srivenkateshwaraa
College of Engineering & Technology
(Approved by AICTE, New Delhi & Affiliated to Pondicherry University, Puducherry)
13-A, Pondy - Villupuram Main Road, Ariyur, Puducherry - 605 102.
ASPIRE TO EXCEL

DEPARTMENT OF ELECTRONICS AND COMMUNICATION ENGINEERING

EC T44 LINEAR AND DIGITAL CONTROL SYSTEM NOTES

II YEAR/ IV SEM

UNIT - I

System Modeling: Introduction to control system-Basic elements of control system-Open and closed loop control systems-Differential equation representation of physical systems-Transfer function-Mathematical modeling of electrical and mechanical systems (Translational and Rotational)-Analogous system-Block diagram reduction techniques-Signal flow graph.

UNIT - II

Time Domain Analysis: Time response analysis-Analysis of transient and steady state behavior of control systems-Standard test signals –Time response of First order system-step, ramp and impulse response analysis-Second order system – step response analysis-steady state error-generalized error co-efficient–Response with P, PI, PD and PID controllers-Analysis using MATLAB.

UNIT – III

Frequency Domain Analysis: Frequency response-Frequency domain specifications-Correlation between time domain and frequency domain specifications-Bode plot-Stability analysis using Bode plot- transfer function from Bode plot-Polar plot-Analysis using MATLAB.

UNIT – IV

Stability Analysis and Root Locus: Concepts of stability-Location of poles on s-plane for stability-Routh-Hurwitz stability criterion-Nyquist stability criterion-Root locus Techniques-Analysis using MATLAB.

UNIT - V

Digital Control System: Basic digital control system-Z transform and its properties-Inverse Z transform-Response of linear discrete time systems-Pulse transfer function-Stability analysis-Jury's stability criterion.

State Space Analysis: State space model of a control system -State space representation using physical, phase and canonical variables-diagonal canonical form-Jordan canonical form.

Text Books:

1. Benjamin. C. Kuo, ―Digital control systems‖, Second Edition, Oxford University Press, 2012.
2. I. J. Nagrath, M. Gopal, ―Control Systems Engineering‖, Fifth Edition, New Age International, New Delhi, 2011.

Reference Books:

1. R. Anandanatarajan, P. Ramesh Babu, ―Control Systems Engineering‖, Second edition, Scitech Publications, 2005.
2. Benjamin C.Kuo, ―Automatic Control Systems‖, Seventh Edition, PHI Learning, New Delhi, 1997.
3. Katsuhiko Ogata, ―Discrete Time Control Systems‖, Second Edition, PHI Learning, New Delhi, 2006.

Web References:

1. http://ctms.engin.umich.edu/CTMS/index.php?aux=Home
2. http://www.mathworks.in/academia/student_center/tutorials/controls-tutorial-launchpad.html
3. http://ctms.engin.umich.edu/CTMS/index.php?example=Introduction&section=ControlDigital
4. http://www.library.cmu.edu/ctms/ctms/
5. http://www.ee.usyd.edu.au/tutorials_online/matlab/PID/PID.html

UNIT-I

MathematicalModelingofPhysicalControlsystems

Conceptofcontrolsystem

A **control system** manages commands, directs or regulates the behavior of other devices or systemsusing controlloops.Itcanrangefromasinglehomeheating controllerusinga thermostatcontrolling a domestic boiler to large Industrial controlsystemswhich are used for controlling processesor machines. A control system is a system, which provides the desired response by controlling the output. The following figure shows the simple block diagram of a control system.

Examples–Trafficlightscontrolsystem,washingmachine

Traffic lights control system is an example of control system. Here, a sequence of input signal is applied to this control system and theoutput is one of the three lights that will be on for some duration of time. During this time, the other two lights will be off. Based onthe traffic study at a particular junction, the on and off times of the lights can be determined. Accordingly, the input signal controls the output. So, the traffic lights control system operates on time basis.

ClassificationofControlSystems

Basedonsomeparameters,wecanclassifythe controlsystemsinto thefollowing ways.

ContinuoustimeandDiscrete-timeControlSystems

- ControlSystemscanbeclassifiedascontinuoustimecontrol systemsanddiscrete time control systems based on the **type of the signal**used.
- In**continuoustime** controlsystems,allthesignalsarecontinuousintime. But, in **discrete time** control systems, there exists one or more discrete time signals.

SISOandMIMOControlSystems

- ControlSystemscanbeclassifiedasSISOcontrolsystemsandMIMOcontrol systems based on the **number of inputs and outputs** present.

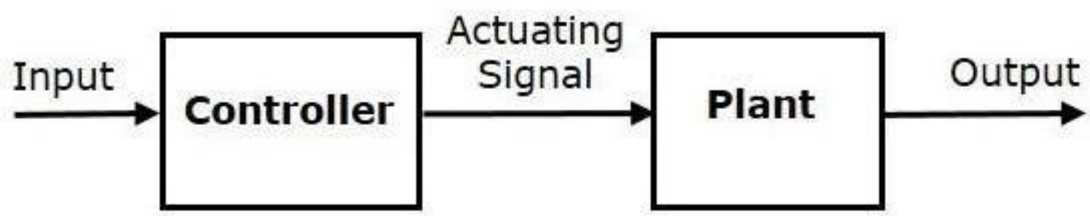
- **SISO** (Single Input and Single Output) control systems have one input and one output. Whereas, **MIMO** (Multiple Inputs and Multiple Outputs) control systems have more than one input and more than one output.

