There are two types of analog design:

i) Butterworth filter
ii) chebyshev Method.

Butterworth filter design..

List of Butterworth polynomials Denominator of $H(s)$

N	
1	$s+1$
2	$s^2+\sqrt{2}s+1$
3	$(s+1)(s^2+s+1)$
4	$(s^2+0.76537s+1)(s^2+1.8477s+1)$
5	$(s+1)(s^2+0.61803s+1)(s^2+1.61803s+1)$
6	$(s^2+1.931855s+1)(s^2+\sqrt{2}s+1)(s^2+0.51764s+1)$
7	$(s+1)(s^2+1.80194s+1)(s^2+1.247s+1)(s^2+0.445s+1)$

Steps to Design an analog. Butterworth low pass filter:.

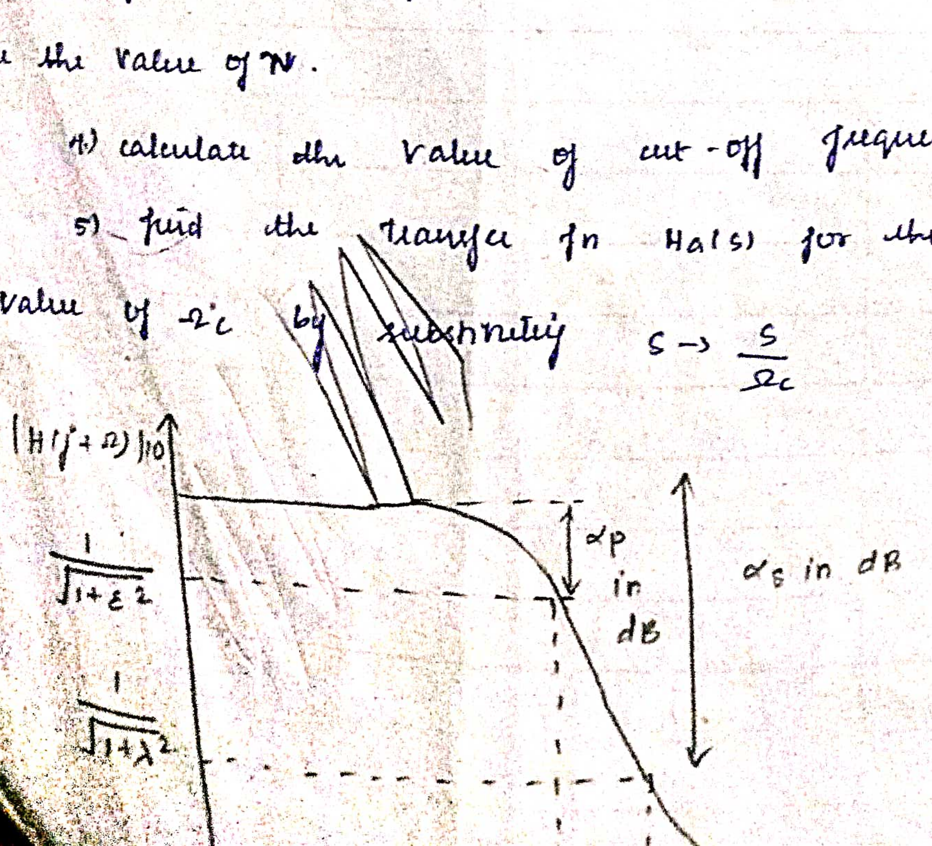
1) From the Given specifications find the order of the filter (N)

2. Round off it to the next higher integer.

3) find the Transfer fn $H(s)$ for $\Omega_c = 1$ rad/sec. for the value of N.

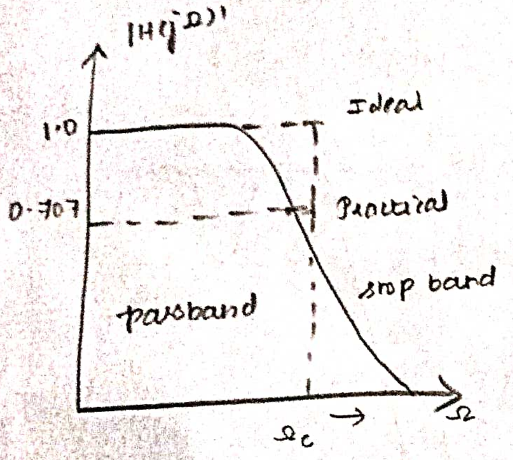
4) calculate the value of cut-off frequency Ω_c.

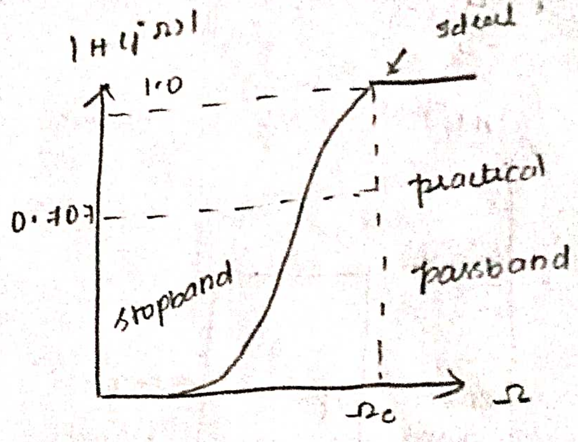
5) find the transfer fn $H_a(s)$ for the above value of Ω_c by substituting $s \to \frac{s}{\Omega_c}$



1. Lowpass filter.

The magnitude response of an ideal lowpass filter allow low frequencies in the pass band $0 < \Omega < \Omega_c$ to pass, whereas the higher frequencies in the stopband $\Omega > \Omega_c$ are blocked. The frequency Ω_c b/w the two band is cut-off frequency.





2. Highpass filter:

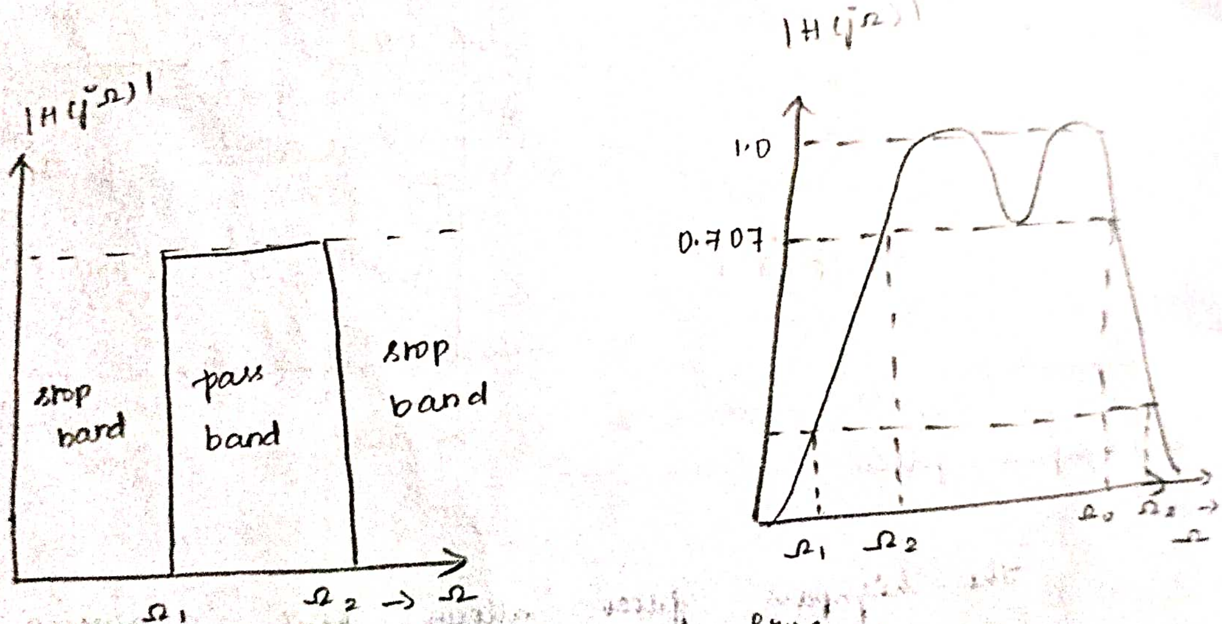
The highpass filter allows high frequencies above Ω & Ω_c and reject the frequencies b/w $\Omega = 0$ and $\Omega = \Omega_c$. The magnitude response of an ideal and practical highpass filter.

3. Band pass filter:

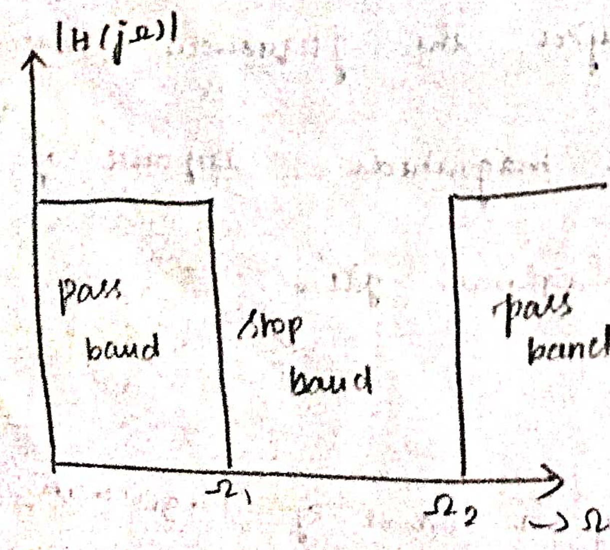
It allows only a band of frequencies, Ω_1 to Ω_2 to pass and stop all other frequencies the ideal and practical response of bandpass filter.

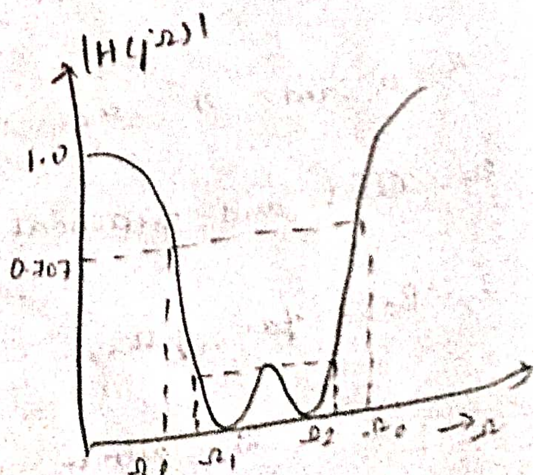
4. Band reject filter :

It reject all the frequencies b/w Ω_1 and Ω_2 and allows remaining frequencies. the magnitude response of an ideal and practical filter.



Bandpass filter :





1) Determine the order and the poles of low pass butterworth filter that has 3 db attenuation at 500 Hz and an attenuation of 40 dB at 1000 Hz.

$\alpha_p = 3dB$

$\alpha_s = 40dB$

$\Omega_p = 2\pi f = 2 \times \pi \times 500$

$\Omega_s = 2\pi f = 2 \times \pi \times 1000$

$\Omega_p = 1000\pi$ rad/sec

$\Omega_s = 2000\pi$ rad/sec.

order of filter

$$N = \frac{\log \sqrt{\dfrac{10^{0.1 \times \alpha_s} - 1}{10^{0.1 \times \alpha_p} - 1}}}{\log \left(\dfrac{\Omega_s}{\Omega_p}\right)}$$

$$N = \frac{\log \sqrt{\dfrac{10^{0.1 \times 40} - 1}{10^{0.1 \times 3} - 1}}}{\log \left(\dfrac{2000\pi}{1000\pi}\right)}$$

$$N = \frac{\log \sqrt{\dfrac{9999}{0.9952}}}{\log 2}$$

$N = 7$

Design a analog ... 2dB pass band attenuation at frequency of 20 rad/sec and atleast -10 dB stopband attenuation at 80 rad/sec.

Soln:

Given,

$\alpha_p = 2dB$, $\Omega_p = 20$ rad/sec.

$\alpha_s = 10 dB$, $\Omega_s = 50$ rad/sec.

$$N \geq \frac{\log \sqrt{\dfrac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}}}{\log \dfrac{\Omega_s}{\Omega_p}}$$

$$\geq \frac{\log \sqrt{\dfrac{10-1}{10^{0.2}-1}}}{\log \dfrac{30}{20}} = 3.37$$

Rounding off N to the next highest integer we get,

$N = 4$

The normalized lowpass butterworth filter for $N = 4$,

$$H(s) = \frac{1}{(s^2+0.765375s+1)(s^2+1.84775s+1)}$$

$$\Omega_c = \frac{\Omega_p}{(10^{0.1\alpha_p}-1)^{1/2N}}$$

$$= \frac{20}{(10^{0.1\times 2}-1)^{1/8}}$$

$$\Omega_c = 21.386$$

The transfer function for $\Omega_c = 21.3868$,

$$s \rightarrow \frac{s}{21.3864} \quad \text{in} \quad H(s)$$

i.e, $$H(s) = \frac{1}{\left[\frac{s}{21.3868}\right]^2 + 0.76537\left[\frac{s}{21.3868}\right]+1}$$

$$\times \frac{1}{\left[\frac{s}{21.3864}\right]^2 + 1.8477\left[\frac{s}{21.3868}\right]+1}$$

2 dB

$$= \frac{0.20921\times 10^6}{(s^2+16.3686s+457.394)(s^2+39.517s+457.394)}$$

$$\frac{1}{s^2+0.76537\times 21.3868}$$

$$H(s) = \frac{1}{\frac{(s^2+16.3686s+457.394)(s^2+39.5176s+457.394)}{(21.386)^2 * (21.386)^2}}$$

$$= \frac{1 \times (21.386)^2 * (21.386)^2}{(s^2+16.3686s+457.394)(s^2+39.5176s+457.394)}$$

$$= \frac{209179.0807}{(s^2+16.3686s+457.394)(s^2+39.5176s+457.394)}$$

3) Problem-3.

for the given specifications design an analog Butterworth filter.

$$0.9 \leq |H(j\Omega)| \leq 1 \text{ for } 0 \ \Omega \leq 0.2\pi$$

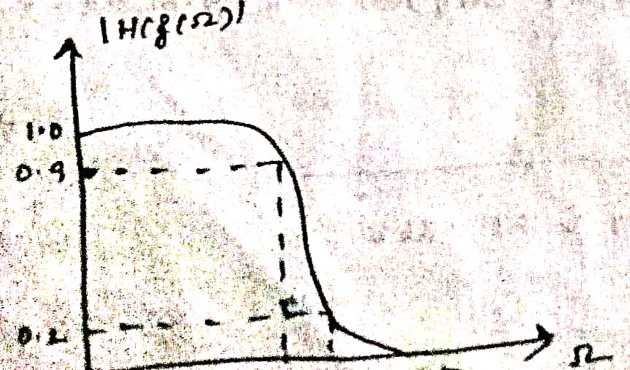
$$|H(j\Omega)| \leq 0.2 \text{ for } 0.4\pi \leq \Omega \leq \pi.$$

Solu:-

from the data we finds,

$\Omega_p = 0.2\pi$; $\frac{1}{\sqrt{1+\varepsilon^2}} = 0.9$

$\Omega_s = 0.4\pi$; $\frac{1}{\sqrt{1+\lambda^2}} = 0.2$



$\varepsilon = 0.484$ and $\lambda = 4.898$.

1) Order of the filter:.

$$N \geq \frac{\log(\lambda/\varepsilon)}{\log\left(\frac{\Omega_s}{\Omega_p}\right)}$$

$$\geq \frac{\log\left(\frac{4.898}{0.484}\right)}{\log\left(\frac{0.4\pi}{0.2\pi}\right)}$$

$$\frac{1}{\sqrt{1+\varepsilon^2}} = 0.9$$

$$\frac{1}{0.9} = \sqrt{1+\varepsilon^2}$$

$$(1.11)^2 = \left(\sqrt{1+\varepsilon^2}\right)^2$$

$$1.2321 = 1 + \varepsilon^2$$

$$= 0.231$$

$N \geq 3.339$

$\boxed{N = 4}$

2) Find the Transfer fn of Butterworth polynomial for $N = 4$

$$H(s) = \frac{1}{(s^2+0.7635+1)(s^2+1.8445+1)}$$

$$\frac{1}{\sqrt{1+\lambda^2}} = 0.2$$

3) Find cut-off frequency.

$$\Omega_c = \frac{\Omega_p}{(\varepsilon)^{1/N}} = \frac{0.2\pi}{(0.484)^{1/4}}$$

$\Omega_c = 0.75$ r/s.

$$0.04 = \frac{1}{1+\lambda^2}$$

$$1+\lambda^2 = \frac{1}{0.04}$$

$$1+\lambda^2 = 25$$

$$\lambda^2 = 1 - 25$$

$$\lambda^2 = 24$$

$$\lambda = \sqrt{24}$$

$$\lambda = 4.898$$

Chebyshev lowpass filter Design :

Step to design an analog chebyshev lowpass filter .

1) From the given specification find the order of the filter N

2) Round off it to the next higher integer.

3) using the following formula find the value of a and b which are minor and major axis of the ellipse respectively .

$$a = \Omega_p \frac{\left[\mu^{1/N} - \mu^{1/N}\right]}{2} \quad ; \quad b = \Omega_p \left[\frac{\mu^{1/N} + \mu^{1/N}}{2}\right]$$

where ,

$$\mu = \varepsilon^{-1} + \sqrt{\varepsilon^{-2} + 1}$$

$$\varepsilon = \sqrt{10^{0.1 \alpha p} - 1}$$

Ωp = passband frequency .

$\alpha p \Rightarrow$ Maximum allowable attenuation in the passband . (normalized chebyshev filter $\Omega p = 1$ rad/ sec) .

4. calculate the poles of chebyshev filter which lie on an ellipse by using the formula ,

$s_k = a \cos \phi_k + jb \sin \phi_k \; ; \; k = 1, 2, \dots, N$

where,

$$\phi_k = \frac{\pi}{2} + \left[\frac{2k-1}{2N}\right]\pi$$

5.) Find the denominator polynomial of the transfer function using the above poles.

6.) The numerator of the transfer function depends on the value of N.

a) For N odd substitute $s = 0$ in the denominator polynomial and find the value. This value is equal to the numerator of the transfer fn.

(for N odd the magnitude response $|H(j\Omega)|$ starts at 1).

b.) For N even substitude $s = 0$ in the denominator polynomial and divide the result by $\sqrt{1+\varepsilon^2}$. This value is equal to the numerator.

$\sqrt{1-\varepsilon^2}$

$s=0$

$s=0$

Chebyshev lowpass

Ex: 4.6 Given the specifications $\alpha_p = 3dB$, $\alpha_s = 16 dB$, $f_p = 1kHz$ and $f_s = 2kHz$. Determine the order of the filter using chebyshev approximation. Find H(s).

Solu:

From the given data,

$\Omega_p = 2\pi \times 1000 Hz = 2000\pi$ rad/sec.

$\Omega_s = 2\pi \times 2000 Hz = 4000\pi$ rad/sec

Step 1:

$$N \geq \frac{\cosh^{-1}\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\cosh^{-1}\frac{\Omega_s}{\Omega_p}}$$

$$= \frac{\cosh^{-1}\sqrt{\frac{10^{1.6}-1}{10^{0.3}-1}}}{\cosh^{-1}\frac{4000\pi}{2000\pi}}$$

$= 1.91$

Step 2:

Rounding N to next higher value we get $N=2$.

Step 3: the values of minor axis and major axis can be found as below.

$$\varepsilon = (10^{0.1\alpha p} - 1)^{0.5}$$

$$= (10^{0.3} - 1)^{0.5}$$

$$= 1$$

$$\mu = \varepsilon^{-1} + \sqrt{1 + \varepsilon^{-2}}$$

$$= 2.414$$

$$a = \Omega p \frac{[\mu^{1/N} - \mu^{1/N}]}{2}$$

$$= 2000\pi \frac{[(2.414)^{1/2} - (2.414)^{1/2}]}{2}$$

$$= 910\ \Omega$$

$$b = \Omega p \frac{[\mu^{1/N} + \mu^{-1/N}]}{2}$$

$$= 2000\pi \frac{[(2.414)^{1/2} + (2.414)^{-1/2}]}{2}$$

$$= 2197\pi$$

<u>Step 4</u>: the poles are given by,

$$s_k = a \cos \phi_k + j b \sin \phi_k \quad ; \ k = 1, 2$$

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N} \quad ; \ k = 1, 2$$

$$\phi_1 = \frac{\pi}{2} + \pi/4 = 135^\circ.$$

$$\phi_2 = \frac{\pi}{2} + \frac{8\pi}{4} = 225°$$

$s_1 = a\cos\phi_1 + jb\sin\phi_1$

$= -643.46\pi + j1554\pi$

$s_2 = a\cos\phi_2 + jb\sin\phi_2$

$= -643.46\pi - j1554\pi.$

Step 5: The Denominator of H(s)

$$H(s) = \frac{(s+643.46\pi)^2 + (1554\pi)^2}{\sqrt{1+\varepsilon^2}}$$

$= (1414.38)^2\pi^2$

The transfer function H(s),

$$H(s) = \frac{(1414.38)^2\pi^2}{s^2 + 1287\pi s + (1682)^2\pi^2}$$

Obtain analog chebyshev filter transfer for that satisfies the constraints.

$$\frac{1}{\sqrt{2}} \le |H(j\Omega)| \le 1 ; \quad 0 \le \Omega \le 2$$

$$|H(j\Omega)| \le 0.1 ; \quad \Omega \ge 4 .$$

Soln:

Step 1: From the given data, we can find that,

$$\frac{1}{\sqrt{1+\varepsilon^2}} = \frac{1}{\sqrt{2}}, \quad \frac{1}{\sqrt{1+\lambda^2}} = 0.1,$$

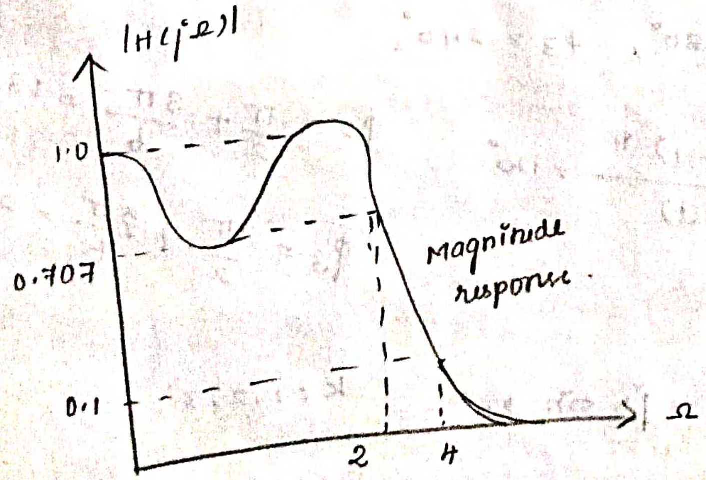
$\Omega_p = 2$ and $\Omega_s = 4$, from which can be obtain,

$\varepsilon = 1$ and $\lambda = 9.95$.

W.K.T

$$N \ge \frac{\cosh^{-1} \lambda/\varepsilon}{\cosh^{-1} \frac{\Omega_s}{\Omega_p}} = \frac{\cosh^{-1} 9.95}{\cosh^{-1} 2} = 2.269 .$$

Step 2: Rounding N to next higher value, $N = 3$.



