactance and the impedance of the circuit are given by

$$X_C=\frac{1}{\omega C}=\frac{1}{(100\text{ rad/s})(50.0\times10^{-6}\text{ F})}=200\ \Omega$$

$$X_L=\omega L=(100\text{ rad/s})(80.0\times10^{-3}\text{ H})=8.00\ \Omega$$

$$Z=\sqrt{R^2+(X_L-X_C)^2}=\sqrt{(40.0\ \Omega)^2+(8.00\ \Omega-200\ \Omega)^2}=196\ \Omega$$

respectively. Therefore, the corresponding maximum current amplitude is

$$I_0=\frac{V_0}{Z}=\frac{150\text{ V}}{196\ \Omega}=0.765\text{ A}$$

The maximum voltage across the resistance would be just the product of maximum current and the resistance:

$$V_{R0}=I_0R=(0.765\text{ A})(40.0\ \Omega)=30.6\text{ V}$$

Similarly, the maximum voltage across the inductor is

$$V_{L0}=I_0X_L=(0.765\text{ A})(8.00\ \Omega)=6.12\text{ V}$$

and the maximum voltage across the capacitor is

$$V_{C0} = I_0 X_C = (0.765\ \text{A})(200\ \Omega) = 153\ \text{V}$$

Note that the maximum input voltage V_0 is related to V_{R0}, V_{L0} and V_{C0} by

$$V_0 = \sqrt{V_{R0}{}^2 + (V_{L0} - V_{C0})^2}$$

(b) From b to d, the maximum voltage would be the difference between V_{L0} and V_{C0}:

$$|V_{bd}| = |\vec{V}_{L0} + \vec{V}_{C0}| = |V_{L0} - V_{C0}| = |6.12\ \text{V} - 153\ \text{V}| = 147\,\text{V}$$

A sinusoidal voltage $V(t) = (200\,\text{V})\sin\omega t$ is applied to a series RLC circuit with $L = 10.0\ \text{mH}$, $C = 100\ \text{nF}$ and $R = 20.0\ \Omega$. Find the following quantities:

(a) the resonant frequency,

(b) the amplitude of the current at resonance,

(c) the quality factor Q of the circuit, and

(d) the amplitude of the voltage across the inductor at the resonant frequency.

Solution:

(a) The resonant frequency for the circuit is given by

$$f = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{1}{LC}} = \frac{1}{2\pi}\sqrt{\frac{1}{(10.0\times10^{-3}\ \text{H})(100\times10^{-9}\ \text{F})}} = 5033\,\text{Hz}$$

(b) At resonance, the current is

$$I_0 = \frac{V_0}{R} = \frac{200\ \text{V}}{20.0\ \Omega} = 10.0\,\text{A}$$

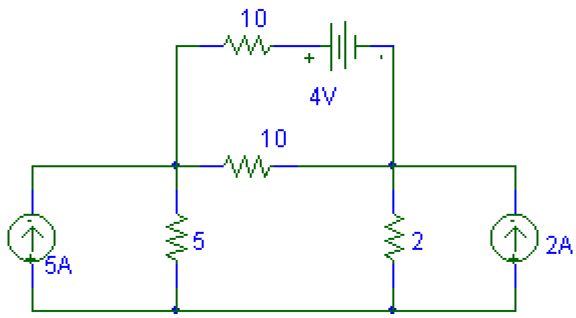
(c) The quality factor Q of the circuit is given by

$$Q = \frac{\omega_0 L}{R} = \frac{2\pi(5033\ \text{s}^{-1})(10.0\times10^{-3}\ \text{H})}{(20.0\ \Omega)} = 15.8$$

(d) At resonance, the amplitude of the voltage across the inductor is

$$V_{L0} = I_0 X_L = I_0\omega_0 L = (10.0\ \text{A})2\pi(5033\ \text{s}^{-1})(10.0\times10^{-3}\ \text{H}) = 3.16\times10^3\,\text{V}$$

Solve for the current through the 5 ohm resistor and the current through the 4V source using Node-Voltage Analysis.



Now write KCL at each node (except the reference):
KCL at V1:
-5A + V1/5 + (V1-V2)/10 + [V1-(V2+4)]/10 = 0

Note that there are four terms in the equation, one for each branch leaving the node. The terms list the current leaving right, down, left, and up.

KCL at V2:

(V2-V1)/10 + V2/2 - 2A + [V2-(V1-4)]/10 = 0

Note that there are four terms in the equation, one for each branch leaving the node. The terms list the current leaving right, down, left, and up.

Now gather terms (multiplying through by 10 to clear up the fractions):

4V1 - 2V2 = 54

-2V1 + 7V2 = 16

Now solve the set of 2 equations with 2 unknowns.

V1 = 17.08V

V2 = 7.17V

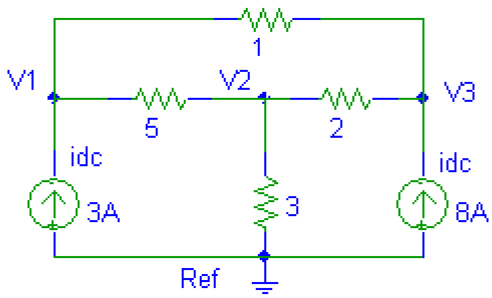
We can now determine the current through the 5 ohm by Ohm's law:

I = V1/5 = 3.41A

The current through the 4V source can be found as:

I = [V1-(V2+4)]/10 = 0.59A

Solve for the current through the 5 ohm resistor and the voltage over the 3A source using Node-Voltage Analysis.



Now write KCL equations for each node except the reference, in terms of the node voltages:

KCL at V1:

-3A + (V1-V2)/5 + (V1-V3)/1 = 0

KCL at V2:

(V2-V1)/5 + V2/3 + (V2-V3)/2 =0

KCL at V3:

(V3-V2)/2 + (V3-V1)/1 - 8A = 0

Now gather terms and clear up the fractions:

6V1 - V2 - 5V3 = 15

-6V1 + 31V2 - 15V3 =0

-2V1 -V2 +3 V3 = 16

Finally, solve the 3 equations in 3 unknowns.

V1 = 48.625V

V2 = 33 V

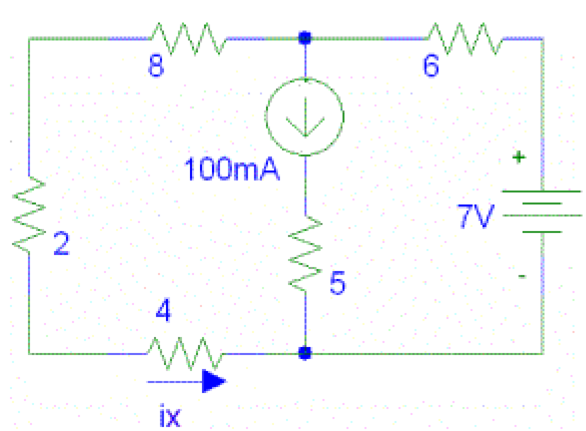
V3 = 48.75V

The current through the 5 ohm resistor can be found by Ohm's law:

I = (V1 - V2)/5 = 3.125A

The voltage over the 3A source is simply V1, or 48.625V.

Solve for the current ix flowing right through the 4 ohm resistor using Mesh-Current Analysis.



Label each mesh (window pane) with a mesh current. Then write the KVL equations for each pane. Note that we were forced to label the voltage over the current source (Vx) in order to write the voltage term there:

$$19i_1 + v_x - 5i_2 = 0$$
$$-5i_1 - v_x + 11i_2 + 7 = 0$$

We now have an extra unknown (Vx), so we need another equation. It is found be relating the two mesh currents to the current source.

$$i_1 - i_2 = 100mA$$

Note that i1 is positive because it is in the same direction of the source. I2 is negative because it is in the opposite direction as the source.

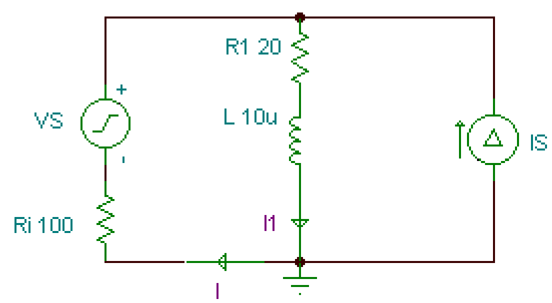
Now solve the three equations in three unknowns. I1 is found to be -320mA. Since ix is in the opposite direction of i1, then ix = 320mA.

In the circuit shown below

R_i = 100 ohm, R_1 = 20 ohm, R_2 = 12 ohm, L = 10 uH, C = 0.3 nF, $v_S(t)=50\cos(\omega t)$ V, $i_S(t)=1\cos(\omega t+30°)$ A, f=400 kHz.

Notice that both sources have the same frequency: we will only work in this chapter with sources all having the same frequency. Otherwise, superposition must be handled differently.

Find the currents i(t) and $i_1(t)$ using the superposition theorem.



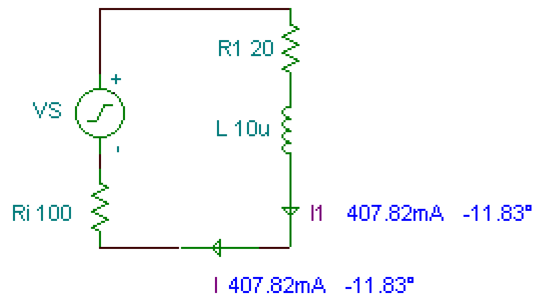
Let's use TINA and hand calculations in parallel to solve the problem.

First substitute an open circuit for the current source and calculate the complex phasors **I', I1'** due to the contribution only from **VS.**

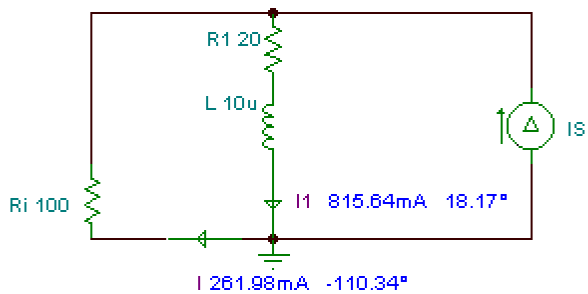
The currents in this case are equal:

$$\mathbf{I'} = \mathbf{I_1'} = \mathbf{V_S}/(R_i + R_1 + \mathbf{j}\cdot\omega\cdot L) = 50/(120+\mathbf{j}2\cdot\pi\cdot 4\cdot 10^5\cdot 10^{-5}) = 0.3992-\mathbf{j}0.0836$$

$$\mathbf{I'} = 0.408\, e^{\mathbf{j}\, 11.83°}\ \text{A}$$



Next substitute a short-circuit for the voltage source and calculate the complex phasors **I'', I1''** due to the contribution only from **IS.**



In this case we can use the current division formula:

$$I'' = -I_{G1}\frac{R_1 + j*\omega*L}{R_1 + R_i + j*\omega*L} = -e^{j30°}\frac{20 + j*2*\pi*4*10^5*10^{-5}}{100+20+j*2*\pi*4*10^5*10^{-5}} = -0.26198\,e^{j69.67°}\,A$$

I'' = -0.091 - **j** 0.246 A

and

$$I_1'' = I_{G1}\frac{R_i}{R_i + R_1 + j*\omega*L} = e^{j30°}\frac{100}{100+20+j*2*\pi*4*10^5*10^{-5}} = 0.8156\,e^{j18.17°}\,A$$

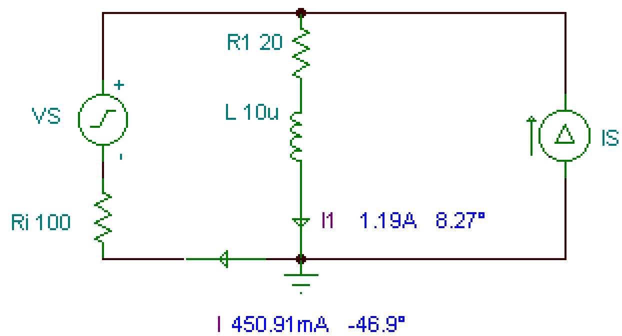
I_1'' = 0.7749 + **j** 0.2545 A

The sum of the two steps:

I = **I'** + **I''** = 0.3082 - **j** 0.3286 = 0.451 $e^{-j\,46.9°}$ A

$\mathbf{I_1} = \mathbf{I_1''} + \mathbf{I_1'}$ = 1.174 + **j** 0.1709 = 1.1865 $e^{j\,8.28°}$ A

These results correspond well with the values calculated by TINA:

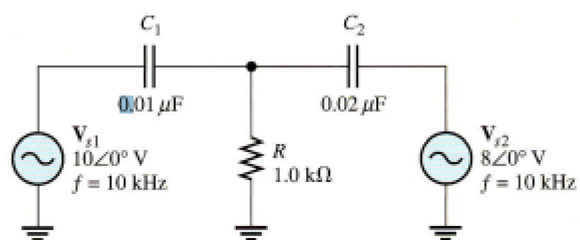


The time functions of the currents:

$$i(t) = 0.451 \cos(\omega \cdot t - 46.9^\circ)\ \text{A}$$

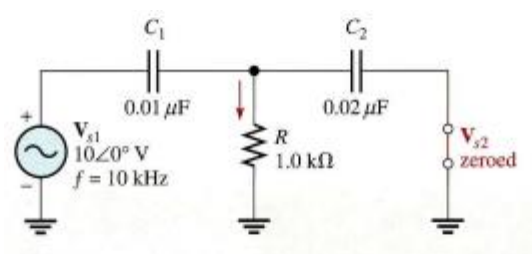
$$i_1(t) = 1.1865 \cos(\omega \cdot t + 8.3^\circ)\ \text{A}$$

Find the current in R using the superposition theorem. Assume the internal source impedances are zero.



Solution

Step 1. Replace V_{s2} with its internal impedance (zero), and find the current in R due to V_{s1}, as indicated in Figure



$$X_{C1} = \frac{1}{2\pi f C_1} = \frac{1}{2\pi(10\ \text{kHz})(0.01\ \mu\text{F})} = 1.59\ \text{k}\Omega$$

$$X_{C2} = \frac{1}{2\pi f C_2} = \frac{1}{2\pi(10\ \text{kHz})(0.02\ \mu\text{F})} = 796\ \Omega$$

Looking from V_{s1}, the impedance is

$$\mathbf{Z} = \mathbf{X}_{C1} + \frac{\mathbf{R}\mathbf{X}_{C2}}{\mathbf{R} + \mathbf{X}_{C2}} = 1.59\angle -90^\circ \text{ k}\Omega + \frac{(1.0\angle 0^\circ \text{ k}\Omega)(796\angle -90^\circ \ \Omega)}{1.0 \text{ k}\Omega - j796 \ \Omega}$$

$$= 1.59\angle -90^\circ \text{ k}\Omega + 622\angle -51.5^\circ \ \Omega$$

$$= -j1.59 \text{ k}\Omega + 387 \ \Omega - j487 \ \Omega = 387 \ \Omega - j2.08 \text{ k}\Omega$$

Converting to polar form yields

$$\mathbf{Z} = 2.12\angle -79.5^\circ \text{ k}\Omega$$

The total current from source 1 is

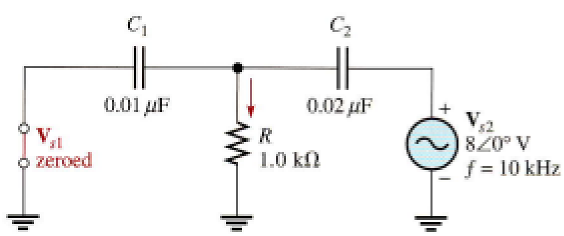
$$\mathbf{I}_{s1} = \frac{\mathbf{V}_{s1}}{\mathbf{Z}} = \frac{10\angle 0^\circ \text{ V}}{2.12\angle -79.5^\circ \text{ k}\Omega} = 4.72\angle 79.5^\circ \text{ mA}$$

Use the current-divider formula. The current through R due to V_{s1} is

$$\mathbf{I}_{R1} = \left(\frac{X_{C2}\angle -90^\circ}{R - jX_{C2}}\right)\mathbf{I}_{s1} = \left(\frac{796\angle -90^\circ \ \Omega}{1.0 \text{ k}\Omega - j796 \ \Omega}\right) 4.72\angle 79.5^\circ \text{ mA}$$

$$= (0.623\angle -51.5^\circ \ \Omega)(4.72\angle 79.5^\circ \text{ mA}) = 2.94\angle 28.0^\circ \text{ mA}$$

Step 2. Find the current in R due to source V_{s2} by replacing V_{s1} with its internal impedance (zero), as shown in Figure 20–3.



Looking from V_{s2}, the impedance is

$$\mathbf{Z} = \mathbf{X}_{C2} + \frac{\mathbf{R}\mathbf{X}_{C1}}{\mathbf{R} + \mathbf{X}_{C1}} = \frac{796\angle -90^\circ \ \Omega + (1.0\angle 0^\circ \text{ k}\Omega)(1.59\angle -90^\circ \text{ k}\Omega)}{1.0 \text{ k}\Omega - j1.59 \text{ k}\Omega}$$

$$= 796\angle -90^\circ \ \Omega + 847\angle -32.2^\circ \ \Omega$$

$$= -j796 \ \Omega + 717 \ \Omega - j451 \ \Omega = 717 \ \Omega - j1247 \ \Omega$$

Converting to polar form yields

$$\mathbf{Z} = 1438\angle -60.1^\circ \ \Omega$$

Use the current-divider formula. The current through R due to V_{s2} is

$$\mathbf{I}_{R2} = \left(\frac{X_{C1}\angle -90^\circ}{R - jX_{C1}}\right)\mathbf{I}_{s2}$$

$$= \left(\frac{1.59\angle -90^\circ \text{ k}\Omega}{1.0 \text{ k}\Omega - j1.59 \text{ k}\Omega}\right)5.56\angle 60.1^\circ \text{ mA} = 4.70\angle 27.9^\circ \text{ mA}$$

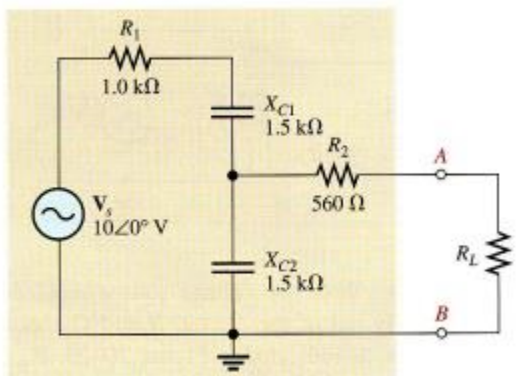
Step 3. Convert the two individual resistor currents to rectangular form and add to get the total current through R.

$$\mathbf{I}_{R1} = 2.94\angle 28.0^\circ \text{ mA} = 2.60 \text{ mA} + j1.38 \text{ mA}$$

$$\mathbf{I}_{R2} = 4.70\angle 27.9^\circ \text{ mA} = 4.15 \text{ mA} + j2.20 \text{ mA}$$

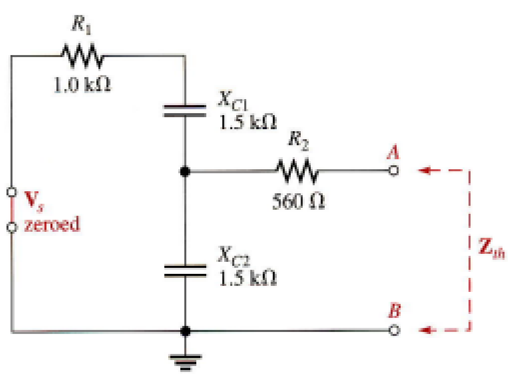
$$\mathbf{I}_R = \mathbf{I}_{R1} + \mathbf{I}_{R2} = 6.75 \text{ mA} + j3.58 \text{ mA} = \mathbf{7.64\angle 27.9^\circ \text{ mA}}$$

For the circuit in Figure, determine Zth, as seen by RL.



Solution:

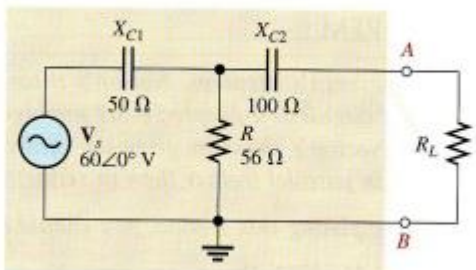
Replace the voltage source with its internal resistance.