OpenLoopandClosedLoopControlSystems

ControlSystemscanbeclassifiedasopenloopcontrolsystemsandclosedloopcontrol systems based on the **feedback path**.

In**openloop controlsystems**,outputisnotfed-backto theinput.So,thecontrolaction is independent of the desired output.

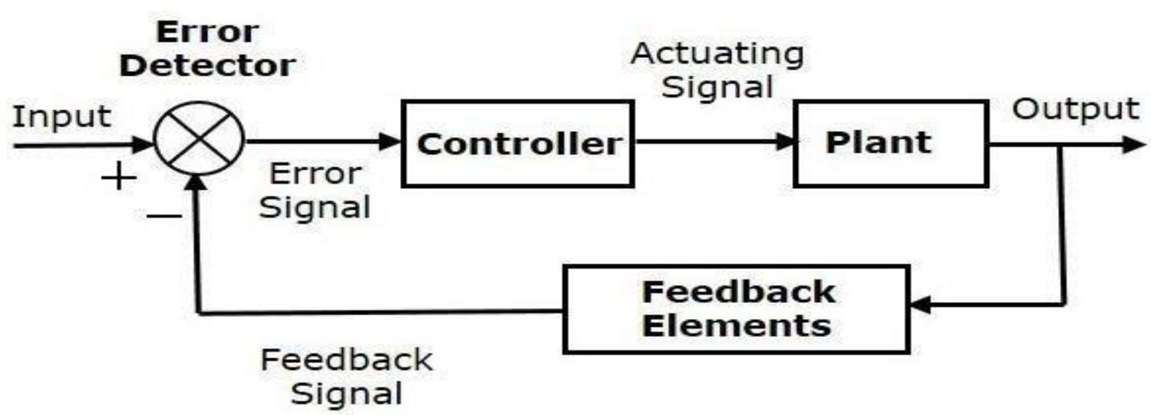
Thefollowingfigureshowstheblockdiagramoftheopenloopcontrol system.



Here, an input is applied to a controller and it produces an actuating signal or controlling signal. This signal is given as an input to a plant or process which is to be controlled. So, the plant produces an output, which is controlled. The traffic lights control system which we discussed earlier is an example of an open loop control system.

In **closed loop control systems**, output is fed back to the input. So, the control action is dependent on the desired output.

Thefollowingfigureshowstheblockdiagramofnegativefeedbackclosedloopcontrolsystem.



Theerrordetectorproducesan errorsignal,whichisthedifference betweentheinputand the feedback signal. This feedback signal is obtained from the block (feedback elements) by considering the output of the overall system as an input to this block. Instead of the direct input, the error signal is applied as an input to a controller.

So, the controller produces an actuating signal which controls the plant. In this combination, the output of the control system is adjusted automatically till we get the desired response. Hence, the closed loop control systems are also called the automatic control systems. Traffic lights control system having sensor at the input is an example of a closed loop control system.

Thedifferencesbetween theopenloop andtheclosed loopcontrol systemsarementioned in the following table.

Open Loop Control Systems	Closed Loop Control Systems
Control action is independent of the desired output.	Control action is dependent of the desired output.
Feedback path is not present.	Feedback path is present.
These are also called as **non-feedback control systems**.	These are also called as **feedback control systems**.
Easy to design.	Difficult to design.
These are economical.	These are costlier.
Inaccurate.	Accurate.

If either the output or some part of the output is returned to the input side and utilized as part of the system input, then itis knownas **feedback**. Feedback plays an importantrole in order to improve the performance of the control systems. In this chapter, let us discuss the types of feedback & effects of feedback.

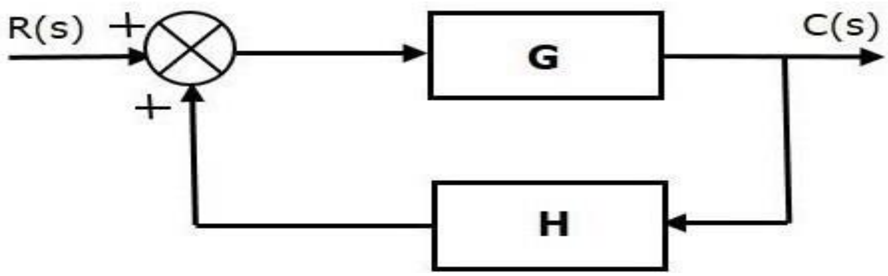
TypesofFeedback

Therearetwotypesoffeedback−

- Positivefeedback
- Negativefeedback

PositiveFeedback

Thepositivefeedbackaddsthereferenceinput,R(s)R(s) andfeedbackoutput.The following figure shows the block diagram of **positive feedback control system**



heconceptoftransferfunctionwillbediscussedinlaterchapters.Forthetimebeing, consider the transfer function of positive feedback control system is,

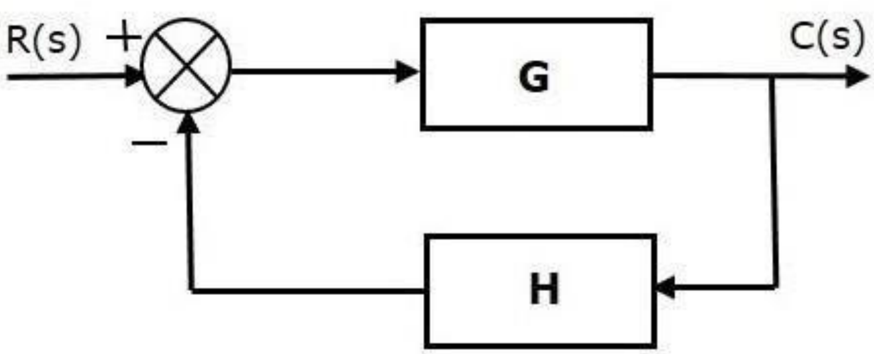
$$T = \frac{G}{1-GH} \quad \text{(Equation 1)}$$

Where,

- **T**isthetransfer functionor overallgainofpositivefeedbackcontrolsystem.
- **G**istheopenloopgain,whichisfunctionoffrequency.
- **H**isthegain offeedbackpath,whichisfunctionoffrequency.