Step 3: Finding the values of a & b:

$$\mu = \varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}} = 2.414$$

$$a = \Omega_p \left[\frac{\mu^{1/N} - \mu^{-1/N}}{2} \right]$$

$$= 2\left[\frac{(2.4114)^{1/3} - (2.4114)^{-1/3}}{2}\right]$$

$a = 0.596$.

$$b = \Omega_p\left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right]$$

$$= 2\left[\frac{(2.4114)^{1/3} + (2.4114)^{-1/3}}{2}\right]$$

$b = 2.087$.

Step 4:

To calculate the poles of chebyshev filter

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N} \qquad k = 1, 2, 3 \ldots$$

$\phi_1 = 120°$, $\phi_2 = 180°$, $\phi_3 = 240°$.

$$\phi_1 = \frac{\pi}{2} + \frac{(2(1)-1)\pi}{2(3)} = 120°$$

$$\phi_2 = \frac{\pi}{2} + \frac{3\pi}{6} = 180°$$

$$\phi_3 = \frac{\pi}{2} + \frac{5\pi}{6} = 240°$$

W.K.T

$$s_k = a\cos\phi_k + jb\sin\phi_k \qquad k = 1, 2, 3 \ldots$$

$$s_1 = a\cos\phi_1 + jb\sin\phi_1 \qquad (s - s_1)$$

$$= 0.596\cos 120° + j\,2.087\sin 120°$$

s.

$$= \underline{-0.298 + j\,1.807}$$

$S_2 = a\cos\phi_2 + jb\sin\phi_2$

$= 0.596\cos 180^\circ + j2.087\sin 180^\circ$

$= \underline{-0.596}$. $(s+0.596)$

$s_3 = a\cos\phi_3 + jb\sin\phi_3 = 0.596\cos 240^\circ + j2.087\sin 240^\circ$.

$= -0.298 - j1.807$ $(s+0.298) + j1.80$ $(a+b)$

Step 5:- The Denominator polynomial is given by,

$$(s+0.596)\{(s+0.298) - j1.807\}\{(s+0.298) + j1.807\}.$$

$$(s+0.596)\left((s+0.298)^2 + (1.807)^2\right).$$

$(1.807)^2$ a^2+b^2-2ab

$\cancel{(s+0.596)\ (s^2+0.08880H}$

$(s+0.596)\left(s^2+0.298s\right) + (1.807)^2)\ \left(s^2+2\times\frac{0.298s}{2}\right)$

$(s+0.596)\left(s^2+0.596s+3.354\right)$ $(s+0.298)$ s^2+2 3.26

Step 6: The numerator $H(s)$ can be obtained by

substituting $s=0$ (for N odd in Dr)

Therefore the Nr $H(s) = 2$

3.2652

$$H(s) = \frac{2}{(s+0.596)(s^2+0.596s+3.354)}$$

Passband attenuation of 2.5 dB at $\Omega p = 20$ rad/sec and the stopband attenuation of 30 dB at Ω

$\Omega_s = 50$ rad/sec.

Solu

Step 1:: From the given data we can find data,

$\alpha_p = 2.5$ dB, $\Omega_p = 20$ rad/sec | $\alpha_s = 30$ dB. $\Omega_s = 50$ rad/sec.

$$N \geq \frac{\cosh^{-1} \lambda/\varepsilon}{\cosh^{-1} \frac{\Omega s}{\Omega p}} \qquad N \geq \frac{\cosh^{-1} \lambda/\varepsilon}{\cosh^{-1} 1/k}$$

$$\lambda = \sqrt{10^{0.1\alpha s} - 1}$$

$$= \sqrt{10^{0.1 \times 30} - 1}$$

$$\boxed{\lambda = 31.606}$$

$$\varepsilon = \sqrt{10^{0.1\alpha p} - 1} = \sqrt{10^{0.1 \times 2.5} - 1}$$

$$\boxed{\varepsilon = 0.8822}$$

$$k = \frac{\Omega p}{\Omega s} = \frac{20}{50} = 0.4$$

$$\boxed{k = 0.4}$$

$$N \geq \frac{\cosh^{-1} \frac{31.606}{0.8822}}{\cosh^{-1} 1/0.4}$$

$$N = \frac{\cosh^{-1} 35.826}{\cosh^{-1} 2.5}$$

$$N = 2.726$$

$$\boxed{N = 3}$$

Step 3: Find the value of a & b.

$$\mu = \varepsilon^{-1} + \sqrt{1 + \varepsilon^{-2}} =$$

$$= 1.133 + \sqrt{1 + 1.2848}$$

$$\mu = 2.644 = 2.65$$

$$a = \Omega_p \left[\frac{\mu^{1/N} - \mu^{-1/N}}{2} \right]$$

$$= 20 \left[\frac{(2.65)^{1/3} - (2.65)^{-1/3}}{2} \right]$$

$$= 20 \left[\frac{1.3833 - 0.7226}{2} \right]$$

$$a = 6.6$$

$$b = \Omega_p \left[\frac{\mu^{1/N} + \mu^{-1/N}}{2} \right] = 20 \left[\frac{(2.65)^{1/3} + (2.65)^{-1/3}}{2} \right]$$

$$= 20 \left[\frac{1.3838 + 0.7226}{2} \right] = 21.064$$

$$b = 21.064$$

Step 4:

To calculate the poles of chebyshev filter.

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N} \quad k = 1, 2, 3 \ldots$$

$$\phi_k = \frac{\pi}{2} + \frac{(2(1)-1)}{2N}\pi = 120^\circ$$

$$\phi_2 = \frac{\pi}{2} + \frac{3\pi}{6} = 180^\circ$$

$$\phi_3 = \frac{\pi}{2} + \frac{5\pi}{6} = 240^\circ$$

W.K.T,

$$S_k = a\cos\phi_k + jb\sin\phi_k$$

$$S_1 = a\cos\phi_1 + jb\sin\phi_1$$

$$= 6.6\cos 120^\circ + j\,21.064\sin 120^\circ$$

$$S_1 = -3.3 + 18.241j$$

$$S_2 = a\cos\phi_2 + jb\sin\phi_2$$

$$= 6.6\cos 180^\circ + j\,21.064\sin 180^\circ$$

$$= -6.6 + 0j$$

$$S_3 = a\cos\phi_3 + jb\sin\phi_3$$

$$= 6.6\cos 240^\circ + j\,21.064\sin 240^\circ$$

$$= -3.3 + (-18.241)j$$

$$= -3.3 - j18.241$$

$(s-s_1)(s-s_2)(s-s_3)$

$(s-(-3.3+18.241j))(s-(-6.6))(s-(-3.3-j18.241))$

$(s+6.6)(s3.33+(18.241j)s)(s3.3+(-18.241)j)$

~~(s+6.6) (3.33~~

$(s+6.6)(s+(3.3-18.241j))(s+(3.3+18.241j))$

$(s+6.6)\ [(s+3.3)-18.241j]\ [(s+3.3)+18.24j]$

$(s+6.6)\ [(s+3.3)^2+(18.241)^2]$

$(s+6.6)\ [s^2+10.89+6.6s]+332.73$

$(s+6.6)(s^2+6.6s+343.62)$

Step 6 :· The numerator H(s) can be obtained by substituting s = 0, for N odd in Dr

Therefore the Nr H(s) = 2267.892

$$H(s) = \frac{2267.892}{(s+6.6)(s^2+6.6s+343.62)}$$

Design of IIR filter using Bilinear transformation:

Let us consider an analog linear filter with system fn,

$$H(s) = \frac{b}{s+a} \rightarrow (a)$$

W.k.t

$$\frac{Y(s)}{X(s)} = \frac{b}{s+a}$$

$$sY(s) + aY(s) = bX(s)$$

This can be characterised by the differential eq,

$$\frac{dy(t)}{dt} + ay(t) = bx(t) \rightarrow (1)$$

y(t) can be approximated by the trapezoidal formula, thus,

$$y(t) = \int_{t_0}^{t} y'(\tau)\, d\tau + y(t_0) \rightarrow (2)$$

where $y'(t)$ denotes the derivative of $y(t)$.

The approximation of the integral in eq (2) by the trapezoidal formula at $t = nT$ $t_0 = nT - T$ yields

$$y(nT) = \frac{T}{2}\left[y'(nT) + y'(nT-T)\right] + y(nT-T) \rightarrow (3)$$

from the differential Eq (1)

$$y'(nT) = -ay(nT) + bx(nT) \rightarrow (4)$$

Subs (4) in (3)

$$y(nT) = \frac{T}{2}\left[-ay(nT) + bx(nT) - ay(nT-T) + bx(nT-T)\right] + y(nT-T)$$

which implies,

$$y(nT) + \frac{aT}{2}y(nT) - \left[1 - \frac{aT}{2}\right].y(nT-T) = \frac{bT}{2}\left[x(nT) + x(nT-T)\right]$$

with $y(n) = y(nT)$

$x(n) = x(nT)$

$$\left[1 + \frac{aT}{2}\right]y(n) - \left[1 - \frac{aT}{2}\right]y(n-1) = \frac{bT}{2}\left[x(n) + x(n-1)\right]$$

$$\left[1 + \frac{aT}{2}\right]y(z) - \left[1 - \frac{aT}{2}\right]z^{-1}Y(z) = \frac{bT}{2}\left[1 + z^{-1}\right]X(z)$$

linear the system fn of the digital filter,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{bT/2\,[1 + z^{-1}]}{1 + \frac{aT}{2} - \left[1 - \frac{aT}{2}\right]z^{-1}}$$

$$= \frac{\frac{bT}{2}\,[1 + z^{-1}]}{(1 - z^{-1}) + \frac{aT}{2}(1 + z^{-1})}$$

Div num and Dnr by $\frac{T}{2}(1+z^{-1})$ we get,

$$H(z) = \frac{b}{\frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right] + a} \rightarrow (6)$$

Comparing eq (5) and (6),

the mapping from s-plane to the z-plane can be obtained as,

$$s = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right] \rightarrow (7)$$

This relationship between s and z is known as bilinear transformation,

$$\left.\begin{array}{l} \text{Let } z = re^{j\omega} \\ s = \sigma + j\Omega \end{array}\right\} \rightarrow (8) \qquad \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$$

Then (7) can be expressed as,

$$s = \frac{2(z-1)}{T(z+1)}$$

$$= \frac{2}{T}\left[\frac{re^{j\omega}-1}{re^{j\omega}+1}\right] = \frac{2}{T}\left[\frac{r\cos\omega - 1 + jr\sin\omega}{r\cos\omega + 1 + jr\sin\omega}\right]$$

$$= \frac{2}{T}\left[\frac{r\cos\omega - 1 + jr\sin\omega}{r\cos\omega + 1 + jr\sin\omega}\right]\left[\frac{r\cos\omega + 1 - jr\sin\omega}{r\cos\omega + 1 - jr\sin\omega}\right]$$

$$= \frac{2}{T}\left[\frac{r^2\cos^2\omega - 1 + r^2\sin^2\omega + j2r\sin\omega}{(r\cos\omega + 1)^2 + r^2\sin^2\omega}\right]$$

$$= \frac{2}{T}\left[\frac{r^2\cos^2\omega - 1 + r^2\sin^2\omega + j2r\sin\omega}{1 + r^2\cos^2\omega + 2r\cos\omega + r^2\sin^2\omega}\right]$$

Separating imaginary and real parts,

$$S = \frac{2}{T}\left[\frac{r^2-1}{1+r^2+2r\cos\omega} + j\frac{2r\sin\omega}{1+r^2+2r\cos\omega}\right] \rightarrow (7).$$

Comparing (6) and (7).

$$\sigma = \frac{2}{T}\left[\frac{r^2-1}{1+r^2+2r\cos\omega}\right] ; \quad \Omega = \frac{2}{T}\left[\frac{2r\sin\omega}{1+r^2+2r\cos\omega}\right]$$

We find that, if $r \le 1$, then $\sigma < 0$ & if $r > 1$, the $\sigma > 0$.

LHP in 's' maps into inside of the unit circle in the z-plane and RHP in 's' map into outside of the unit circle,

when $r = 1$, then $\sigma = 0$

$$\Omega = \frac{2}{T}\frac{\sin\omega}{1+\cos\omega} = \frac{2}{T}\frac{2\sin\omega/2\cos\omega/2}{2\cos^2\omega/2}$$

$$= \frac{2}{T}\tan\frac{\omega}{2} \rightarrow (8)$$

$$\omega = 2\tan^{-1}\frac{\Omega T}{2}$$

Warping effect: Let Ω and ω represent the frequency variables in the analog filter and the derived digital filter respectively. From eq (8) we have, (by using additional filter).

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2} \quad ; \text{for small value of } \omega$$

$$\Omega = \frac{2}{T}\frac{\omega}{2} = \frac{\omega}{T} \quad ; \omega = \Omega T. \qquad \boxed{\tan\theta = \theta}$$

Prewarping: The warping effect can be eliminated by prewarping the analog filter. This can be done by finding prewarping analog frequency using the for...

iii) Bilinear Transformation

iv) Matched z transformation technique.

Design of IIR filter using Impulse Invariance Technique:

In Impulse invariance method the IIR filter is designed such that the unit impulse response h(n) of digital filter is sampled version of the impulse response of analog filter.

⇒ The z-transform of an infinite impulse response is given by,

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n} \rightarrow ①$$

$$H(z)\Big|_{z=e^{ST}} = \sum_{n=0}^{\infty} h(n) e^{-STn}$$

Let us consider the mapping of points from the s-plane to the z-plane implied by the relation,

$$z = e^{ST} \rightarrow ②$$

If we sub $s = \sigma + j\Omega$ and express the complex variable z in polar form as $z = re^{j\omega}$ we get,

$$re^{j\omega} = e^{(\sigma+j\Omega)T}$$

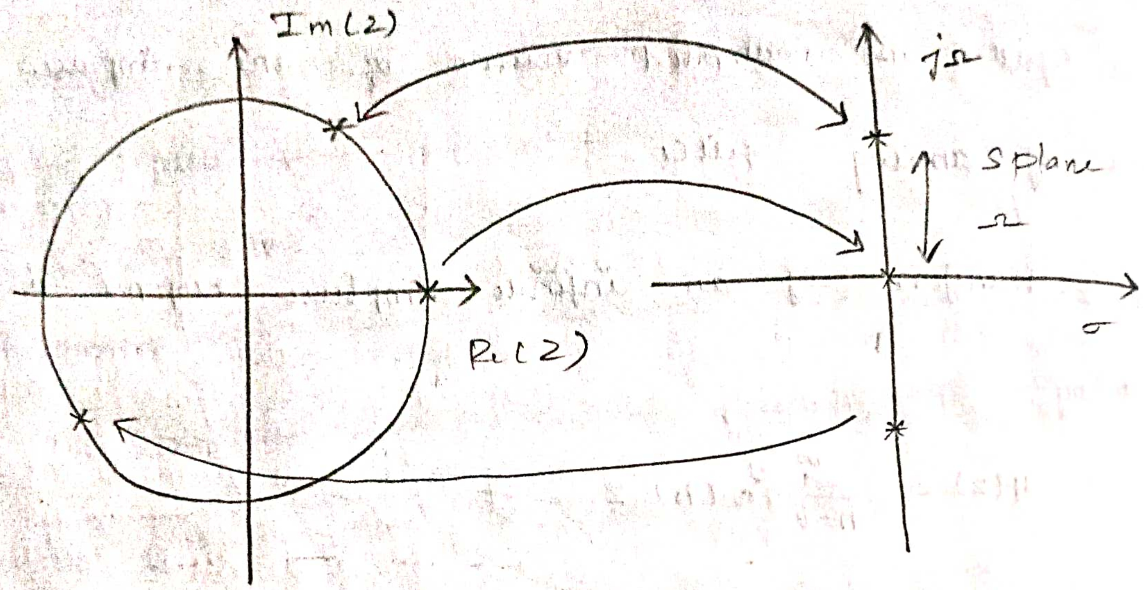
$$= e^{\sigma T} \cdot e^{j\Omega T} \rightarrow ③$$

Which gives,

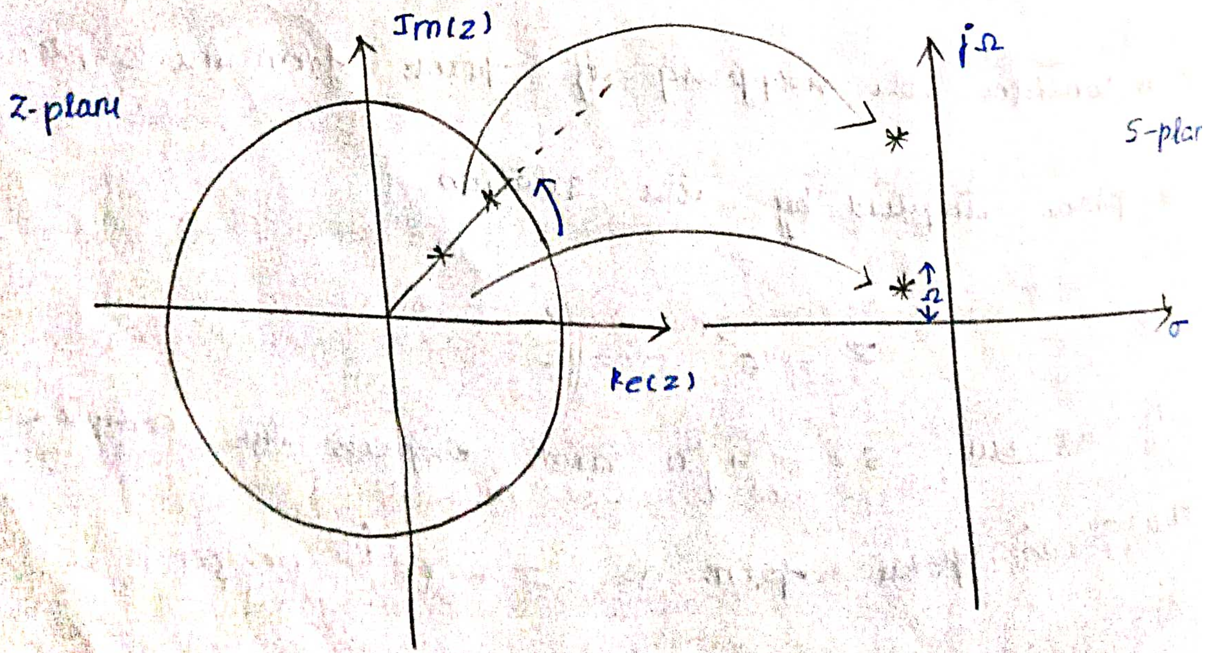
$$r = e^{\sigma T} \rightarrow ④$$

$\omega = \Omega T \rightarrow$ (5)

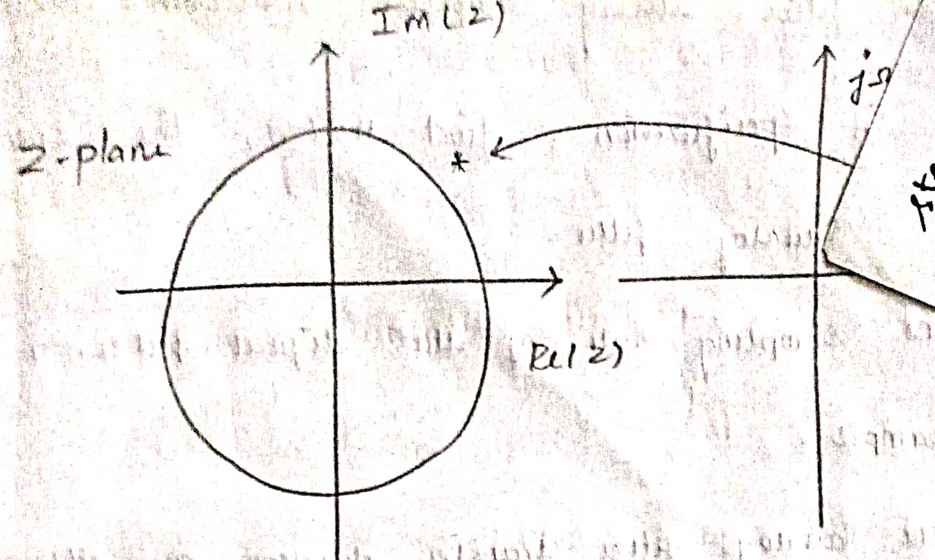
The first item in the product in eq (3), $e^{\sigma T}$, has a magnitude of $e^{\sigma T}$ and an angle of 0 - a real number.



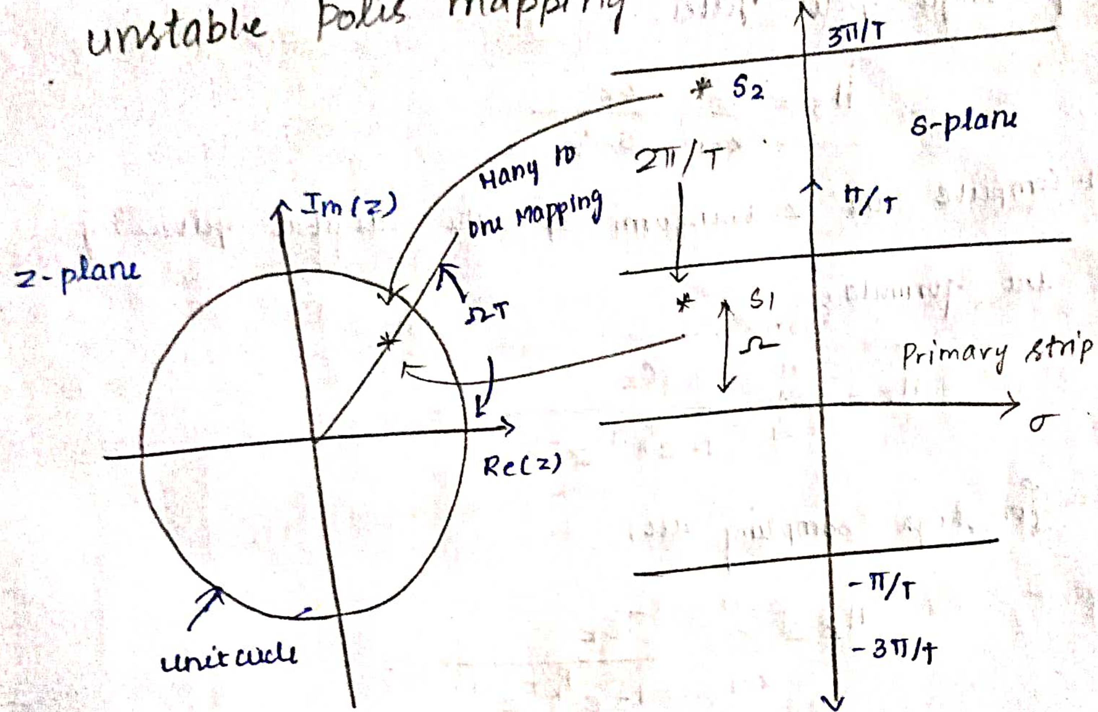
jΩ - axis mapping to unit circle.



Stable poles mapping inside the circle.



unstable poles mapping outside the circle.



Impulse invariant pole mapping

Let $H_a(s)$ is the system function of an analog filter this can be expressed in partial fraction from as,

$$H_a(s) = \sum_{K=1}^{N} \frac{C_K}{s - P_K} \rightarrow (b)$$

where $\{P_K\}$ are the poles of analog filter and $\{C_K\}$ are the co-efficient in the partial fraction expansion.

$$\boxed{H(z) = \sum_{K=1}^{N} \frac{T C_K}{1 - e^{P_K T} z^{-1}}}$$

steps to design filter using Impulse Invariant method

1. for the Given Specification, find Ha(s), the transfer function of an analog filter.