Looking from terminals A and B, C_2 appears in parallel with the series combination of R_1 and C_1. This entire combination is in series with R_2. The calculation for $\mathbf{Z}_{th}$ is as follows:

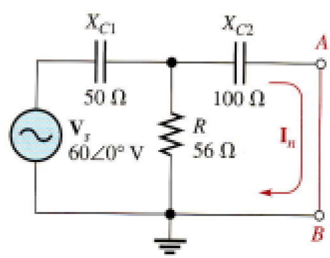
$$\mathbf{Z}_{th} = R_2\angle 0° + \frac{(X_{C2}\angle -90°)(R_1 - jX_{C1})}{R_1 - jX_{C1} - jX_{C2}}$$
$$= 560\angle 0°\ \Omega + \frac{(1.5\angle -90°\ \text{k}\Omega)(1.0\ \text{k}\Omega - j1.5\ \text{k}\Omega)}{1.0\ \text{k}\Omega - j3\ \text{k}\Omega}$$
$$= 560\angle 0°\ \Omega + \frac{(1.5\angle -90°\ \text{k}\Omega)(1.8\angle -56.3°\ \text{k}\Omega)}{3.16\angle -71.6°\ \text{k}\Omega}$$
$$= 560\angle 0°\ \Omega + 854\angle -74.7°\ \Omega = 560\ \Omega + 225\ \Omega - j824\ \Omega$$
$$= 785\ \Omega - j824\ \Omega = \mathbf{1138\angle -46.4°\ \Omega}$$

Using Norton's theorem, determine the current through R_L.



Solution:

Short the terminal AB as shown below.



$\mathbf{I}_n$ is the current through the short and is calculated as follows. First, the total impedance viewed from the source is

$$\mathbf{Z} = \mathbf{X}_{C1} + \frac{\mathbf{R}\mathbf{X}_{C2}}{\mathbf{R} + \mathbf{X}_{C2}} = 50\angle -90°\ \Omega + \frac{(56\angle 0°\ \Omega)(100\angle -90°\ \Omega)}{56\ \Omega - j100\ \Omega}$$
$$= 50\angle -90°\ \Omega + 48.9\angle -29.3°\ \Omega$$
$$= -j50\ \Omega + 42.6\ \Omega - j23.9\ \Omega = 42.6\ \Omega - j73.9\ \Omega$$

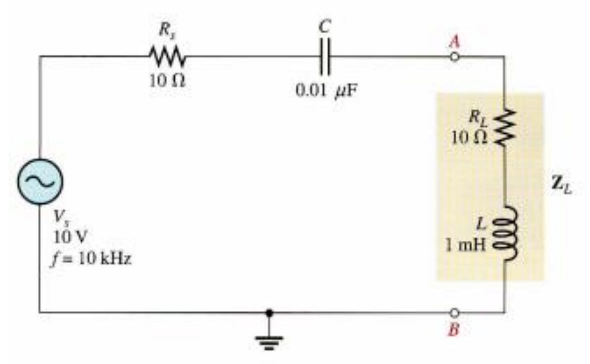
Converting to polar form yields

$$\mathbf{Z} = 85.3\angle -60.0°\ \Omega$$

Next, the total current from the source is

$$\mathbf{I}_s = \frac{\mathbf{V}_s}{\mathbf{Z}} = \frac{60\angle 0°\ \text{V}}{85.3\angle -60.0°\ \Omega} = 703\angle 60.0°\ \text{mA}$$

In the following circuit, calculate the power delivered to the load for each of the following frequencies 10 kHz, 30 kHz, 50 kHz, 80 kHz, and 100 kHz.



Solution:

For $f = 10$ kHz,

$$X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi(10\text{ kHz})(0.01\ \mu\text{F})} = 1.59\text{ k}\Omega$$

$$X_L = 2\pi fL = 2\pi(10\text{ kHz})(1\text{ mH}) = 62.8\ \Omega$$

The magnitude of the total impedance is

$$Z_{tot} = \sqrt{(R_s + R_L)^2 + (X_L - X_C)^2} = \sqrt{(20\ \Omega)^2 + (1.53\text{ k}\Omega)^2} = 1.53\text{ k}\Omega$$

The current is

$$I = \frac{V_s}{Z_{tot}} = \frac{10\text{ V}}{1.53\text{ k}\Omega} = 6.54\text{ mA}$$

The load power is

$$P_L = I^2R_L = (6.54\text{ mA})^2(10\ \Omega) = \mathbf{428\ \mu W}$$

For $f = 30$ kHz,

$$X_C = \frac{1}{2\pi(30\text{ kHz})(0.01\ \mu\text{F})} = 531\ \Omega$$

$$X_L = 2\pi(30\text{ kHz})(1\text{ mH}) = 189\ \Omega$$

$$Z_{tot} = \sqrt{(20\ \Omega)^2 + (342\ \Omega)^2} = 343\ \Omega$$

$$I = \frac{V_s}{Z_{tot}} = \frac{10\text{ V}}{343\ \Omega} = 29.2\text{ mA}$$

$$P_L = I^2R_L = (29.2\text{ mA})^2(10\ \Omega) = \mathbf{8.53\ mW}$$

For $f = 50$ kHz,

$$X_C = \frac{1}{2\pi(50\text{ kHz})(0.01\ \mu\text{F})} = 318\ \Omega$$
$$X_L = 2\pi(50\text{ kHz})(1\text{ mH}) = 314\ \Omega$$

Note that X_C and X_L are very close to being equal which makes the impedances approximately complex conjugates. The exact frequency at which $X_L = X_C$ is 50.3 kHz.

$$Z_{tot} = \sqrt{(20\ \Omega)^2 + (4\ \Omega)^2} = 20.4\ \Omega$$
$$I = \frac{V_s}{Z_{tot}} = \frac{10\text{ V}}{20.4\ \Omega} = 490\text{ mA}$$
$$P_L = I^2R_L = (490\text{ mA})^2(10\ \Omega) = \mathbf{2.40\ W}$$

For $f = 80$ kHz,

$$X_C = \frac{1}{2\pi(80\text{ kHz})(0.01\ \mu\text{F})} = 199\ \Omega$$
$$X_L = 2\pi(80\text{ kHz})(1\text{ mH}) = 503\ \Omega$$
$$Z_{tot} = \sqrt{(20\ \Omega)^2 + (304\ \Omega)^2} = 305\ \Omega$$
$$I = \frac{V_s}{Z_{tot}} = \frac{10\text{ V}}{305\ \Omega} = 32.8\text{ mA}$$
$$P_L = I^2R_L = (32.8\text{ mA})^2(10\ \Omega) = \mathbf{10.8\ mW}$$

For $f = 100$ kHz,

$$X_C = \frac{1}{2\pi(100\text{ kHz})(0.01\ \mu\text{F})} = 159\ \Omega$$
$$X_L = 2\pi(100\text{ kHz})(1\text{ mH}) = 628\ \Omega$$
$$Z_{tot} = \sqrt{(20\ \Omega)^2 + (469\ \Omega)^2} = 469\ \Omega$$

$$I = \frac{V_s}{Z_{tot}} = \frac{10\text{ V}}{469\ \Omega} = 21.3\text{ mA}$$
$$P_L = I^2R_L = (21.3\text{ mA})^2(10\ \Omega) = \mathbf{4.54\ mW}$$

UNIT- IIITransient Analysis

Preliminary definitions:

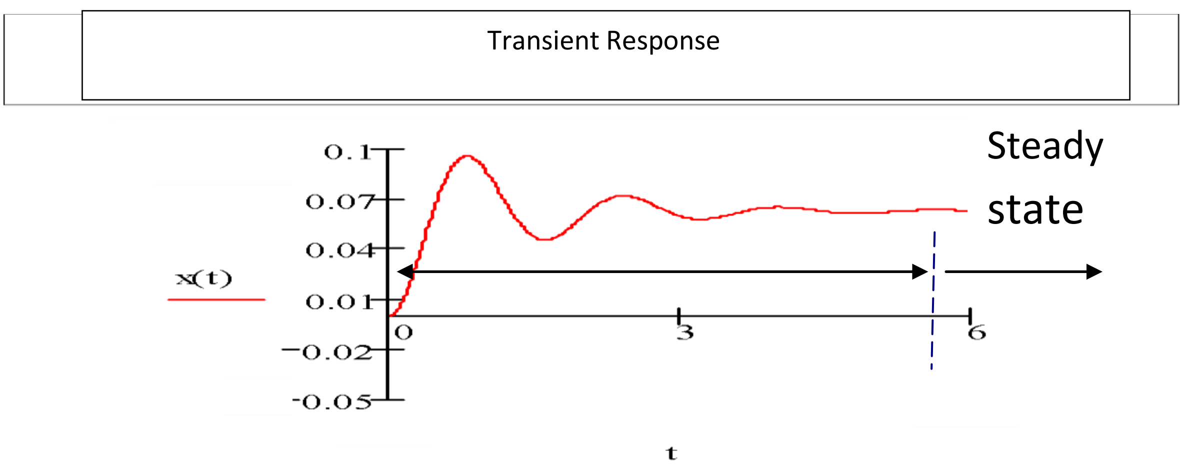
Total Response = natural response + forced response

Natural response: solution of equation of motion of the system when the excitation is zero. The expression for natural response contains constants.

***Forced response*:** any solution of equation of motion of the system for non zero excitation.

If the natural response tends to zero when time tends to infinity and the limit of the forced response as time goes to infinity exists and is bounded (not infinite), then the limit is called *steady state response*.

***Transient response*:** Processof going from initial state to steady state.



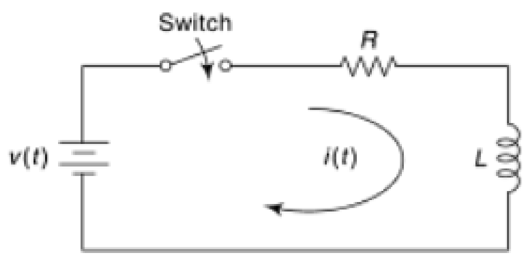
Transient response is due to both the application of the force and the non zero initial conditions

Transient response of RC, RL and RLC circuits to excitation by DC and exponential sources

The RL Series Circuit

The voltage as a function of time across an inductor in an RL series circuit is observed on an oscilloscope and compared to the theoretically calculated plot when the parameters of the circuit are known. When a square wave generator is connected to an inductor and resistor in series, the circuit looks as shown in Figure 1. The inductor in the circuit has an inductance L and resistance RL, the generator has an output resistanceR_G,and the additional resistance from a resistance box is *R*.The square wave generator acts like a battery switching into the circuit with a voltage 6 then shorting out periodically.

RL Series Circuit with Step Input We consider an *RL* series circuit as shown in the figure.



R-L series circuit

If the switch is closed at time $t = 0$, the voltage across the *RL* combination would be $v(t)$ which is a step of magnitude V [or $Vu(t)$] and not a constant as is the supply voltage V.

$$v(t) = 0, \text{ for } t \le 0$$
$$= V, \text{ for } t \ge 0$$

Thus the differential equation governing the behaviour of the circuit would be

$$Ri(t) + L\frac{di(t)}{dt} = Vu(t)$$

Taking Laplace transform, we get

$$RI(s) + L[sI(s) - i(0-)] = \frac{V}{s}$$

or,
$$I(s) = \frac{\frac{V}{L}}{s\left(s+\frac{R}{L}\right)} + \frac{i(0-)}{s+\frac{R}{L}} = \frac{V}{R}\left(\frac{1}{s} - \frac{1}{s+\frac{R}{L}}\right) + \frac{i(0-)}{s+\frac{R}{L}}$$

Taking inverse Laplace transform,

$$i(t) = \frac{V}{R}\left(1 - e^{-\left(\frac{R}{L}\right)t}\right) + i(0-)\,e^{-\left(\frac{R}{L}\right)t} = \frac{V}{R}\left(1 - e^{-\left(\frac{R}{L}\right)t}\right) \text{ with } i(0-) = 0.$$

The transient part of the current response, $i_{tr} = [i(t) - i_s] = -\frac{V}{R}e^{-\frac{R}{L}t}$

From the current equation at $t = \tau = \frac{L}{R}$, $i = \frac{V}{R}(1 - e^{-1}) = 0.63\frac{V}{R} = 0.63i_s$

When the switch is first closed, the voltage across the inductor will immediately jump to battery voltage (acting as though it were an open-circuit) and decay down to zero over time (eventually acting as though it were a short-circuit). Voltage across the inductor is determined by calculating how much voltage is being dropped across R, given the current through the inductor, and subtracting that voltage value from the battery. When the switch is first closed, the current is zero, then it increases over time until it is equal to the battery voltage divided by the series resistance. This behavior is precisely opposite that of the series resistor-capacitor circuit, where current started at a maximum and capacitor voltage at zero.

The steady state part of the current response, $i_s = \frac{V}{R}$

The variation of the current is shown in Figure 6.12.

The quantity $\tau = \frac{L}{R}$ is known as the Time-constant of the circuit and it is defined as follows.

Definitions of Time-constant (τ)

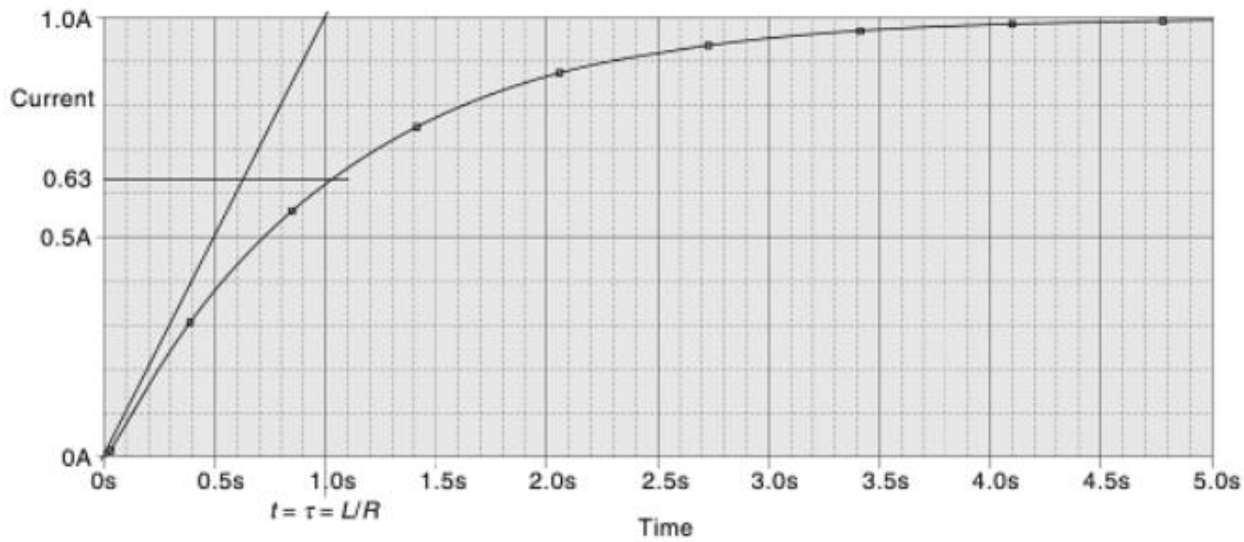
1. It is the time taken for the current to reach 63% of its final value. Thus, it is a measure of the rapidity with which the steady state is reached.
   Also, at $t = 5\tau$, $i = 0.993i_s$; the transient is therefore, said to be practically disappeared in five time constants.
2. The tangent to the equation $i = \frac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right)$ at $t = 0$, intersects the straight line, $i = \frac{V}{R}$ at $t = \tau = \frac{L}{R}$. Thus, time-constant is the time in which steady state would be reached if the current increases at the initial rate.

Physically, time-constant represents the speed of the response of a circuit. A low value of time-constant represents a fast response and a high value of time-constant represents a sluggish response.

Calculations of the Voltage Across Elements

Voltage across the resistor, $V_R = Ri(t) = V\left(1 - e^{-\frac{R}{L}t}\right)$

Voltage across the inductor, $V_L = L\frac{di(t)}{dt} = L\frac{d}{dt}\left[\frac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right)\right] = Ve^{-\frac{R}{L}t}$

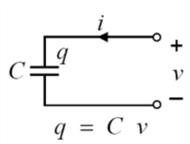


Variation of current with time RL series circuit with step input

First-Order RC Circuits

- Used for filtering signal by blocking certain frequencies and passing others. e.g. low-pass filter
- Any circuit with a single energy storage element, an arbitrary number of sources and an arbitrary number of resistors is a circuit of order 1.
- Any voltage or current in such a circuit is the solution to a 1st order differential equation.

Ideal Linear Capacitor



$$i(t) = \frac{dq}{dt} = c\frac{dv}{dt} \qquad v_c(t+) = v_C(t\ \ )$$

Energy stored $w = \int pdt = \int cvdv = \frac{1}{2}cv^2$

A capacitor is an energy storage device → memory device.

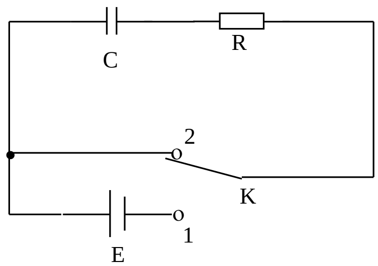
RC DECAY

$$v_c + i_c R = 0$$

$$i_c = C\frac{dv_c}{dt}$$

$$\rightarrow \quad v_c + RC\frac{dv_c}{dt} = 0$$

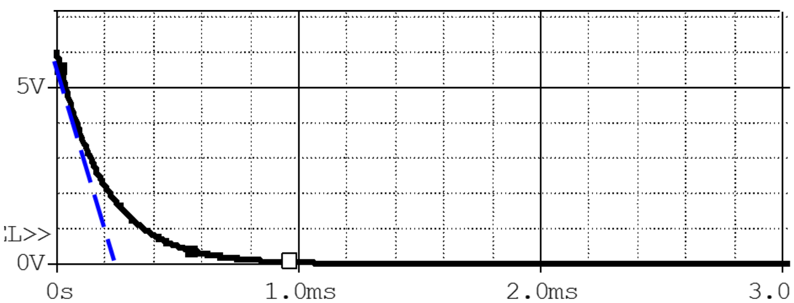
$$v_c = Ae^{-\frac{t}{RC}}$$



Initial condition $v_C(0+) = v_C(0-) = E$

$$\rightarrow \quad v_c = Ee^{-t/RC} = Ee^{-t/\tau}$$

$$i_c = -\frac{E}{R}e^{-t/\tau}$$



R=2k

C=0.1μF

$$v_C(t) = Ee^{-\frac{t}{RC}} = Ee^{-\frac{t}{\tau}}$$

$$\left.\frac{dv_C}{dt}\right|_{t=0} = -\frac{E}{\tau}$$

$$\tau = -\frac{E}{\left.\frac{dv_C}{dt}\right|_{t=0}}$$

Summary

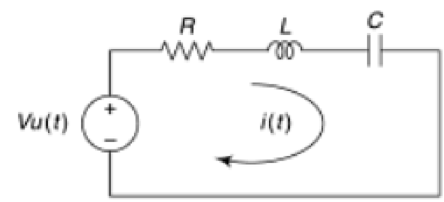
		Initial Value ($t=0$)	Steady Value ($t\to\infty$)	time constant τ
RL Circuits	Source (0 state)	$i_0=0$	$i_L=\frac{E}{R}$	L/R
	Source-free (0 input)	$i_0=\frac{E}{R}$	$i=0$	L/R
RC Circuits	Source (0 state)	$v_0=0$	$v=E$	RC
	Source-free (0 input)	$v_0=E$	$v=0$	RC

RLC Series Circuit with Step Input With zero initial conditions, the Kirchhoff's voltage law equation becomes,

$$Ri(t)+L\frac{di(t)}{dt}+\frac{1}{C}\int_0^t i(t)dt=Vu(t)$$

or
$$RI(s)+sLI(s)+\frac{1}{Cs}I(s)=\frac{V}{s}$$

or
$$I(s)=\frac{\frac{V}{L}}{s^2+\frac{R}{L}s+\frac{1}{LC}}$$



RLC series circuit

The roots of the denominator polynomial of equation are,

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

or $$s_1 = -\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \quad \text{and,} \quad s_2 = -\frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$$

Let $$\omega_0 = \frac{1}{\sqrt{LC}} \quad \text{and} \quad \xi\omega_0 = \frac{R}{2L} \text{ i.e. } \xi = \frac{R}{2}\sqrt{\frac{C}{L}} = \text{Damping Ratio}$$

Then, $$s_1 = -\xi\omega_0 + \omega_0\sqrt{\xi^2 - 1} \quad \text{and} \quad s_2 = -\xi\omega_0 - \omega_0\sqrt{\xi^2 - 1}$$

So, $$I(s) = \frac{\frac{V}{L}}{(s - s_1)(s - s_2)} = \frac{A}{s - s_1} + \frac{B}{s - s_2}$$

$\therefore$ $$A = (s - s_1)\left.\frac{\frac{V}{L}}{(s - s_1)(s - s_2)}\right|_{s = s_1} = \frac{\frac{V}{L}}{(s_1 - s_2)} = \frac{V}{2\omega_0 L\sqrt{\xi^2 - 1}}$$

and, therefore $$B = (s - s_2)\left.\frac{\frac{V}{L}}{(s - s_1)(s - s_2)}\right|_{s = s_2} = \frac{\frac{V}{L}}{(s_2 - s_1)} = -\frac{V}{2\omega_0 L\sqrt{\xi^2 - 1}}$$

Putting these values of A and B, we get,

$$I(s) = \frac{V}{2\omega_0 L\sqrt{\xi^2 - 1}}\left[\frac{1}{s - s_1} - \frac{1}{s - s_2}\right]$$