Negative Feedback

Negative feedback reduces the error between the reference input, R(s)R(s) and system output.Thefollowingfigureshowstheblockdiagramofthe **negativefeedbackcontrol system**.



Transferfunctionofnegativefeedbackcontrolsystemis,

$$T = \frac{G}{1+GH} \qquad \text{(Equation 2)}$$

Where,

- **T**isthetransferfunctionoroverallgainofnegativefeedbackcontrolsystem.
- **G**istheopenloopgain,whichisfunctionoffrequency.
- **H**isthegain offeedbackpath,whichisfunctionoffrequency.

Thederivationoftheabovetransferfunctionispresentinlater chapters.

EffectsofFeedback

Letusnow understand theeffectsoffeedback.

Effectof FeedbackonOverallGain

- From Equation 2, we can say that the overall gain of negative feedback closed loop control system is the ratio of 'G' and (1+GH). So, the overall gain may increase or decrease depending on the value of (1+GH).
- Ifthevalueof(1+GH)islessthan1, thentheoverall gainincreases.Inthis case,'GH' value is negative because the gain of the feedback path is negative.
- If the value of (1+GH) is greater than 1, then the overall gain decreases. In this case, 'GH' value is positive because the gain of the feedback path ispositive.

Ingeneral,'G'and'H'arefunctionsoffrequency.So,thefeedbackwillincreasetheoverall gain of the system in one frequency range and decrease in the other frequency range.

EffectofFeedbackonSensitivity

Sensitivity oftheoverallgainofnegativefeedbackclosedloopcontrolsystem(**T**)tothe variation in open loop gain (**G**) is defined as

$$S_G^T = \frac{\frac{\partial T}{T}}{\frac{\partial G}{G}} = \frac{Percentage\ change\ in\ T}{Percentage\ change\ in\ G} \qquad \text{(Equation 3)}$$

Where, **∂T** is the incremental change in T due to incremental change in G.

We can rewrite Equation 3 as

$$S_G^T = \frac{\partial T}{\partial G}\frac{G}{T} \qquad \text{(Equation 4)}$$

Do partial differentiation with respect to G on both sides of Equation 2.

$$\frac{\partial T}{\partial G} = \frac{\partial}{\partial G}\left(\frac{G}{1+GH}\right) = \frac{(1+GH).1-G(H)}{(1+GH)^2} = \frac{1}{(1+GH)^2} \qquad \text{(Equation 5)}$$

From Equation 2, you will get

$$\frac{G}{T} = 1 + GH \qquad \text{(Equation 6)}$$

Substitute Equation 5 and Equation 6 in Equation 4.

$$S_G^T = \frac{1}{(1+GH)^2}(1+GH) = \frac{1}{1+GH}$$

So, we got the **sensitivity** of the overall gain of closed loop control system as thereciprocal of (1+GH). So, Sensitivity may increase or decrease depending on the value of (1+GH).

- If the value of (1+GH) is less than 1, then sensitivity increases. In this case, 'GH' value is negative because the gain of feedback path is negative.
- If the value of (1+GH) is greater than 1, then sensitivity decreases. In this case, 'GH' value is positive because the gain of feedback path ispositive.

In general, 'G' and 'H' are functions of frequency. So, feedback will increase the sensitivity of the system gain in one frequency range and decrease in the other frequency range. Therefore,wehaveto choosethevaluesof'GH'insuchawaythatthesystemisinsensitive or less sensitive to parameter variations.

EffectofFeedbackonStability

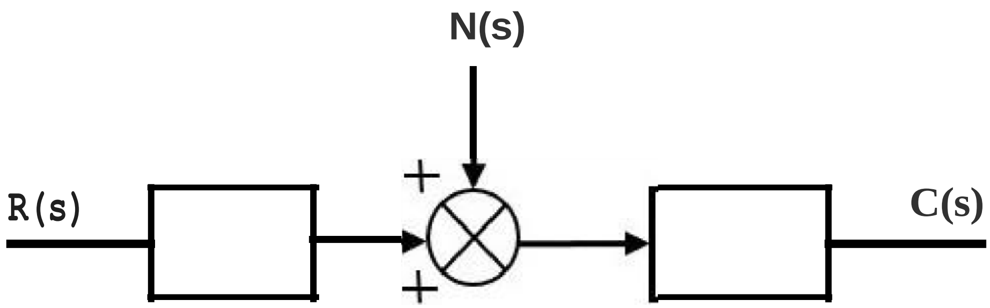
- A system issaid to be stable, if itsoutput isunder control. Otherwise, it issaid to be unstable.
- In Equation 2, if the denominator value is zero (i.e., GH = -1), then the output of the control system will be infinite. So, the control system becomes unstable.

Therefore,wehavetoproperlychoosethefeedbackinordertomakethecontrolsystem stable.