2. Select the sampling rate of the digital filter, T seconds per sample.

3. Express the analog filter transfer function as the sum of single pole filter.

$$H_S = \sum_{K=1}^{N} \frac{C_K}{S.P_K}$$

4. Compute the z-transform of the digital filter by using the formula.

$$H_Z = \sum_{K=1}^{N} \frac{C_K}{1-e^{P_K.T} z^{-1}}$$

for high sampling use,

$$H_Z = \sum_{K=1}^{N} \frac{TC_K}{1-e^{P_K T} z^{-1}} .$$

Ex:: 4.11. for the analog transfer fn $H(S) = \frac{2}{(S+1)(S+2)}$

Determine H(z) using Impulse Invariance method. Assume T = 1 sec.

4 $\frac{1}{2}$.

Solu:

Given,

$$H(S) = \frac{2}{(S+1)(S+2)}$$

Using partial fraction we can write.

$$H(S) = \frac{A}{S+1} + \frac{B}{S+2}$$

$$A = (s+1)\frac{2}{(s+1)(s+2)}\Bigg|_{s=-1}$$

$s=-1$

$= 2$

$$B = (s+2)\frac{2}{(s+1)(s+2)}\Bigg|_{s=-2}$$

$= -2$

subs A and B

$$\frac{2}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)}$$

$$H(s) = \frac{2}{s+1} - \frac{2}{s+2}$$

$$= \frac{2}{s-(-1)} - \frac{2}{s-(-2)}$$

$s - P_k \qquad 1-e^{P_k T} z^{-1} \qquad \frac{C_k}{s-P_k}$

using impulse invariance technique,

$$H(s) = \sum_{k=1}^{N} \frac{C_k}{s-P_k} \quad \text{then,} \qquad H(s) = \sum_{k=1}^{N} \frac{C_k}{s-P_k}$$

$$H(z) = \sum_{k=1}^{N} \frac{C_k}{1-e^{P_k T} z^{-1}}$$

(i.e) $(s-P_k)$ is transformed to $1-e^{P_k T} z^{-1}$.

$1-e^{P_k T} z^{-1}$

there are two poles,

$P_1 = -1$

$P_2 = -2$.

$$H(z) = \frac{2}{1-e^{-T}z^{-1}} - \frac{2}{1-e^{-2T}z^{-1}}$$

e^{-1}

0.36

for $T = 1$ sec

$$H(z) = \frac{2}{1-e^{-1}z^{-1}} - \frac{2}{1-e^{-2}z^{-1}}$$

e^{-2}

$$= \frac{2}{1-0.3678z^{-1}} - \frac{2}{1-0.1353z^{-1}}$$

$$H(z) = \frac{2}{1-0.3678z^{-1}} - \frac{2}{}$$

$$2(1-0.1353z^{-1}) - 2(1-0.3678z^{-1}) \qquad 1.7294z^{-1}$$

$$(2-0.1353z^{-1}) - (2 - \qquad 1.2644$$

$$= \frac{2(1-0.1353z^{-1}) - 2(1-0.3678z^{-1})}{1-0.503z^{-1}+0.04976z^{-2}}$$

$$H(z) = \frac{0.465z^{-1}}{1-0.503z^{-1}+0.04976z^{-2}}$$

Apply impulse invariant method and find $H(z)$ for

$$H(s) = \frac{s+a}{(s+a)^2+b^2}$$

Soln. the inverse laplace transform of given fn is,

$$h(t) = \begin{cases} e^{-at}\cos(bt) & \text{for } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Simpling the fn produces,

$$h(nT) = \begin{cases} e^{-anT}\cos(bnT) & \text{for } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$H(z) = \sum_{n=0}^{\infty} e^{-anT}\cos[bnT]\,z^{-n}.$$

$$= \sum_{n=0}^{\infty}\left[e^{-anT}z^{-n}\left[\frac{e^{jbnT}+e^{-jbnT}}{2}\right]\right]$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left[\left(e^{-(a-jb)T}z^{-1}\right)^n + \left(e^{-(a+jb)T}z^{-1}\right)^n\right]$$

$$= \frac{1}{2}\left[\frac{1}{1-e^{-(a-jb)T}z^{-1}} + \frac{1}{1-e^{-(a+jb)T}z^{-1}}\right]$$

$$= \frac{1-e^{-aT}\cos(bT)z^{-1}}{1-2e^{-aT}\cos(bT)z^{-1}+e^{-2aT}z^{-2}}$$

Ex 4.15 :: An analog filter has a transfer fn $H(S) = \frac{10}{S^2+7S+10}$

Design a digital filter equivalent to this using impulse invariant method for $T = 0.2$ sec

solu:: Given

$$H(S) = \frac{10}{S^2+7S+10}$$

$$= \frac{-3.33}{S+5} + \frac{3.33}{S+2}$$

$$= \frac{-3.33}{S-(-5)} + \frac{3.33}{S-(-2)}$$

$H(S)$ $P_1 = -5$

$P_2 = -2$

$$H(z) = \frac{-3.33}{1-e^{-5T}z^{-1}} - \frac{3.33}{1-e^{-2T}z^{-1}}$$

$T=1$ sec:

$T=0.2$

$$= \frac{3.33}{1-e^{-5\times 0.2}z^{-1}} - \frac{3.33}{1-e^{-2\times 0.2}z^{-1}}$$

$$= \left[\frac{0.66}{1-0.3678z^{-1}} + \frac{0.666}{1-0.67z^{-1}}\right]$$

$$= \frac{0.2012z^{-1}}{1-1.0378z^{-1}+0.247z^{-2}}$$

Ex 4.16. Apply bilinear Transformation to $H(s) = \frac{2}{(s+1)(s+2)}$ with T = 1 sec and find H(z)

Soln:

Given, $H(s) = \frac{2}{(s+1)(s+2)}$

subs $s = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$ in H(s) to get H(z)

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}$$

$$= \frac{2}{(s+1)(s+2)}\Bigg|_{s=\frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}$$

T = 1 sec

$$H(z) = \frac{2}{\left[2\left\{\frac{1-z^{-1}}{1+z^{-1}}\right\}+1\right]\left[\frac{2}{\cancel{T}}\left\{\frac{1-z^{-1}}{1+z^{-1}}\right\}+2\right]}$$

$$= \frac{2(1+z^{-1})^2}{(3-z^{-1})(4)}$$

$$2\left[\frac{1-z^{-1}+1+z^{-1}}{1+z^{-1}}\right]$$

$$= \frac{(1+z^{-1})^2}{6-2z^{-1}}$$

$$= \cancel{2}\,4\,[$$

$$= \frac{0.1666(1+z^{-1})^2}{(1-0.33z^{-1})}$$

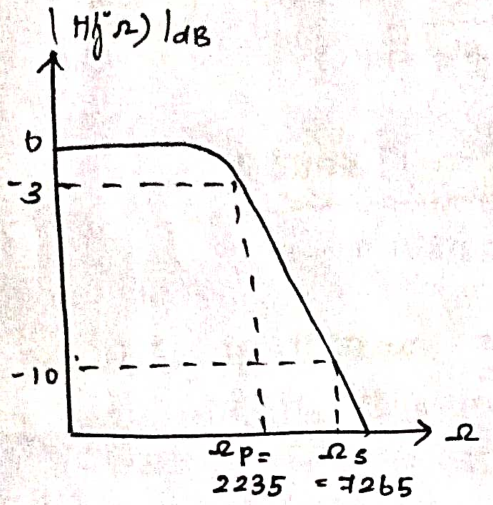
Ex. Using the bilinear transform, design a HPF filter monotonic in passband with cut-off frequency of 1000 Hz and down 10 dB at 350 Hz. The sampling frequency is 5000 Hz.

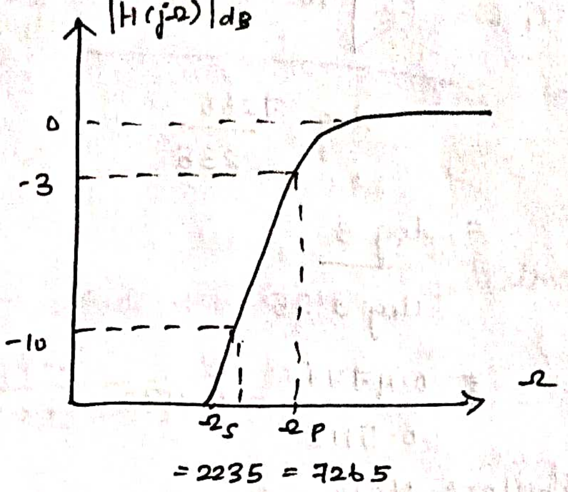
Solu:

Given $\alpha_p = 3dB$; $\omega_c = \omega_p = 2 \times \pi \times 1000 = 2000\pi$ rad/sec

$\alpha_s = 10dB$; $\omega_s = 2 \times \pi \times 350 = 700\pi$ rad/sec

$$T = \frac{1}{f} = \frac{1}{5000} = 2 \times 10^{-4} \text{ sec}.$$





The characteristics are monotonic in both passband and stopband. Therefore the filter is Butterworth filter.

Prewarping the digital frequencies we have,

$$\left[\frac{1-z^{-1}}{1+z^{-1}} + \frac{2(1+z^{-1})}{}\right]$$

$$\Omega_p = \frac{2}{T} \tan\frac{\omega_p T}{2} = \frac{2}{2 \times 10^{-4}} \tan\frac{(2000\pi \times 2 \times 10^{-4})}{2}$$

$$\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$$

$$= 10^4 \tan(0.2\pi)$$

$$= 7265 \text{ rad/sec}.$$

1 -

$$\Omega_s = \frac{2}{T} \tan\frac{\omega_s T}{2} = \frac{2}{2 \times 10^{-4}} \tan\frac{(700\pi \times 2 \times 10^{-4})}{2}$$

$$= 10^4 \tan(0.07\pi)$$

$$= 2235 \text{ rad/sec}$$

First we design a LPF for the given specifications and use suitable transformation to obtain transfer fn of HPF,

the order of the filter,

6.9 Frequency sampling method of designing FIR filters

Let $h(n)$ is the filter coefficients of an FIR filter and $H(k)$ is the DFT of $h(n)$. Then we have

$$h(n) = \frac{1}{N}\sum_{k=0}^{N-1} H(k)e^{j2\pi kn/N} \quad n = 0, 1, \ldots N-1 \tag{6.110}$$

and

$$H(k) = \sum_{k=0}^{N-1} h(n)e^{-j2\pi kn/N} \quad k = 0, 1, \ldots N-1 \tag{6.111}$$

The DFT samples $H(k)$ for an FIR sequence can be regarded as samples of the filter z-transform evaluated at N-points equally spaced around the unit circle. That is,

$$H(k) = H(z)\Big|_{z=e^{j2\pi k/N}} \tag{6.112}$$

The transfer function $H(z)$ of an FIR filter with impulse response is given by

$$H(z) = \sum_{n=0}^{N-1} h(n)z^{-n} \tag{6.113}$$

Substituting Eq.(6.110) in Eq.(6.113) we get

$$\begin{aligned} H(z) &= \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{k=0}^{N-1} H(k)e^{j2\pi kn/N}\right] z^{-n} \\ &= \sum_{k=0}^{N-1}\frac{H(k)}{N}\sum_{n=0}^{N-1} H(k)\left(e^{j2\pi k/N}z^{-1}\right)^n \\ &= \sum_{k=0}^{N-1}\frac{H(k)}{N}\frac{1-\left(e^{j2\pi k/N}z^{-1}\right)^N}{1-e^{j2\pi k/N}z^{-1}} \\ &= \frac{1-z^{-N}}{N}\sum_{k=0}^{N-1}\frac{H(k)}{1-e^{j2\pi k/N}z^{-1}} \end{aligned} \tag{6.114}$$

We know

$$H(e^{j\omega})\Big|_{\omega=\frac{2\pi k}{N}} = H(e^{j2\pi kn/N}) = H(k) \tag{6.115}$$

That is $H(k)$ is kth DFT component obtained by sampling the frequency response $H(e^{j\omega})$. As such this approach for designing FIR filter is called the frequency sampling method.

6.9.1 *Frequency Sampling Realization*

The Eq. (6.114) can be written as

$$H(z) = \frac{1 - z^{-N}}{N} \sum_{k=0}^{N-1} G_k(z) \tag{6.116}$$

where

$$G_k(z) = \frac{H(k)}{1 - e^{j2\pi k/N} z^{-1}} \quad 0 \le k \le N-1 \tag{6.117}$$

is the transfer function of first order FIR filters, where poles lie on the unit circle at equidistant points. The Eq.(6.116) can be realized as shown in Fig. 6.56.

$G_k(z)$ in Eq. (6.116) are sometimes called resonant filters, because they are resonant at the sample values of k^{th} frequency.

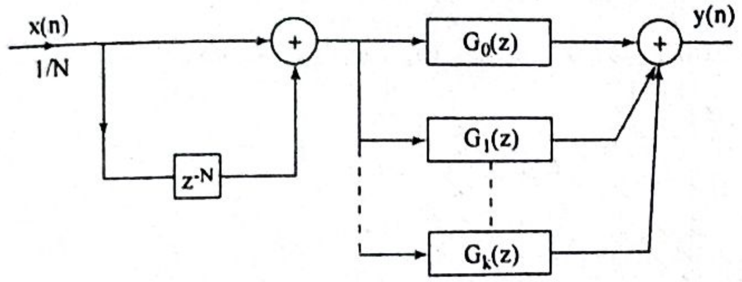


Fig. 6.56 Frequency sampling realization of Eq.(6.116)

6.9.2 *Frequency response*

The frequency response of the FIR filter can be obtained by setting $z = e^{j\omega}$ in Eq.(6.116)

$$\begin{aligned}
H(e^{j\omega}) &= \frac{1 - e^{-j\omega N}}{N} \sum_{k=0}^{N-1} \frac{H(k)}{1 - e^{j2\pi k/N} e^{-j\omega}} \\
&= \frac{e^{-j\omega N/2}(e^{j\omega N/2} - e^{-j\omega N/2})}{N} \sum_{k=0}^{N-1} \frac{H(k)}{1 - e^{-j(\omega - 2\pi k/N)}} \\
&= \frac{e^{-j\omega N/2}}{N} \sum_{k=0}^{N-1} \frac{H(k)(e^{j\omega N/2} - e^{-j\omega N/2})}{e^{-j(\omega/2 - \pi k/N)} \left[e^{j(\omega/2 - \pi k/N)} - e^{-j(\omega/2 - \pi k/N)}\right]} \\
&= \frac{e^{-j\omega(N-1)/2}}{N} \sum_{k=0}^{N-1} \frac{H(k) e^{-j\pi k/N} \sin \omega N/2}{\sin(\omega/2 - \pi k/N)}
\end{aligned}$$

$$= \frac{e^{-j\omega(N-1)/2}}{N} \sum_{k=0}^{N-1} \frac{H(k)(-1)^k e^{-j\pi k/N} \sin N(\omega/2 - k\pi/N)}{\sin(\omega/2 - \pi k/N)} \tag{6.118}$$

$$\boxed{\begin{aligned} \because \sin\left(\tfrac{\omega N}{2} - k\pi\right) \\ = (-1)^k \sin \tfrac{\omega N}{2} \end{aligned}}$$

6.9.3 *Design*

We exploit the basic symmetry property of the sampled frequency response to simplify the computations in designing an FIR filter. Based on the set of samples that we choose from the frequency response, there are two types of design.

Type 1 design

In this type of design the frequency samples of the desired response $H_d(e^{j\omega})$ are determined, using the relation

$$H(k) = H_d(e^{j\omega})\Big|_{\omega=\frac{2\pi}{N}k} \quad k = 0, 1, \ldots N-1 \tag{6.119}$$

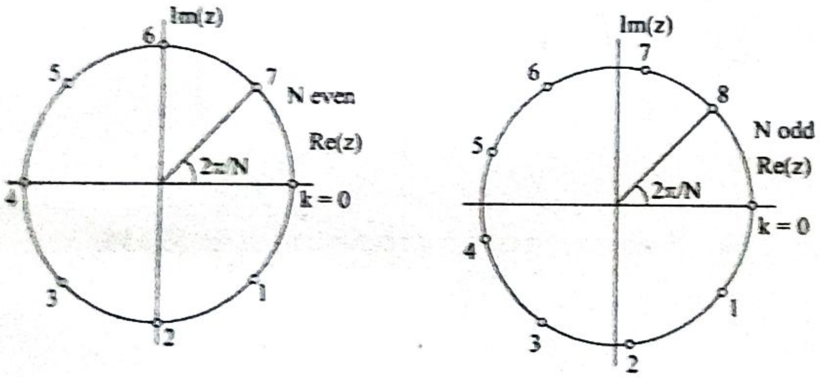


Fig. 6.57 Location of DFT samples on the unit circle for type 1 design

Fig. 6.57 illustrate exactly where the frequency samples are located. The initial point is located at $\omega = 0$ and the spacing between two points is $\frac{2\pi}{N}$.

The frequency samples can be expressed in the form

$$H(k) = |H(k)|e^{j\theta(k)} \tag{6.120}$$

For linear phase

$$\begin{aligned} \theta(k) &= -\alpha\,\omega\Big|_{\omega=\frac{2\pi}{N}k} \quad k = 0, 1, \ldots N-1 \\ &= -\left(\frac{N-1}{2}\right)\frac{2\pi}{N}k \\ &= -\left(\frac{N-1}{N}\right)\pi k \quad k = 0, 1, \ldots, N-1 \end{aligned} \tag{6.121}$$

The filter coefficients $h(n)$ can be obtained by finding IDFT of $H(k)$, i.e.,

$$h(n) = \frac{1}{N}\sum_{k=0}^{N-1} H(k)e^{j2\pi nk/N} \quad n = 0, 1, \ldots N-1 \tag{6.122}$$

If $h(n)$, the impulse response of the filter is to be a real valued signal, the frequency samples $H(k)$ must satisfy the symmetry requirement, i.e.,

for N odd or even

$$H(N-k) = H^*(k) \quad k = 0, 1, \ldots N-1 \tag{6.123}$$

in addition, for N even $H\left(\frac{N}{2}\right) = 0$

With the frequency response $H(k)$, the magnitude response is an even function

$$|H(k)| = |H(N-k)| \quad k = 0, 1, \ldots N-1 \tag{6.124}$$

and the phase is an odd function

$$\theta(k) = -\theta(N-k) \quad k = 0, 1, \ldots N-1 \tag{6.125}$$

In Eq.(6.121) replacing k by $N-k$, we get

$$\begin{aligned}\theta(N-k) &= -\left(\frac{N-1}{N}\right)\pi(N-k)\\ &= -(N-1)\pi + \left(\frac{N-1}{N}\right)\pi k\end{aligned}$$

To satisfy the requirements of Eq.(6.125), $\theta(k)$ for N odd is given by

$$\begin{aligned}\theta(k) &= -\left(\frac{N-1}{N}\right)\pi k \quad k = 0, 1, 2, \ldots, \frac{N-1}{2}\\ &= (N-1)\pi - \left(\frac{N-1}{N}\right)\pi k \quad k = \frac{N+1}{2}, \ldots N-1\end{aligned} \tag{6.126}$$

Similarly for N even

$$\begin{aligned}\theta(k) &= -\left(\frac{N-1}{N}\right)\pi k \quad k = 0, 1, 2, \ldots \frac{N}{2}-1\\ &= (N-1)\pi - \left(\frac{N-1}{N}\right)\pi k \quad k = \frac{N}{2}+1, \ldots, N-1\\ &= 0 \qquad k = \frac{N}{2}\end{aligned} \tag{6.127}$$

Substituting Eq. (6.126) in Eq. (6.120) we get for N odd

$$\begin{aligned}H(k) &= |H(k)|e^{-j(N-1)\pi k/N} \quad k = 0, 1, \ldots \frac{N-1}{2}\\ &= |H(k)|e^{j[(N-1)\pi-(N-1)\pi k/N]} \quad k = \frac{N+1}{2}, \ldots, N-1\end{aligned} \tag{6.128}$$

Substituting Eq. (6.127) in Eq. (6.120) we get for N even

$$\begin{aligned} H(k) &= |H(k)|e^{-j(N-1)\pi k/N} \quad k = 0, 1, \ldots \frac{N}{2} - 1 \\ &= |H(k)|e^{j[(N-1)\pi - (N-1)\pi k/N]} \quad k = \frac{N}{2} + 1, \ldots, N - 1 \\ &= 0 \quad k = \frac{N}{2} \end{aligned} \tag{6.129}$$

If the filter is to be linear phase, the $h(n)$ must also satisfy the symmetry condition

$$h(n) = h(N - 1 - n) \tag{6.130}$$

This symmetry condition, along with the symmetry condition for $H(k)$, can be used to reduce the frequency specifications from N points to $\frac{N+1}{2}$ points for N odd, and $\frac{N}{2}$ points for N even. Substituting Eq.(6.123) in Eq.(6.122) the filter coefficients can be written as

$$h(n) = \frac{1}{N}\left\{H(0) + 2\sum_{k=1}^{(N-1)/2} Re\left[H(k)e^{j2\pi kn/N}\right]\right\} \quad N \text{ odd} \tag{6.131}$$

and

$$h(n) = \frac{1}{N}\left\{H(0) + 2\sum_{k=1}^{\frac{N}{2}-1} Re\left[H(k)e^{j2\pi kn/N}\right]\right\} \quad N \text{ even} \tag{6.132}$$

Once the filter coefficients $h(n)$ have been determined, the system function of the filter is given by

$$H(z) = \sum_{n=0}^{N-1} h(n)z^{-n} \tag{6.133}$$

Type 2 design

In this type of design the frequency samples $H(k)$ are obtained using the relation

$$\begin{aligned} H(k) &= H_d(e^{j\omega})\Big|_{\omega = \frac{2\pi}{N}\left(k+\frac{1}{2}\right)} \\ &= H_d\left(e^{j\pi(2k+1)/N}\right) \quad k = 0, 1, \ldots, N - 1 \end{aligned} \tag{6.134}$$

Fig. 6.58 shows exactly where the frequency samples are located. The initial point is located at $\omega = \frac{\pi}{N}$ and the spacing between two points is $\frac{2\pi}{N}$.