Taking inverse Laplace transform,

$$i(t) = \frac{V}{2\omega_0 L\sqrt{\xi^2 - 1}}[e^{s_1 t} - e^{s_2 t}] = \frac{V}{2\omega_0 L\sqrt{\xi^2 - 1}} e^{-\xi\omega_0 t}[e^{(\omega_0\sqrt{\xi^2 - 1})t} - e^{-(\omega_0\sqrt{\xi^2 - 1})t}]$$

Depending upon the values of R, L and C, three cases may appear:

(a) $\frac{R}{2L} > \frac{1}{\sqrt{LC}}$ (Overdamped condition)

(b) $\frac{R}{2L} < \frac{1}{\sqrt{LC}}$ (Underdamped condition)

(c) $\frac{R}{2L} = \frac{1}{\sqrt{LC}}$ (Critically Damped condition)

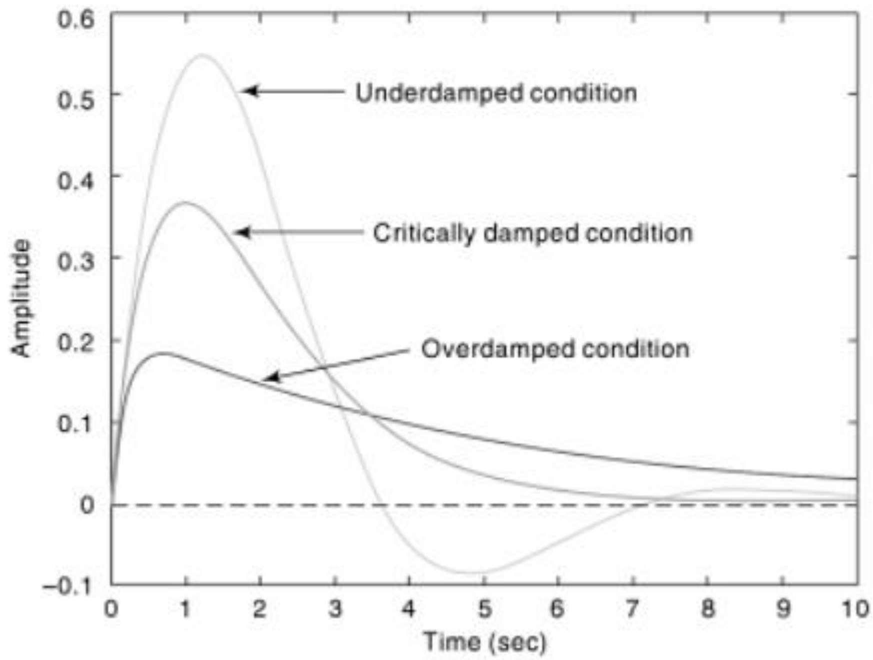
A. Overdamped Condition The condition is, $\frac{R}{2L} > \frac{1}{\sqrt{LC}}$ or, $\xi > 1$ or $Q < \frac{1}{2}$

$\left(\text{Since, Quality Factor, } Q = \frac{\omega_0 L}{R} \text{ and } \omega_0 = \frac{1}{\sqrt{LC}}\right)$

Under this condition, the current becomes,

$$i(t) = \frac{V}{2\omega_0 L\sqrt{\xi^2 - 1}} e^{-\xi\omega_0 t}\left[e^{(\omega_0\sqrt{\xi^2-1})t} - e^{-(\omega_0\sqrt{\xi^2-1})t}\right] = \frac{V}{\omega_0 L\sqrt{\xi^2-1}} e^{-\xi\omega_0 t} \sinh(\omega_0\sqrt{\xi^2-1})t$$

The graphical plot for the current is shown in Figure



Current response in RLC series circuit for three different damping conditions

B. Critically Damped Condition The condition is, $\frac{R}{2L} = \frac{1}{\sqrt{LC}}$ or, $\xi = 1$ or $Q = \frac{1}{2}$

From equation (6.1),

$$I(s) = \frac{\frac{V}{L}}{s^2 + 2\omega_0 s + \omega_0^2} = \frac{V}{L}\left(\frac{1}{(s+\omega_0)^2}\right)$$

Taking inverse Laplace transform,

$$i(t) = \frac{V}{L} t e^{-\omega_0 t}$$

The graphical plot for the current is shown in Figure

Underdamped Condition The condition is, $\frac{R}{2L} < \frac{1}{\sqrt{LC}}$ or, $\xi < 1$ or $Q > \frac{1}{2}$

So, the current becomes,

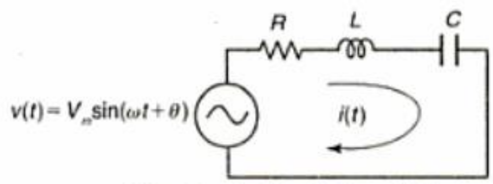
$$i(t) = \frac{V}{2\omega_0 L\sqrt{\xi^2 - 1}} e^{-\xi\omega_0 t}\left[e^{(\omega_0\sqrt{\xi^2-1})t} - e^{-(\omega_0\sqrt{\xi^2-1})t}\right]$$

$$= \frac{V}{\omega_0 L\sqrt{1-\xi^2}} e^{-\xi\omega_0 t}\left[\frac{e^{\left(j\omega_0\sqrt{1-\xi^2}\right)t} - e^{-\left(j\omega_0\sqrt{1-\xi^2}\right)t}}{2j}\right]$$

$$= \frac{V}{\omega_0 L\sqrt{1-\xi^2}} e^{-\xi\omega_0 t}\sin(\omega_0\sqrt{1-\xi^2})t$$

So, the circuit is oscillatory. When $R = 0$, $\xi = 0$, the oscillations are undamped or sustained. The frequency of the undamped oscillation (ω_0) is known as *undamped natural frequency*.

RLC Series Circuit with Sinusoidal Input Sinusoidal voltage $v(t) = V_m \sin(\omega t + \theta)$ is applied to a series RLC circuit at time $t = 0$. We want to find the complete solution for the current $i(t)$ using Laplace transform method.



RLC series circuit with sinusoidal input

By KVL, $Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int_{-\infty}^{t} i(t)dt = V_m \sin(\omega t + \theta)$

Taking Laplace transform with zero initial conditions,

$$I(s)\left[R + sL + \frac{1}{Cs}\right] = V_m \frac{(s\sin\theta + \omega\cos\theta)}{s^2 + \omega^2}$$

or,

$$I(s) = \frac{V_m s(s\sin\theta + \omega\cos\theta)}{L(s^2+\omega^2)\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)} = \frac{V_m}{L}\frac{s(s\sin\theta + \omega\cos\theta)}{(s+j\omega)(s-j\omega)(s-s_1)(s-s_2)}$$

where, s_1, s_2 are the roots of the quadratic equation:

$$\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right) = 0$$

Thus, $s_1 = -\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$ and, $s_2 = -\frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$

Now, let $\dfrac{s(s\sin\theta+\omega\cos\theta)}{(s+j\omega)(s-j\omega)(s-s_1)(s-s_2)}=\dfrac{K_1}{s-s_1}+\dfrac{K_2}{s-s_2}+\dfrac{K_3}{s+j\omega}+\dfrac{K_4}{s-j\omega}$

So, by residue method, multiplying by $(s-s_1)$ and putting $s = s_1$,

$$K_1=\frac{s_1(s_1\sin\theta+\omega\cos\theta)}{(s_1+j\omega)(s_1-j\omega)(s_1-s_2)} \quad \text{and} \quad K_2=\frac{s_2(s_2\sin\theta+\omega\cos\theta)}{(s_2+j\omega)(s_2-j\omega)(s_2-s_1)}$$

Similarly, multiplying by $(s+j\omega)$ and putting $s=-j\omega$,

$$K_3=\frac{-j\omega(-j\omega\sin\theta+\omega\cos\theta)}{(-j\omega-j\omega)(-j\omega-s_1)(-j\omega-s_2)}=\frac{\omega(\cos\theta-j\sin\theta)}{2(s_1+j\omega)(s_2+j\omega)}$$

and,

$$K_4=\frac{j\omega(-\omega\sin\theta+\omega\cos\theta)}{(j\omega+j\omega)(j\omega-s_1)(j\omega-s_2)}=\frac{\omega(\cos\theta+j\sin\theta)}{2(s_1-j\omega)(s_2-j\omega)}$$

Hence the current response becomes,

$$i(t)=\frac{V_m}{L}\left[K_1e^{s_1t}+K_2e^{s_2t}\right]+\frac{V}{L}\left[K_3e^{-j\omega t}+K_4e^{j\omega t}\right]=I_{tr}+I_{ss}$$

Thus, the *transient part of the total current* is

$$I_{tr}=\frac{V_m}{L}\left[\frac{s_1(s_1\sin\theta+\omega\cos\theta)}{(s_1^2+\omega^2)\sqrt{\frac{R^2}{L^2}-\frac{4}{LC}}}e^{s_1t}-\frac{s_2(s_2\sin\theta+\omega\cos\theta)}{(s_2^2+\omega^2)\sqrt{\frac{R^2}{L^2}-\frac{4}{LC}}}e^{s_2t}\right]$$

The steady-state part of the total current is obtained as follows.

$$I_{ss}=\frac{V_m}{2L}\left[\frac{\omega e^{-j\theta}e^{-j\omega t}}{(s_1+j\omega)(s_2+j\omega)}+\frac{\omega e^{j\theta}e^{j\omega t}}{(s_1-j\omega)(s_2-j\omega)}\right]=\frac{V_m\omega}{2L}\left[\frac{e^{-j(\omega t+\theta)}}{(s_1+j\omega)(s_2+j\omega)}+\frac{e^{j(\omega t+\theta)}}{(s_1-j\omega)(s_2-j\omega)}\right]$$

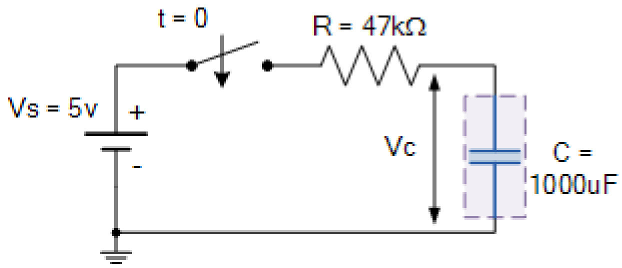
or,

$$I_{ss}=\frac{V_m}{\sqrt{R^2+\left(\omega L-\frac{1}{\omega C}\right)^2}}\sin\left\{\omega t+\theta-\tan^{-1}\left(\frac{\omega L-\frac{1}{\omega C}}{R}\right)\right\}$$

This gives the steady-state current of the series *RLC* circuit to a sinusoidal voltage.

Problems:

Calculate the RC time constant, τ of the following circuit.



The time constant, τ is found using the formula T = R x C in seconds.

Therefore the time constant τ is given as:
T = R x C = 47k x 1000uF = 47 Secs

a) **What value will be the voltage across the capacitor at 0.7 time constants?**

At 0.7 time constants (0.7T) Vc = 0.5Vs. Therefore, Vc = 0.5 x 5V = 2.5V

b) **What value will be the voltage across the capacitor at 1 time constant?**

At 1 time constant (1T) Vc = 0.63Vs. Therefore, Vc = 0.63 x 5V = 3.15V

c) **How long will it take to "fully charge" the capacitor?**

The capacitor will be fully charged at 5 time constants.

1 time constant (1T) = 47 seconds, (from above). Therefore, 5T = 5 x 47 = 235 secs

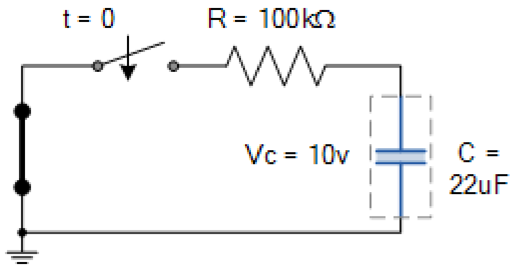
d) **The voltage across the Capacitor after 100 seconds?**

The voltage formula is given as $Vc = V(1 – e^{-t/RC})$

which equals: $Vc = 5(1-e^{-100/47})$ RC = 47 seconds from above, Therefore, Vc = 4.4 volts

We have seen that the charge on a capacitor is given by the expression: Q = CV and that when a voltage is firstly applied to the plates of the capacitor it charges up at a rate determined by its time constant, τ. In the next tutorial we will examine the current-voltage relationship of a discharging capacitor and look at the curves associated with it when the capacitors plates are shorted together.

Calculate the RC time constant, τ of the following RC discharging circuit.



The time constant, τ is found using the formula T = R x C in seconds.

Therefore the time constant τ is given as:
T = R x C = 100k x 22uF = <u>2.2 Seconds</u>

a) **What value will be the voltage across the capacitor at 0.7 time constants?**

At 0.7 time constants (0.7T) Vc = 0.5Vc. Therefore, Vc = 0.5 x 10V = <u>5V</u>

b) **What value will be the voltage across the capacitor after 1 time constant?**

At 1 time constant (1T) Vc = 0.37Vc. Therefore, Vc = 0.37 x 10V = <u>3.7V</u>

c) **How long will it take for the capacitor to "fully discharge" itself, (equals 5 time constants)**

1 time constant (1T) = 2.2 seconds. Therefore, 5T = 5 x 2.2 = <u>11 Seconds</u>

<u>Example</u> A Dc voltage of 100 volts is applied to a series RL circuit with R=25ohm. What is the current in the circuit aat twice the time constant?

E= 100 V

R= 25Ω

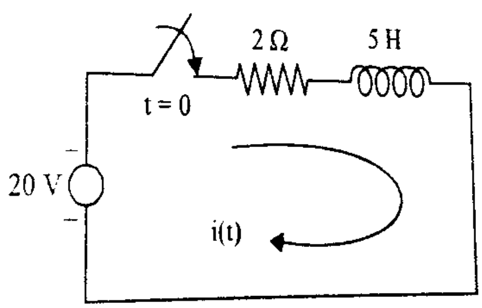
i(t) = $\frac{E}{R}(1 - e^{-\frac{R}{L}t})$

time constant τ= L/R

i(t) = $\frac{E}{R}(1 - e^{-\frac{t}{\tau}})$ Given t= 2τ

therefore i(t) = $\frac{100}{25}(1 - e^{-\frac{2\tau}{\tau}})$ $= 4(1 - e^{-2})$ = **3.45 A**

<u>Example</u> find the expression for transient after switched is closed at t= 0 assuming zero initial conditions



Applying KVL for the Loop

$$2i(t) + 5\frac{di(t)}{dt} = 20$$

Taking Laplace on both sides

$$2I(s) + 5(sI(s) - i(0)) = \frac{20}{s}$$

Since i(0) = 0
We have

$$2I(s) + 5sI(s) = \frac{20}{s}$$

$$(2 + 5s)I(s) = \frac{20}{s}$$

$$I(s) = \frac{20}{s(2+5s)}$$

$$I(s) = \frac{4}{s(s+0.4)}$$

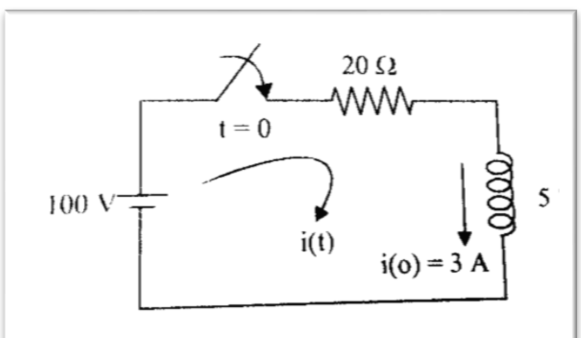
Taking partial fraction

$$I(s) = \frac{4}{s(s+0.4)} = \frac{A}{s} + \frac{B}{s+0.4} = \frac{10}{s} - \frac{10}{s+0.4}$$

$$I(s) = \frac{10}{s} - \frac{10}{s+0.4}$$

Taking inverse we get L^{-1} [I(s)] = **i(t) = 10 – 10 $e^{-0.4t}$**

<u>Example</u> find the expression for transient voltage across R and L after switch is closed at t= 0 assuming initial current through inductor as 3A before it is closed



Applying KVL for the Loop

$$20i(t) + 5\frac{di(t)}{dt} = 100$$

Taking Laplace on both sides

$$20I(s) + 5(sI(s) - i(0)) = \frac{100}{s}$$

Since i(0) = 3A
We have

$$20I(s) + 5sI(s) - 15 = \frac{100}{s}$$

$$(20+5s)I(s)=\frac{100}{s}+15$$

$$I(s)=\frac{100+15s}{5s(4+s)}$$

$$I(s)=\frac{20+3s}{s(s+4)}$$

Taking partial fraction

$$I(s)=\frac{20+3s}{s(s+4)} \;=\frac{A}{s}+\frac{B}{s+4}= \;\frac{5}{s}-\frac{2}{s+4}$$

$$I(s)=\frac{5}{s}-\frac{2}{s+4}$$

Taking inverse we get L^{-1} [I(s)] = **i(t) = 5 – 2 e^{-4t} Amps**

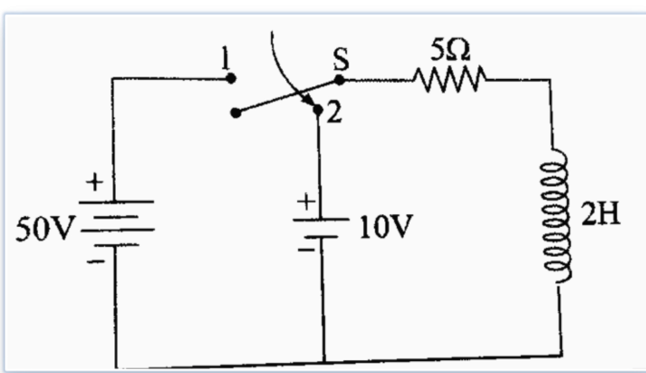
Voltage across Resistor E_R= 20 x (5 – 2 e^{-4t} **)** = 100 – 40 e^{-4t} **Volts**

Voltage across Inductor e_L = L di/dt

$$e_L=5\,\frac{d}{dt}(5-2\,e^{-4t})$$

$$\boldsymbol{e_L=40\;e^{-4t}\;Volts}$$

Example : In the Circuit shown below switch S is in Position 1 for a long time and brought to position 2 at time t = 0 . determine the circuit current.



After closing the switch to position 2 and applying the KVL equation

$$5i(t)+2\frac{di(t)}{dt}=10$$

Taking Laplace on both sides

$$5I(s) + 2(sI(s) - i(0)) = \frac{10}{s}$$

I(0) is the initial current in L. Since inductor does not allow sudden change in current it's the steady state current flowing before switch comes to position 2.
i.e. i(0) = 50 /5 = 10 A

therefore we get

$$5I(s) + 2sI(s) - 2x10 = \frac{10}{s}$$

$$(2s + 5)I(s) = \frac{10}{s} + 20$$

$$I(s) = \frac{20s + 10}{s(2s + 5)}$$

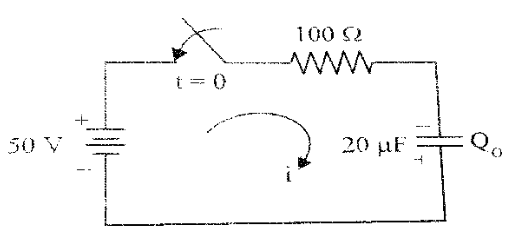
Taking partial fraction

$$I(s) = \frac{20s + 10}{s(2s + 5)} = \frac{A}{s} + \frac{B}{2s + 5} = \frac{2}{s} + \frac{8}{s + 2.5}$$

$$I(s) = \frac{2}{s} + \frac{8}{s + 2.5}$$

Taking inverse we get L^{-1} [I(s)] = **i(t) = 2+8 $e^{-2.5t}$ Amps**

Example the 20 uF capacitor in the circuit has an intial charge of Q=0.001 C as shown . the switch is closed at t=0. Find the transient.