EffectofFeedbackon Noise

Toknowtheeffectoffeedbackonnoise,letuscomparethetransferfunctionrelations with and without feedback due to noise signal alone.

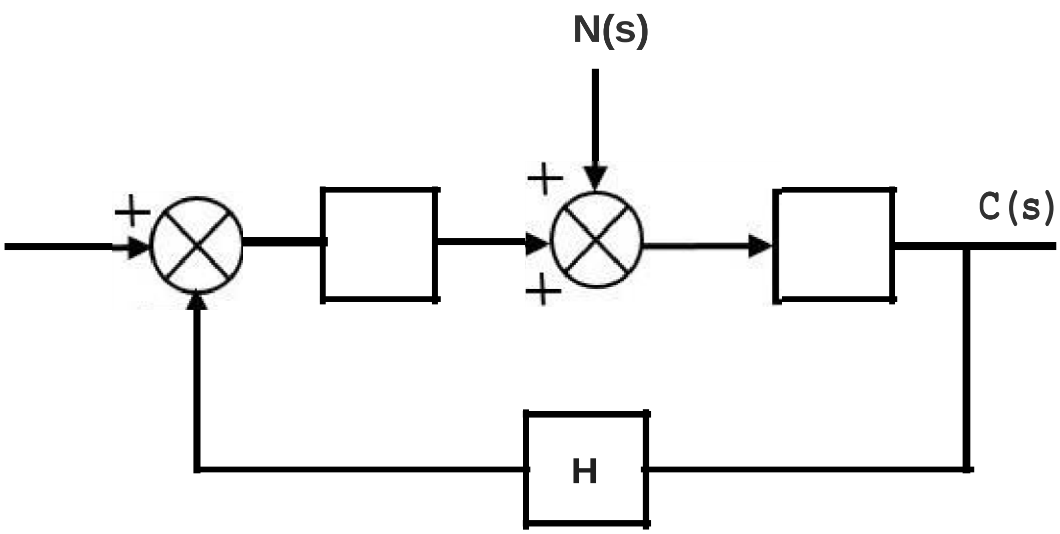
Consideran**openloopcontrolsystem**withnoisesignalasshownbelow.



The**openlooptransferfunction**duetonoisesignalaloneis

$$\frac{G(s)}{N(s)} = G_b \qquad \text{(Equation7)}$$

ItisobtainedbymakingtheotherinputR(s)equaltozero.



The**closedlooptransferfunction**duetonoisesignalaloneis

$$\frac{O(s)}{N(s)} \quad \frac{G_b}{1+G_a G_b H} \qquad \text{(Equation8)}$$

Itis obtained bymaking the other input R(s) equal tozero.

Compare Equation 7 and Equation 8,

Intheclosedloopcontrolsystem,thegainduetonoisesignalisdecreasedbya factorof$(1+G_aG_bH)$providedthatthe term$(1+G_aG_bH)$isgreaterthan one.

Thecontrolsystemscanberepresentedwithasetofmathematicalequations knownas **mathematical model**. These models are useful for analysis and design of control systems. Analysis of control system means finding the output when we know the input and mathematical model. Design of control system means finding the mathematical model when we know the input and the output.

Thefollowingmathematicalmodelsaremostlyused.

- Differentialequationmodel
- Transferfunction model
- Statespacemodel

Unit-II

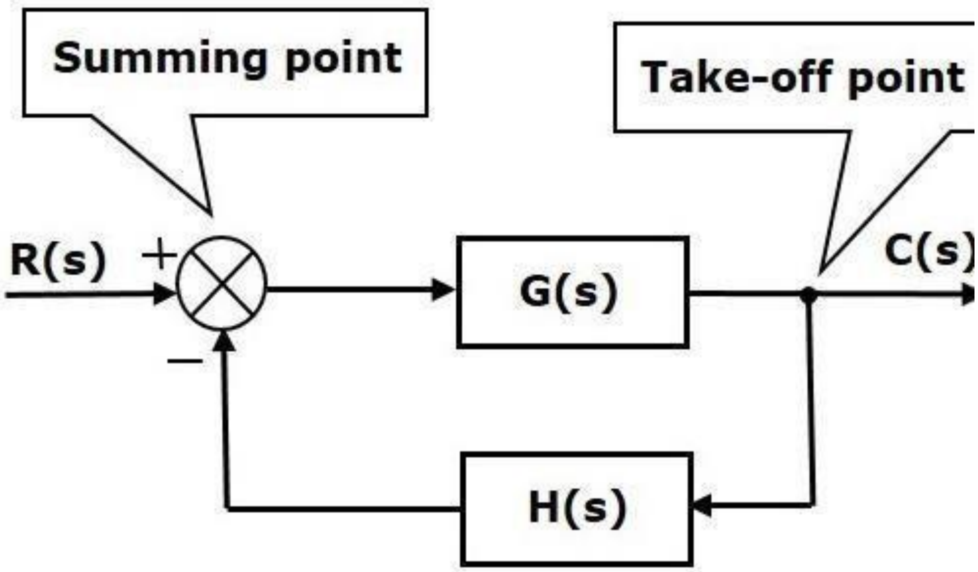
TRANSFERFUNCTIONREPRESENTATION

BlockDiagrams

Block diagrams consist of a single block or a combination of blocks. These are used to represent the control systems in pictorial form.

BasicElementsofBlock Diagram

The basic elements of a block diagram are a block, the summing point and the take-off point. Let us consider the block diagram of a closed loop control system as shown in the following figure to identify these elements.