The filter coefficients can be obtained using the relation

$$h(n) = \sum_{k=0}^{N-1} H(k)e^{j2\pi kn/N} \quad k = 0, 1, \ldots N - 1 \tag{6.135}$$

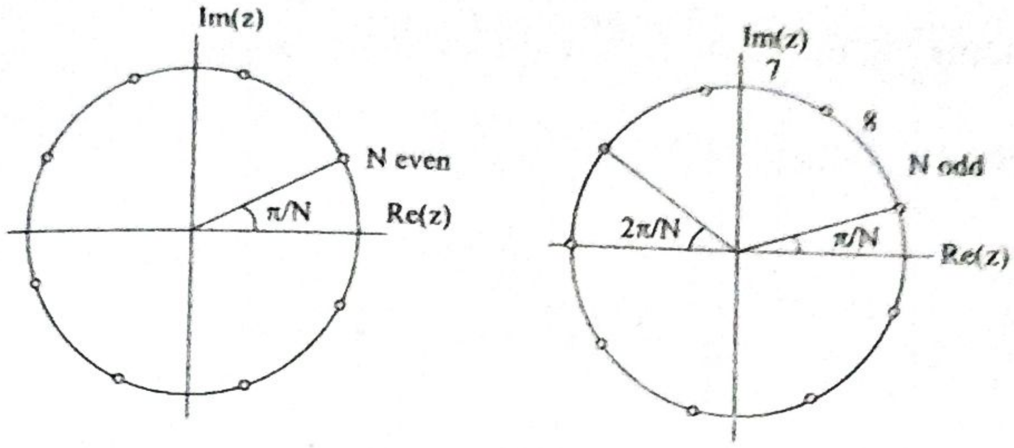


Fig. 6.58 Location of DFT samples on the unit circle for type 2 design

The condition that $h(n)$ be real is,
for N odd

$$H(N-k-1) = H^*(k) \quad k = 0, 1, \ldots \frac{N-1}{2} - 1$$

$$H\left(\frac{N-1}{2}\right) = 0 \tag{6.136}$$

and for N even

$$H(N-k-1) = H^*(k) \quad k = 0, 1, \ldots \frac{N}{2} - 1 \tag{6.137}$$

When these conditions are satisfied, the filter coefficients for N odd are

$$h(n) = \frac{2}{N} \sum_{k=0}^{\frac{N-3}{2}} Re\left[H(k)e^{jn\pi(2k+1)/N}\right] \tag{6.138}$$

and for N even

$$h(n) = \frac{2}{N} \sum_{k=0}^{\frac{N}{2}-1} Re\left[H(k)e^{jn\pi(2k+1)/N}\right] \tag{6.139}$$

Exampl

Example 6.5 Design an ideal lowpass filter with a frequency response

$$H_d(e^{j\omega}) = 1 \text{ for } -\frac{\pi}{2} \leq \omega \leq \frac{\pi}{2}$$
$$= 0 \text{ for } \frac{\pi}{2} \leq |\omega| \leq \pi$$

using Blackman window

Find the values of $h(n)$ for $N = 11$. Find $H(z)$. Plot the magnitude response.

Solution

The frequency response of lowpass filter with $\omega_c = \frac{\pi}{2}$ is shown in Fig. 6.8.
Given

$$H_d(e^{j\omega}) = 1 \text{ for } \frac{-\pi}{2} \leq \omega \leq \frac{\pi}{2}$$
$$= 0 \text{ for } \frac{\pi}{2} \leq |\omega| \leq \pi$$

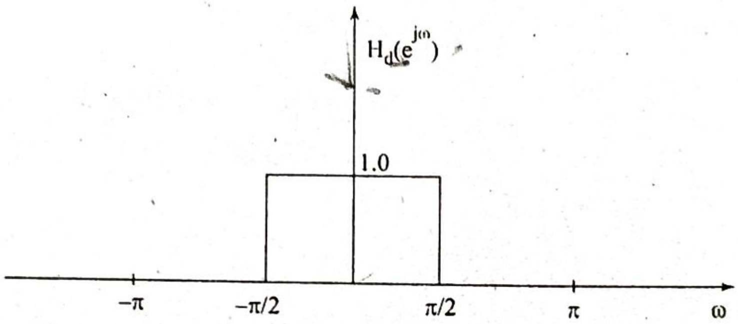


Fig. 6.8 Ideal frequency response of Example 6.5

From the frequency response we can find that $\alpha = 0$. Therefore, we get a non-causal filter coefficients symmetrical about $n = 0$, i.e., $h_d(n) = h_d(-n)$. The filter coefficients can be obtained by using the formula given in table 6.2 for zero phase frequency response (or) we can proceed as follows.

We know

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}d\omega$$
$$= \frac{1}{2\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} e^{j\omega n}d\omega \qquad (6.54)$$
$$= \frac{1}{2\pi jn} e^{j\omega n}\Big|_{-\pi/2}^{\pi/2}$$
$$= \frac{1}{\pi n(2j)}\left[e^{j\pi n/2} - e^{-j\pi n/2}\right]$$
$$= \frac{\sin\frac{\pi}{2}n}{\pi n} \qquad -\infty \leq n \leq \infty \qquad (6.55)$$

Truncating $h_d(n)$ to 11 samples, we have

$$h(n) = \frac{\sin\frac{\pi}{2}n}{\pi n} \quad \text{for} \quad |n| \le 5$$
$$= 0 \quad \text{otherwise} \tag{6.56}$$

For $n = 0$ Eq. (6.56) becomes indeterminate. So

$$h(0) = \lim_{n\to 0} \frac{\sin\frac{\pi}{2}n}{\pi n} = \frac{1}{2} \lim_{n\to 0} \frac{\sin\frac{\pi}{2}n}{\frac{\pi n}{2}} = \frac{1}{2} \qquad \boxed{\therefore \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1}$$

(or)

Substitute $n = 0$ in Eq. (6.54) we get

$$h(0) = h_d(0) = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} d\omega = \frac{1}{2\pi} \omega \Big|_{-\pi/2}^{\pi/2} = \frac{\pi}{2\pi} = \frac{1}{2}.$$

For $n = 1$

$$h(1) = h(-1) = \frac{\sin\frac{\pi}{2}}{\pi} = \frac{1}{\pi} = 0.3183.$$

Similarly

$$h(2) = h(-2) = \frac{\sin\pi}{2\pi} = 0$$
$$h(3) = h(-3) = \frac{\sin\frac{3\pi}{2}}{3\pi} = -\frac{1}{3\pi} = -0.106$$
$$h(4) = h(-4) = \frac{\sin\frac{4\pi}{2}}{4\pi} = 0$$
$$h(5) = h(-5) = \frac{\sin\frac{5\pi}{2}}{5\pi} = \frac{1}{5\pi} = 0.06366.$$

The transfer function of the filter is given by

$$H(z) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} \left[h(n)\left(z^n + z^{-n}\right)\right]$$
$$= 0.5 + \sum_{n=1}^{5} h(n)\left(z^n + z^{-n}\right)$$
$$= 0.5 + 0.3183\left(z^1 + z^{-1}\right) - 0.106\left(z^3 + z^{-3}\right) + 0.06366\left(z^5 + z^{-5}\right).$$

The transfer function of the realizable filter is

$$H'(z) = z^{-(N-1)/2} H(z)$$

$$= z^{-5}\left[0.5 + 0.3183\left(z + z^{-1}\right) - 0.106\left(z^{3} + z^{-3}\right) + 0.06366\left(z^{5} + z^{-5}\right)\right]$$

$$= 0.06366 - 0.106z^{-2} + 0.3183z^{-4} + 0.5z^{-5} + 0.3183z^{-6}$$

$$- 0.106z^{-8} + 0.06366z^{-10} \tag{6.57}$$

From the above Eq. (6.57) the filter coefficients of causal filter are given by

$$h(0) = h(10) = 0.06366; \quad h(1) = h(9) = 0; \quad h(2) = h(8) = -0.106$$

$$h(3) = h(7) = 0; \quad h(4) = h(6) = 0.3183; \quad h(5) = 0.5$$

The frequency response is given by

$$\overline{H}(e^{j\omega}) = \sum_{n=0}^{5} a(n)\cos\omega n \quad \text{where}$$

$$a(0) = h\left(\frac{N-1}{2}\right) = h(5) = 0.5$$

$$a(n) = 2h\left(\frac{N-1}{2} - n\right)$$

$$a(1) = 2h(5-1) = 2h(4) = 0.6366$$

$$a(2) = 2h(5-2) = 2h(3) = 0$$

$$a(3) = 2h(5-3) = 2h(2) = -0.212$$

$$a(4) = 2h(5-4) = 2h(1) = 0$$

$$a(5) = 2h(5-5) = 2h(0) = 0.127$$

$$\overline{H}(e^{j\omega}) = 0.5 + 0.6366\cos\omega - 0.212\cos 3\omega + 0.127\cos 5\omega \tag{6.58}$$

The magnitude in dB is calculated by varying ω from 0 to π and tabulated below. The magnitude $|H(e^{j\omega})|_{dB} = 20\ \log|\overline{H}(e^{j\omega})|$.

ω(in degrees)	0	10	20	30	40	50	60	70	80
$\left\|H(e^{j\omega})\right\|_{dB}$	0.4	0.21	-0.26	-0.517	-0.21	0.42	0.77	0.21	-1.79

90	100	110	120	130	140	150	160	170	180
-6	-14.56	-31.89	-20.6	-26	-32	-24.7	-30.55	-32	-26

The frequency response plot is shown in Fig. 6.9.

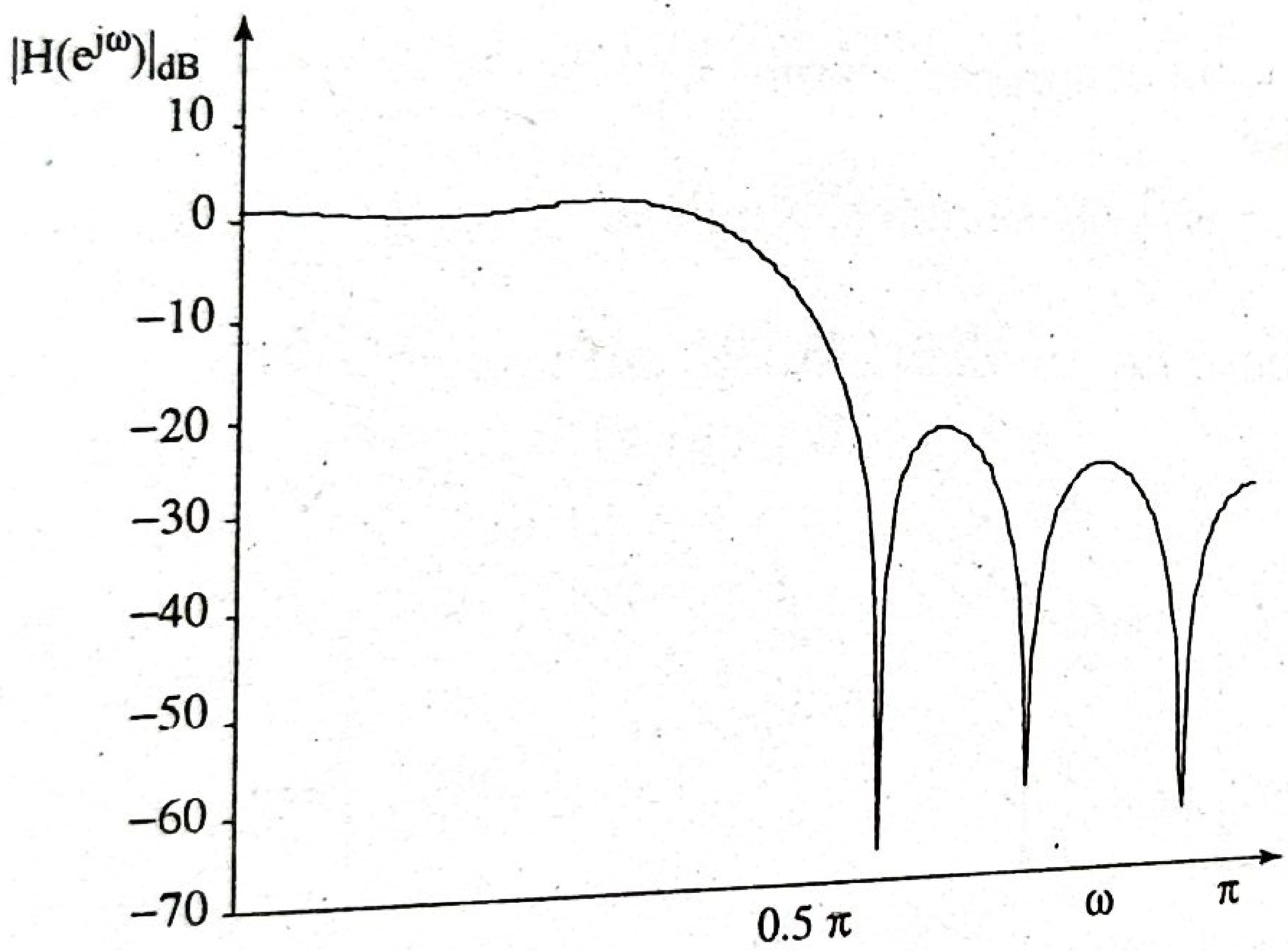


Fig. 6.9 Frequency response of lowpass filter of example 6.5.

Example 6.7 Design an ideal bandpass filter with a frequency response

$$H_d(e^{j\omega}) = 1 \quad \text{for} \quad \frac{\pi}{4} \leq |\omega| \leq \frac{3\pi}{4}$$
$$= 0 \quad \text{otherwise}$$

Find the values of $h(n)$ for $N = 11$ and plot the frequency response.

Solution

The ideal frequency response of the filter shown in Fig. 6.12. We know

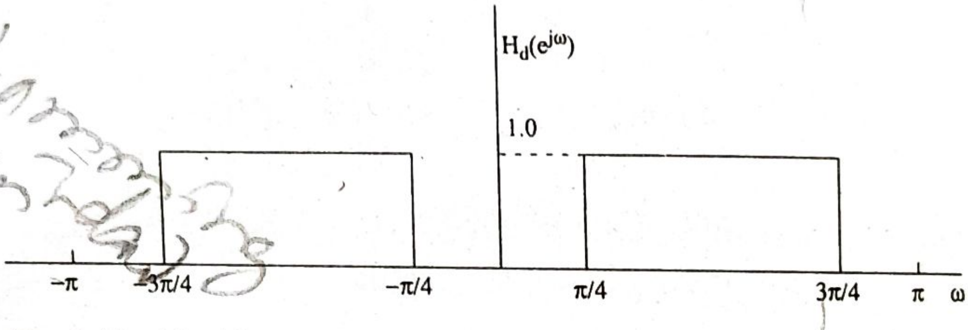


Fig. 6.12 Ideal frequency response of Bandpass filter of example 6.7.

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}$$
$$= \frac{1}{2\pi}\left[\int_{\frac{-3\pi}{4}}^{\frac{-\pi}{4}} e^{j\omega n}d\omega + \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} e^{j\omega n}d\omega\right]$$
$$= \frac{1}{2\pi jn}\left[e^{-j\pi n/4} - e^{-j3\pi n/4} + e^{j3\pi n/4} - e^{j\pi n/4}\right]$$
$$= \frac{1}{\pi n}\left[\sin\frac{3\pi}{4}n - \sin\frac{\pi}{4}n\right] \qquad -\infty \leq n \leq \infty \tag{6.63}$$

Truncating $h_d(n)$ to 11 samples, we have

$$h(n) = h_d(n) \quad \text{for} \quad |n| \leq 5$$
$$= 0 \quad \text{otherwise}$$

The filter coefficients are symmetrical about $n = 0$ satisfying the condition $h(n) = h(-n)$.

For $n = 0$

$$h(0) = \frac{1}{2\pi}\left[\int_{\frac{-3\pi}{4}}^{\frac{-\pi}{4}} d\omega + \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} d\omega\right]$$
$$= \frac{1}{2\pi}\left[-\frac{\pi}{4} + \frac{3\pi}{4} + \frac{3\pi}{4} - \frac{\pi}{4}\right] = \frac{1}{2} = 0.5$$

WR(0) = 1

$$h(1) = h(-1) = \frac{\sin\frac{3\pi}{4} - \sin\frac{\pi}{4}}{\pi} = 0$$

$$h(2) = h(-2) = \frac{\sin\frac{3\pi}{2} - \sin\frac{\pi}{2}}{2\pi} = \frac{-2}{2\pi} = -0.3183$$

$$h(3) = h(-3) = \frac{\sin\frac{9\pi}{4} - \sin\frac{3\pi}{4}}{3\pi} = 0$$

$$h(4) = h(-4) = \frac{\sin 3\pi - \sin\pi}{4\pi} = 0$$

$$h(5) = h(-5) = \frac{\sin\frac{15\pi}{4} - \sin\frac{5\pi}{4}}{5\pi} = 0$$

The transfer function of the filter is

$$\begin{aligned} H(z) &= h(0) + \sum_{n=1}^{\frac{N-1}{2}} \left[h(n)\left(z^n + z^{-n}\right)\right] \\ &= 0.5 - 0.3183(z^2 + z^{-2}) \end{aligned} \tag{6.64}$$

The transfer function of the realizable filter is

$$\begin{aligned} H'(z) &= z^{-5}\left[0.5 - 0.3183(z^2 + z^{-2})\right] \\ &= -0.3183z^{-3} + 0.5z^{-5} - 0.3183z^{-7} \end{aligned} \tag{6.65}$$

The filter coefficients of the causal filters are

$$\begin{aligned} &h(0) = h(10) = h(1) = h(9) = h(2) = h(8) = h(4) = h(6) = 0 \\ &h(3) = h(7) = -0.3183 \\ &h(5) = 0.5 \end{aligned}$$

$$\overline{H}(e^{j\omega}) = \sum_{n=1}^{\frac{N-1}{2}} a(n)\cos\omega n$$

$$\begin{aligned} a(0) &= h\left(\frac{N-1}{2}\right) = h(5) = 0.5 \\ a(n) &= 2h\left(\frac{N-1}{2} - n\right] \\ a(1) &= 2h(5-1) = 2h(4) = 0 \\ a(2) &= 2h(5-2) = 2h(3) = -0.6366 \\ a(3) &= 2h(5-3) = 2h(2) = 0 \\ a(4) &= 2h(5-4) = 2h(1) = 0 \\ a(5) &= 2h(5-5) = 2h(0) = 0 \end{aligned} \tag{6.66}$$

$$\overline{H}(e^{j\omega}) = 0.5 - 0.6366\cos 2\omega$$

ω (in degrees)	0	20	30	45	60	75	90	105	120	135	150	160	180
$\overline{H}(e^{j\omega})$	−0.1366	0.012	0.1817	0.5	0.818	1.05	1.1366	1.05	0.818	0.5	0.1817	0.012	−0.1366
$\lvert H(e^{j\omega})\rvert_{dB}$	−17.3	−38.17	−14.8	−6.02	−1.74	0.4346	1.11	0.4346	−1.74	−6.02	−14.8	−38.17	−17.3

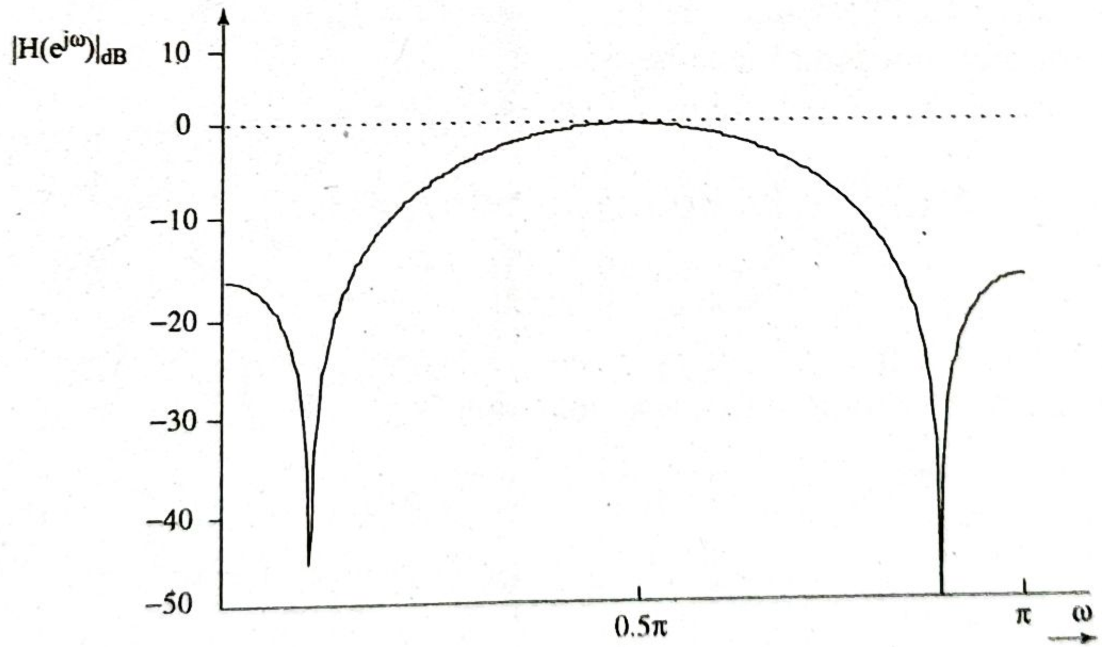


Fig. 6.13 Frequency response of Bandpass filter of example 6.7.

ideal bandreject filter with a desired frequency response

6.7 Digital Differentiator

The frequency response of an ideal digital differentiator is linearly proportional to frequency. It is given by

$$H_d(e^{j\omega}) = j\omega \qquad -\pi \le \omega \le \pi$$

The phase response of an ideal differentiator is 0.5π for all frequencies. If linear phase is desired, the frequency response of an ideal differentiator can be written as

$$H_d(e^{j\omega}) = j\omega\, e^{-j\alpha\omega}$$

where $\alpha = \dfrac{N-1}{2}$

The ideal impulse response of a digital differentiator with linear phase is given by

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} j\omega e^{-j\alpha\omega}\, e^{j\omega n}\, d\omega$$

$$= \frac{j}{2\pi}\int_{-\pi}^{\pi} \omega\, e^{j\omega(n-\alpha)}\, d\omega$$

$$= \frac{j}{2\pi}\left[\omega \frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\bigg|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]$$

$$= \frac{j}{2\pi}\left[\frac{\pi\, e^{j\pi(n-\alpha)} + \pi\, e^{-j\pi(n-\alpha)}}{j(n-\alpha)} + \frac{e^{j\omega(n-\alpha)}}{(n-\alpha)^2}\Bigg|_{-\pi}^{\pi}\right]$$

$$= \frac{j}{2\pi}\left[\frac{\pi\left(e^{j\pi(n-\alpha)} + e^{-j\pi(n-\alpha)}\right)}{j(n-\alpha)} + \frac{e^{j\pi(n-\alpha)} - e^{-j\pi(n-\alpha)}}{(n-\alpha)^2}\right]$$

$$= \frac{\cos\pi(n-\alpha)}{n-\alpha} - \frac{\sin(n-\alpha)\pi}{\pi(n-\alpha)^2}$$

If N is odd, α is an integer and we have $\sin(n-\alpha)\pi = 0$ for any integer n. If N is even, then $\cos\left[\frac{2n-(N-1)}{2}\pi\right] = 0$ for any integer. Thus we have, for N odd

$$\begin{aligned} h_d(n) &= \frac{\cos[(n-\alpha)\pi]}{n-\alpha} && \text{for } n \neq \alpha \\ &= 0 && \text{for } n = \alpha \end{aligned}$$

and for N even

$$h_d(n) = \frac{-\sin[(n-\alpha)\pi]}{\pi(n-\alpha)^2}$$

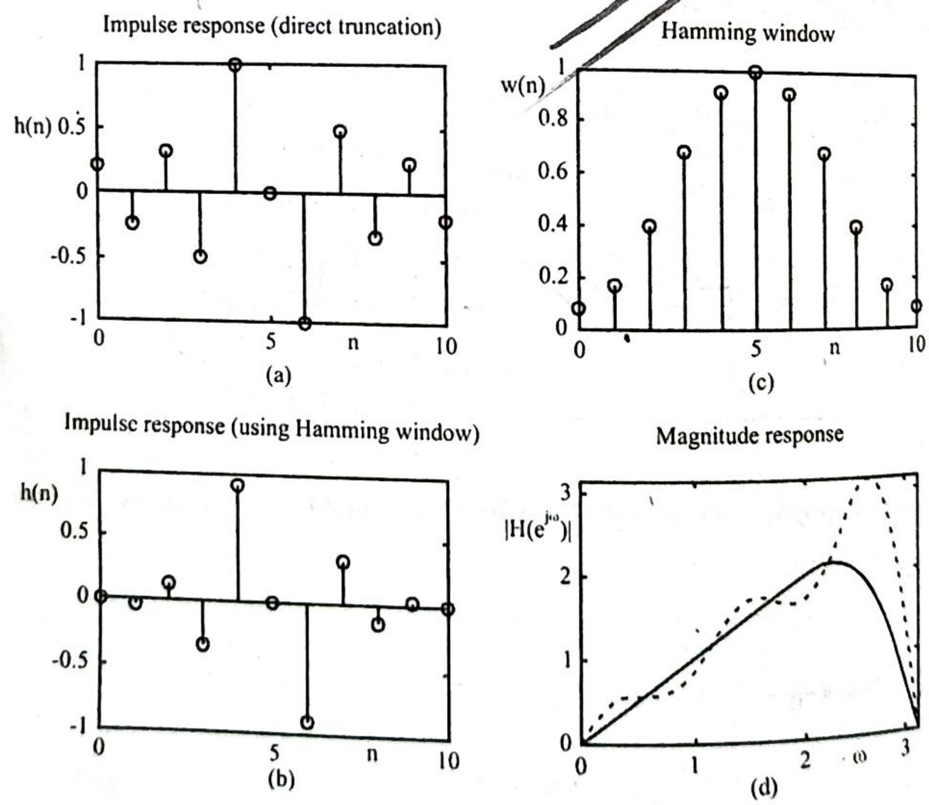


Fig. 6.47 a) Impulse response of a differentiator for $N = 11$ with direct truncation, b) Impulse response of differentiator with Hamming window truncation, c) Hamming window sequence, d) Magnitude response using rectangular window (dotted line) using Hamming window (solid line)

Both have the property $h_d(n) = -h_d(N-1-n)$. The coefficients are asymmetric and of infinite length. The finite impulse response can be obtained by truncating them by using a window of length N. Then we obtain

Fig. 6.48 d) shows the magnitude response of a type IV differentiator with N=12. The magnitude response of the differentiator (dotted line) obtained by direct truncation shows only small ripples and the magnitude response obtained using a Hamming window does not have any ripple (see example 6.13). Note that the magnitude response of type IV differentiator is better than that of type III differentiator. The reason is that the type IV filters have zeros only at $\omega = 0$; thus they are more suitable than type III for designing differentiators. That is we should use type IV FIR filter to approximate the ideal differentiator.