The differential equation of the circuit is given by

$$100\, i + \frac{1}{C}\int i\, dt - \frac{Q}{C} = 50$$

$$100\, i + \frac{1}{C}\int i\, dt = 50 + \frac{0.001}{20 \times 10^{-6}}$$

$$100\,i + \frac{1}{C}\int i\,dt = 100$$

Taking Laplace on both sides $\quad 100\,I\,(s) + \frac{1}{C}\frac{I(s)}{s} = \frac{100}{s}$

$$(100 + \frac{1}{Cs})I(s) = \frac{100}{s}$$

$$I\,(s) = \frac{1}{s + \frac{1}{100C}}$$

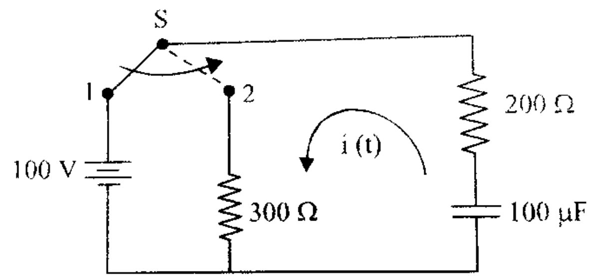
Taking inverse transform

$i(t) = e^{(-1/100C)\,t}$

C=20uf **therefore i(t) = e^{-500t}**

EXAMPLE

Switch moves from position 1 to 2 at t=0. Find the energy dissipated across the two resistors



Applying KVL in the loop after the switch is closed

$$500\,i + \frac{1}{C}\int i\,dt = 100$$

Taking Laplace

$$500\,I(s) + \frac{I(s)}{Cs} = 100\,/s$$

$$(500 + \frac{1}{Cs})\, I(s) \;= \frac{100}{s}$$

$$I(s) \;= \frac{100}{500s + \frac{1}{C}}$$

$$I(s) \;= \frac{0.2}{s + 20}$$

Taking inverse we get i(t) = $0.2e^{-20t}$

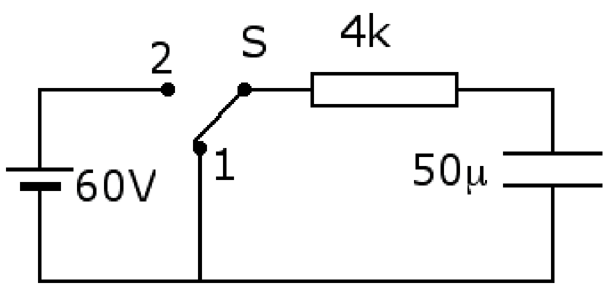
The energy dissipated in the resistors

$$E = \int_0^{\infty} i^2 R dt$$

$E = \int_0^{\infty} (0.2e^{-20t})^2 500\, dt = \int_0^{\infty} 20e^{-40t} \; dt$ = 0.5 J

EXAMPLE

For the circuit shown below, find the charge on the capacitor and the current in the circuit 0.03 s after the switch is closed.



$$\tau = RC = 10 \times 10^{3} \times 5 \times 10^{-6} = 0.05 \text{ s}$$

$$q(t) = CV\left(1 - e^{-\frac{t}{\tau}}\right)$$

$$= 5 \times 10^{-6} \times 20 \times \left(1 - e^{-\frac{t}{0.05}}\right)$$

$$q(0.3s) = 0.1 \times 10^{-3} \times \left(1 - e^{-\frac{0.03}{0.05}}\right)$$

$$= 0.1 \times 10^{-3} \times (1 - 0.55)$$

$$= 45 \times 10^{-6} \text{ C}$$

$$= 45 \; \mu\text{C}$$

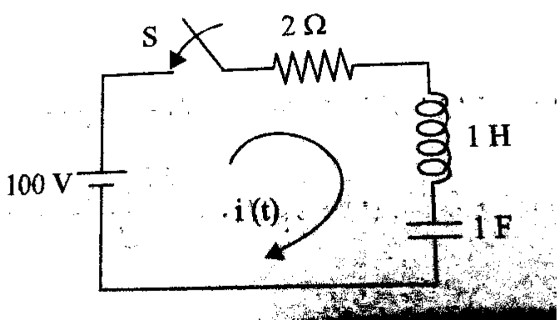
$$i = \frac{V}{R} e^{-\frac{t}{\tau}}$$

$$= \frac{20}{10 \times 10^3} \cdot e^{-\frac{0.03}{0.05}}$$

$$= 1.1 \times 10^{-3} \text{ A}$$

$$= 1.1 \text{ mA}$$

<u>EXAMPLE</u> find the current in the circuit when the switch is closed at t= 0.



Applying KVL

$$2\,i + \frac{di}{dt} + \frac{1}{C}\int i\,dt = 100$$

Taking Laplace Transform $2I(s) + sI(s) + \frac{I(s)}{s} = \frac{100}{s}$

$$I(s)(2 + s + \frac{1}{s}) = \frac{100}{s}$$

$$I(s) = \frac{100}{s^2 + 2s + 1}$$

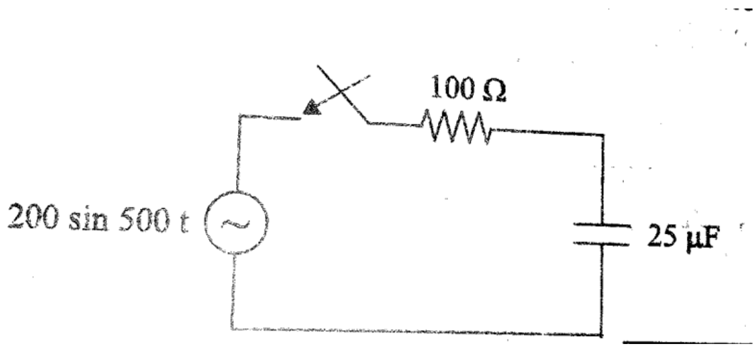
$$I(s) = \frac{100}{(s+1)^2}$$

Taking inverse on both sides , we get

i(t) = 100 t e^{-t} Amps

<u>Example</u>

Find the current i(t)   assuming no initial charges



Applying KVL

$$100\, i + \frac{1}{25u}\int i\, dt = 200 \sin 500t$$

Taking Laplace transform

$$100I(s) + 40000\frac{I(s)}{s} = 200\ \frac{500}{s^2 + 500^2}$$

$$I(s)[100 + \frac{40000}{s}] = \frac{100000}{s^2 + 500^2}$$

$$I(s) = \frac{100000}{(s^2 + 500^2)(100 + \frac{40000}{s})}$$

$$I(s) = \frac{1000\, s}{(s^2 + 500^2)(s + 400)}$$

Taking partial fraction

$$\frac{1000\, s}{(s^2 + 500^2)(s + 400)} = \frac{A}{(s + 400)} + \frac{B}{s + j500} + \frac{C}{s - j500}$$

A= -0.96 B = $\frac{5(4+j5)}{41}$ C = $\frac{5(4-j5)}{41}$

Therefore i(t) = -0.976 e^{-400t}+ 1.546 sin(500t+38.7)

UNIT- IV MAGNETICALLY COUPLED CIRCUITS

Self-Inductance

A current-carrying coil produces a magnetic field that links its own turns. If the current in the coil changes the amount of magnetic flux linking the coil changes and, by Faraday's law, an emf is produced in the coil. This emf is called a *self-induced* emf.

Let the coil have N turns. Assume that the same amount of magnetic flux Φ links each turn of the coil. The net flux linking the coil is then $N\Phi$. This net flux is proportional to the magnetic field, which, in turn, is proportional to the current I in the coil. Thus we can write $N\Phi \propto I$. This proportionality can be turned into an equation by introducing a constant. Call this constant L, the ***self-inductance*** (or simply ***inductance***) of the coil:

$$N\Phi = LI \quad \text{or} \quad L = \frac{N\Phi}{I}$$

As with mutual inductance, the unit of self-inductance is the henry.

The self-induced emf can now be calculated using Faraday's law:

$$\mathscr{E} = -N\frac{\Delta\Phi}{\Delta t} = -\frac{\Delta(N\Phi)}{\Delta t} = -\frac{\Delta(LI)}{\Delta t} = -L\frac{\Delta I}{\Delta t}$$

$$\boxed{\mathscr{E} = -L\frac{\Delta I}{\Delta t}}$$

The above formula is the ***emf due to self-induction***.

Example

Find the formula for the self-inductance of a solenoid of *N* turns, length *l*, and cross-sectional area *A*.

Assume that the solenoid carries a current *I*. Then the magnetic flux in the solenoid is

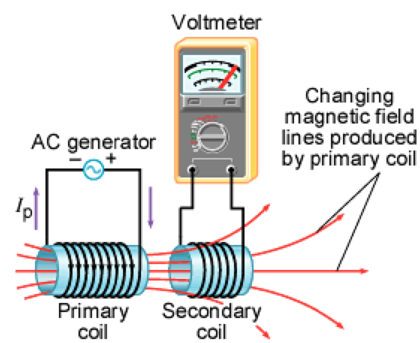
$$\Phi = \mu_0 \frac{NI}{l} A. \ \ L = \frac{N\Phi}{I} = \frac{N}{\cancel{I}}\mu_0\frac{N\cancel{I}}{l}A$$

$$L = \mu_0 \frac{N^2}{l} A \ \text{ or } \ \boxed{L = \mu_0 n^2 A l} \ \text{ where } n = \frac{N}{l}.$$

(Note how L is independent of the current I.)

Mutual Inductance

Suppose we hook up an AC generator to a solenoid so that the wire in the solenoid carries AC. Call this solenoid the *primary coil*. Next place a second solenoid connected to an AC voltmeter near the primary coil so that it is coaxial with the primary coil. Call this second solenoid the *secondary coil*. See the figure at the right.



The alternating current in the primary coil produces an alternating magnetic field whose lines of flux *link* the secondary coil (like thread passing through the eye of a needle). Hence the secondary coil encloses a *changing* magnetic field. By Faraday's law of induction this changing magnetic flux induces an emf in the secondary coil. This effect in which changing current in one circuit induces an emf in another circuit is called ***mutual induction***.

<u>Mutual Inductance</u>

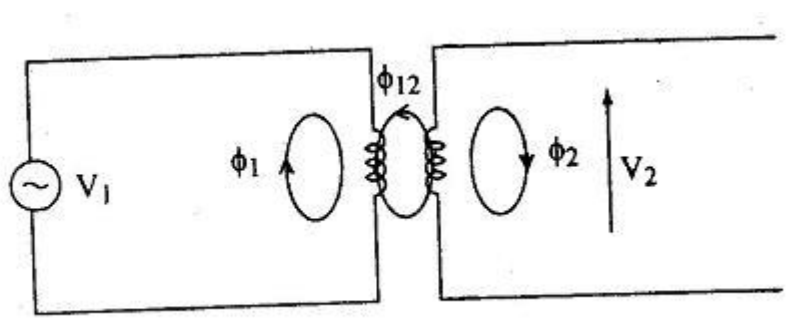


Fig. 8.1.

Consider the circuit shown in fig. 8.1, the changing current produces a variable flux in the first coil. For the purpose of analysis, is divided into two components

Here ϕ_1 is the total flux established by i_1, ϕ_{11} = a part of ϕ_1. It links with coil 1 only but not with coil 2.

ϕ_{12} = it is a part of ϕ_1. It links with both coils 2 and 1.

As, the flux linking with coil 2 changes, an e.m.f. is induced in the coil ϕ_2 and is given by

$$e_2 = N_2 \frac{d\phi_{12}}{dt} \quad \dots (7)$$

Also, e_2 is proportional to time rate of change of i_1. It is because ϕ_{12} is produced by i_1, therefore,

$$e_2 = M \frac{di_1}{dt} \quad \dots (8)$$

From equations (7) and (8), we can write that,

$$M = \frac{N_2 \, d\phi_{12}}{di_1} \quad \dots (9)$$

If the permeability is constant, the above equation becomes

$$M = \frac{N_2 \phi_{12}}{i_1} \quad \dots (10)$$

Suppose that the second coil is connected to a voltage source. Let i2 be the current flow and 42 be the total flux.

$$\phi_2 = \phi_{22} + \phi_{21}$$

$$e_1 = \frac{N_1 \, d\phi_{21}}{dt} \quad \dots (11)$$

$$\text{also } e_1 = M \frac{di_2}{dt} \quad \dots (12)$$

$$\text{Hence } M \frac{di_2}{dt} = N_1 \frac{d\phi_{21}}{dt}$$

$$\text{Hence } M = N_1 \frac{\phi_{21}}{i_2} \quad \dots (13)$$

In equations (10 & 13) M is called mutual inductance.

Definition for Mutual Inductance

The mutual inductance between 2 coils is defined as the weber turns in one coil per ampere current in other coil. It is measured in henrys.

The mutual inductance is also defined as the ability of one coil to produce e.m.f. in other coil by induction when the current in the first changes.

Coefficient of coupling (K) or coefficient of magnetic coupling (KM).

Consider the fig. 8.1, the fraction of the total flux produced by coil 1 linking coil 2 is

$$\frac{\phi_{12}}{\phi_1}$$

It is called coefficient of coupling. Thus

$$K = \frac{\phi_{12}}{\phi_1} \qquad \dots (14)$$

$$\text{Also} \quad K = \frac{\phi_{21}}{\phi_2} \qquad \dots (15)$$

Multiplying equations (10) & (13), we get

$$M^2 = \frac{N_2 \phi_{12}}{i_1} \cdot \frac{N_1 \phi_{21}}{i_2}$$

$$= \frac{N_2 K\phi_1}{i_1} \cdot \frac{N_1 K\phi_2}{i_2}$$

$$= K^2 \left(\frac{N_1 \phi_1}{i_1} \cdot \frac{N_2 \phi_2}{i_2}\right)$$

$$M^2 = K^2 L_1 \cdot I_2 \qquad \dots (16)$$

$$\therefore M = K\sqrt{L_1 L_2}$$

$$K = \frac{M}{\sqrt{L_1 L_2}} \qquad \dots (17)$$

From equation (16), we write that,

$$K^2 = \frac{M^2}{L_1 L_2} = \left(\frac{M}{L_1}\right)\left(\frac{M}{L_2}\right) \qquad \dots (18)$$

From the above expression, we can say that

$\frac{M}{L_1}$, K and $\frac{M}{L_2}$

are in geometric progression.

COUPLING COEFFICIENT

Coupling

Important- voltage is multiplied or divided directly by the transformer ratio, but impedance is multiplied or divided by the ratio squared. Remember that transformers are frequency and level sensitive, and that measurement conditions should match operating conditions for accurate results.

For mutual inductance, measure the inductance of the primary and secondary in series, and then interchange the connections of one winding for a second reading. Apply the equation below:

$$M = \frac{1}{4}\left(L_{series+} - L_{series-}\right)$$

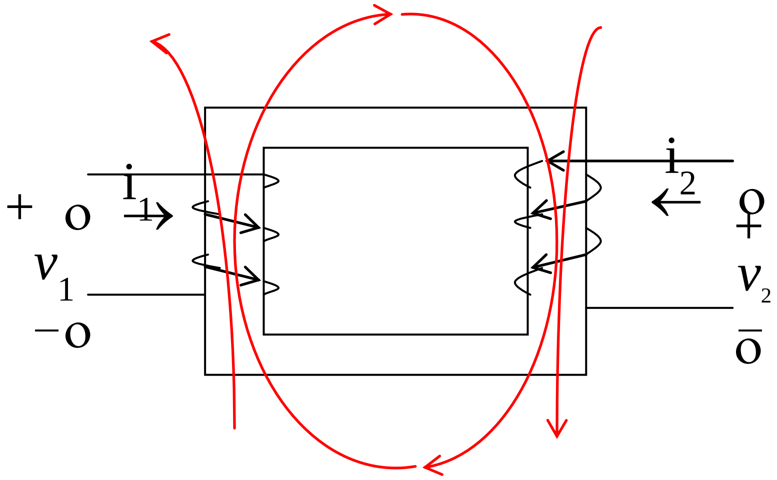
For coupling, measure the primary and secondary separately then apply the equation below:

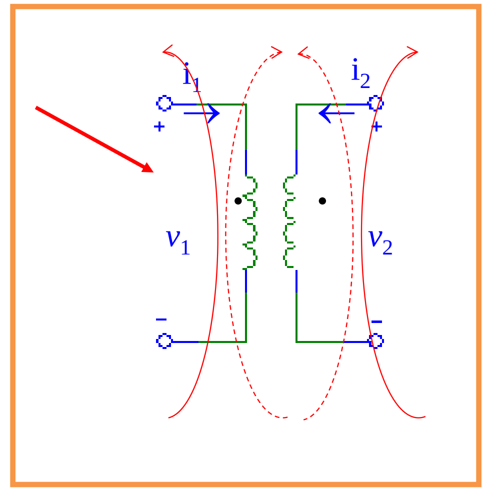
$$k = \frac{M}{\sqrt{L_p L_s}}$$

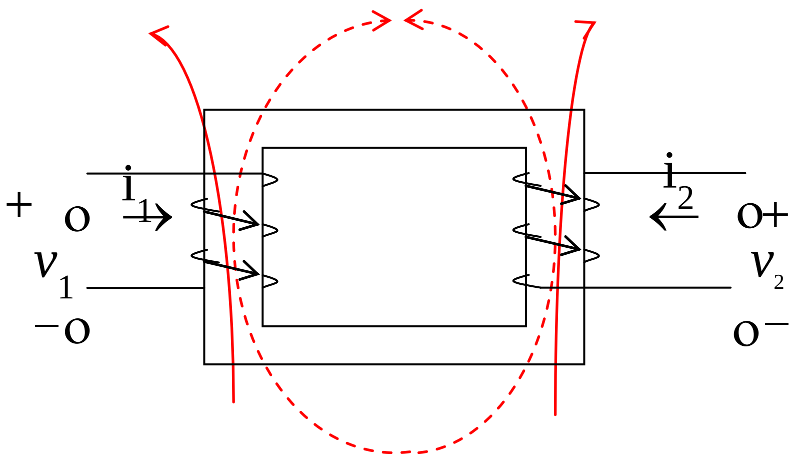
k is the coefficient of coupling, zero to one.

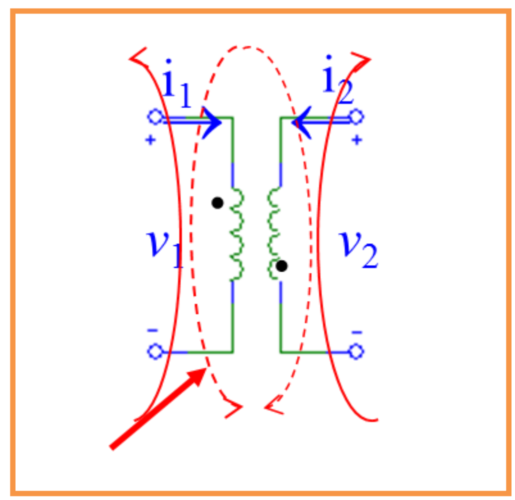
DOT RULE

A current entering the dotted terminal of one coil produces an open circuit voltage with a positive voltage reference at the dotted terminal of the second coil.

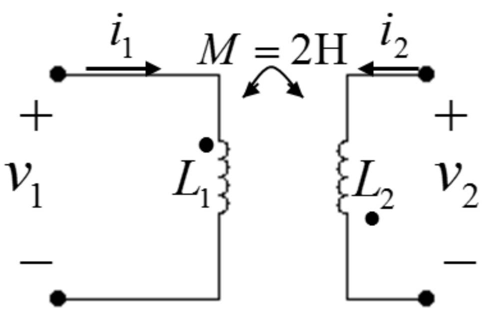








Example : If $i_2 = 5\sin 45t$A and $i_1 = 0$ Apply dot convention , M creates a negative potential at the dot position of the primary mesh



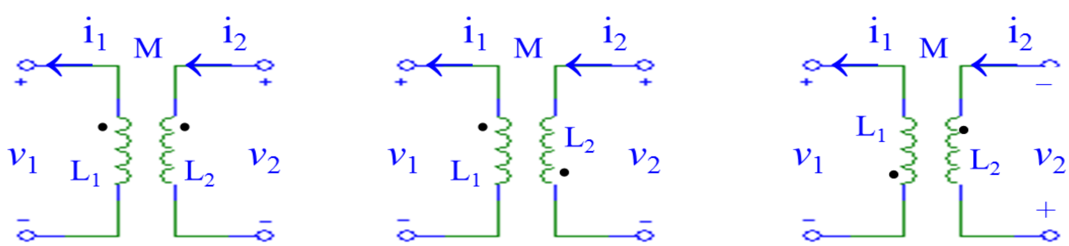
$$v_1 = -M\frac{di_2}{dt} = -2\cdot 5\cdot 45\cdot \cos 45t = -450\cos 45t \text{ V}$$

If $i_1 = -8e^{-t}$A and $i_2 = 0$

Apply dot convention , M creates a negative potential at the dot position of the secondary mesh

$$v_2 = -M\frac{di_1}{dt} = (-2)\cdot(-8)\cdot(-1)e^{-t} = -16e^{-t}$$

For the circuit shown in following figures, determine v_1 and v_2.