The above block diagram consists of two blocks having transfer functions G(s) and H(s). It is also having one summing point and one take-off point. Arrows indicate the direction of the flow of signals. Let us now discuss these elements one by one.

Block

Thetransferfunctionofacomponentisrepresentedbyablock.Blockhassingleinputand single output.

Thefollowing figureshowsablockhaving inputX(s),outputY(s)and thetransfer function G(s).

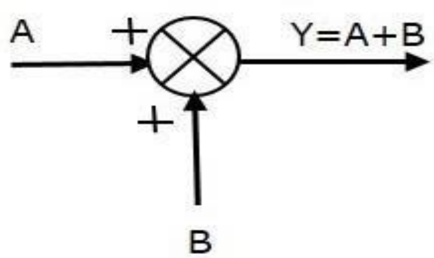
Transfer Function, $G(s) = \frac{Y(s)}{X(s)}$

$$\Rightarrow Y(s) = G(s)X(s)$$

SummingPoint

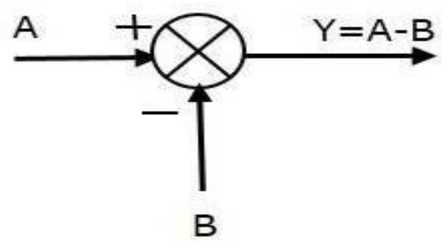
The summing point is represented with a circle having cross (X) inside it. It has two or more inputs and single output. It produces the algebraic sum of the inputs. It alsoperforms the summation or subtraction or combination of summation and subtraction of the inputs based on the polarity of the inputs. Let us see these three operations one byone.

The following figure shows the summing point with two inputs (A, B) and one output (Y). Here, the inputs A and B have a positive sign. So, the summing point produces the output,Y as **sum of A and B** i.e. = A + B.



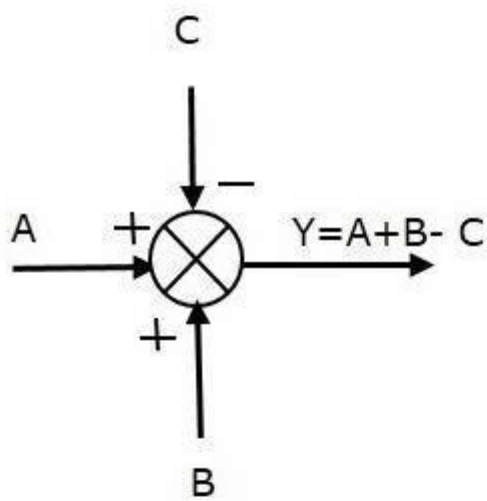
The following figure shows the summing point with two inputs (A, B) and one output (Y). Here, the inputs A and B are having opposite signs, i.e., A is having positive sign and B is having negative sign. So, the summing point produces the output **Y** as the **difference of A and B** i.e

Y=A+ (-B)= A-B.



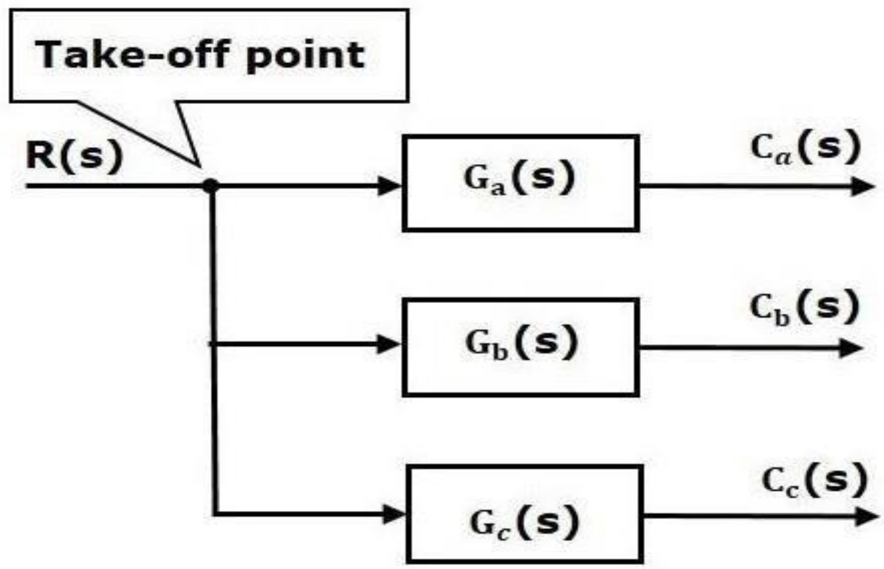
Thefollowingfigureshowsthesummingpointwiththreeinputs(A,B,C)andoneoutput (Y). Here, the inputs A and B are having positive signs and C is having a negative sign. So, the summing point produces the output **Y** as

Y=A+B+(−C)= A+B−C.