Example 6.13 Design an ideal differentiator with frequency response

$$H(e^{j\omega}) = j\omega \qquad -\pi \le \omega \le \pi$$

using (a) rectangular window (b) Hamming window with $N = 8$. Plot frequency response in both cases.

Solution

The frequency response of the ideal differentiator is shown in Fig. 6.49.

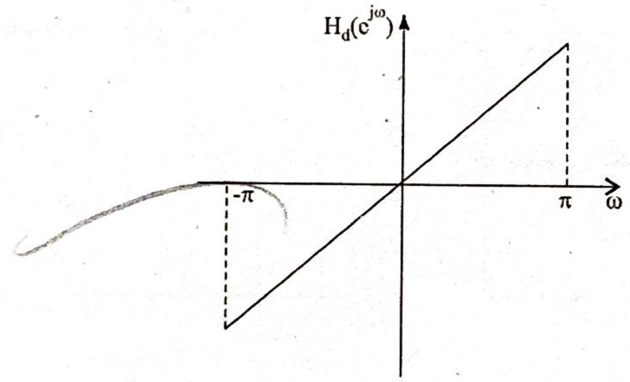


Fig. 6.49 Frequency response of ideal differentiator of example 6.13

Given

$$H(e^{j\omega}) = j\omega$$
$$= e^{j\pi/2}\omega$$
$$= e^{j\pi/2}e^{-j\alpha\omega}\overline{H}(e^{j\omega})$$

where $\alpha = 0$

and

$$\overline{H}\left(e^{j\omega}\right) = \omega$$

To get a linear phase differentiator the frequency response

$$H_d\left(e^{j\omega}\right) = j\omega e^{-j\alpha\omega} \qquad \text{for} \qquad -\pi \le \omega \le \pi$$

Since $N = 8$, the differentiator is a type IV FIR filter. The impulse response of a differentiator with N even is given by

$$h_d(n) = -\frac{\sin(n-\alpha)\pi}{\pi(n-\alpha)^2}$$

$$\alpha = \frac{N-1}{2} = \frac{8-1}{2} = \frac{7}{2}$$

The impulse response

$$h_d(n) = \frac{-\sin\left(n-\frac{7}{2}\right)\pi}{\pi\left(n-\frac{7}{2}\right)^2}$$

The impulse response satisfy the antisymmetry property. That is

$$h_d(n) = -h_d(N-1-n)$$

Therefore

$$h_d(0) = -h_d(7) = \frac{-\sin\left(-\frac{7}{2}\right)\pi}{\pi\left(-\frac{7}{2}\right)^2} = -0.026$$

$$h_d(1) = -h_d(6) = \frac{-\sin\left(-\frac{5}{2}\right)\pi}{\pi\left(-\frac{5}{2}\right)^2} = 0.0509$$

$$h_d(2) = -h_d(5) = \frac{-\sin\left(-\frac{3}{2}\right)\pi}{\pi\left(-\frac{3}{2}\right)^2} = --0.1415$$

$$h_d(3) = -h_d(4) = \frac{-\sin\left(-\frac{\pi}{2}\right)}{\pi\left(-\frac{1}{2}\right)^2} = 1.27$$

(a) **Rectangular window**

The rectangular window sequence for $N = 8$ is given by

$$\begin{aligned} w_R(n) &= 1 \quad \text{for} \quad 0 \le n \le 7 \\ &= 0 \quad \text{otherwise} \end{aligned}$$

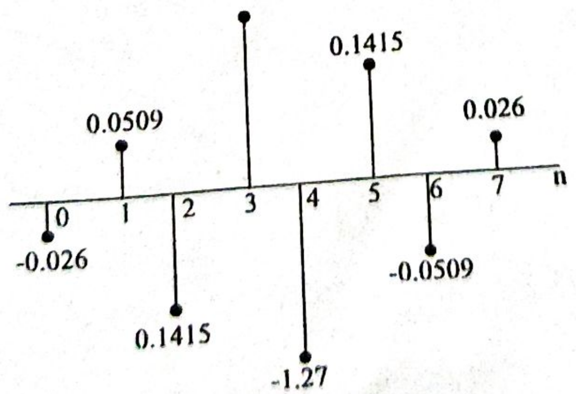


Fig. 6.50 Impulse response of differentiator

The filter coefficients using rectangular window are

$$h(n) = h_d(n)w_R(n) = h_d(n) \quad \text{for} \quad 0 \le n \le 7$$
$$= 0 \qquad \text{otherwise}$$

$$\Rightarrow h(0) = -h(7) = -0.026$$
$$h(1) = -h(6) = 0.0509$$
$$h(2) = -h(5) = -0.1415$$
$$h(3) = -h(4) = 1.27$$

The impulse response for a seven coefficient differentiator is shown in Fig. 6.50.

The transfer function of the differentiator is

$$H(z) = -0.026 + 0.0509z^{-1} - 0.1415z^{-2} = 1.27z^{-3} - 1.27z^{-4}$$
$$+ 0.1415z^{-5} - 0.0509z^{-6} + 0.026z^{-7}$$

$$\overline{H}\left(e^{j\omega}\right) = \sum_{n=1}^{\frac{N}{2}} d(n) \sin \omega \left(n - \frac{1}{2}\right)$$

where $d(n) = 2h\left(\dfrac{N}{2} - n\right)$

$$d(1) = 2h(3) = 2.54$$
$$d(2) = 2h(2) = -0.283$$
$$d(3) = 2h(1) = 0.1018$$
$$d(4) = 2h(0) = -0.052$$

$$\overline{H}\left(e^{j\omega}\right) = 2.54 \sin \frac{\omega}{2} - 0.283 \sin \frac{3\omega}{2} + 0.1018 \sin \frac{5\omega}{2} - 0.052 \sin \frac{7\omega}{2}$$

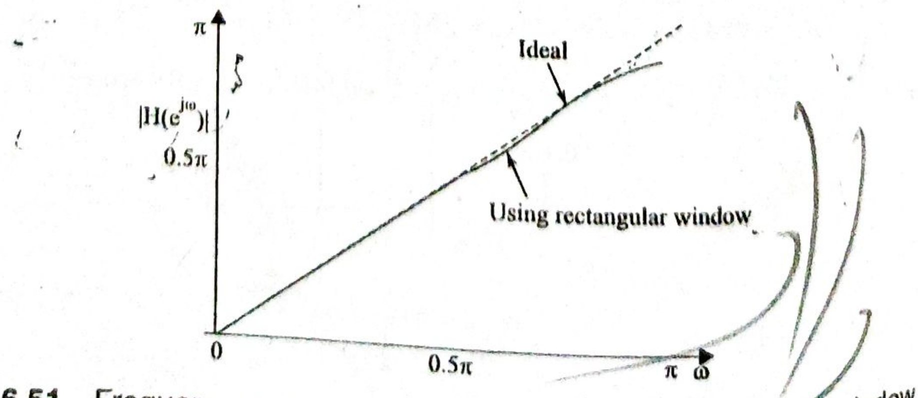


Fig. 6.51 Frequency response of differentiator using rectangular window

6.8 Hilbert Transformers

The frequency response of an ideal Hilbert transformer is given by

$$H_d(e^{j\omega}) = \begin{cases} -j & \text{for } 0 < \omega < \pi \\ j & \text{for } -\pi < \omega < 0 \end{cases}$$

The magnitude response of a Hilbert transformer is 1 for all frequencies; thus it is an all-pass filter. But it introduces a phase shift of -90° for $\omega > 0$ and 90° for $\omega < 0$. Hilbert transformers are used in communication systems, particularly in the generation of single-sideband modulated signals, radar signal processing, and speech signal processing.

The impulse response of an ideal Hilbert transformer is

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}\,d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} j\,e^{j\omega n}\,d\omega - \int_{0}^{\pi} j\,e^{j\omega n}\,d\omega\right]$$

$$= \frac{1}{2\pi}\left[j\left.\frac{e^{j\omega n}}{jn}\right|_{-\pi}^{0} - j\left.\frac{e^{j\omega n}}{jn}\right|_{0}^{\pi}\right]$$

$$= \frac{1}{2\pi n}\left[1 - e^{-j\pi n} - e^{j\pi n} + 1\right]$$

$$= \frac{1}{2\pi n}\left[2 - (e^{j\pi n} + e^{-j\pi n})\right]$$

$$= \frac{1}{2\pi n}\left[2 - 2\cos\pi n\right] = \frac{1 - \cos\pi n}{\pi n}$$

$$= \frac{2\sin^2\left(\frac{\pi n}{2}\right)}{\pi n} \qquad \text{for } n \neq 0$$

$$= 0 \qquad \text{for } n = 0$$

$$h_d(n) = \begin{cases} 2\sin^2\left(\frac{\pi n}{2}\right) & \text{for } n \neq 0 \\ 0 & \text{for } n = 0 \end{cases}$$

Note that $h_d(n)$ is infinite in duration and non-causal.
The frequency response a linear phase Hilbert transformer is given by

$$H_d(e^{j\omega}) = -j\, e^{-j\alpha\omega} \quad \text{for } 0 < \omega < \pi$$

$$= j\, e^{-j\alpha\omega} \quad \text{for } -\pi < \omega < 0$$

where $$\alpha = \frac{N-1}{2}$$

The linear phase introduces a delay of α samples.
The impulse response of a linear phase Hilbert transformer is

$$h_d(n) = \frac{2\sin^2\frac{(n-\alpha)\pi}{2}}{\pi(n-\alpha)}$$

For N odd, the truncated $h_d(n)$ is a type III FIR filter. We know that for a type III FIR filter $\overline{H}(e^{j\omega})$ is zero at $\omega = 0$ and $\omega = \pi$. Then it is impossible to design an all-pass Hilbert transformer. But in practical signal processing applications an all pass Hilbert transformer is unnecessary.

Example 6.14 Design an ideal Hilbert transformer having frequency response

$$H(e^{j\omega}) = j \quad \text{for} \quad -\pi \leq \omega \leq 0$$

$$= -j \quad \text{for} \quad 0 \leq \omega \leq \pi$$

Using (a) rectangular window (b) Blackman window
For $N = 11$ plot the frequency response in both cases. (AU ECE May'07)

Solution

The ideal frequency response is shown in Fig. 6.53.

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} je^{j\omega n}d\omega + \int_{0}^{\pi} -je^{j\omega n}d\omega\right]$$

$$= \frac{1 - \cos\pi n}{\pi n}$$

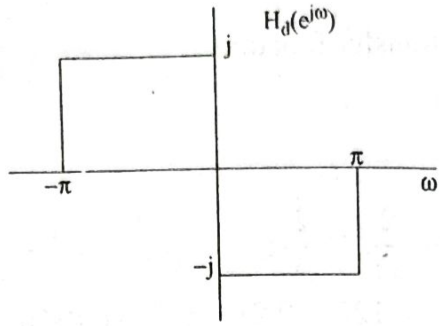


Fig. 6.53 Frequency response of ideal Hilbert transformer

The filter coefficients are antisymmetrical about $n = 0$, satisfying $h_d(n) = -h_d(-n)$. For $n = 0$

$$h_d(0) = \frac{j}{2\pi}\left[\int_{-\pi}^{0} d\omega + \int_{0}^{\pi} d\omega\right]$$

$$= \frac{j}{2\pi}(0 + \pi - \pi - 0) = 0$$

$$h_d(1) = -h_d(-1) = \frac{1 - \cos\pi}{\pi} = \frac{2}{\pi}$$

$$h_d(2) = -h_d(-2) = \frac{1 - \cos 2\pi}{2\pi} = 0$$

$$h_d(3) = -h_d(-3) = \frac{1 - \cos 3\pi}{3\pi} = \frac{2}{3\pi}$$

$$h_d(4) = -h_d(-4) = \frac{1 - \cos 4\pi}{4\pi} = 0$$

$$h_d(5) = -h_d(-5) = \frac{1 - \cos 5\pi}{5\pi} = \frac{2}{5\pi}$$

(a) Rectangular window

$$h(n) = h_d(n)w_R(n) = h_d(n) \quad \text{for} \quad -5 \le n \le 5$$

$$h(1) = -h(-1) = \frac{2}{\pi}$$

$$h(2) = -h(-2) = 0$$

$$h(3) = -h(-3) = \frac{2}{3\pi}$$

$$h(4) = -h(-4) = 0$$

$$h(5) = -h(-5) = \frac{2}{5\pi}$$

The transfer function of the Hilbert-transformer is

$$H(z) = \frac{2}{\pi}(z - z^{-1}) + \frac{2}{3\pi}(z^3 - z^{-3}) + \frac{2}{5\pi}(z^5 - z^{-5})$$

The realizable transfer function

$$
\begin{aligned}
H'(z) &= z^{-5}H(z) \\
&= \frac{2}{5\pi} + \frac{2}{3\pi}z^{-2} + \frac{2}{\pi}z^{-4} - \frac{2}{\pi}z^{-6} - \frac{2}{3\pi}z^{-8} - \frac{2}{5\pi}z^{-10} \\
&= 0.127 + 0.2122z^{-2} + 0.6366z^{-4} - 0.6366z^{-6} \\
&\quad - 0.2122z^{-8} - 0.127z^{-10}
\end{aligned}
$$

The causal sequence of Hilbert transformer is

$$
\begin{aligned}
h(0) &= -h(10) = 0.1273 \\
h(1) &= h(9) = h(3) = h(7) = h(5) = 0 \\
h(2) &= -h(8) = 0.2122 \\
h(4) &= -h(6) = 0.6366
\end{aligned}
$$

$$
\overline{H}(e^{j\omega}) = \sum_{n=1}^{5} c(n)\sin\omega n
$$

$$
\begin{aligned}
c(n) &= 2h\left(\frac{N-1}{2} - n\right) \\
c(1) &= 2h(4) = 1.2732 \\
c(2) &= 2h(3) = 0 \\
c(3) &= 2h(2) = 0.4244 \\
c(4) &= 2h(1) = 0 \\
c(5) &= 2h(0) = 0.2546
\end{aligned}
$$

$$
\begin{aligned}
\overline{H}(e^{j\omega}) &= 1.2732\sin\omega + 0.4244\sin 3\omega + 0.2546\sin 5\omega \\
H(e^{j\omega}) &= -j\overline{H}(e^{j\omega}) \\
H(e^{j\omega}) &= -j[1.2732\sin\omega + 0.4244\sin 3\omega + 0.2546\sin 5\omega]
\end{aligned}
$$

60 75

Example 5.16 Determine the coefficients of a linear phase FIR filter of length $M = 15$ has a symmetric unit sample response and a frequency response that satisfies the conditions

$$H\left(\frac{2\pi k}{15}\right) = 1 \quad k = 0, 1, 2, 3$$
$$= 0 \quad k = 4, 5, 6, 7$$

Solution

$$|H(k)| = 1 \quad \text{for} \quad 0 \le k \le 3 \quad \text{and} \quad 12 \le k \le 14$$
$$= 0 \quad \text{for} \quad 4 \le k \le 11$$

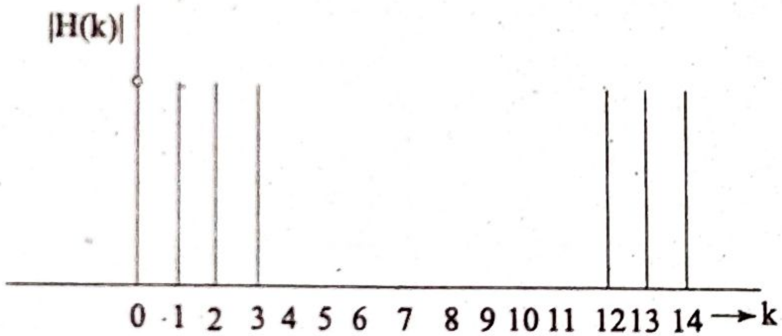


Fig. 5.60 Ideal magnitude response with samples for example 5.16

$$\theta(k) = -\left(\frac{N-1}{N}\right)\pi k$$
$$= \frac{-14}{15}\pi k \quad 0 \le k \le 7$$

and

$$\theta(k) = 14\pi - \frac{14\pi k}{15} \quad \text{for } 8 \le k \le 14$$

$$\begin{aligned} H(k) &= e^{-j14\pi k/15} && \text{for } k = 0, 1, 2, 3 \\ &= 0 && \text{for } 4 \le k \le 11 \\ &= e^{-j14\pi(k-15)/15} && \text{for } 12 \le k \le 14 \end{aligned}$$

$$h(n) = \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{\frac{N-1}{2}} Re\left(H(k)e^{j2\pi nk/15}\right)\right]$$
$$= \frac{1}{15}\left[1 + 2\sum_{k=1}^{7} Re\left(e^{-j14\pi k/15}e^{j2\pi nk/15}\right)\right]$$
$$= \frac{1}{15}\left[1 + 2\sum_{k=1}^{3} \cos\frac{2\pi k(7-n)}{15}\right]$$
$$= \frac{1}{15}\left[1 + 2\cos\frac{2\pi(7-n)}{15} + 2\cos\frac{4\pi(7-n)}{15} + 2\cos\frac{6\pi(7-n)}{15}\right]$$

$h(0) = h(14) = -0.05;\quad h(1) = h(3) = 0.041\quad h(4) = h(10) = -0.1078$

$h(2) = h(12) = 0.0666;\quad h(3) = h(11) = -0.0365\quad h(5) = h(9) = 0.034$

$h(6) = h(8) = 0.3188\quad h(7) = 0.466$

Comparison between FIR and IIR filters.

	FIR filter	IIR filter
1.	The impulse response of this filter is restricted to finite number of samples.	The impulse response of this filter extends over an infinite duration.
2.	FIR filters can have precisely linear phase.	These filters do not have linear phase.
3.	Closed-form design equations do not exist.	A variety of frequency selective filters can be designed using closed-form design formulas.
4.	Most of the design methods are iterative procedures, requiring powerful computational facilities for their implementation.	These filter can be designed using only a hand calculator and tables of analog filter design parameters.
5.	Greater flexibility to control the shape of their magnitude response.	Less flexibility specially for obtaining non-standard frequency responses.

6.	In these filters, the poles are fixed at the origin, high selectivity can be achieved by using a relatively high order for the transfer function.	The poles are placed anywhere inside the unit circle, high selectivity can be achieved with low-order transfer functions.
7.	Always stable.	Not always stable.
8.	Errors due to roundoff noise are less severe.	IIR filters are more susceptible to errors due to roundoff noise.

Practice Problem 5.7 Using frequency sampling method design a band reject filter

5.11 Realization of FIR Filters

5.11.1 *Transversal Structure*

The system function of an FIR filter can be written as

$$H(z) = \sum_{n=0}^{N-1} h(n)z^{-n}$$

$$= h(0) + h(1)z^{-1} + h(2)z^{-2} \ldots + h(N-1)z^{-(N-1)} \quad (5.155)$$

$$Y(z) = h(0)X(z) + h(1)z^{-1}X(z) + h(2)z^{-2}X(z) + \ldots$$
$$+ h(N-1)z^{-(N-1)}X(z) \quad (5.156)$$

The Eq. (5.156) can be realized as shown in Fig. 5.68

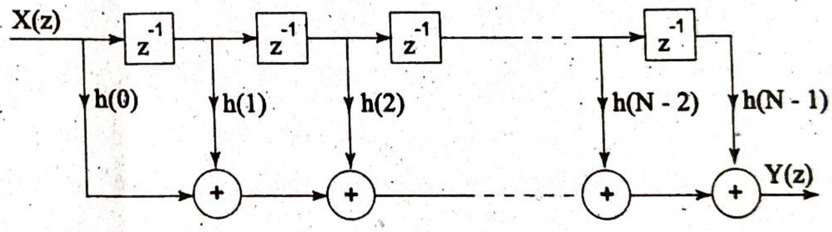


Fig. 5.68 Direct form realization of Eq. (5.156)

This structure is known as transversal structure or direct form realization. The transversal structure requires N multipliers, $N-1$ adders, and $N-1$ delay elements.

Cascade realization

The Eq. (5.155) can be realized in cascade form from the factored form of $H(z)$. For N odd

$$H(z) = \prod_{k=1}^{\frac{N-1}{2}} (b_{k0} + b_{k1}z^{-1} + b_{k2}z^{-2})$$

$$= \left(b_{10} + b_{11}z^{-1} + b_{12}z^{-2}\right)\left(b_{20} + b_{21}z^{-1} + b_{22}z^{-2}\right)\ldots$$
$$\times\left(b_{\frac{(N-1)}{2}0} + b_{\frac{(N-1)}{2}1}z^{-1} + b_{\frac{(N-1)}{2}2}z^{-2}\right) \qquad (5.157)$$

For N odd, $N-1$ will be even and $H(z)$ will have $(N-1)/2$ second order factors. Each second order factored form of $H(z)$ is realized in direct form and is cascaded to realize $H(z)$ as shown in Fig. 5.69.