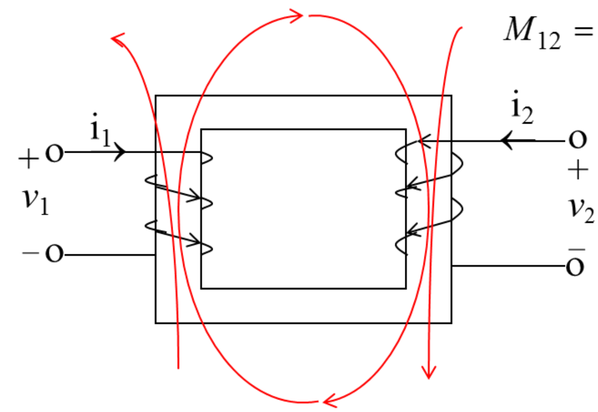


$$v_1 = -L_1\frac{di_1}{dt} + M\frac{di_2}{dt}$$
$$v_2 = L_2\frac{di_2}{dt} - M\frac{di_1}{dt}$$

$$v_1 = -L_1\frac{di_1}{dt} - M\frac{di_2}{dt}$$
$$v_2 = L_2\frac{di_2}{dt} + M\frac{di_1}{dt}$$

$$v_1 = -L_1\frac{di_1}{dt} - M\frac{di_2}{dt}$$
$$v_2 = -L_2\frac{di_2}{dt} - M\frac{di_1}{dt}$$

Coupled Circuitsand *v* ~ *i* relationship



$$M_{12} = M_{21} = M \qquad \Psi_1(t) = L_1 i_1(t) + M_{12} i_2(t)$$

$$\Psi_2(t) = M_{21} i_1(t) + L_2 i_2(t)$$

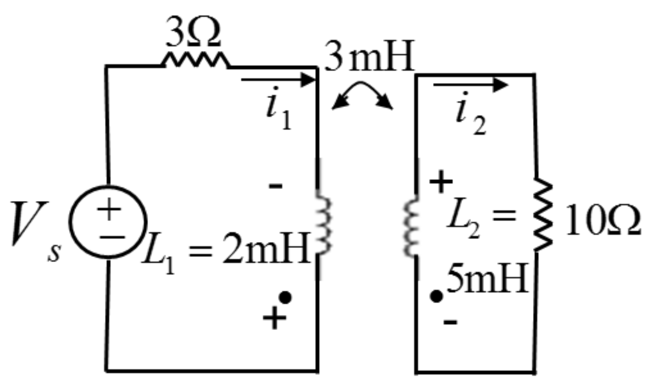
$$\begin{cases} v_1 = \dfrac{d\psi_1}{dt} = L_1 \dfrac{di_1}{dt} + M \dfrac{di_2}{dt} \\ v_2 = \dfrac{d\psi_2}{dt} = M \dfrac{di_1}{dt} + L_2 \dfrac{di_2}{dt} \end{cases}$$

For sinusoidal circuit, $\dot{V}_1 = j\omega L_1 \dot{I}_1 + j\omega M \dot{I}_2$

$\dot{V}_2 = j\omega M \dot{I}_1 + j\omega L_2 \dot{I}_2$

Example 1.

$V_s = 20e^{-1000t}\,\text{V}$



Primary mesh :

$$-V_s + 3\Omega \cdot i_1 + L_1 \frac{di_1}{dt} - M\frac{di_2}{dt} = 0$$

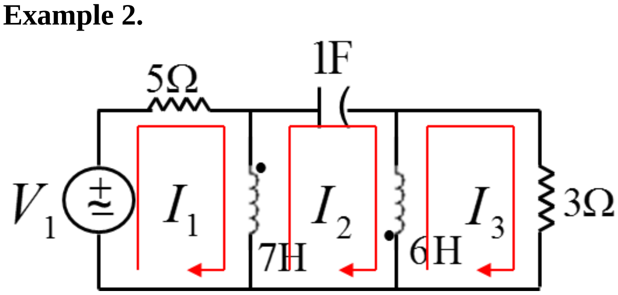
$$\left(-V_s + 3\Omega \cdot \dot{I}_1 + j\omega L_1 \dot{I}_1 - j\omega M \dot{I}_2 = 0\right)$$

Secondary mesh :

$$-M\frac{di_1}{dt} + L_2\frac{di_2}{dt} + 10\Omega \cdot i_2 = 0$$

$$\left(-j\omega M \dot{I}_1 + j\omega L_2 \dot{I}_2 + 10\Omega \cdot \dot{I}_2 = 0\right)$$

Example 2.



Mesh 1 :

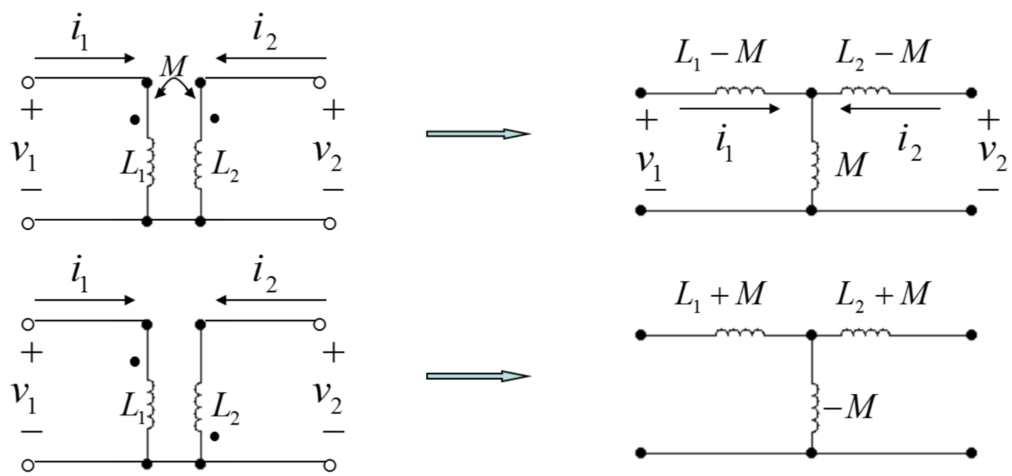
$$-V_1 + 5\cdot\dot{I}_1 + j7\omega\left(\dot{I}_1 - \dot{I}_2\right) + j2\omega\left(\dot{I}_3 - \dot{I}_2\right) = 0$$

Mesh 2 :

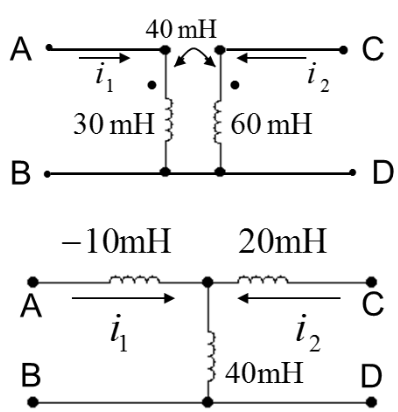
$$j7\omega\left(\dot{I}_2 - \dot{I}_1\right) + j2\omega\left(\dot{I}_2 - \dot{I}_3\right) + \frac{1}{j\omega}\cdot\dot{I}_2 + j6\omega\left(\dot{I}_2 - \dot{I}_3\right) + j2\omega\left(\dot{I}_2 - \dot{I}_1\right) = 0$$

Mesh 3 : $j6\omega\left(\dot{I}_3 - \dot{I}_2\right) + j2\omega\left(\dot{I}_1 - \dot{I}_2\right) + 3\cdot\dot{I}_3 = 0$

Transformer



In the equivalent network, mutual inductance no longer exists. And the dot convention has been removed, and are also treated as self-inductance.



Example 3.

$$L_1 = 30\text{mH},\quad L_2 = 60\text{mH} \quad and \quad M = 40\text{mH}$$

Let $v_1 = 10\cos 100t\ \text{V}$

Apply the original transformer:

$$i_1 = \frac{1}{30\times10^{-3}}\int 10\cos(100t)dt = 3.33\sin 100t \text{ A}$$

$$v_2 = M\frac{di_1}{dt} = 40\times10^{-3}\times3.33\times100\cos 100t$$

$$= 13.33\cos 100t \text{ V}$$

Apply the T equivalent network:

$$i_1 = \frac{1}{(-10+40)\times10^{-3}}\int 10\cos(100t)dt = 3.33\sin 100t \text{ A}$$

$$v_2 = 40\times10^{-3}\times3.33\times100\cos 100t$$

$$= 13.33\cos 100t \text{ V}$$

Analysis of multi winding coupled circuits

For more windings the flux in each coil are

$$\phi_1 = L_{11}I_1 + L_{12}I_2 + L_{13}I_3 + ..$$

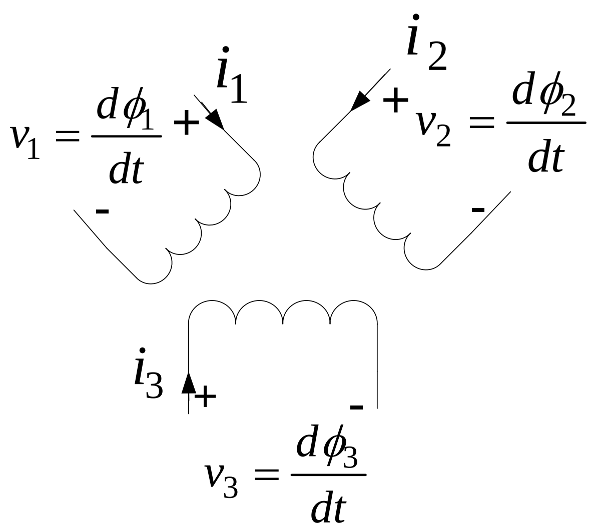
$$\phi_2 = L_{21}I_1 + L_{22}I_2 + L_{23}I_3 + ..$$

$$\phi_3 = L_{31}I_1 + L_{32}I_2 + L_{33}I_3 + ..$$

L_{11}, L_{22}, L_{33} are self inductances and

$L_{12} = L_{21}, L_{13} = L_{31}, L_{23} = L_{32}$ Are mutual inductances.

$$\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} \qquad i = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} \qquad L = \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix}$$



<u>Analysis of Coupled Circuits</u>

Consider the coupled circuits.

Each circuit contains a voltage source. As both currents i1 and i2 enter the coils through the dotted ends, M is taken as positive. By applying KVL, the two loop equations may be written as below :

In the sinusoidal steady state the above equations become,

$$R_1 i_1 + L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} = e_1 \quad \ldots (19)$$

$$R_2 i_2 + L_2 \frac{di_2}{dt} + M \frac{di_1}{dt} = e_2 \quad \ldots (20)$$

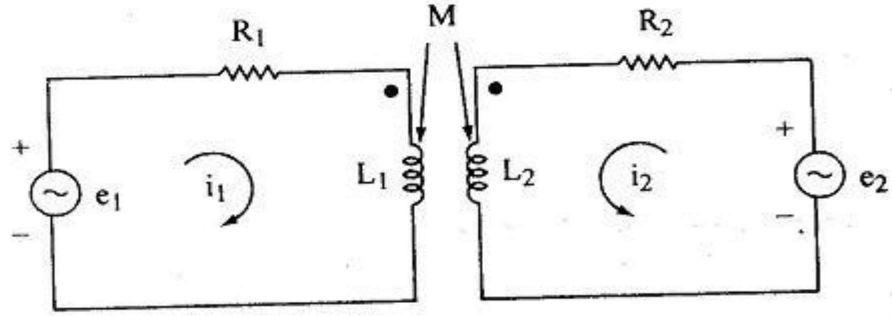


Fig. 8.2.

$$(R_1 + j\omega L_1) I_1 + j\omega M I_2 = E_1 \quad \ldots (21)$$

$$j\omega M I_1 + (R_2 + j\omega L_2) I_2 = E_2 \quad \ldots (22)$$

In the matrix form, the last two equations may be written as,

$$\begin{bmatrix} R_1 + j\omega L_1 & j\omega M \\ j\omega M & R_2 + j\omega L_2 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \quad \ldots (23)$$

The equations (21) & (22) may be written as

$$[R_1 + j\omega (L_1 - M + M) I_1] + j\omega M I_2 = E_1 \quad \ldots (24)$$

and $$j\omega M I_1 + [R_2 + j\omega (L_2 - M + M)] I_2 = E_2 \quad \ldots (25)$$

The coupled circuit of fig. 8.2 may be now re-drawn as in fig. 8.3. It is called conductively coupled equivalent circuit of the mutually coupled circuit. It is so called because of the common conducting element M.

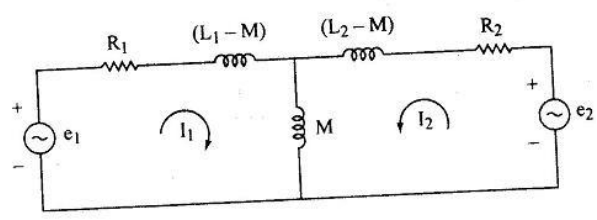
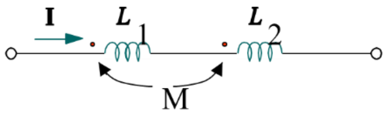


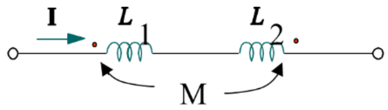
Fig. 8.3.

Series, Parallel connection of coupled inductors

Series connection



(a)mutually coupled coils in series-aiding connection



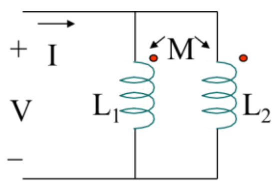
(b)mutually coupled coils in series–opposing connection

Total inductance

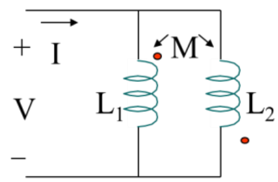
$$L_T = L_1 + L_2 + 2M$$

$$L_T = L_1 + L_2 - 2M$$

Parallel Connection



(a)mutually coupled coils in parallel-aiding connection



(b)mutually coupled coils in parallel–opposing connection

Equivalent inductance

$$L_e = \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M}$$

$$L_e = \frac{L_1 L_2 - M^2}{L_1 + L_2 + 2M}$$

<u>Combination of Conductively Connected Mutually Coupled Coils</u>

Consider two coils of self inductances L1 and L2. Let M be the mutual inductance between them. These two coils can be connected in the following two ways :

1. Series connection,

2. Parallel connection

Again, series connection can be (a) series aiding or cumulative and (b) series opposition or differential. Similarly, the parallel connection can be (a) parallel aiding or cumulative and (b) parallel opposition or differential.

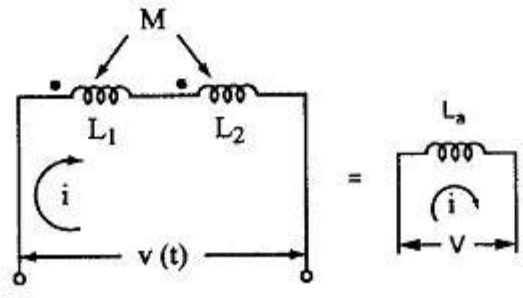


Fig. (a)

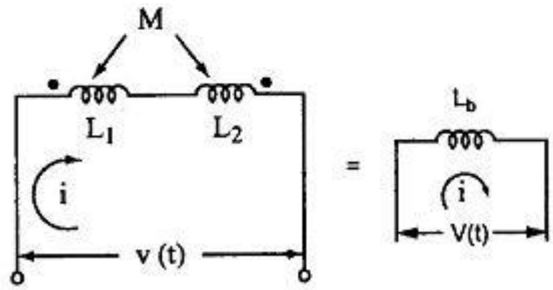


Fig. 8.7 (b)

1. (a) Series connection (aiding)

Refer fig. 8.7 (a), the current is entering both the coils at the dotted terminal. So, it is called series aiding combination. For this circuit, we can write that

$$L_1 \frac{di}{dt} + M \frac{di}{dt} + L_2 \frac{di}{dt} + M \frac{di}{dt} = v(t)$$

or $$(L_1 + L_2 + 2M) \frac{di}{dt} = v(t) \qquad \ldots (26)$$

Let La be the equivalent inductance of the combination shown in fig. 8.7 (a),

Then $$L_a \times \frac{di}{dt} = v(t) \qquad \ldots (27)$$

From equations (26) & (27), we can obtain that,

$$L_a = L_1 + L_2 + 2M \qquad \ldots (28)$$

(b) Series Opposition : (bucking)

Refer fig. 8.7 (b), the current is entering first coil at dotted terminal and leaving the other coil at dotted terminal. So the mesh equation for this circuit is

$$L_1 \frac{di}{dt} - M\frac{di}{dt} + L_2\frac{di}{dt} - M\frac{di}{dt} = v(t)$$

or

$$(L_1 + L_2 - 2M)\frac{di}{dt} = v(t) \quad \dots (29)$$

Let Lb be the equivalent inductance of the combination shown in fig. 8.7 (b),

$$\text{Then} \quad L_b\frac{di}{dt} = v(t) \quad \dots (30)$$

From equations (29) & (30), we find that

$$L_b = L_1 + L_2 - 2M \quad \dots (31)$$

[Note : Equivalent inductance in the series aiding combination is more than that in series opposing combination by an amount = 4M.]

2. (a) Parallel Combination (aiding) :

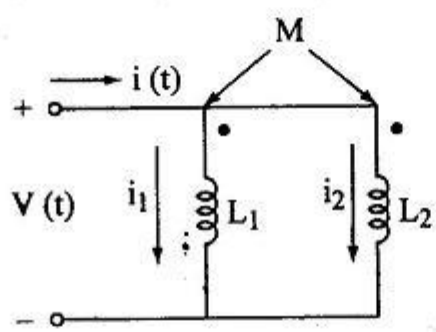


Fig. 8.8. (a)

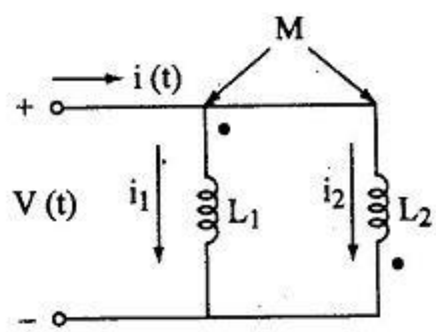


Fig. 8.8 (b)

Here, both the currents it and i2 enter the coils at the dotted terminals. Then, the equations are

$$L_1\frac{di_1}{dt} + M\frac{di_2}{dt} = v(t)\,. \quad \dots (32)$$

$$M\frac{di_1}{dt} + L_2\frac{di_2}{dt} = v(t) \quad \dots (33)$$

Assume that the excitations are sinusoidal for convenience. Then, the above equations can be written as

$$j\omega L_1 I_1 + j\omega M I_2 = V \quad \dots (34)$$

$$j\omega M I_1 + j\omega L_2 I_2 = V \quad \dots (35)$$

Solving above equations for II and 12, we get

$$I_1 = \frac{j\omega (L_2 - M) V}{\omega^2 (M^2 - L_1 L_2)}, \text{ and}$$

$$I_2 = \frac{j\omega (L_1 - M) V}{\omega^2 (M^2 - L_1 L_2)}$$

Therefore, the total current

$$I = I_1 + I_2$$

$$I = \frac{j\omega (L_1 + L_2 - 2M) V}{\omega^2 (M^2 - L_1 L_2)}$$

Therefore, the input impedance

$$= \frac{V}{I}$$

$$= \frac{\omega^2 (M^2 - L_1 L_2)}{j\omega (L_1 + L_2 - 2M)} = \frac{j\omega (L_1 L_2 - M^2)}{(L_1 + L_2 - 2M)} \quad \ldots (36)$$

Let La be the equivalent of the combination of inductances then

$$\frac{V}{I} = j\omega (L_a) \quad \ldots (37)$$

From equations (36) & (37), we write that

$$L_a = \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M} \quad \ldots (38)$$

(b) Parallel Opposition

Let Lb be the equivalent inductance in this case, by derivation, we can get that

$$L_b = \frac{L_1 L_2 - M^2}{L_1 + L_2 + 2M} \quad \ldots (39)$$

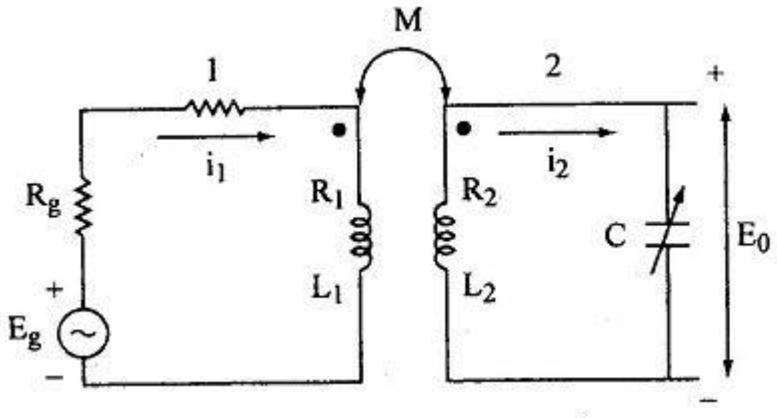
Note : On observing equations (38) and (39), we can conclude that the equivalent inductance in the parallel aiding is more than that in parallel opposition. It is because the denominator of equation (38) is less than that of equation (39)

SINGLE TUNED AND DOUBLE TUNED COUPLED CIRCUITS

Single Tuned Coupled Circuits

Consider the circuit shown in the fig. 8.15. A parallel resonant circuit on the secondary is inductively coupled to coil 1. This coil 1 is excited by a source Eg. Let Rg be the source resistance.

Let R1, R2 be the resistances of coils 1 and 2 respectively and let L , L2 be the self-inductances of the coils 1 and 2 respectively.



Assume that Rg >> R1 >> jcoLi i.e., Ignore R1 and jo3L1 in comparison with Rg.