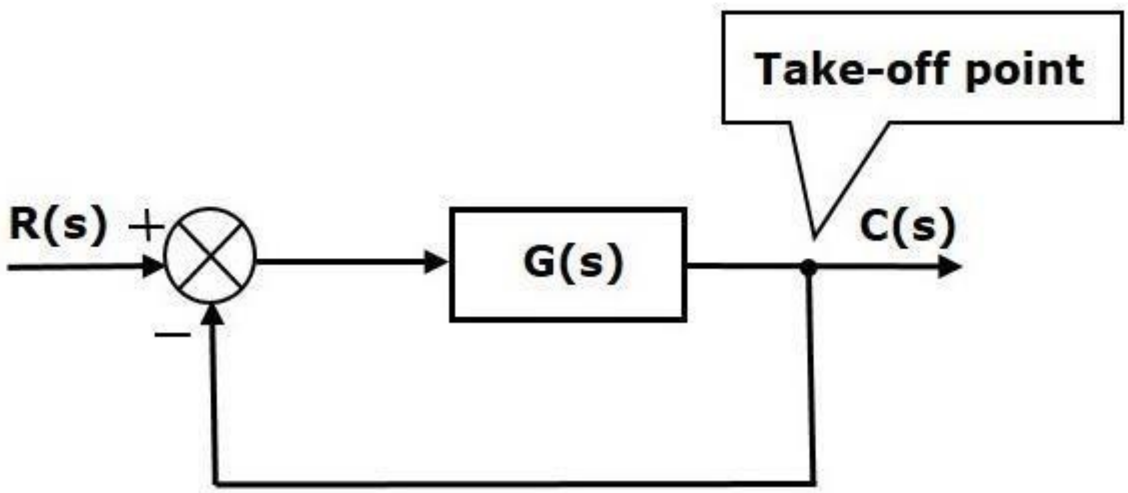


Take-offPoint

Thetake-offpointis apointfrom which thesameinputsignal can be passed through more than one branch. That means with the help of take-off point, we can apply the same input to oneor moreblocks,summing points. Inthefollowing figure,thetake-offpointisusedto connect the same input, R(s) to two more blocks.



Inthefollowing figure,thetake-offpointisusedto connecttheoutputC(s),asoneofthe inputs to the summing point.



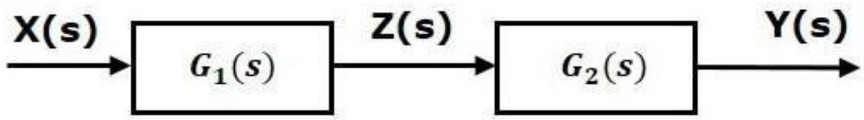
Blockdiagramalgebraisnothingbutthealgebrainvolvedwiththebasicelementsoftheblock diagram. This algebra deals with the pictorial representation of algebraic equations.

BasicConnectionsforBlocks

Therearethreebasictypesofconnectionsbetweentwoblocks.

SeriesConnection

Seriesconnectionisalsocalled **cascadeconnection**.Inthefollowingfigure,twoblocks having transfer functions G1(s)G1(s) and G2(s)G2(s) are connected in series.



For this combination, we will get the output $Y(s)$ as

$$Y(s) = G_2(s)Z(s)$$

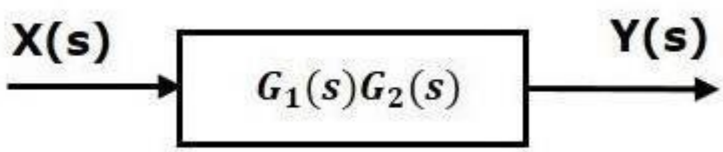
Where, $Z(s) = G_1(s)X(s)$

$$\Rightarrow Y(s) = G_2(s)[G_1(s)X(s)] = G_1(s)G_2(s)X(s)$$

$$\Rightarrow Y(s) = \{G_1(s)G_2(s)\}X(s)$$

Compare this equation with the standard form of the output equation, $Y(s) = G(s)X(s)$. Where, $G(s) = G_1(s)G_2(s)$.

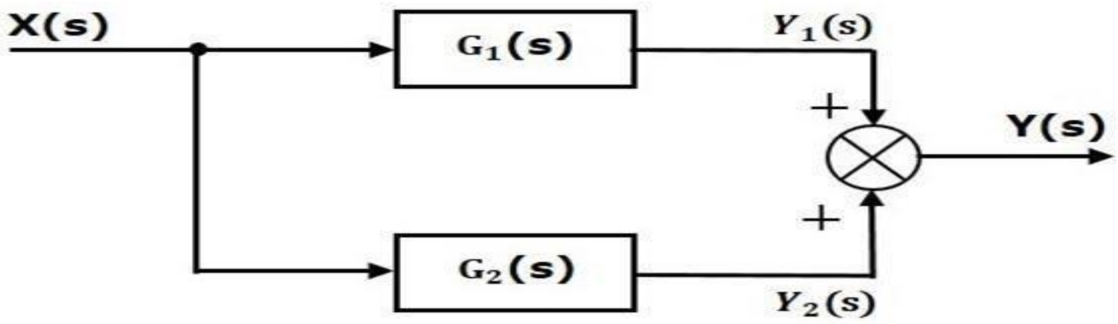
Thatmeanswecan representthe **seriesconnection**oftwo blockswith asingleblock.The transferfunctionofthissingleblockisthe **productof thetransferfunctions** ofthosetwo blocks. The equivalent block diagram is shown below.



Similarly, you can represent series connection of ‘n’ blocks with a single block. Thetransfer function of this single block is the product of the transfer functions of all those ‘n’ blocks.

ParallelConnection

The blocks which are connected in **parallel** will have the **same input**. In the following figure, two blocks having transfer functions G1(s)G1(s) and G2(s)G2(s) are connected in parallel. The outputs of these two blocks are connected to the summing point.