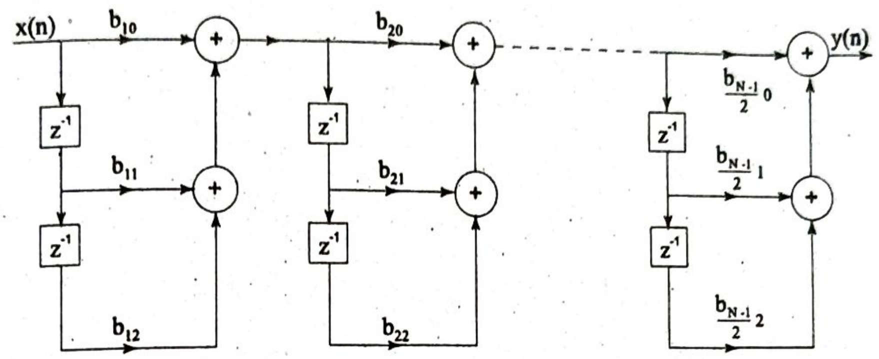


Fig. 5.69 Cascade realization of Eq. (5.157)

For N even

$$H(z) = \left(b_{10} + b_{11}z^{-1}\right)\prod_{k=2}^{N/2}\left(b_{k0} + b_{k1}z^{-1} + b_{k2}z^{-2}\right) \qquad (5.158)$$

when N is even, $N-1$ is odd and $H(z)$ will have one first order factor and $\frac{(N-2)}{2}$ second order factors.

$$H(z) = \left(b_{10} + b_{11}z^{-1}\right)\left(b_{20} + b_{21}z^{-1} + b_{22}z^{-2}\right)\left(b_{30} + b_{31}z^{-1} + b_{32}z^{-2}\right)$$
$$\times\left(b_{\frac{N}{2}0} + b_{\frac{N}{2}1} + b_{\frac{N}{2}2}z^{-2}\right) \qquad (5.159)$$

Now each factored form in $H(z)$ is realized in direct form and is cascaded to obtain the realization of $H(z)$ as shown in Fig. 5.70.

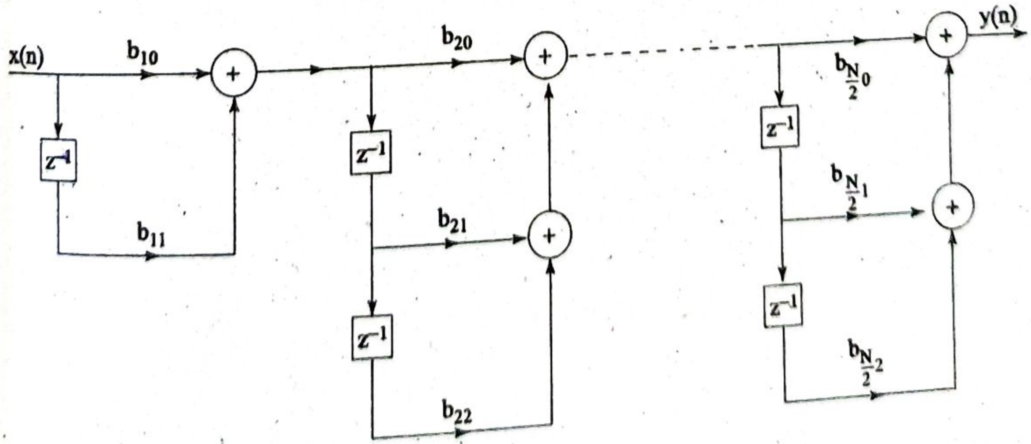


Fig. 5.70 Cascade realization of Eq. 5.158.

5.11.4 *Polyphase realization of FIR filter*

Let us consider an FIR filter with impulse response having N coefficients. The system function of such a filter is given by

$$H(z) = \sum_{n=0}^{N-1} h(n)z^{-n} \tag{5.188}$$

If $N = 11$ then

$$\begin{aligned}H(z) &= h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} \\ &\quad + h(6)z^{-6} + h(7)z^{-7} + h(8)z^{-8} + h(9)z^{-9} + h(10)z^{-10}.\end{aligned} \tag{5.189}$$

The above transfer function can be partitioned into two sub-signals, one sub signal containing even indexed coefficients and the other containing odd-indexed coefficients. That is

$$\begin{aligned}H(z) &= h(0) + h(2)z^{-2} + h(4)z^{-4} + h(6)z^{-6} + h(8)z^{-8} + h(10)z^{-10} \\ &\quad + h(1)z^{-1} + h(3)z^{-3} + h(5)z^{-5} + h(7)z^{-7} + h(9)z^{-9}\end{aligned} \tag{5.190}$$

$$\begin{aligned}&= h(0) + h(2)z^{-2} + h(4)z^{-4} + h(6)z^{-6} + h(8)z^{-8} + h(10)z^{-10} \\ &\quad + z^{-1}[h(1) + h(3)z^{-2} + h(5)z^{-4} + h(7)z^{-6} + h(9)z^{-8}]\end{aligned} \tag{5.191}$$

$$= P_0(z^2) + z^{-1}P_1(z^2) \tag{5.192}$$

where

$$\begin{aligned}P_0(z) &= h(0) + h(2)z^{-1} + h(4)z^{-2} + h(6)z^{-3} + h(8)z^{-4} + h(10)z^{-5} \\ P_1(z) &= h(1) + h(3)z^{-1} + h(5)z^{-2} + h(7)z^{-3} + h(9)z^{-4}\end{aligned} \tag{5.193}$$

Now $H(z)$ can be written as

$$H(z) = \sum_{m=0}^{M-1} z^{-m} P_m(z^M) \tag{5.194}$$

where

$$P_m(z^M) = \sum_{n=0}^{\lfloor (N+1)/M \rfloor} h(Mn+m)z^{-n} \quad 0 \le m \le M-1 \tag{5.195}$$

If $H(z)$ is partitioned into three sub-signals, then

$$\begin{aligned} H(z) &= h(0) + h(3)z^{-3} + h(6)z^{-6} + h(9)z^{-9} + h(1)z^{-1} + h(4)z^{-4} \\ &\quad + h(7)z^{-7} + h(10)z^{-10} + h(2)z^{-2} + h(5)z^{-5} + h(8)z^{-8} \end{aligned} \tag{5.196}$$

$$\begin{aligned} &= h(0) + h(3)z^{-3} + h(6)z^{-6} + h(9)z^{-9} + z^{-1}[h(1) + h(4)z^{-3} \\ &\quad + h(7)z^{-6} + h(10)z^{-9}] + z^{-2}[h(2) + h(5)z^{-3} + h(8)z^{-6}] \end{aligned} \tag{5.197}$$

$$= P_0(z^3) + P_1(z^3) + P_2(z^3) \tag{5.198}$$

where

$$P_0(z) = h(0) + h(3)z^{-1} + h(6)z^{-2} + h(9)z^{-3} \tag{5.199}$$

$$P_1(z) = h(1) + h(4)z^{-1} + h(7)z^{-2} + h(10)z^{-3} \tag{5.200}$$

$$P_2(z) = h(2) + h(5)z^{-1} + h(8)z^{-2} \tag{5.201}$$

Here $M = 3$

$$H(z) = \sum_{m=0}^{M-1} z^{-m} P_m(z^M) \tag{5.202}$$

$$P_m(z^M) = \sum_{n=0}^{\lfloor (N+1)/M \rfloor} h(Mn+m)z^{-n} \quad 0 \le m \le M-1 \tag{5.203}$$

$\left\lfloor \dfrac{(N+1)}{M} \right\rfloor$ denotes the integer part of $\dfrac{N+1}{M}$.

The decomposition of $H(z)$ in the form of Eq. (5.192) and Eq. (5.198) is known as type 1 polyphase decomposition of the transfer function of order N. We know that for a general case of M-sub signals.

$$H(z) = \sum_{m=0}^{M-1} z^{-m} P_m(z^M) \tag{5.204}$$

where

$$P_m(z) = \sum_{n=0}^{[(N+1)/M]} h(Mn+m)z^{-n} \quad 0 \le m \le M-1 \tag{5.205}$$

Now Eq. (5.204) can be realized as shown in the Fig. 5.82 where each $P_m(z^M)$ can be realized in direct form.

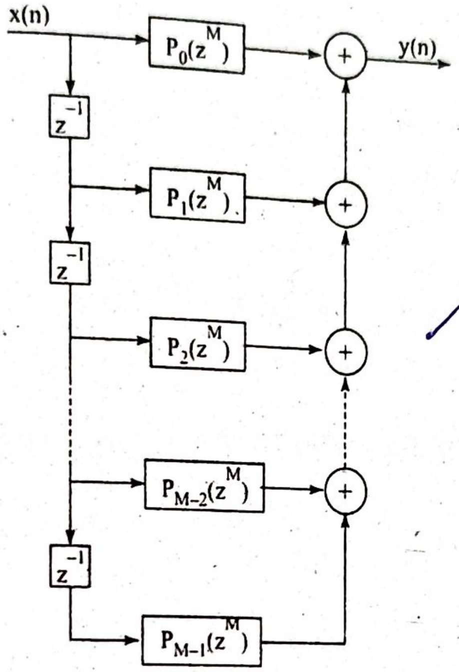


Fig. 5.82

If we replace m by $M-1-m$ in Eq. (5.204) we get the type 2 Polyphase decomposition.

$$H(z) = \sum_{m=0}^{M-1} P_{M-1-m}(z^M) \tag{5.206}$$

$$= \sum_{m=0}^{M=1} Q_m(z^M) \tag{5.207}$$

where

$$Q_m(z^M) = P_{M-1-m}(z^M)$$

Replacing m by $-m$ in Eq. (5.204) we obtain the type 3 polyphase representation

$$H(z) = \sum_{m=0}^{m-1} z^m R_m(z^M) \tag{5.208}$$

where

$$R_0(z^M) = P_0(z^M)$$

$$R_m(z^M) = z^{-1} P_{M-m}(z^M)$$

unit - 3.

FINITE WORD LENGTH EFFECTS IN DIGITAL FILTERS.

⇒ Digital signal processing algorithms are realized either with special purpose digital hardware or as programs for a general purpose digital computer.

⇒ In both cases the numbers and coefficients are stored in finite - length registers. Therefore co efficients and numbers are quantized by trucation or rounding off when they are stored.

The following errors arise due to quantization of numbers.

1. Input quantization error.
2. Product Quantization error.
3. co-efficient Quantization error.

1. The conversion of a continuous - time input signal into digital value produces an error, which is known as input quantization error. This error arises due to the representation of the input signal by a fixed number of digits in A/D conversion process.

2. Product Quantization error arise at the output of a multiplier. Multiplication of a b-bit data with a b bit co.efficient results a product having 2b bits. Since a b bit register is used, the multiplier output must be rounded or truncated to b bits which produces an error.

3. The filter co.efficients are computed to infinite precision in theory. If they are quantized, the frequency response of the resulting filter may differ from the desired response and sometimes the filter

may fail to meet the desired specifications.

If the poles of the desired filters are close to the unit circle, then those of the filter with quantized coefficients may lie just outside the unit circle, leading to instability.

⇒ The other error arising from quantization are roundoff noise and limit cycle oscillations.

Number Representation :

We can represent a number N to any desired accuracy by a finite series.

$$N = \sum_{i=n_1}^{n_2} c_i r^i$$

where r is called as radix.

If r=10, the representation is known as decimal representation having numbers from 0 to 9.

In this number representation,

$$30.285 = \sum_{i=-3}^{1} c_i 10^i$$

$$= 3 \times 10^1 + 0 \times 10^0 + 2 \times 10^{-1} + 8 \times 10^{-2} + 5 \times 10^{-3}$$

To convert from decimal to binary, we divide the integer part of the number (left to the decimal point) repeatedly by 2 and arrange the remainder in reverse order.

The fractional part (right to the decimal point) is repeatedly multiplied by 2, each time removing the integer part, and writing in normal order.

Types of Number Representation :-

There are three common forms that are used to represent the numbers in a digital computer or any other digital hardware.

2M 1. Fixed point representation.

2. Floating point representation.

3. Block Floating point representation.

Fixed point representation :

⇒ In fixed point arithmetic the position of the binary point is fixed. The bit to the right represent the fractional part of number and those to the left represent the integer part.

For example, the binary number 01.1100 has the value 1.75 in decimal.

⇒ The manner in which negative numbers are represented gives three different forms for fixed point arithmetic

1. Sign - magnitude form.
2. One's - complement form.
3. Two- complement form.

One's complement form : In one's complement from the positive number is represented as in the sign - magnitude notation.

⇒ But the negative number is obtained by complementing all the bits of the positive numbers.

for example,

the decimal number -0.875 can be represented as follows,

$$(0.875)_{10} = (0.111000)_2$$

$\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow$ complementing each bit

1.000111

$$(-0.875)_{10} = (1.000111)_2$$

This is same as subtracting the magnitude from $2-2^b$, where b is the number of bits,

$$2-2^b = 10.000000 - 0.000001$$

$$2-2^b = 1.111111$$

Now subtract.

$$0.875 = (0.111000)_2$$

$$\begin{array}{r} 1.111111 \\ 0.111000 \\ \hline 1.000111 \\ \hline \end{array} = (-0.875)_{10} \text{ in one's complement form.}$$

In one's complement form the magnitude of the negative number is given by,

$$1 - \sum_{i=1}^{b} c_i \, 2^{-i} - 2^{-b}$$

$$\therefore 1 - (2^{-4} + 2^{-5} + 2^{-6}) - 2^{-6} = 0.875$$

Two's complement form :

In two's complement representation positive numbers are represented as sign - magnitude and one's complement.

The negative number is obtained by complementing all the bits of the positive numbers and adding one at the least significant bit.

for example,

$$(0.875)_{10} = (0.111000)_2$$

$$\begin{array}{rl} & 1.000111 \\ +) & 0.000001 \\ \hline & 1.001000 \\ \hline \end{array}$$

$(0.875)_{10} = (1.001000)_2$ in two's complement form.

The magnitude of the negative number is given by,

$$1 - \sum_{i=1}^{b} c_i 2^{-i}$$

Floating point Numbers :

In floating point representation a positive number is represented as $F = 2^{e} \cdot M$ where M called mantissa, is a fraction such that $1/2 \leq M \leq 1$ and e, the exponent can be either positive or negative.

The decimal numbers 4.5, 1.5, 6.5 and 0.625 having floating point representations as $2^{3} \times 0.5625$, $2^{1} \times 0.75$, $2^{3} \times 0.8125$, $2^{0} \times 0.625$ respectively.

$$2^{3} \times 0.5625 = 2^{011} \times 0.1001$$

$$2^{1} \times 0.75 = 2^{001} \times 0.1100$$

Negative floating point Numbers are generally represented by considering the mantissa, is a fixed point number. The sign of the floating point number is obtained from the first bit of mantissa. In floating point arithmetic multiplications are carried out as follows. Let

$$F_1 = 2^{c_1} \times M_1$$

$$F_2 = 2^{c_2} \times M_2$$

Then the product $F_3 = F_1 \times F_2 = (M_1 \times M_2) 2^{C_1 + C_2}$,

that is, the mantissas are multiplied using fixed point arithmetic and exponents are added.

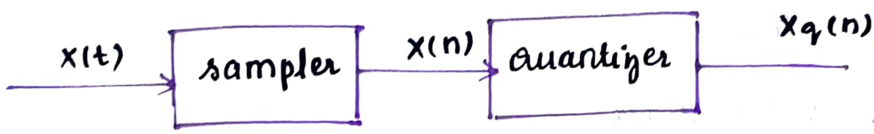
The product $(M_1 \times M_2)$ must be in the range of 0.25 to 1.0. To correct this problem the exponent $(C_1 + C_2)$ must be altered.

Block Floating Point Numbers :-

A compromise between fixed and floating point systems is the block-floating point arithmetic, Here, the set of signals to be handled is divided into blocks. Each block have the same value for the exponent. The arithmetic operations with in the block uses fixed point arithmetic and only one exponent per block is stored, Thus saving memory. This representation of numbers is most suitable in certain FFT flow graphs and in digital audio applications.

Quantization Noise :

The process of converting an analog signal to a digital is shown in above fig.



At first the signal x(t) is sampled at regular intervals t = nT,

where, n = 0, 1, 2 . . . to create a sequence x(n). This is done by a sampler.

Then numeric equivalent of each sample x(n) is expressed by a finite number of bits giving the sequence xq(n). The difference signal e(n) = xq(n) - x(n) is called Quantization noise or A/D conversion noise

If ADC is used to convert the sinusoidal signal it employs (b+1) bits including sign bit.

Then the number of levels available for quantizing x(n) is 2^{b+1}.

Thus the interval between successive levels is,

$$q = \frac{2}{2^{b+1}} = 2^{-b}$$

where, q is known as quantization step size.

If b = 3 bits,

$q = 2^{-3} = 0.125$

The common methods of Quantization are,

1. Truncation
2. Rounding.

Truncation :- (2m)

Truncation is a process of discarding all bits less significant than least significant bit that is retained. Suppose if we truncate the following binary numbers from 8 bits to 4 bits, we obtain.

0.00110011 to 0.0011

8 bits 4 bits.

When we truncate a number, the signal value is approximated by the highest Quantization level that is not greater than the signal.

Rounding :: (Q.2m)

Rounding of a numbers of b bits is accomplished by choosing the rounded result as the b bit number closest to the original number unrounded.

For example : 0.11010 rounded to three bits is either 0.110 (or) 0.111

Error Due to truncation and rounding ::

If the Quantization method is truncation, the number is approximated by the nearest level that does not exceed it. In this case the error $x_T - x$ is negative or zero where,

x_T is truncation value of x and it is assumed $|x| \leq 0$.

The error made by truncating a number to b bits following the binary point satisfies the inequality,

$$0 \geq x_T - x > -2^{-b}$$

Consider first the two's complement representation,

$$x = 1 - \sum_{i=1}^{b} c_i 2^{-i}$$

If we truncate the number to N bits,

$$x_T = 1 - \sum_{i=1}^{N} c_i 2^{-i}$$

The change in magnitude,

$$x_T - x = \sum_{i=1}^{b} c_i 2^{-i} - \sum_{i=1}^{N} c_i 2^{-i}$$

$$x_T - x = \sum_{i=N}^{b} c_i 2^{-i}$$

$$0 \geq x_T - x > -2^{-b}.$$

For one's complement representation the magnitude of negative number with b bits is given by,

$$x = 1 - \sum_{i=1}^{b} c_i 2^{-i} - 2^{-b}.$$

when the number is truncated to N bits, then

$$x_T = 1 - \sum_{i=1}^{N} c_i 2^{-i} - 2^{-N}$$

The change in magnitude due to truncation is,

$$x_T - x = \sum_{i=N}^{b} c_i 2^{-i} - (2^{-N} - 2^{-b})$$

$$< 0.$$

Therefore the magnitude decreases with truncation which implies that error is positive and satisfy the inequality,

$$0 \leq x_T - x < 2^{-b}$$

If $x = 2^c \cdot M$,

Then $x_T = 2^c \cdot M_T$

Error,

$$e = x_T - x = 2^c (M_T - M)$$

If $M = 1/2$, the relative error is maximum.

Therefore,

$$0 \geq \varepsilon \geq -2 \cdot 2^{-b}$$

If $M = -1/2$, the relative error range is,

$$0 \leq \varepsilon < 2 \cdot 2^{-b}.$$

$$0 \geq M_T - M > -2^{-b}$$

$$e = \varepsilon x = \varepsilon 2^c \cdot M \quad \text{and} \quad M = 1/2.$$

$$0 \geq \varepsilon > -2 \cdot 2^{-b}$$

For negative mantissa values the error is,

$$0 \leq M_T - M < 2^{-b}$$

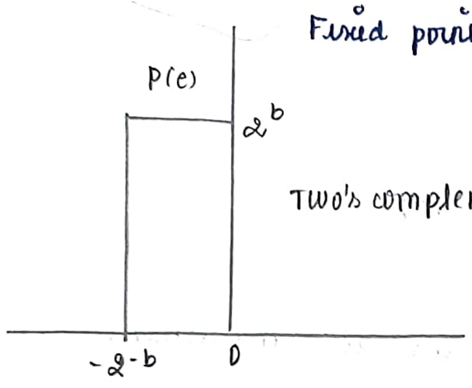
$$0 \leq e < 2^c \cdot 2^{-b}.$$

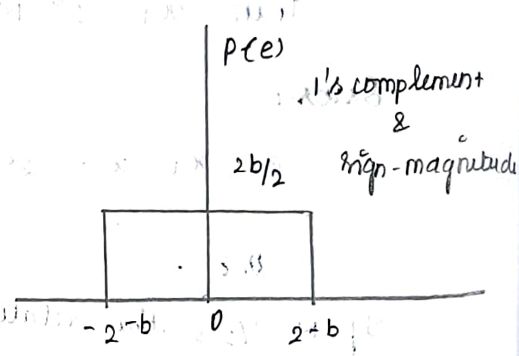
with $M = -1/2$. The maximum range of the relative error for negative M is,

$$0 \geq \varepsilon > -2 \cdot 2^{-b}$$

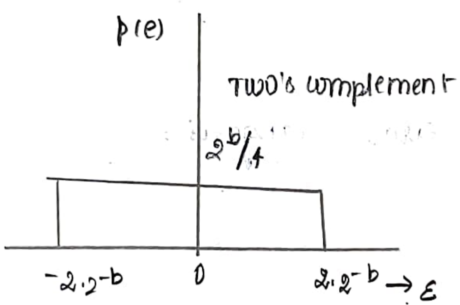
The probability density function $p(e)$ for truncation of fixed point and floating point numbers.

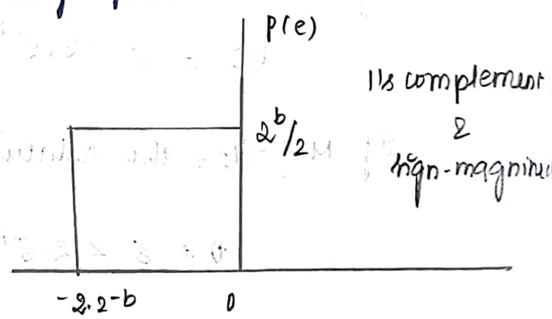
Fixed point





Floating point





In floating point arithmetic, only the mantissa is affected by quantization.

If $x = M \cdot 2^c$

$$x_T = M_T \cdot 2^c$$

Then $e = x_T - x = (M_T - M)\, 2^c$.

But for rounding,

$$\frac{-2^{-b}}{2} \le M_T - M \le \frac{2^{-b}}{2}$$

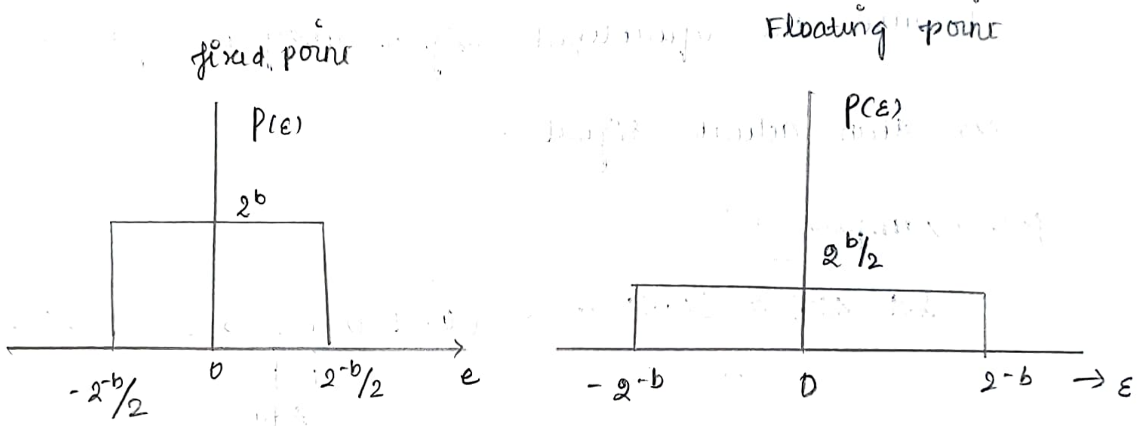
$$-2^c \frac{-2^{-b}}{2} \le x_T - x \le 2^c \frac{2^{-b}}{2}$$

The mantissa satisfies,

$$\tfrac{1}{2} \le M < 1$$

If $M = 1/2$ we get the maximum range of relative error,

$$-2^{-b} \le \varepsilon < 2^{-b}$$



(fig) probability density fn $P(\varepsilon)$ for rounding.