Then, the mesh equations are

$$I_1 R_g - j\omega M I_2 = E_g$$

$$-j\omega M I_1 + \left(R_2 - \omega L_2 + j\frac{j}{\omega C}\right) I_2 = 0$$

Solving equations (i) & (ii), we get

$$I_2 = \frac{j E_g \omega M}{R_g\left(R_2 + j\,\omega\, L_2 - \frac{j}{\omega C}\right)\omega^2 M^2}$$

$$\text{The output voltage } E_0 = I_2 \times \left(\frac{-j}{\omega C}\right) = I_2 \left(\frac{1}{j\omega C}\right)$$

$$E_0 = \frac{j E_g \omega M}{j\omega C\left[\left\{R_g\left(R_2 + \left(\overline{j\omega L_2 - \frac{j}{\omega C}}\right)\right)\right\} + \omega^2 M^2\right]}$$

$$E_0 = \frac{E_g M}{C\left[\left\{R_g\left(R_2 + \left(\overline{j\omega L_2 - \frac{j}{\omega C}}\right)\right)\right\} + \omega^2 M^2\right]}$$

$\therefore$ The voltage transfer function = Voltage amplification.

$$A = \frac{E_0}{E_g} = \frac{M}{C\left[\left\{R_g\left(R_2 + \left(\overline{j\omega L_2 - \frac{j}{\omega C}}\right)\right)\right\} + \omega^2 M^2\right]}$$

When the secondary side is tuned i.e., when the values of the frequency co, is such that

$$\omega L_2 = \frac{1}{\omega C},$$

$$A = \frac{E_0}{E_g} = \frac{M}{C\,[R_g R_2 + \omega_r^2 M^2]}$$

$$\omega_r^2 = \frac{1}{L_2 C}$$

From equation (iii) the current I2 at resonance is obtained by putting

$$\omega L_2 = \frac{1}{\omega C}$$

and replacing

ω by ω_r.

Therefore I2 at resonance.

$$= \frac{j E_g \omega_r M}{R_g R_2 + \omega_r^2 M^2}$$

From equations (vi) and (viz) it is observed that at resonance frequency E0, 12 and A depend on M. The maximum value of E0 or A dE0 depends upon M. To get the condition for maximum Eo,

$$\frac{dE_0}{dM} = 0$$

$$\Rightarrow \frac{d}{dM}\left[\frac{E_g M}{C\,[R_g R_2 + \omega_r^2 M^2]}\right] = 0$$

From this, on simplification, we get

$$M = \frac{\sqrt{R_g R_2}}{\omega_r}$$

$$\text{When } M = \frac{\sqrt{R_g R_2}}{\omega_r},$$

the output voltage is maximum.

Therefore, maximum output voltage

$$= E_{0M} = \frac{\frac{\sqrt{R_g R_2}}{\omega_r} E_g}{C\,(R_g R_2 + R_g R_2)} = \frac{E_g}{2\omega_r C\sqrt{R_g R_2}}$$

Maximum amplification

$$= A_M = \frac{E_{0M}}{E_g} = \frac{1}{2\omega_r C\sqrt{R_g R_2}} \quad \dots (x)$$

Maximum value of current

$$= \frac{E_g}{2\sqrt{R_g R_2}}$$

These maximum values are obtained by substituting

$$M = \frac{\sqrt{R_g R_2}}{\omega_r}$$

in expressions E0, A, and 12 at resonance.

We know that

$$M = K\sqrt{L_1 L_2}$$

By changing the coupling factor K, we can vary M. The variation of amplification factor or output voltage with the coefficient of the coupling is shown in the fig. 8.16.

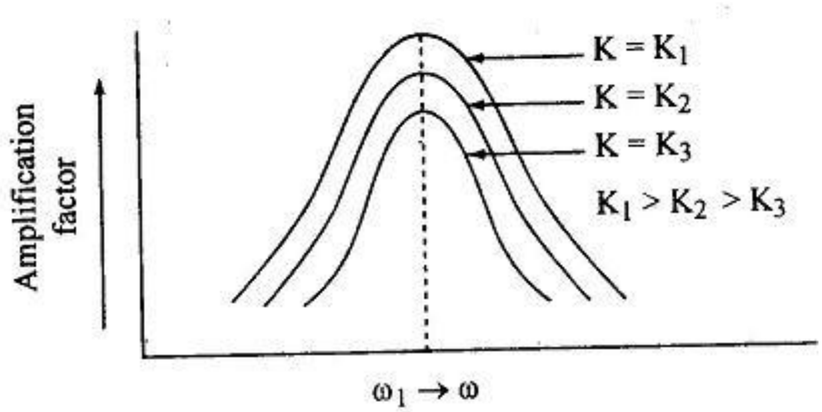


Fig. 8.16.

Double Tuned Coupled Circuits

Double tuned circuits are generally parallel fed in the primary but it is simpler to consider the series fed circuit.

For the circuit shown in the fig., we consider, a special case where the primary and secondary resonate at the same frequency,

$$i.e., \quad \omega_r^2 = \frac{1}{L_1 C_1} = \frac{1}{L_2 C_2}$$

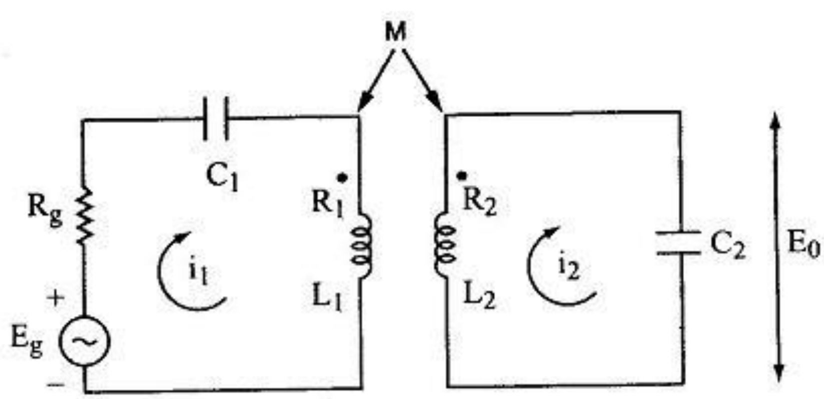


Fig. 8.17.

The mesh equations are :

$$\left[R_g + R_1 + j\omega L_1 - \frac{1}{j\omega C_1} \right] I_1 - j\omega M I_2 = E_g$$

$$-j\omega M I_1 + I_2 \left[R_2 + j\omega L_2 - \frac{1}{j\omega C_2} \right] = 0$$

From equations (ii) and we get

$$I_2 = \frac{E_g j\omega M}{\left[(R_g + R_1) + j\left(\omega L_1 - \frac{1}{\omega C_1}\right)\right]\left[R_2 + j\left(\omega L_2 - \frac{1}{\omega C_2}\right)\right] + \omega^2 M^2}$$

At resonance,

$$\omega_r = \frac{1}{\sqrt{L_1 C_1}} = \frac{1}{\sqrt{L_2 C_2}}$$

Hence, at resonance current

$$I_2 = \frac{E_g\, j\, \omega_r\, M}{(R_g + R_1)\, R_2 + \omega_r^2\, M^2}$$

Hence, output voltage

$$E_0 = I_2 \frac{-j}{C\omega_r}$$

$$= \frac{E_g\,(j\,\omega_r - M)}{(R_g + R_1)\, R_2 + \omega_r^2\, M^2}$$

$$E_0 = \frac{E_g\, M}{C\,[(R_g + R_1)\, R_2 + \omega_r^2\, M^2]}$$

$$= A\, E_g$$

$$\therefore A = \frac{M}{C\,[(R_g + R_1)\, R_2 + \omega_r^2\, M^2]}$$

The maximum value of A or the maximum value of Eo can be obtained by taking the first derivative of A or E0 with respect to M and equating it to 0.

$$\textit{i.e.,}\quad \frac{dE_0}{dM} = 0$$

$$\text{or}\quad \frac{dA}{dM} = 0$$

$$\frac{dA}{dM} = (R_1 + R_g)\, R_2 + \omega_r^2\, M^2 - 2M^2\, \omega_r^2 = 0$$

$$\omega_r^2\, M^2 = R_2\,(R_1 + R_g)$$

$$\therefore M_c = \frac{\sqrt{R_2\,(R_1 + R_g)}}{\omega_r}$$

Mc is the critical value of mutual inductance. The maximum values of E0 and I2 are obtained by substituting the value of Mc in equations of Eo and I2.

From definition,

$$M = K \sqrt{L_1 L_2},$$

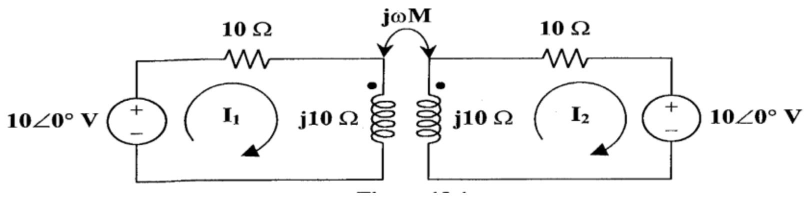
the coefficient of coupling K at M = Mc is called the critical coefficient of coupling. It is given by

$$K_c = M_c \sqrt{L_1 L_2}$$

The critical coupling causes i2 to have the maximum possible value. At resonance, the maximum value of A is obtained by changing M, or by changing the coupling coefficient for given values of L1 and L2.

PROBLEMS

1. Find I1 and I2 of the circuit for K=1.



Given K= 1

$$k = \frac{M}{\sqrt{L_1 L_2}} = 1$$

therefore $M = \sqrt{L_1 L_2}$ that is

$$\omega L_1 = \omega L_2 = 10 \text{ and } L_1 = L_2 = L.$$

$$M = \sqrt{L_1 L_2} = L \text{ and } j\omega M = j\omega L = j10.$$

Now, using mesh analysis,

Loop 1 :
$$-10 + 10I_1 + j10I_1 - j10I_2 = 0$$
$$(10 + j10)I_1 - j10I_2 = 10$$
$$(1 + j)I_1 - jI_2 = 1$$

Loop 2 :
$$-j10I_1 + j10I_2 + 10I_2 + 10 = 0$$
$$-j10I_1 + (10 + j10)I_2 = -10$$
$$-jI_1 + (1 + j)I_2 = -1$$

In matrix form,

$$\begin{bmatrix} 1+j & -j \\ -j & 1+j \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

or

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \frac{\begin{bmatrix} 1+j & j \\ j & 1+j \end{bmatrix}}{\Delta} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

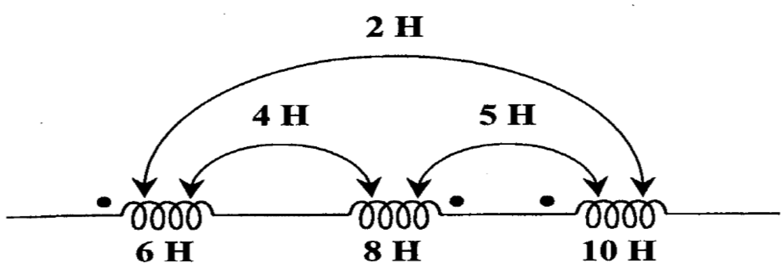
where $\Delta = (1 + j)^2 - (-j)^2 = (1 + j2 + j^2) - j^2 = 1 + j2$.

Therefore,

$$I_1 = \frac{1 + j - j}{1 + j2} = \frac{1}{1 + j2} = \frac{1\angle 0°}{\sqrt{5}\angle 63.43°} = 0.4472\angle -63.43° \text{ A}$$

$$I_2 = \frac{j - (1 + j)}{1 + j2} = \frac{-1}{1 + j2} = \frac{1\angle 180°}{\sqrt{5}\angle 63.43°} = 0.4472\angle 116.57° \text{ A}$$

2. Find the equivalent inductance of the three inductors using dot rule.



For coil 1, $L_1 - M_{12} + M_{13} = 6 - 4 + 2 = 4$

For coil 2, $L_2 - M_{21} - M_{23} = 8 - 4 - 5 = -1$

For coil 3, $L_3 + M_{31} - M_{32} = 10 + 2 - 5 = 7$

$$L_T = 4 - 1 + 7 = \underline{\mathbf{10\ H}}$$

or

$$L_T = L_1 + L_2 + L_3 - 2M_{12} - 2M_{23} + 2M_{12}$$
$$L_T = 6 + 8 + 10 - (2)(4) - (2)(5) + (2)(2)$$
$$L_T = 6 + 8 + 10 - 8 - 10 + 4 = \underline{\mathbf{10\ H}}$$

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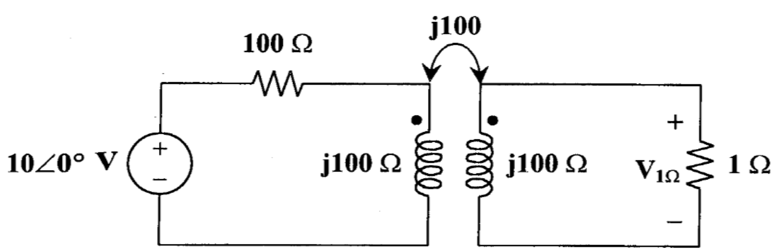
3. For the given circuit fing K and the voltage across the 1 ohm resistor.



Ans:

$k = \underline{\mathbf{1}}$ $\qquad V_{1\Omega} = \underline{\mathbf{0.1\angle 0^\circ\ V}}$

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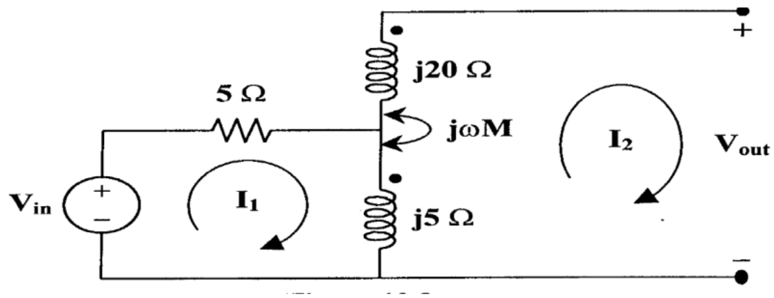
4. **For the given citcuit find V_{out} if $V_{in}(t) = 10\cos(377t)$ and the value of K=0.8**



Form the value of K=0.8 we can get

$$k = \frac{\omega M}{\sqrt{(\omega L_1)(\omega L_2)}}.$$

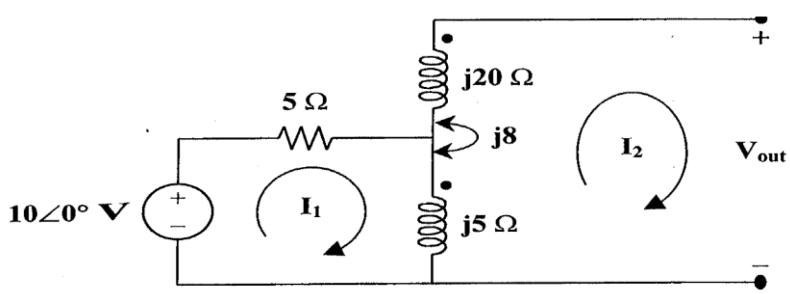
$$\omega M = k\sqrt{(\omega L_1)(\omega L_2)} = (0.8)\sqrt{(5)(20)} = 8$$

The in put can be written as

$$A\cos(377t + \phi).$$

$$V_{in} = 10\angle 0°.$$

The circuit can be redrawn as



Loop #1 : $-10 + 5I_1 + j5(I_1 - I_2) - j8I_2 = 0$

Loop #2 : $j8I_2 + j5(I_2 - I_1) - j8(I_1 - I_2) + j20I_2 + V_{out} = 0$

Due to open circuit $I_2 = 0$

Therefore $(5+j5)\,I_1 = 10$ and $V_{out} = j\,13\,I_1$ clearly

$$I_1 = \frac{10}{5 + j5} = \frac{10\angle 0°}{5\sqrt{2}\angle 45°} = \sqrt{2}\angle -45°$$

$$V_{out} = j13I_1 = (13\angle 90°)(\sqrt{2}\angle -45°) = 13\sqrt{2}\angle 45°$$

In the time domain it can be written as

Vout = 13√2 cos(377 t + 45) Volts

UNIT -V

Network Topology: Network terminology - Graph of a network - Incidence and reduced incidence matrices – Trees –Cutsets - Fundamental cutsets - Cutset matrix – Tiesets – Link currents and Tieset schedules -Twig voltages and Cutset schedules, Duality and dual networks.

Graph(orlineargraph): Anetworkgraphisanetworkinwhichallnodesandloopsarere-tainedbutitsbranchesarerepresentedbylines.Thevoltagesourcesarereplacedbyshortcircuits andcurrentsourcesarereplacedbyopencircuits.(Sourceswithoutinternalimpedancesorad-mittancescanalsobetreatedinthesamewaybecausetheycanbeshiftedtootherbranchesby E-shiftand/orI-shiftoperations.)

Branch: Alinesegmentreplacingoneormorenetworkelementsthatareconnectedinseriesor parallel.

Node: Interconnectionoftwoormorebranches.Itisaterminalofabranch.Usuallyintercon-nectionsofthreeormorebranchesarenodes.

Path: Asetofbranchesthatmaybetraversedinanorderwithoutpassingthroughthesamenode morethanonce.

Loop: Anyclosedcontourselectedinagraph.

Mesh: Aloopwhichdoesnotcontainanyotherloopwithinit.

Planargraph: Agraphwhichmaybedrawnonaplanesurfaceinsuchawaythatnobranch passesoveranyotherbranch.

Non-planargraph: Anygraphwhichisnotplanar.

Orientedgraph: Whenadirectiontoeachbranchofagraphisassigned,theresultinggraphis calledanorientedgraphoradirectedgraph.

Connectedgraph: Agraphisconnectedifandonlyifthereisapathbetweeneverypairofnodes.

Subgraph: Anysubsetofbranchesofthegraph.

Tree: Aconnectedsub-graphcontainingallnodesofagraphbutnoclosedpath.i.e.itisaset ofbranchesofgraphwhichcontainsnoloopbutconnectseverynodetoeveryothernodenot necessarilydirectly.Anumberofdifferenttreescanbedrawnforagivengraph.

Link: Abranchofthegraphwhichdoesnotbelongtotheparticulartreeunderconsideration. Thelinksformasub-graphnotnecessarilyconnectedandiscalledtheco-tree.

Treecompliment: Totalityoflinksi.e.Co-tree.

Independentloop: Theadditionofeachlinktoatree,oneatatime,resultsoneclosedpathcalled anindependentloop.Suchaloopcontainsonlyonelinkandothertreebranches.Obviously,the numberofsuchindependentloopsequalsthenumberoflinks.

Tieset: Asetofbranchescontainedinaloopsuchthateachloopcontainsonelinkandthe remainderaretreebranches.

Treebranchvoltages: Thebranchvoltagesmaybeseparatedintotreebranchvoltagesandlink voltages.Thetreebranchesconnectallthenodes.Thereforeifthetreebranchvoltagesareforced tobezero,thenallthenodepotentialsbecomecoincidentandhenceallbranchvoltagesareforced tobezero.Astheactofsettingonlythetreebranchvoltagestozeroforcesallvoltagesinthe networktobezero,itmustbepossibletoexpressallthelinkvoltagesuniquelyintermsoftree branchvoltages.Thustreebranchformanindependentsetofequations.

Cutset: Asetofelementsofthegraphthatdissociatesitintotwomainportionsofanetworksuch thatreplacinganyoneelementwilldestroythisproperty.Itisasetofbranchesthatifremoved dividesaconnectedgraphintotwoconnectedsub-graphs.Eachcutsetcontainsonetreebranch andtheremainingbeinglinks.

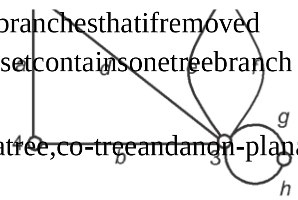
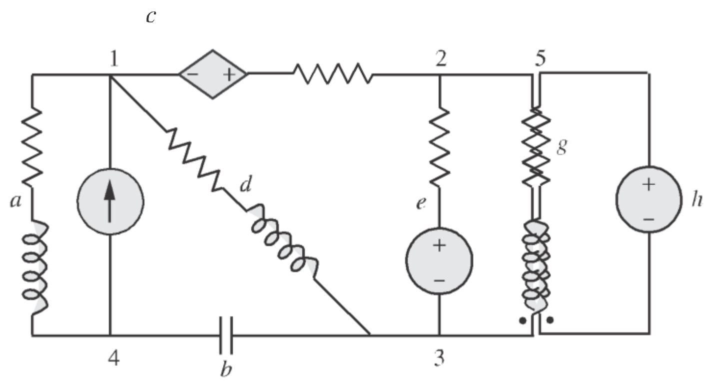
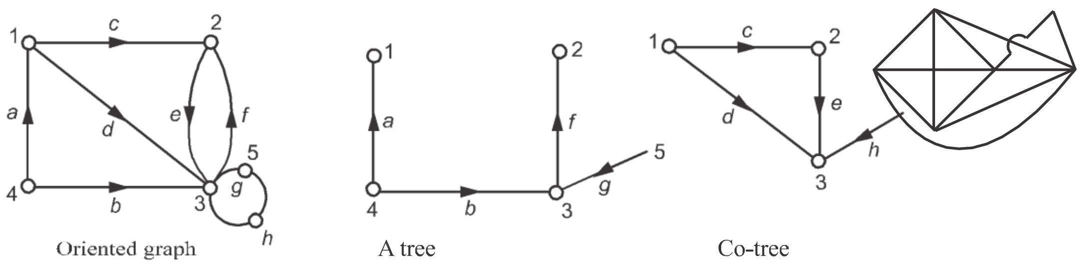


Fig.2.1showsatypicalnetworkwithitsgraph,orientedgraph,atree,co-treeandanon-planar graph.