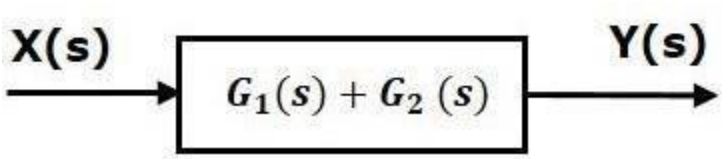
$$Y(s) = Y_1(s) + Y_2(s)$$

$$Y_1(s) = G_1(s)X(s) \text{ and } Y_2(s) = G_2(s)X(s)$$

$$\Rightarrow Y(s) = G_1(s)X(s) + G_2(s)X(s) = \{G_1(s) + G_2(s)\}X(s)$$

$$G(s) = G_1(s) + G_2(s).$$

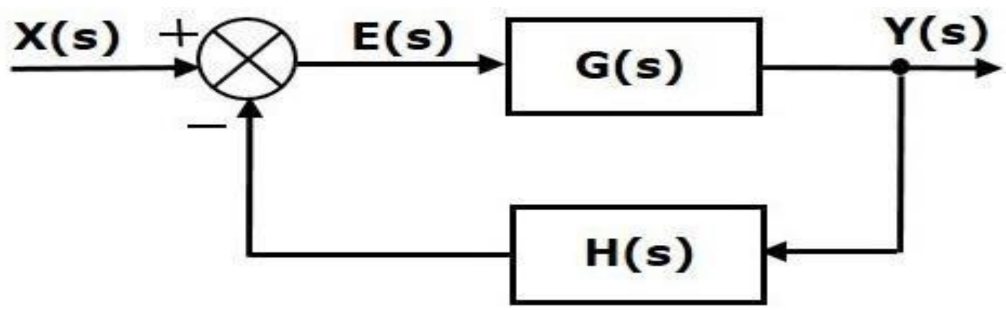
That means we can represent the **parallel connection** of two blocks with a single block. The transfer function of this single block is the **sum of the transfer functions** of thosetwo blocks. The equivalent block diagram is shown below.



Similarly, you can represent parallel connection of 'n' blocks with a single block. The transfer function of this single block is the algebraic sum of the transfer functions of all those 'n' blocks.

FeedbackConnection

As we discussed in previous chapters, there are two types of **feedback** — positive feedback and negative feedback. The following figure shows negative feedback control system. Here, two blocks having transfer functions G(s)G(s) and H(s)H(s) form a closed loop.



The output of the summing point is -

$$E(s) = X(s) - H(s)Y(s)$$

The output $Y(s)$ is -

$$Y(s) = E(s)G(s)$$

Substitute $E(s)$ value in the above equation.

$$Y(s) = \{X(s) - H(s)Y(s)\}G(s)\}$$

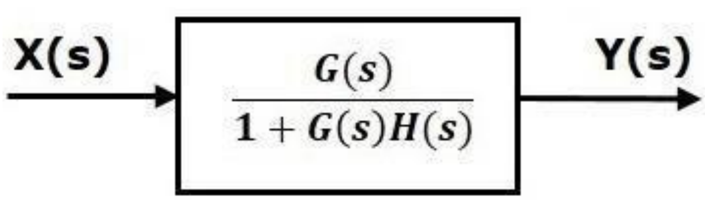
$$Y(s)\{1 + G(s)H(s)\} = X(s)G(s)\}$$

$$\Rightarrow \frac{Y(s)}{X(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

Therefore,thenegativefeedbackclosedlooptransferfunctionis:

$$\frac{G(s)}{1+G(s)H(s)}$$

Thismeanswecan represent thenegativefeedback connection of two blockswith asingle block. The transfer function of this single block is the closed loop transfer function of the negative feedback. The equivalent block diagram is shown below.



Similarly, you can represent the positive feedback connection of two blocks with a single block. The transfer function of this single block is the closed loop transfer function of the positive feedback, i.e.,

$$\frac{G(s)}{1-G(s)H(s)}$$

BlockDiagramAlgebraforSummingPoints

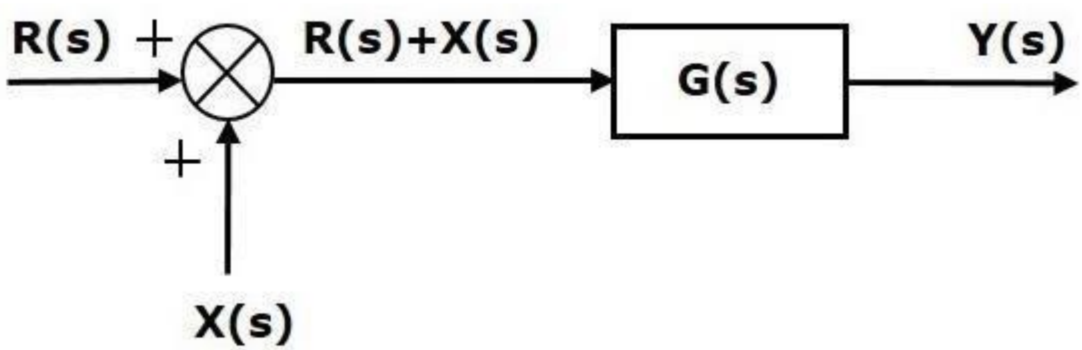
Therearetwopossibilitiesofshiftingsummingpointswithrespectto blocks−

- Shiftingsummingpointaftertheblock
- Shiftingsummingpointbeforetheblock

Letusnowseewhatkindofarrangementsneed tobedoneintheabovetwocasesone by one.

ShiftingtheSummingPointbeforeaBlocktoaftera Block

Considertheblockdiagramshowninthefollowingfigure.Here,thesummingpointis present before the block.