Input Quantization Error :- ⊛ 11M.

The Quantization error arises when a continuous signal is converted into digital value.

The Quantization error is given by,

$$e(n) = x_q(n) - x(n).$$

where,

$x_q(n)$ = sampled quantized value

$x(n)$ = sample unquantized value.

The Quantization,

If rounding of a number is used to get $x_q(n)$. Then the error signal satisfies the relation,

$$\frac{-q}{2} \leq e(n) \leq \frac{q}{2}$$

because the quantized signal may be greater or less than actual signal.

for example,

let $x(n) = (0.70)_{10} = (0.1011\,0011\ldots)_2$

Add.

After rounding $x(n)$ to 3 bits,

we have,

$$x_q(n) = 0.101$$
$$\underline{\quad\quad 1}$$
$$\underline{0.110}$$

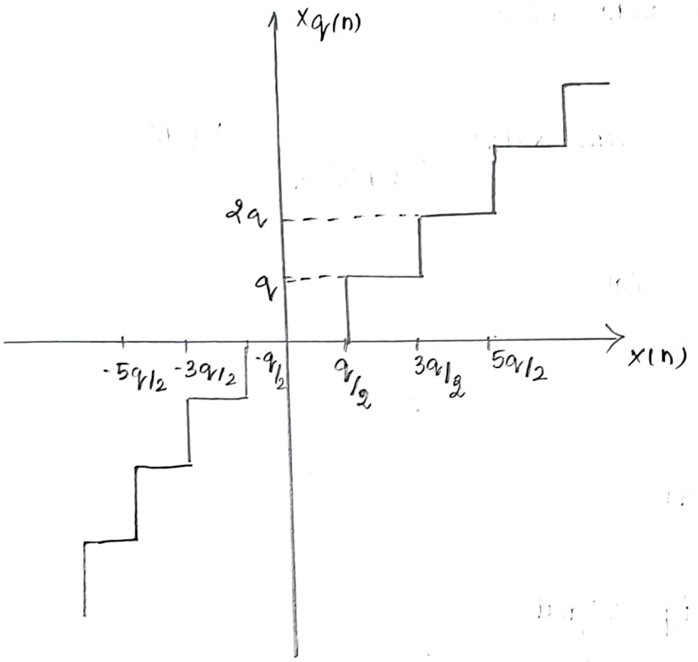
$$x_q(n) = (0.75)_{10}$$

Now the error,

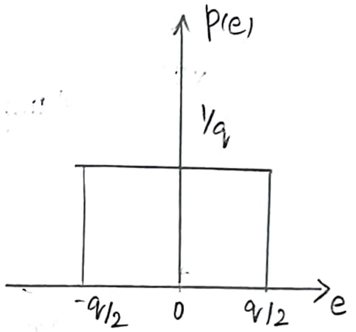
$$e(n) = x_q(n) - x(n) = 0.05$$

In truncation the signal is represented by the highest quantization level that is not greater than the signal. Therefore, in two's complement truncation, the error $e(n)$ is always negative and satisfies the inequality.

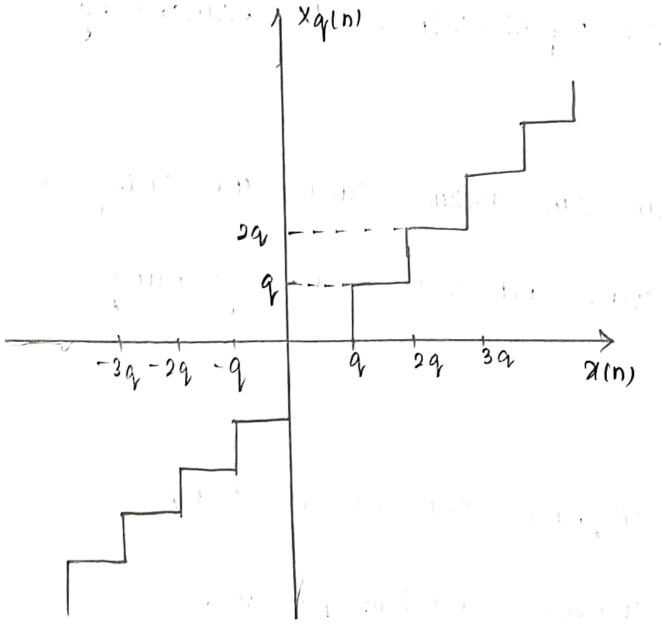
$$-q \leq e(n) < 0.$$



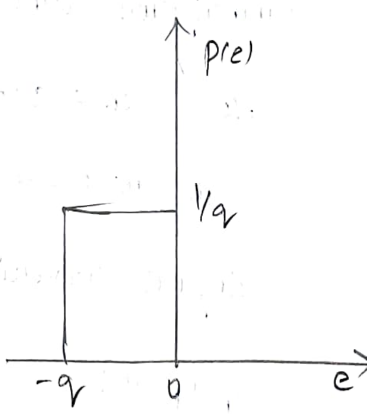
a) Quantizer characteristics with rounding



b) pdf for roundoff error



a) Quantizer characteristics with 2's complement truncation.

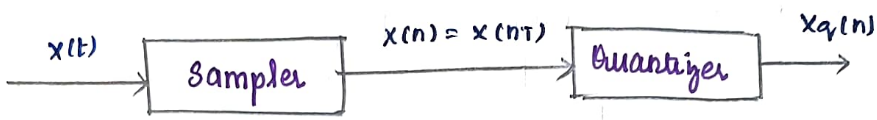


b) pdf of truncating error.

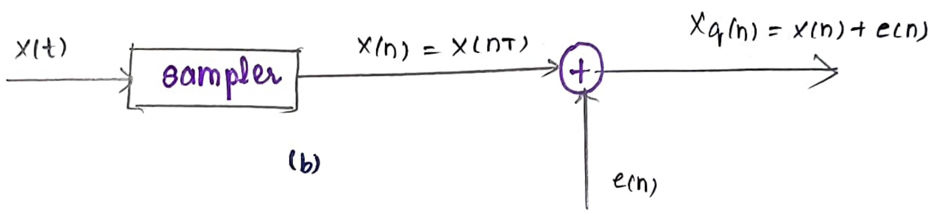
Steady State Input Noise power :

In Digital processing of analog signals, the Quantization error is commonly viewed as an additive noise signal, That is,

$$x_q(n) = x(n) + e(n)$$



(a)



(b)

$x(n) \Rightarrow$ i/p signal

$e(n)$ error signal.

If rounding is used for quantization then the Quantization error $e(n) = x_q(n) - x(n)$. is bounded by $-q/2 \leq e(n) \leq q/2$.

In most cases, we can assume that the analog-to-digital conversion error $e(n)$ has the following properties.

1. The error sequence $e(n)$ is a sample sequence of a stationary random process.

2. the error sequence is uncorrelated with $x(n)$ and other signal in the system

3. The error is a white noise process with uniform amplitude probability distribution over range of quantization error.

In case of rounding The $e(n)$ lies b/w $-q/2$ and $q/2$ with equal probability. The variance of $e(n)$ is given by,

$$\sigma_e^2 = E[e^2(n)] - E^2[e(n)]$$

where $E[e^2(n)]$ is the average of $e^2(n)$ and $E[e(n)]$ is mean value of $e(n)$.

Therefore, for rounding,

$$\sigma_e^2 = \frac{2^{-2b}}{12}$$

for two's complement truncation,

$$p(e) = \frac{1}{q} \quad \text{for } -q \leq e(n) \leq 0$$

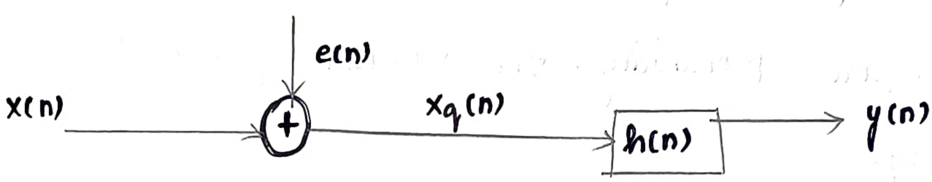
In both cases the value $\sigma_e^2 = \frac{2^{-2b}}{12}$, which is also known as the steady state noise power due to input Quantization.

Thus, to obtain

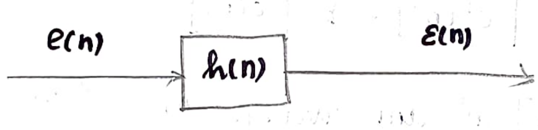
SNR $\geq$ 80 dB requires $b = 14$ bits.

Steady state output Noise power :

Due to A/D conversion noise one can represent the quantized input to a digital system with impulse response $h(n)$ as shown in fig:



(a)



(b)

Representation of A/D Conversion Noise.

$$\mathcal{E}(n) = e(n) * h(n)$$

$$= \sum_{k=0}^{n} h(k)\, e(n-k).$$

The variance of any term in the above sum is equal to $\sigma_e^2 h^2(n)$.

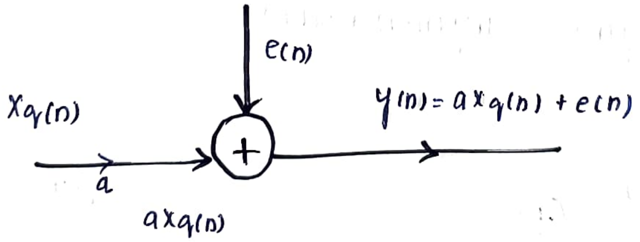
Using Parseval's theorem the steady state output noise variance due to the quantization error is given by,

$$\sigma_{\mathcal{E}}^2 = \sigma_e^2 \sum_{n=0}^{\infty} h^2(n) = \frac{\sigma_e^2}{2\pi j} \oint_c H(z)\, H(z^{-1})\, z^{-1}\, dz$$

Product Quantization error :-

In fixed point arithmetic the product of two b bit numbers 2b bits long. In digital signal processing applications, it's necessary to round this product to a b-bit number, which produce an error known as product quantization error or product roundoff noise.

The multiplication is modeled as an infinite precision multiplier followed by an adder where round off noise is added to the product so that overall results equals some quantization levels.



The roundoff noise sample is a zero mean random variable with a variance $\frac{2^{-2b}}{12}$, where b is the number of bits used to represent the variables.

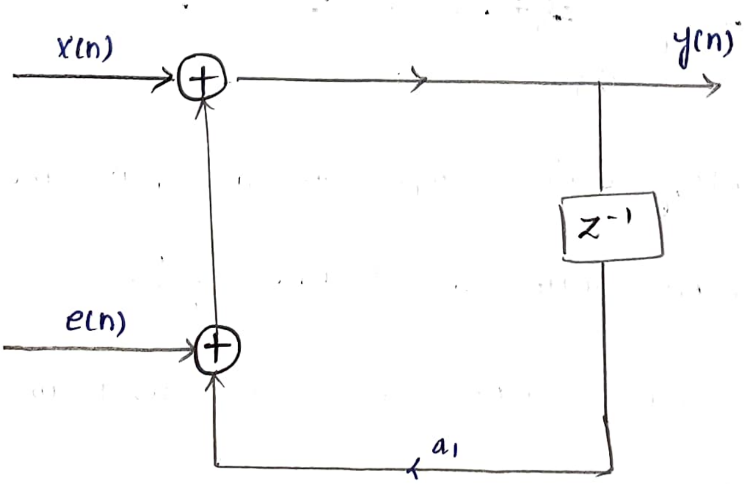
1. For any n, the error sequence e(n) is uniformly distributed over the range $-q/2$ and $q/2$. This implies that mean value of e(n) is zero and its variance is $\sigma_e^2 = \frac{2^{-2b}}{12}$.

2. The error sequence e(n) is a stationary white noise sequence.

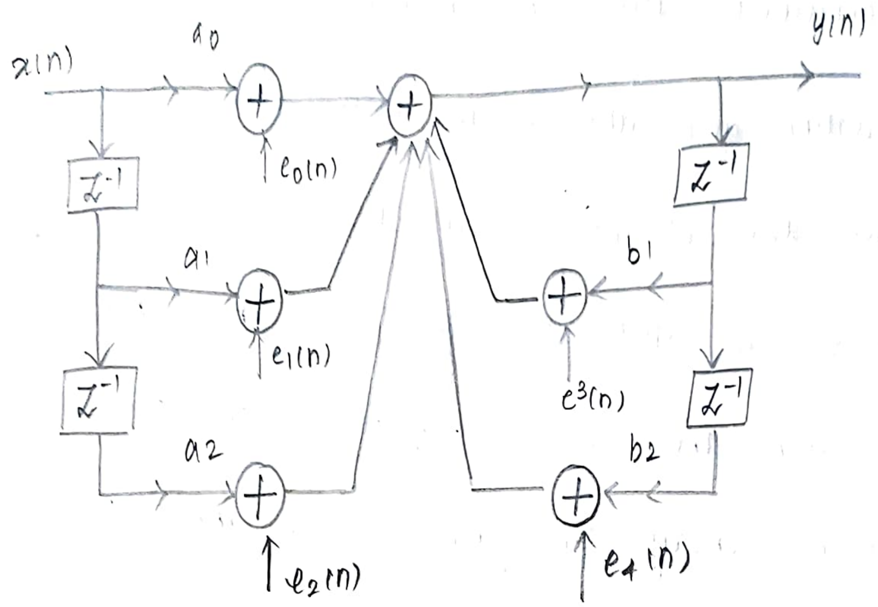
3. The error sequence e(n) is uncorrelated with the signal sequence x(n). Then each noise source is modeled as a discrete stationary white random process with a power density spectrum of $\frac{2^{-2b}}{12}$.

IIR system represented by the following difference equation,

$$y(n) = a_1 y(n-1) + x(n)$$



Quantization noise model for a first order system.



b) Quantization noise model for a second-order system with five noise sources.

Since all the noise sources are added at the same point in the filter, all these sources can be replaced by single noise source $e(n) = e_1(n) + e_2(n) \ldots + e_4(n)$ with zero mean and variance $\sigma^2 = \sigma_0^2 + \sigma_1^2 + \ldots + \sigma_4^2$

where,

$$\sigma_e^2 = \frac{2^{-2b}}{12}.$$

$$\sigma_{0k}^2 = \sigma_e^2 \frac{1}{2\pi j} \oint_c H_k(z)\, H_k(z^{-1})\, z^{-1} dz.$$

where,

$H_k(z)$ is defined as the noise transfer fn.

Co-efficient Quantization error:-

The frequency response of the actual filter deviates from that which would have been obtained with an infinite word length representation and the filter may actually fail to meet the desired specifications. If the poles of the desired filter are close to the unit circle, then those of the filter with quantized co-efficients may lie just outside the unit circle leading to instability.

Zero-input Limit cycle oscillations:-

When a stable IIR digital filter is excited by a finite input sequence, that is constant, the output will ideally decay to zero. However, the non linearities due to the finite-precision arithmetic operations often cause periodic oscillations to occur in the output.

Consider a first order IIR filter with difference equation,

$$y(n) = x(n) + \alpha y(n-1)$$

Let us assume,

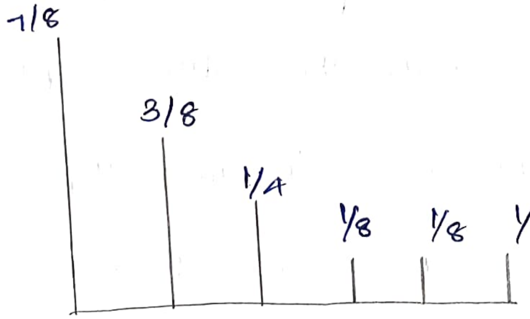
$\alpha = 1/2$ and the data register length is 3 bits plus a sign bit. If the input is,

$$x(n) = \begin{cases} 0.875 & \text{for } n=0 \\ 0 & \text{otherwise} \end{cases}$$

and rounding applied after the arithmetic operation Then the table illustrates the limit cycle behaviour. Here $Q[\cdot]$ represents the rounded operation.

It can find that for $n \geq 3$ the output remains constant and gives $1/8$ as steady output causing limit cycle behaviour.

n	$x(n)$	$y(n-1)$	$\alpha y(n-1)$	$Q[\alpha y(n-1)]$	$y(n) = x(n) + Q[\alpha y(n-1)]$
0	0.875	0.0	0.0	0.000	7/8
1	0	7/8	7/16	0.100	1/2



Dead band :-

The limit cycles occur as a result of the quantization effects in multiplications. The amplitudes of the output during a limit cycle are confined to a range of values that is called the dead band of the filter.

$$y_q(n) = Q\left[\alpha y(n-1)\right] + x(n)]$$

$$y(n-1) \leq \frac{\frac{1}{2} 2^{-b}}{1-|\alpha|}$$ defines the dead band

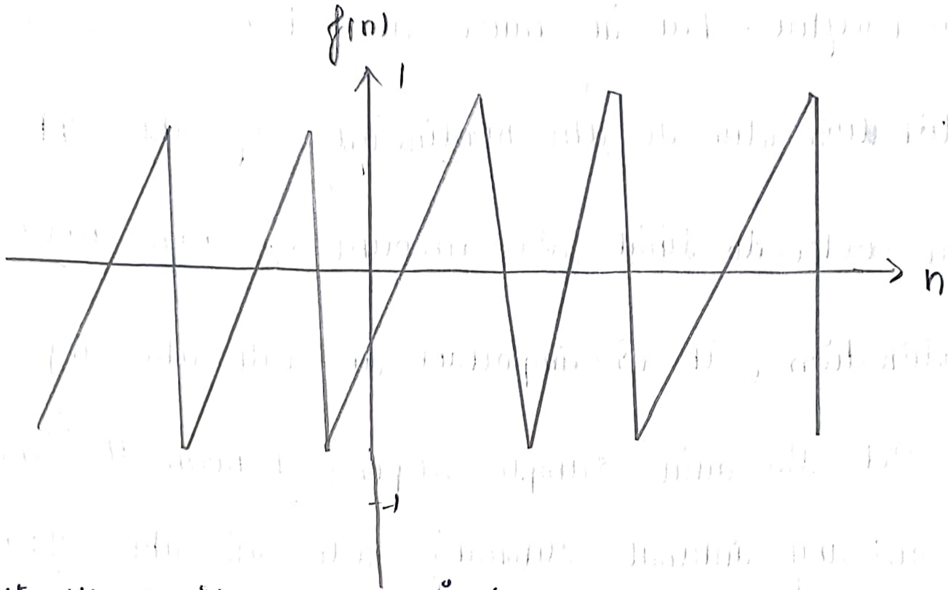
for the first order filter.

Overflow limit cycle Oscillations :

In addition to limit cycle oscillations causing by rounding the result of multiplication, There are several types of limit cycle oscillations caused by addition, which make the filter output oscillate between maximum and minimum amplitudes. such limit cycles have been referred to as overflow oscillations.

An overflow an addition of two or more binary numbers occurs when the sum exceeds the would size available in the digital implementation of the system.

Transfer characteristics of an adder.



Let us consider two positive numbers n_1, and n_2

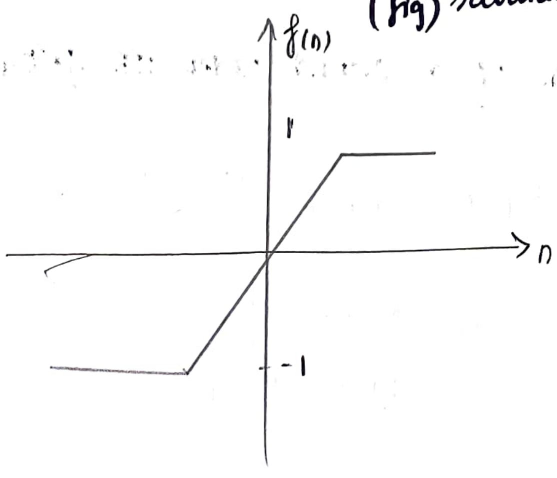
$n_1 = 0.111 \rightarrow 7/8$

$n_2 = 0.110 \rightarrow 6/8$

$n_1 + n_2 = 1.101 \rightarrow -5/8$ in sign magnitude

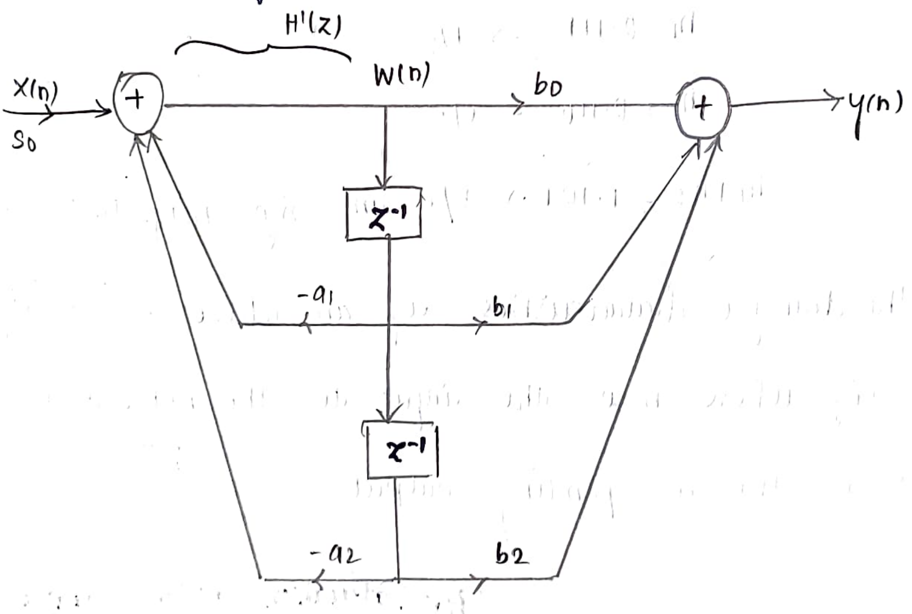
The transfer characteristics of an adder as shown in fig. where n is the input to the adder and f(n) is the corresponding output.

(fig) Saturation adder transfer characteristics.



Signal scaling:

Saturation arithmetic eliminates limit cycles due to overflow, but it causes undesirable signal distortion due to the nonlinearity of the clipper. In order to limit the amount of non-linear distortion, it is important to scale the input signal and the unit sample response between the input and any internal summing node in the system such that overflow becomes a rare event.



Realization of a second order IIR filter.

The overall input - output transfer fn,

$$H(z) = S_0 \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

$$H(z) = \delta_D \frac{N(z)}{D(z)}$$

which gives us.

$$\delta_D^2 = \frac{1}{\frac{1}{2\pi j} \oint_C S(z)\, S(z^{-1}) z^{-1} dz}$$

$$= \frac{1}{\frac{1}{2\pi j} \oint_C \frac{z^{-1} dz}{D(z)\, D(z^{-1})}}$$

where,

$$\boxed{I = \frac{1}{2\pi j} \oint_C \frac{z^{-1} dz}{D(z)\, D(z^{-1})}}$$

Architectural features :-

Though most of the digital signal processors available today have good architectural features the key features of interest include size of on-chip memory, special instructions and I/O capability. In applications where large memory is required on-chip memory is essential. It helps in accessing the data at high speeds and executing the program rapidly.