Network

Graph



Oriented graph A tree Co-tree

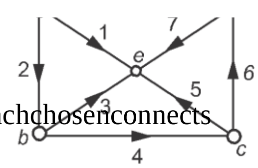
Figure2.1

Relationbetweennodes,links,andbranches

Let B=Totalnumberofbranchesinthegraphornetwork

N=totalnodes

L=linkbranches



ThenN−1branchesarerequiredtoconstructatreebecausethefirstbranchchosenconnects twonodesandeachadditionalbranchincludesonemorenode.

Thereforenumberofindependentnodepairvoltages=N−1=numberoftreebranches.

Then$L=B-(N-1)=B-N+1$

Numberofindependentloops$=B-N+1$

Proposition: Consider a directed graph containing n nodes and e links. When any tree is chosen, the number of branches is: b = n-1;

- the number of cords is: l = e-n+1;
- the number of fundamental circuits is: m = e-n+1;
- the number of fundamental cuts is: c = n-1;
- the chosen orientation
  - of a circuit: that of the associated cord;
  - Of a cut: that of the associated branch.

Figures 1 illustrate the concepts on the graph. Figures 1a, b, c, and d respectively show the network representation by a directed graph, a tree with cords and branches, fundamental circuits, and fundamental cuts.

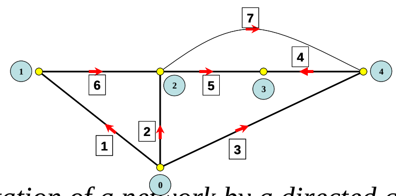


Figure 1a Representation of a network by a directed graph.

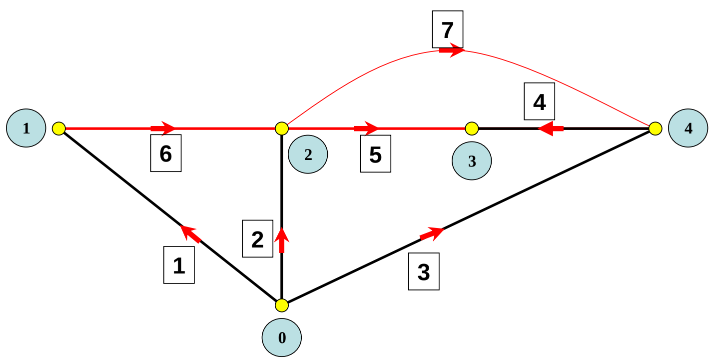


Figure 1b Tree with branches (1-4), and cords (5-7).

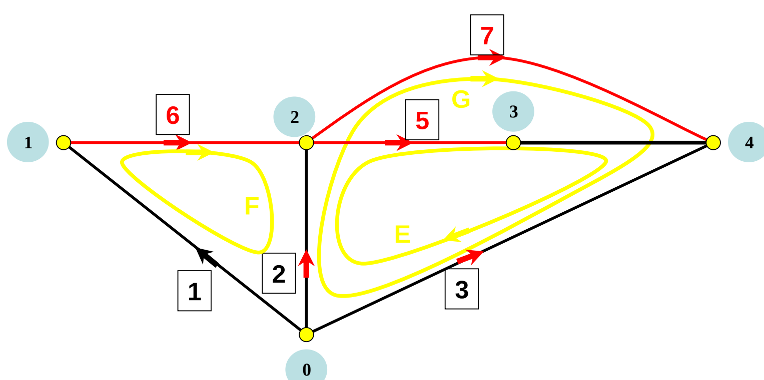


Figure 1c Fundamental circuits (E, F, G).

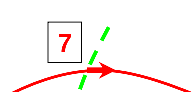


Figure 1d Fundamental cuts (A, B, C, D).

Starting from a description of the network by a unifilar diagram and extraction of the graph which is the topological representation, it is possible to seek by specialized algorithms possible trees and associated cords, branches and circuits. As will be seen in the sections that follow, this description will allow the derivation of the network equations.

Matrix representation of networks

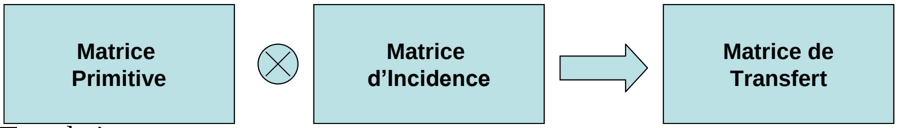
The formulation of the equations of network is based on the definition of a coherent and exact mathematical model which describes the characteristics of the individual components (machines, lines, transformers, loads) and the interconnection between these components. The matrix equation is a suitable model adapted to the mathematical treatment and processing under a systemic aspect. The matrix elements can be either *impedances* (when node voltages are written in terms of injected currents), or *admittances* (when injected currents are written in terms of node voltages).

Network Matrices

The network can be described by three types of matrices:

- *Elementary matrices* (or primitive): these matrices describe the individual components by taking into account, if necessary, their electromagnetic (capacitive and inductive) couplings for lines having common or partial right-of-ways. They are of diagonal structure except for the components whose coupling is represented by non-diagonal elements;
- *Incidence matrices*: these matrices describe the interconnections between the various components of the network. The terms of these matrices are binary digits 1, 0, - 1, which represent the bond between branches and nodes of the network with their orientation;
- *Transfer matrices*: these matrices describe in a mathematical way the electric behavior of the mesh network. They are essentially impedance or admittance matrices which correspond to the nodes of the network (nodal matrices).

The relation between the above three matrices can be described by the operational equation of Figure 2.2. The figure shows that the transfer matrix is obtained from a complex operation using the elementary matrix and the incidence matrix. This operation will studied in the following sections.



Translation:

Matrice primitive: Elementary Matrix

Matriced'incidence: Incidence Matrix

Matrice de Transfer: Transfer Matrix

Incidence Matrix

As indicated above, the incidence matrices characterize the relation between the network elements (generally called branches) and the nodes connecting these elements.

Incidence Matrix *branches-nodes*: «*A*»

Definition: It is a matrix *A* with general term $\{a_{ij}\}$ and dimension (e x n) such as:

- $a_{ij}=1$ if branch i is incident with node j and is directed towards this node;
- $a_{ij}= -1$ if branch i is incident with node j and is directed away from this node;
- $a_{ij}=0$ if branch i is non-incident with node j.

Properties – For every line i:

$$\sum_{j=0}^{n-1} a_{ij} = 0$$

Indeed on the same line corresponding to the branch referred by i, there are only two nonzero elements: The first corresponds to the starting node with value 1, and the second corresponds to the arrival node with the value - 1. The above property indicates that the number of rows of the matrix is lower than n.

Incidence matrix branches-access: «*A'*»

This corresponds to the incidence matrix branch-node in which the choice of a node of reference (for voltage) led to the removal of a column of the matrix «*A*» (in general the first). This matrix is of row n - 1.

Incidence matrix branches-fundamental cuts: «*B*»

Definition: It is a matrix *B* of general term $\{b_{ij}\}$ and dimension (e x b) such as:

- $b_{ij}= +1$ if the i^{th} branch belongs to the j^{th} fundamental cut with same orientation;
- $b_{ij}= -1$ if the i^{th} branch belongs to the j^{th} fundamental cut with opposite orientation;
- $b_{ij}= 0$ if the i^{th} branch does not belong to the j^{th} fundamental cut.

Properties: Let the following sub-matrices of «*A*» and «*B*» be denoted by:

- A_b: branches/access,
- A_c: cords/access.
- B_b: fundamental branches/cuts,
- B_c: cords/fundamental cuts.

Since there is an identity between the branches and the fundamental cuts, then the sub-matrix B_b *is equal to the unity matrix* I. Moreover one can notice that the product:

B_c*A_b = incidence matrix cords/access

Which is precisely the sub-matrix A_c, i.e.,

$B_c * A_b = A_c$

The above yields

$B_c = A_c * A_b^{-1}$

Thus, one can build the matrix B from sub- matrices A_b and A_c of matrix A by the formula:

$$B = [A_c A_b^{-1}]^{-1}$$

Incidence matrix links-fundamentalcircuits*: «C»*

Definition: It is a matrix C of general term $\{c_{ij}\}$ and of dimension (e x m) such as:

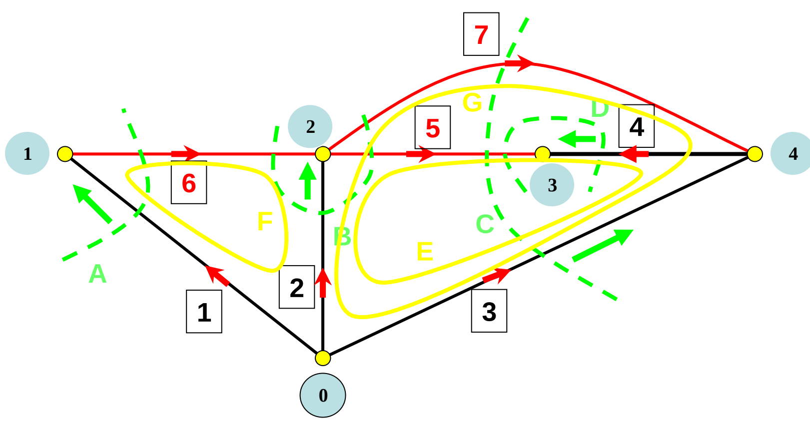
- $c_{ij} = +1$ if the i^{th} link belongs to the j^{th} fundamental circuit with same orientation;
- $c_{ij} = -1$ if the i^{th} link belongs to the j^{th} fundamental circuit with opposite orientation;
- $c_{ij} = 0$ if the i^{th} does not belong to the j^{th} fundamental circuit.

Properties: Let the following sub-matrices of «*C*» be denoted as follows:

- C_b: branches/fundamental circuits;
- C_c: cords/fundamental circuits.

Since there is identity between the cords and fundamental circuit, the sub-matrix C_c *is equal to the unity matrix I.*

Example of incidence matrices: If the graphs of Figures 2.1a - 2.1c are condensed into one graph as displayed in Figure 2.3 which shows the branches, cords, fundamental circuits and fundamental , one can easily build matrices *A*, *B*, and *C* corresponding to this graph:



Graph for the matrices A, B, C, of network.

$A =$

Link/Node	0	1	2	3	4
1	-1	+1	0	0	0
2	-1	0	+1	0	0
3	-1	0	0	0	+1
4	0	0	0	+1	-1
5	0	0	-1	+1	0
6	0	-1	+1	0	0
7	0	0	-1	0	+1

Rows 1–4: A_b; rows 5–7: A_c

$B =$

Link/ Fund. Cut	A	B	C	D	
1	+1	0	0	0	
2	0	+1	0	0	B_b
3	0	0	+1	0	
4	0	0	0	+1	
5	0	-1	+1	+1	
6	-1	+1	0	0	B_c
7	0	-1	+1	0	

$C =$

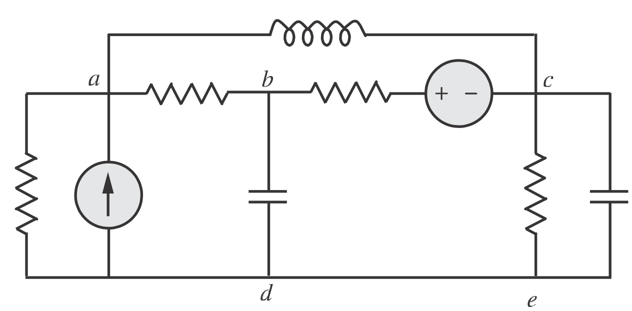
Link/Fund. Ckt.	E	F	G
1	0	+1	0
2	+1	-1	+1
3	+1	0	-1
4	-1	0	-1
5	+1	0	0
6	0	+1	0
7	0	0	+1

REDUCED INCIDENCE MATRIX

Let G be a connected digraph with " n" nodes and " b " branches. Let Aa be the Incidence Matrix of G . The (n -1) x b matrix A obtained by deleting any one row of A a is called a Reduced-Incidence Matrix of G.

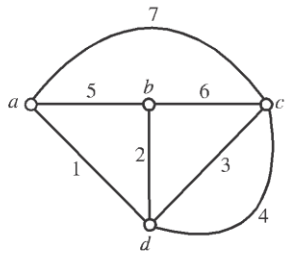
EXAMPLE:

ReferthecircuitshowninFig.Drawthegraph,onetreeanditsco-tree.

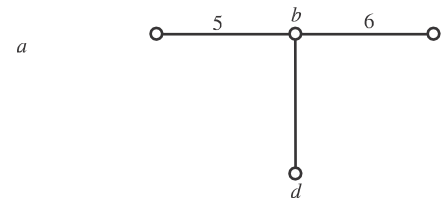


SOLUTION

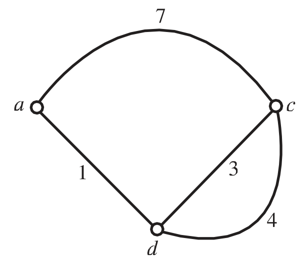
Wefindthattherearefournodes(N=4)andsevenbranches(B=7).
Thetreeismadeupofbranches2,5and6.Theco-treeforthetree
isshown.Theco-treehasL=B−N+1 = 7−4+1 = 4links.



Graph



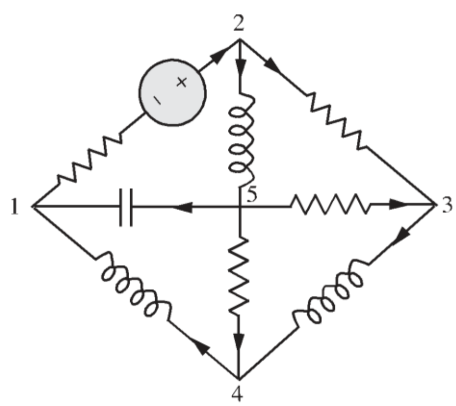
Tree

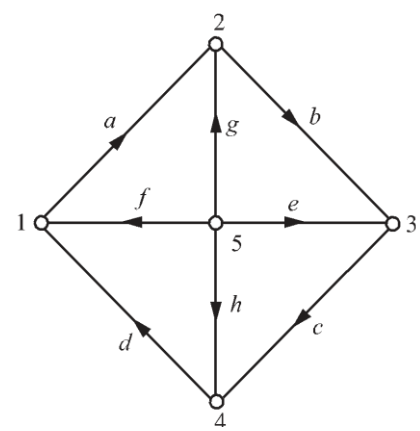


Co-tree

EXAMPLE

ReferthenetworkshowninFig. Obtainthecorrespondingincidencematrix.





SOLUTION

ThenetworkshowninFig(a)hasfivenodesandeightbranches.Thecorrespondinggraph appearsasshowninFig.(b).

Theincidencematrixisformedbyfollowingtherule:Theentryoftheincidencematrix,

=1,ifthecurrentofbranchleavesthenode

= 1,ifthecurrentofbranchentersnode

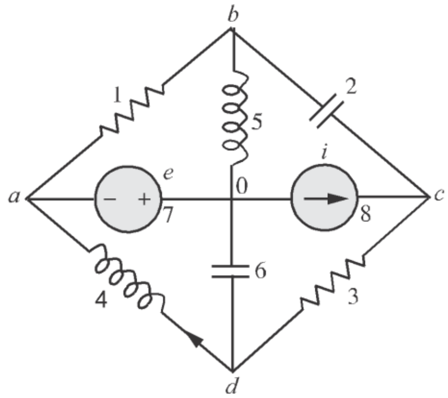
=0,ifthebranchisnotconnectedwithnode.

Incidencematrix:

Nodes	Branchnumbers							2
1	+1	0	0	1	0	1	0	0
2	1	+1	0	0	0	0	1	0
3	0	1	+1	0	1	0	0	0
4	0	0	1	+1	0	0	0	1
5	0	0	0	0	+1	+1	+1	+1

EXAMPLE

ForthenetworkshowninFig.(a),determinethenumberofallpossibletrees.Foratree consistingof(1,2,3)(i)drawtiesetmatrix(ii)drawcut-setmatrix.



Figure(a)

SOLUTION

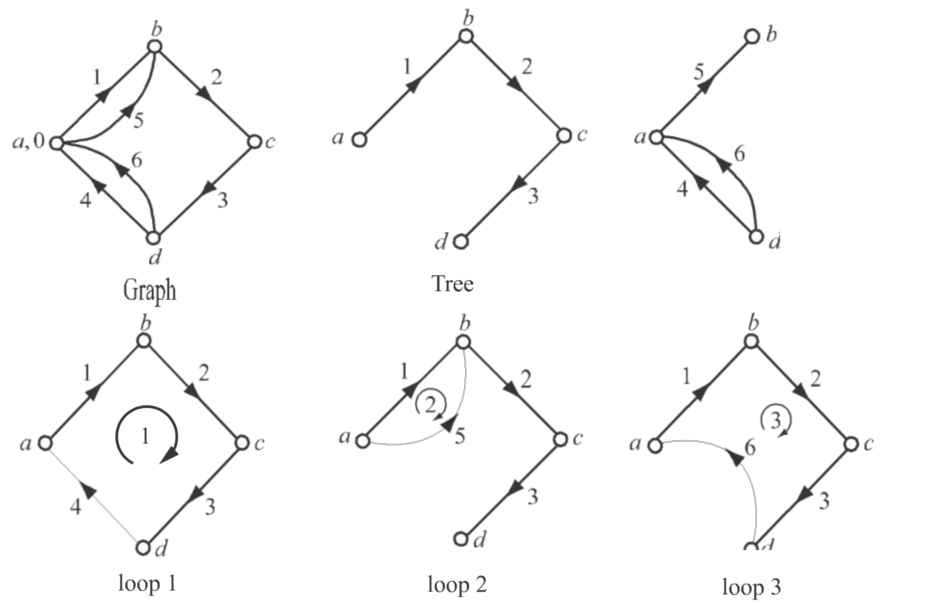
Iftheintentionistodrawatreeonlyforthepurposeoftie-setandcut-setmatrices,theideal currentsourceisopencircuitedandidealvoltagesourceisshortcircuited.Theorientedgraphis drawnforwhichdisthereference.ReferFig.2.12(b),

$$A = \begin{matrix} & 1 & 2 & 3 & 4 & 5 & 6 \\ a & 1 & 0 & 0 & -1 & 1 & -1 \\ b & -1 & 1 & 0 & 1 & -1 & 0 \\ c & 0 & -1 & 1 & 0 & 0 & 0 \end{matrix}$$

$$\text{Det } \mathbf{AA}^T = \begin{vmatrix} 1 & 0 & 0 & -1 & 1 & 1 \\ -1 & 1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix} \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} -1 & 1 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 0 \end{vmatrix} = 12$$

Therefore,possiblenumberoftrees=12.



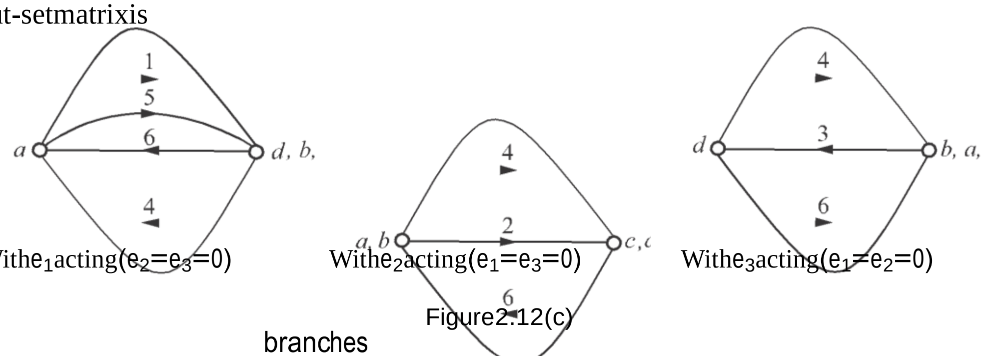
Figure(b)

(i)Tie-setmatrixfortwigs(1,2,3)is

Loop Currents	1	2	3	4	5	6
$i_1 = J_4$	1	1	1	1	0	0
$i_2 = J_5$	1	0	0	0	−1	0
$i_3 = J_6$	1	1	1	0	0	1

(branches)

(ii)Cut-setmatrixis

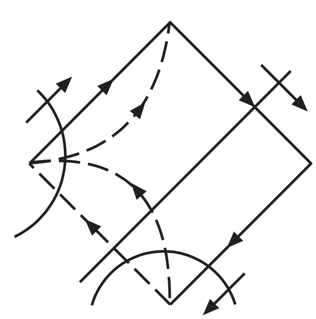


Withe$_1$acting(e_2=e_3=0) Withe$_2$acting(e_1=e_3=0) Withe$_3$acting(e_1=e_2=0)

Figure2.12(c)

Node-pairvoltage	1	2	3	4	5	6
e_1=v_1	1	0	0	1	1	1
e_2=v_2	0	1	0	1	0	1
e_3=v_3	0	0	1	1	0	1

(branches)



EXAMPLE

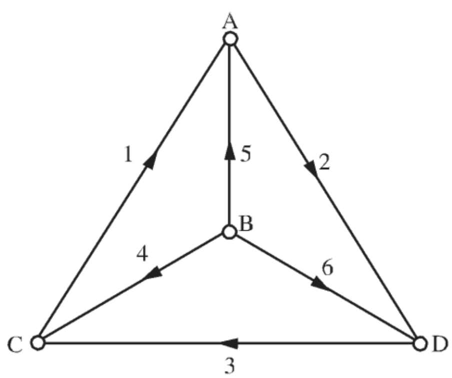
ForthenetworkshowninFig.(a),writeatie-setscheduleandthenfindallthebranchcurrents andvoltages.