For memory hungry applications (e.g digital audio - Dolby AC-2, FAX/Modem, MPEG coding/decoding), the size of internal RAM should be high. For applications that require fast and efficient communication or data flow with the outside world, I/O features such interface to ADC and DACs, DMA capability and support for multiprocessing may be important.

Execution speed :

The execution speed of digital signal processor plays an important role in selecting the processor. The execution speed is measured.

in terms of the clock speed of the processor, in MHz and the number of instructions performed, in millions of instructions per second (MIPS) or in the case of floating point digital signal processors, in millions of floating point operations per second (MFLOPS). Thus, an alternative measure is based on the execution speed of benchmark algorithms such as FFT, FIR and IIR filter.

Type of arithmetic :-

The two most common type of arithmetic used in modern digital signal processors are fixed and floating point arithmetic. Fixed point processors are favoured in low cost, high volume applications (e.g. cellular phones and computer disk drives).

floating arithmetic is the natural choice for applications with wide and variable dynamic range requirements range may be defined as the difference between the largest and smallest signal levels that can be represented).

Word length :-

Processor Data word length is an important parameter in DSP as it can have a significant impact on signal quality. In general, the longer the data word the lower the errors that are introduced by digital signal processing. Fixed point digital signal processors aimed at telecommunications market tend to use a 16-bit word length.

Applications of PDSPs :-

In this section, we will study the applications of PDSPs in both real-world and proto-typing applications. The applications are divided into three categories :

communication system

multimedia

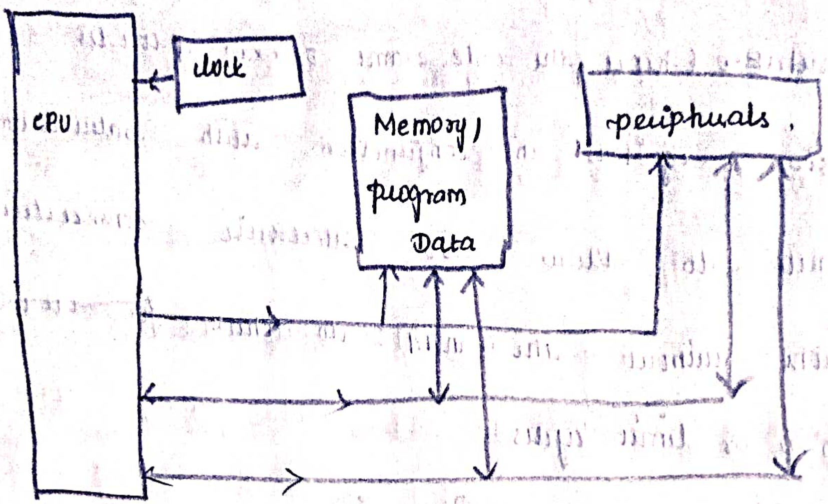
Control / Data acquisitions .

Communication systems :-

PDSPs have been applied to implement various communication systems. Examples include caller ID, cordless handset, and many others. In voice communication, an acoustic-echo canceller based for hands-free wireless system is developed using TMS320C2x. TMS320C50 fixed-point processor can be used to implement a low bit rate (1.2 kbps), real time vocoder (voice coder).

These standards are often implemented in modern digital cameras and digital camcorders - where PDSPs will play an important role.

Von - Neumann Architecture:



In 1946, John Von Neumann developed the first computer architecture that allowed the computer to be programmed by codes residing in memory.

In this, program instructions were stored in Read only Memory (ROM) //

The Von Neumann architecture is most widely used in majority of microprocessors

In a computer with Von-Neumann architecture, the CPU can be either reading an instruction or reading / writing data from / to the memory.

Both cannot occur at the same time since the instruction and data use the same signal pathways and memory.

The Von Neumann architecture consists of three buses: the Data bus, the address bus and control bus.

Data bus:

Transports data between CPU and its peripherals. It is bidirectional. The CPU can read or write data in the peripherals.

The Address Bus:

The CPU uses the address bus to indicate which peripherals it wants to access and within each peripheral which specific register. The address bus is unidirectional.

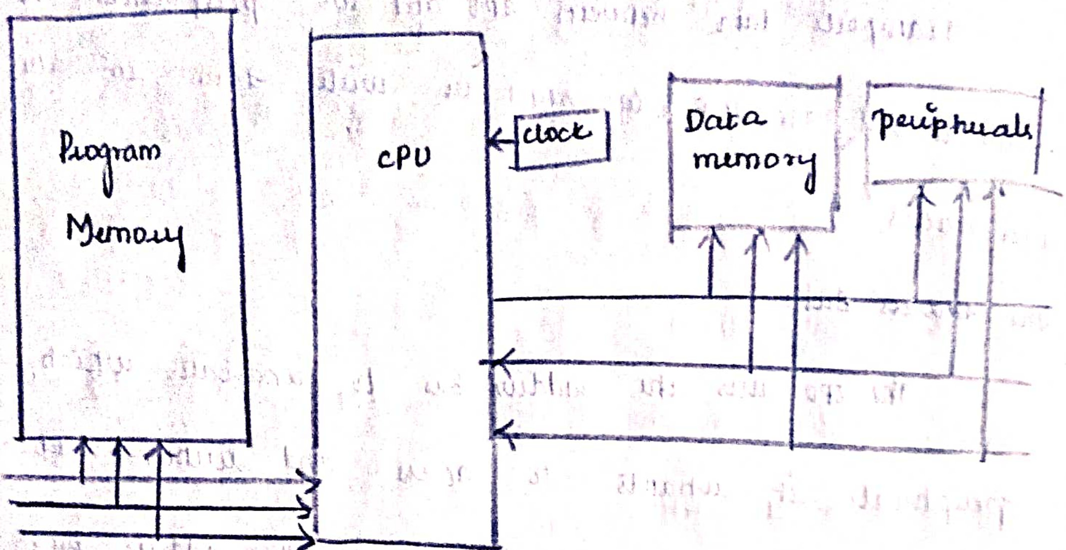
The CPU always writes the address, which is read by the peripherals.

Control Bus:

The bus carries signals that are used to manage and synchronize the changes between the CPU and its peripherals, as well as that indicates if the CPU wants to read and write the peripheral

→ The main characteristics of the Von Neumann architecture is that it only possesses bus system. The same bus carries all the information exchanged between the CPU and the peripherals including the instruction codes as well as the data processed by the CPU.

Harvard Architecture :-



The Harvard architecture physically separates memories for their instructions and data, requiring dedicated buses for each of them.

Instructions and operands can therefore be fetched simultaneously.

Most DSP processors use a modified Harvard architecture with two or three memory buses, allowing access to filter co-efficients and input signals in the same cycle.

Since it possesses two independent bus systems, the Harvard architecture is co or peripheral as part of the execution of the previous instruction. Since it has two memories, it is not possible for the CPU to mistakenly write codes into the program memory and therefore compute the code while it is executing.

However it is less flexible.

It needs two independent memory banks. These two resources are not interchangeable.

. The modified Harvard architecture used DSPs multiport memory that has separate bus system for program memory and Data memory and i/p and o/p peripherals.

These multiple bus system increases complexity of the CPU, but allow it to access several memory locations simultaneously, thus by increasing the data throughput between memory and CPU.

VLIW Architecture:

The new Architecture that has attracted a great deal of attention in the DSP community is the Very Long Instruction Word (VLIW).

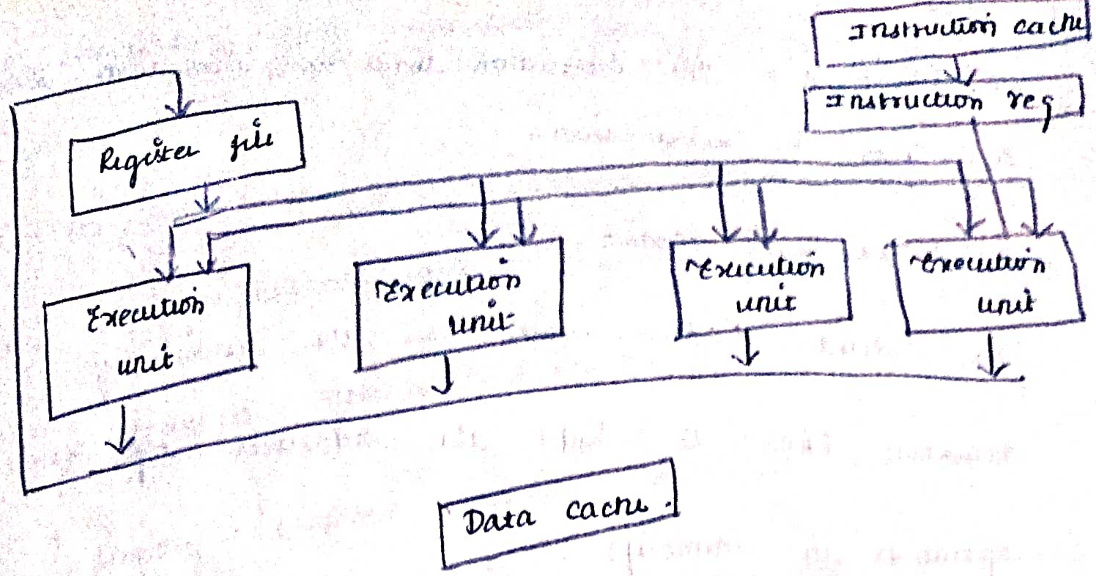
The Very Long Instruction Word processing increase the number of Instructions that are processed per cycle. It is essentially a concatenation of several short instructions and require multiple execution units, running in parallel, to carry out the instructions in a single cycle.

VLIW architecture executes multiple instructions/cycle and use simple, regular instruction sets.

VLIW processor consists of architecture that reads a relatively large group of instructions and executes them at the same time. The VLIW processor combines many simple instructions into a single long instruction word that uses different registers.

A language compiler or pre-processor separates program instructions into basic operations that are performed by the processor in parallel.

These operations are placed into "Very long instruction word" that the processor can then disassemble and then transfer each operation to an appropriate execution unit.



Addressing Mode :

* Immediate Addressing Mode.
* Indirect Addressing Mode.
* Direct Addressing Mode.
* Memory mapped Addressing mode.
* circular Addressing Mode.
* Register Addressing mode.

Immediate Addressing Mode :

Immediate addressing mode is used to handle constant data. It allows the programmer operate an actual value. The Data can be either a 16-bit constant or constant length 7, 9 and 13. Depending on the length of the data, the addressing mode is refered to as long immediate or short immediate addressing mode.

LD #80h, A :. The instruction loads an immediate va
80h into the accumulator.

Indirect Addressing Mode:.

The indirect Address mode uses the auxiliary registers (ARs) to hold the addresses of operands in memory).

Indirect addressing, any location in the 64-k word data memory space can be accessed using a 16-bit address contained in AR.

They are seven types of indirect addressing :

i) auto increment.
ii) auto decrement.
iii) Post indexing by adding the contents of AR0.
iv) Post indexing by subtracting the contents of AR0
v) Single indirect addressing with no increment.
vi) Single indirect addressing with no decrement.
v) Bit reversed addressing.

Register Addressing :-

The register addressing mode uses operands in CPU registers either explicitly, such as with a direct reference to a specific register, or simplicity, with instruction that uses certain registers.

The block move address register (BMAR) and the Dynamic bit manipulation registers (DBMR).

Memory mapped register Addressing:

Memory mapped register addressing is used to access efficiently the CPU and on chip peripheral registers. It operates like the direct addressing expect that the upper 9-bits of the address that is accessed are assumed to be 0s.

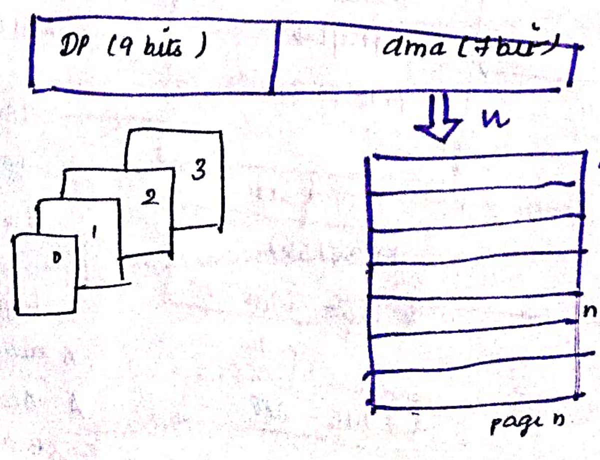
LAMM - Load Accumulator with Memory Mapped register.

LMMR - Load Memory Mapped Register.

SAMM - Store Accumulator in Memory Mapped Register.

SMMR - Store Memory Mapped Register.

Direct Addressing mode:



specify actual location inside 128 word page.

circular Addressing mode :

Circular addressing is the most sophisticated 'C5x addressing mode. Many algorithms such as convolution, correlation and FIR filtering can use circular buffers in memory to implement a sliding window.

CBSR 1 - circular Buffer 1 Start Register

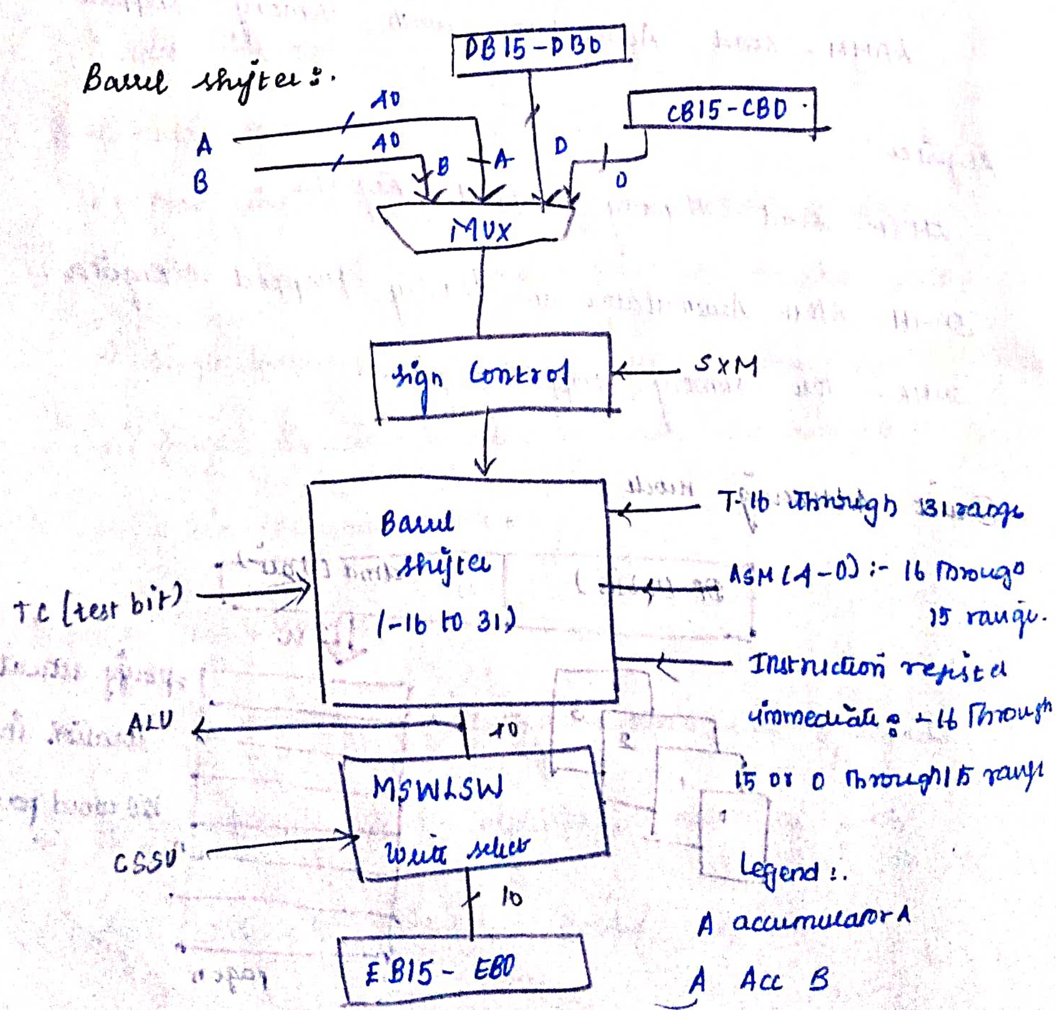
CBSR 2 - circular Buffer 2 Start Register.

CBER 1 - circular Buffer 1 End register.

CBER 2 - circular Buffer 2 End register.

CBCR - circular buffer control register.

Barrel shifter :



Legend :

A accumulator A

B Acc B

C CB databus

D DB data bus

T T reg.

=> 40 bit barrel shifter of C54x can perform arithmetic and logical shifts by upto to 31 bits left or by up to 16-bits right in a single instruction cycle.

The shifter inputs come directly from Data memory or from either of the two accumulators.

Shifter outputs can be sent to ALU or stored in memory. The shift count Determines how many bits to shift.

positive shift Values correspond to left shifts, whereas negative values correspond to right shifts.

The shift count are specified as a 2's complement value in several ways, depending on the instruction type:

The barrel shifter is also used for scaling operations such as:

i) Prescaling an input Data memory operand or the accumulator value before an ALU operation.

ii) Performing a logical or arithmetic shift of the accumulator value.

iii) Normalizing the accumulator

Instruction set :.   Inst

Arithmetic Instructions :.

Addition :

1. ADCB :

The contents of the accumulator buffer (ACCB) and the value of the C bit are added to the contents of the accumulator (ACC).

Example :

Before Instruction   After Instruction

Acc $\boxed{1}$ $\boxed{2345h}$ Acc $\boxed{0}$ $\boxed{2349h}$

ACCB $\boxed{3h}$ ACCB $\boxed{3h}$

Examples :.

ADD 98h, 5 :

The contents of Acc is added with the contents of data memory with dma 98h in the current page after shifting it left by four positions.

Subtraction :.

SBB : The contents of the accumulator buffer (ACCB) are subtracted from the contents of the accumulator.

The result is stored in the ACC and the contents of the ACCB are unaffected.

ACC	X (C)	3000 000h	ACC	1 (C)	1000 000h
ACCB		2000 000h	ACC B		2000 0000h

SUB dma [, shift]: The contents of the data memory address (dma) or a 16 bit constant are shifted left, as defined by the shift code and subtracted from the contents of ACC and the result is stored in ACC.

Multiplication Instructions:

MPY dma (direct addressing): The contents of TREG0 are multiplied by the contents of data memory address and the result is stored in the product register.

Before Instruction

TREG0 | 3h |

After Instruction

TREG0 | 3h |

Logical Instructions:

AND:

Direct addressing AND dma

Indirect addressing AND {ind} [, ARn]

Long immediate AND #lk [, shift] .

The contents of the dma are ANDed with the contents of the ACCL : and the results is stored in ACCL .

AND # 1234, 2

Before execution | After execution

ACC $\boxed{1458h}$ | ACC $\boxed{0050h}$

OR :-

Direct addressing : OR dma

Indirect addressing : OR {ind} [, ARn]

Long immediate : OR #lk, [, shift]

The contents of the dma are ORed with the contents of the ACCL and the result is stored in ACCL .

OR * 1234, 2

Example,

OR *, AR3

Before execution

ARP $\boxed{2}$

AR2 $\boxed{1020h}$

After execution

ARP $\boxed{3}$

AR2 $\boxed{1020h}$

Shift Instructions :

ROL :- Rotate accumulator left

The contents of the accumulator are rotated left 1 bit. The value of the C bit is shifted into LSB of the Acc. The MSB of the original Acc is shifted into the C bit

ROR -> Rotate accumulator right.

Load / store Instructions :.

LACB Load accumulator with ACCB

LACC Load Accumulator with shift.

Ex :

LT dma

LT {ind} [, ARn]

The contents of the dma are loaded into TREG0

Before Instruction		After Instruction	
300h	60h	300h	60h
TREG0	2h	TREG0	60h

Move Instructions :.

DMOV : Data move in Data Memory.

The contents of the data memory address (dma) are copied to the next higher dma.

AR Model Parameters :-

In the Yule-Walker method we simply estimate the autocorrelation from the data and use the estimates in (14.3.7) to solve for the AR model parameters.

In this method it is desirable to use the biased form of the autocorrelation estimate.

$$r_{xx}(m) = \frac{1}{N} \sum_{n=0}^{N-m-1} x^*(n)\, x(n+m) \qquad m \geq 0.$$

to ensure that the autocorrelation matrix is positive semidefinite.

The Levinson-Durbin algorithm described in previous step, with $r_{xx}(m)$ substituted for $\gamma_{xx}(m)$ yields the AR parameters. The corresponding power spectrum estimate is,

$$P_{xx}^{YW}(f) = \frac{\hat{\sigma}^2_{wp}}{\left|1 + \sum_{k=1}^{p} \hat{a}_p(k)\, e^{-j2\pi fk}\right|^2}$$

where, $\hat{a}_p(k)$ are estimates of the AR parameters obtained from the Levinson-Durbin recursions and

$$\hat{\sigma}^2_{wp} = \hat{E}^f_p = r_{xx}(0) \prod_{k=1}^{p} \left[1 - |\hat{a}_k(k)|^2\right]$$

is the estimated of the AR parameters obtained

from the

is the estimated minimum mean-square

value for the p-th order predictor. An example

illustrating the frequency resolution capability

of the estimator is given in section.

In estimating the power spectrum of

sinusoidal signals via AR models. Lacoss

(1971) showed that spectral peaks in

an AR spectrum estimate are proportional

to the square of the power of the sinusoidal

signal.

On other hand, the area under the

peak in the power Density spectrum

is linearly proportional to the power

of the sinusoid. This characteristic

behavior holds for all AR model-based

estimation methods.

MA Model for power spectrum Estimation :

The parameters in an MA(q) model are related to the statistical autocorrelation $\gamma_{xx}(m)$ by

(14.3.10)

However,

$$B(z)\,B(z^{-1}) = D(z) = \sum_{m=-q}^{q} d_m z^{-m}.$$

where the coefficients $\{d_m\}$ are related to the MA parameters by the expression,

$$d_m = \sum_{k=0}^{q-|m|} b_k b_{k+m}, \quad |m| \le q$$

clearly, then,

$$\gamma_{xx}(m) = \begin{cases} \sigma_w^2 d_m, & |m| \le q \\ 0, & |m| > q \end{cases}$$

power spectrum for the MA(q) process is,

$$\Gamma_{xx}^{MA}(f) = \sum_{m=-q}^{q} \gamma_{xx}(m)\, e^{-j2\pi fm}$$

It is apparent from these expressions that we do not have to solve for the MA parameters $\{b_k\}$ to estimate the power spectrum.

The estimates of the autocorrelation $\gamma_{xx}(m)$ for $|m| \le q$ suffice. From such estimates we compute the estimated MA power spectrum, gi

$$P_{xx}^{MA}(f) = \sum_{m=-q}^{q} r_{xx}(m) e^{-j2\pi fm}.$$

which is identical to the classical power spectrum estimate.

=) There is an alternative method for determining {bk} based on high-order AR approximation to the MA process.

=) To be specific, let the MA(q) process be modeled by an AR(p) model, where p>>q.

Then $B(z) = 1/A(z)$ (or) $B(z)\,A(z) = 1$.

Thus the parameters {bk} and {ak} are related by a