C

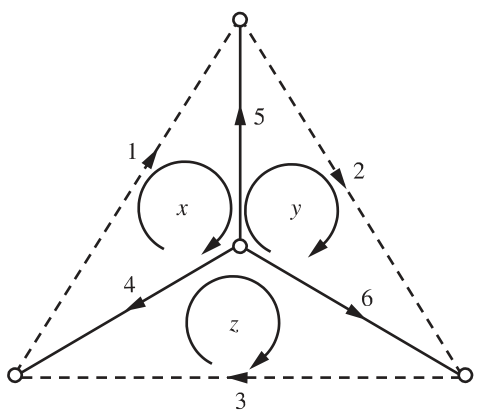
Figure(a)

SOLUTION

Fig.(b)showsthegraphforthenetwrokshowninFig.(a).Also,apossibletreeand co-treeareshowninFig.(c).Co-treeisindottedlines.



Figure(b)



Figure(c)

First, the tie-set schedule is formed and then the tie-set matrix is obtained.
Tie-set schedule:

Loop currents	Branch numbers					
	1	2	3	4	5	6
x	+1	0	0	+1	−1	0
y	0	+1	0	0	+1	−1
z	0	0	+1	−1	0	+1

Tie-set matrix is

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{bmatrix}$$

The branch impedance matrix is

$$\mathbf{Z}_B = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix} \qquad \mathbf{E}_B = \begin{bmatrix} 50 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The loop impedance matrix is

$$\mathbf{Z}_L = \mathbf{M}\mathbf{Z}_B\mathbf{M}^{\mathrm{T}}$$

$$= \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc:ccc} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{array}\right] \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hdashline 0 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{array}\right]$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$+ \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 10 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 5 \end{bmatrix} + \begin{bmatrix} 15 & -5 & -10 \\ -5 & 10 & -5 \\ -10 & -5 & 15 \end{bmatrix} = \begin{bmatrix} 20 & -5 & -10 \\ -5 & 20 & -5 \\ -10 & -5 & 20 \end{bmatrix}$$

$$\mathbf{ME}_B = \begin{bmatrix} 50 \\ 0 \\ 0 \end{bmatrix}$$

The loop equations are obtained using the equation,

$$\mathbf{Z}_L\mathbf{I}_L = \mathbf{ME}_B$$

$$\Rightarrow \qquad \begin{bmatrix} 20 & -5 & -10 \\ -5 & 20 & -5 \\ -10 & -5 & 20 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 50 \\ 0 \\ 0 \end{bmatrix}$$

Solving by matrix method, we get

$x = 4.1666$ A, $\qquad y = 1.16666$ A, $\qquad z = 2.5$ A

The branch currents are computed using the equations:

$$\mathbf{I}_B = \mathbf{M}^{\mathrm{T}}\mathbf{I}_L$$

$$\Rightarrow \qquad \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Hence,

I_1=x=4.1666A, I_2=y=1.6666A, I_3=z=2.5A,
I_4=x−z=1.6666A, I_5=−x+y=−2.5A, I_6=−y+z=0.8334A

Thebranchvoltagesarecomputedusingtheequation:

$$V_B=Z_BI_B-E_B$$

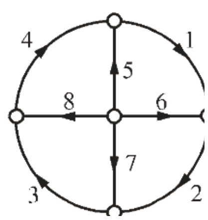
$$\Rightarrow \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{bmatrix} = \left[\begin{array}{ccc|ccc} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{array}\right] \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \end{bmatrix} - \begin{bmatrix} -50 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence,

$V_1=5I_1-50$ =29.167V, $V_2=10I_2$=16.666V, $V_3=5I_3$=12.50V,
$V_4=10I_4$=16.666V, $V_5=5I_5$=−12.50V, $V_6=5I_6$=4.167V

MPLE

Fortheorientedgraphshown,expressloopcurrentsintermsofbranch currentsforanindependentsetofcolumnsasthosepertinenttothelinks ofatree:



(i)Composedof5,6,7,8

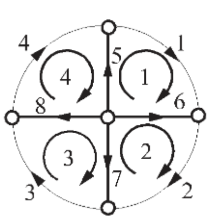
(ii)Composedof1,2,3,6

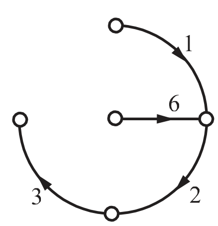
Verifywhetherthetwosetsofrelationsfor’sintemsof’sareequivalent.Constructatie-set schedulewiththecurrentsinthelinks4,5,7,8asloopcurrentsandfindthecorrespondingsetof closedpaths.

SOLUTION

Forthefirstset

Loop No:	1	2	3	4	5	6	7	8
	Branchnumbers							
1	+1	0	0	0	+1	1	0	0
2	0	+1	0	0	0	+1	1	0
3	0	0	+1	0	0	0	+1	1
4	0	0	0	+	1	0	0	+1





B=5678andLink=1234.

$i_1=J_1$, $i_2=J_2$, $i_3=J_3$, & $i_4=J_4$.

Thenforthesecondset,ofthemeshcurrentsindicatedforthefirstset,wehave

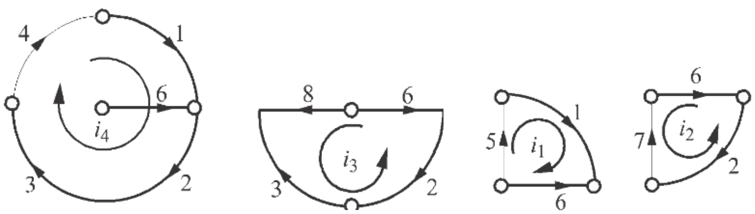
$J_4=i_4$ $\qquad$ $_4=J_4$

$J_5=i_1$ $+i_4$ $\qquad$ $_1=J_1+J_5$

$J_7=i_3$ $+i_2$ $\qquad$ $_3=J_4$ $+J_8$

$J_8=i_4$ $+i_3$ $\qquad$ $_2=J_4$ $+J_7$ $+J_8$

Loop No:↓	Branchnumbers 1	2	3	4	5	6	7	8
1	+1	0	0	0	+1	−1	0	0
2	0	−1	0	0	0	−1	+1	0
3	0	−1	−1	0	0	−1	0	+1
4	+1	+1	+1	+1	0	0	0	0



EXAMPLE

InthegraphshowninFigure(a),theidealvoltagesource $e=1$V.Fortheremainingbrancheseachhasaresistanceof1 ΩwithOasthereference.Obtainthenodevoltagee_1,$e_2$and $e_3$usingnetworktopology.

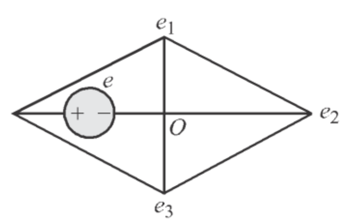
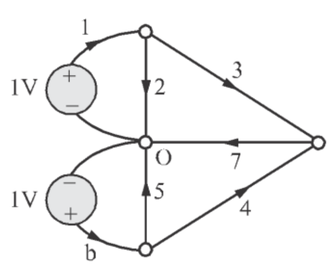


Figure2.16(a)

SOLUTION

Witheshift,graphisasshowninFigure2.16(b).Branchesarenumberedwithorientation.



With $T = (2, 5, 7)$ the cut set matrix is

$$\mathbf{Q} = \begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 \end{bmatrix}$$

$$\mathbf{Y}_B\mathbf{Q}^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{Q}\mathbf{Y}_B\mathbf{Q}^{\mathrm{T}} = \begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

$$\mathbf{Q}\mathbf{Y}_B\mathbf{E}_B = \begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$$

According to the equation $\mathbf{Q}\mathbf{Y}_B\mathbf{Q}^{\mathrm{T}}\mathbf{E}_N = -\mathbf{Q}\mathbf{Y}_B\mathbf{E}_B$, we have

$$\begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Therefore,

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{7} \\ \frac{2}{7} \\ \frac{3}{7} \end{bmatrix}$$

DUAL Networks

Circuits are said to be dual when the characterizing equations of one network can be obtained from the other by simply interchanging v and i and interchanging G and R.

Duality pairs

Resistance <> Conductance

Current <> Voltage

Series <>Parralel

Capacitor <> inductor

The duals of planar networks could be obtained by a graphicaltechnique known as the dot method. The dot method has the following procedure:

1. Put a dot in each independent loop of the network. These dots correspond to independent nodes in the dual network.1 Planar networks are those that can be laid on a plane without branches crossing one another.

2. Put a dot outside the network. This dot corresponds to the reference node in the dual network.

3. Connect all internal dots in the neighbouring loops by dahsed lines cutting the common branches. These branches that are cut by dashed lines will form the branches connecting the corresponding independent nodes in the dual network. As an example, if a common branch contains & and in series, then the parallel combination of ' and should be put between the corresponding independent nodes in the dual network.

4. Join all internal dots to the external dot by dashed lines cutting all external branches. Dualsof these branches cut by dashed lines will form the branches connecting the independent nodes and the reference node.

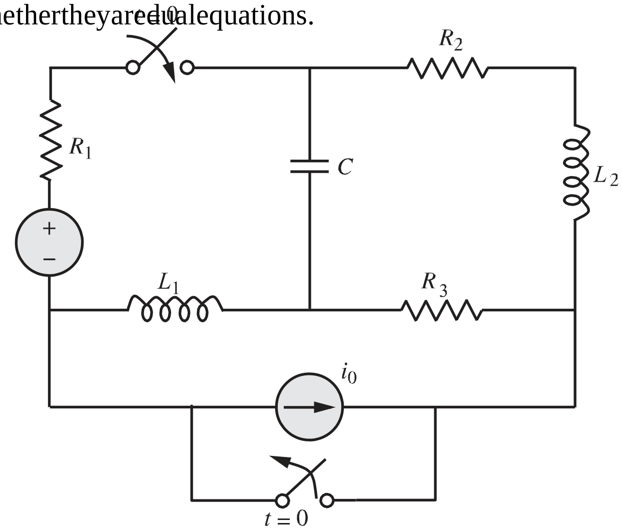
5. Convention for sources in the dual network:

(i) a clockwise current source in a loop corresponds to a voltage source with a positive polarity at the dual independent node.

(ii) a voltage rise in the direction of a clockwise loop current corresponds to a current flowing toward the dual independent node.

Example

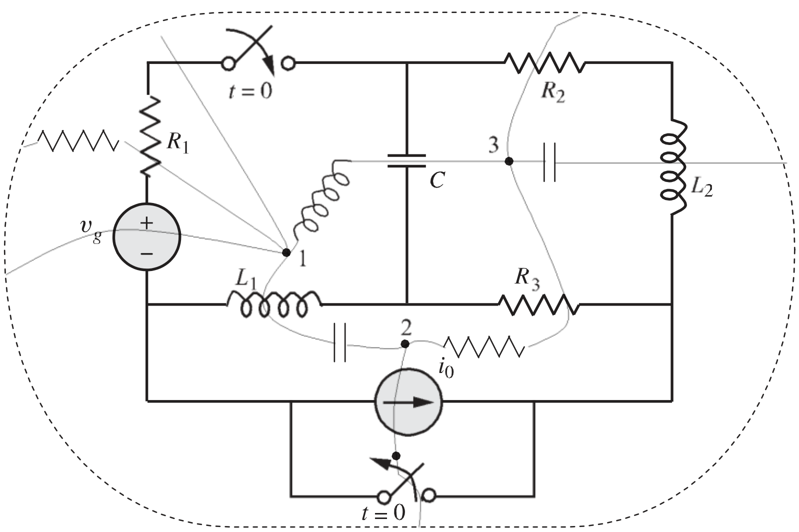
DrawthedualofthecircuitshowninFig.Writethemeshequationsforthegivennetworkandnodeequationsfo ritsdual.Verifywhethertheyaredualequations.



SOLUTION

Forthegivennetwork,themeshequationsare

Thedualnetwork,asperthetheproceduredescribedinthetheoryispreparedasshowninFig. andisdrawnasshownas . Thenodeequationsforthisnetworkare



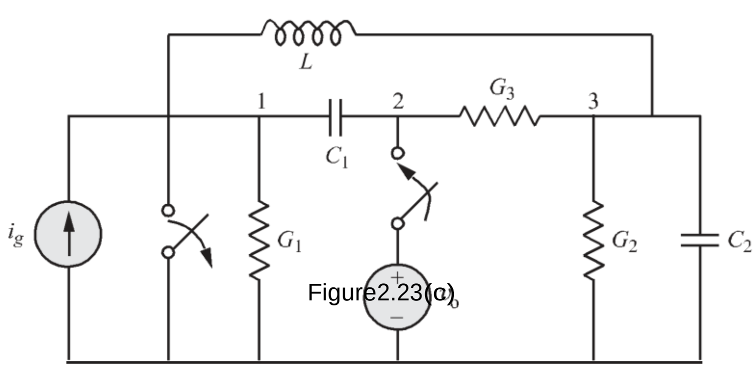


Figure2.23(c)

$$R_1 i_1 + L_1 D\,(i_1 - i_2) + \frac{1}{C}\int (i_1 - i_3)\,dt = v_g$$

$$i_2 = -i_0$$

$$R_2 i_3 + L_2 D i_3 + R_3\,(i_3 - i_2) + \frac{1}{C}\int (i_3 - i_2)\,dt = 0$$

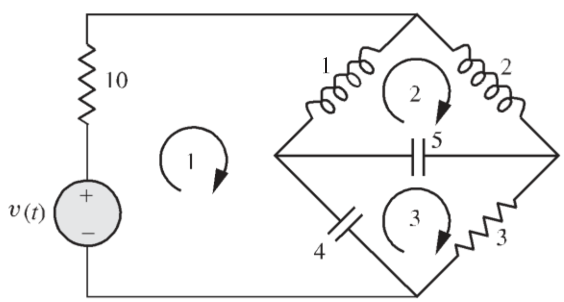
Dual equation

$$G_1 V_1 + C_1 D\,(v_1 - v_2) + \frac{1}{L}\int (v_1 - v_3)\,dt = i_g$$

$$G_2 v_3 + C_2 D v_3 + G_3\,(v_3 - v_2) + \frac{1}{r}\int (v_3 - v_2)\,dt = 0$$

EXAMPLE

ForthebridgenetworkshowninFigdrawitsdual.Writetheintegro-differentialformof themeshequationsforthegivennetworkandnodeequationsforitsdual.Thevaluesforresistors oneinohms,capacitorsareinfaradsandinductorsareinHenrys.



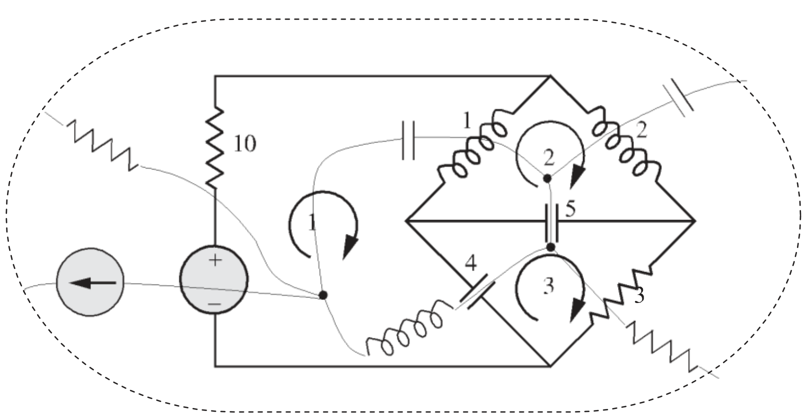
Figure

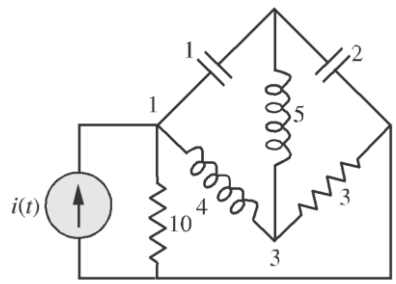
SOLUTION

ThedualforthegivennetworkisshowninFig.2.24(c)usingtheprocedureshowninFig.2.24(b).

Theintegro-differentialformforthenetworkis

– –





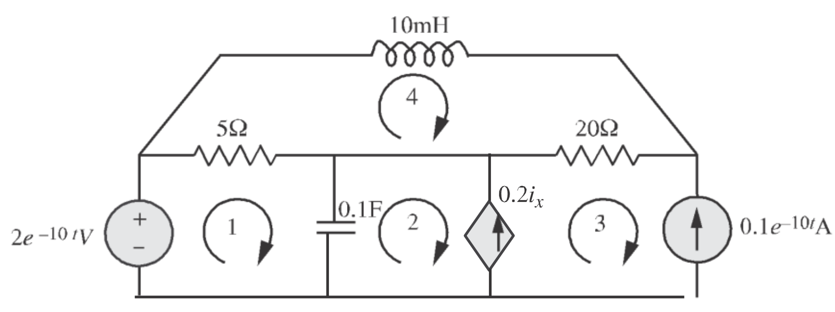
Thenodeequationsforthedualnetworkare

$$10i_1 + D(i_1 - i_2) + \frac{1}{4}\int (i_1 - i_3)\,dt = 10\sin 50t$$
$$D(i_2 - i_1) + 2Di_2 + \frac{1}{5}\int (i_2 - i_3)\,dt = 0$$
$$3i_3 + \frac{1}{4}\int (i_3 - i_1)\,dt + \frac{1}{5}\int (i_3 - i_2)\,dt = 0$$

DUAL

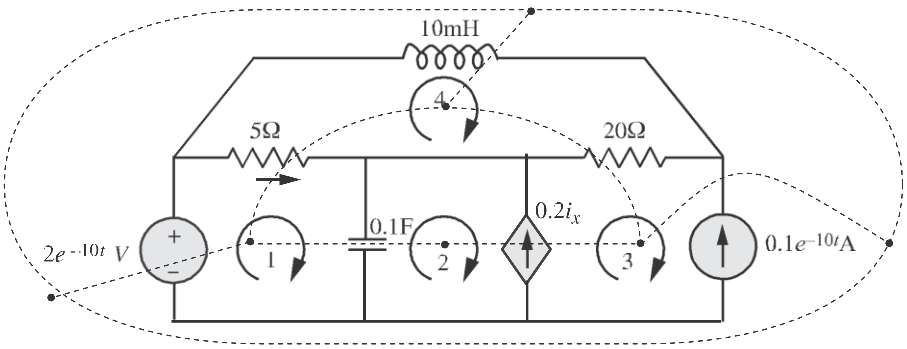
$$10v_1 + D(v_1 - v_2) + \frac{1}{4}\int (v_1 - v_3)\,dt = 10\sin 50t$$
$$D(v_2 - v_1) + 2Dv_2 + \frac{1}{5}\int (v_2 - v_3)\,dt = 0$$
$$3v_3 + \frac{1}{4}\int (v_3 - v_1)\,dt + \frac{1}{5}\int (v_3 - v_2)\,dt = 0$$

EXAMPLE 3



SOLUTION

ThedualforthegivennetworkisshowninFig.2.25(c)usingtheproceduregiveninFig.2.25(b).



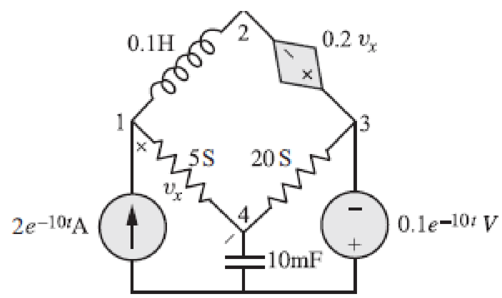


Figure 2.25(c)

Mesh equations for the given network are $i_x = i_1 - i_4$

$$5i_x + 10\int (i_1 - i_2)\, dt = 2e^{-10t}$$

$$i_2 - i_3 = -0.2i_x$$

$$i_3 = -0.1e^{-10t}$$

$$-5i_x + (i_4 - i_3)\,20 + 10 \times 10^{-3} D i_4 = 0$$

The node equations for the dual network are $v_x = v_1 - v_4$

$$5v_x + 10\int (v_1 - v_2)\, dt = 2e^{-10t}$$

$$v_2 - v_3 = -0.2v_x$$

$$v_3 = -0.1e^{-10t}$$

$$-5v_x + (v_4 - v_3)\,20 + 10 \times 10^{-3} D v_4 = 0$$