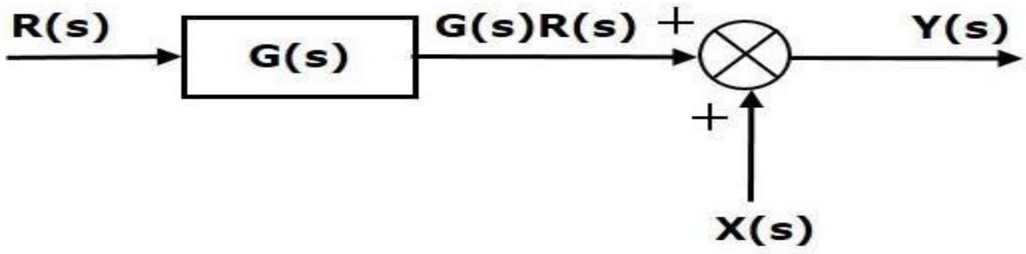
Summing point has two inputs $R(s)$ and $X(s)$

TheoutputofSumming pointis $\{R(s) + X(s)\}$.

So, the input to the block $G(s)$ is $\{R(s) + X(s)\}$ and the output of it is –

$$Y(s) = G(s)\{R(s) + X(s)\}$$

$$\Rightarrow Y(s) = G(s)R(s) + G(s)X(s) \qquad \text{(Equation 1)}$$



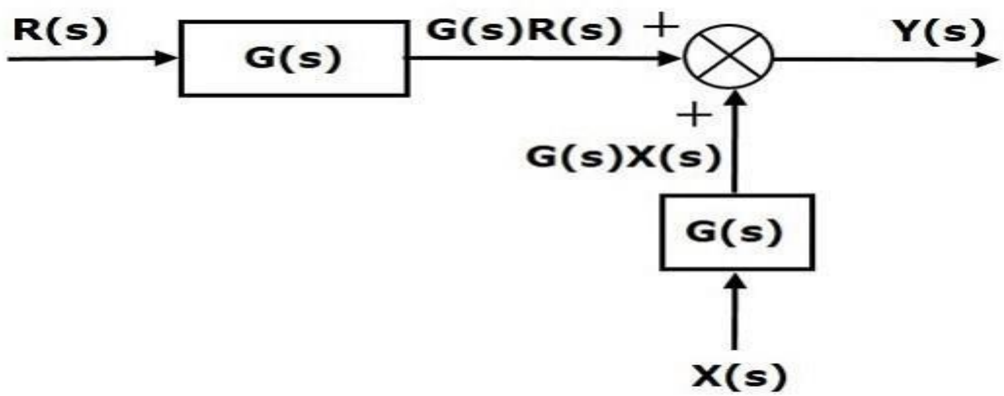
Output of the block $G(s)$ is $G(s)R(s)$.

The output of the summing point is

$$Y(s) = G(s)R(s) + X(s) \qquad \text{(Equation 2)}$$

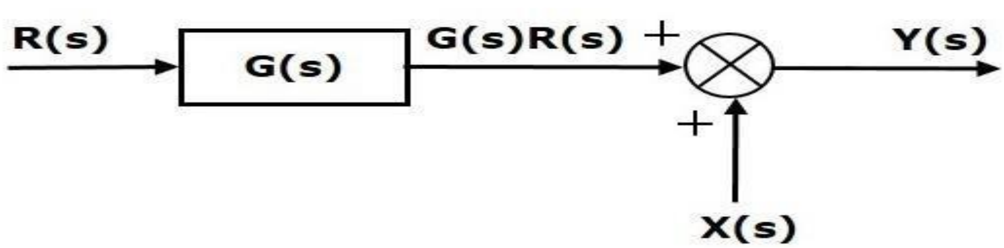
CompareEquation1 andEquation2.

The first term ‘G(s)R(s)′‘G(s)R(s)′ is same in both the equations. But, there is difference in the second term. In order to get the second term also same, we require one more block G(s)G(s). It is having the input X(s)X(s) and the output of this block is given as input to summing point instead of X(s)X(s). This block diagram is shown in the following figure.



Shifting Summing Point Before the Block

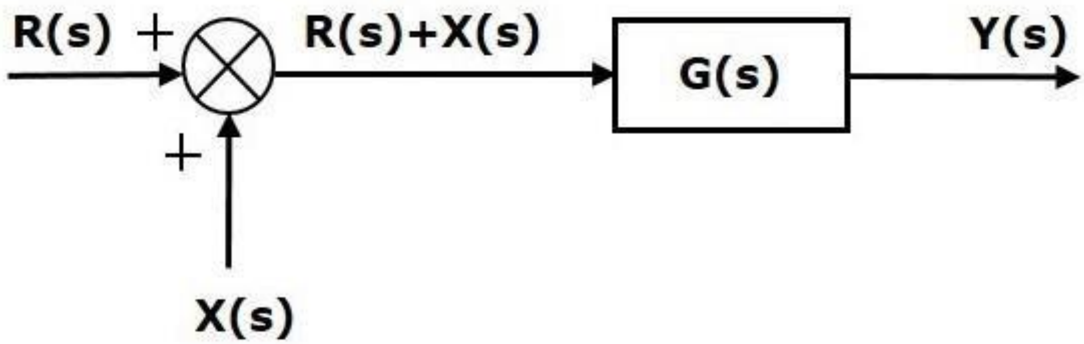
Consider the block diagram shown in the following figure. Here, the summing point is present after the block.



Output of this block diagram is -

$$Y(s) = G(s)R(s) + X(s) \qquad \text{(Equation 3)}$$

Now, shift the summing point before the block. This block diagram is shown in the following figure.

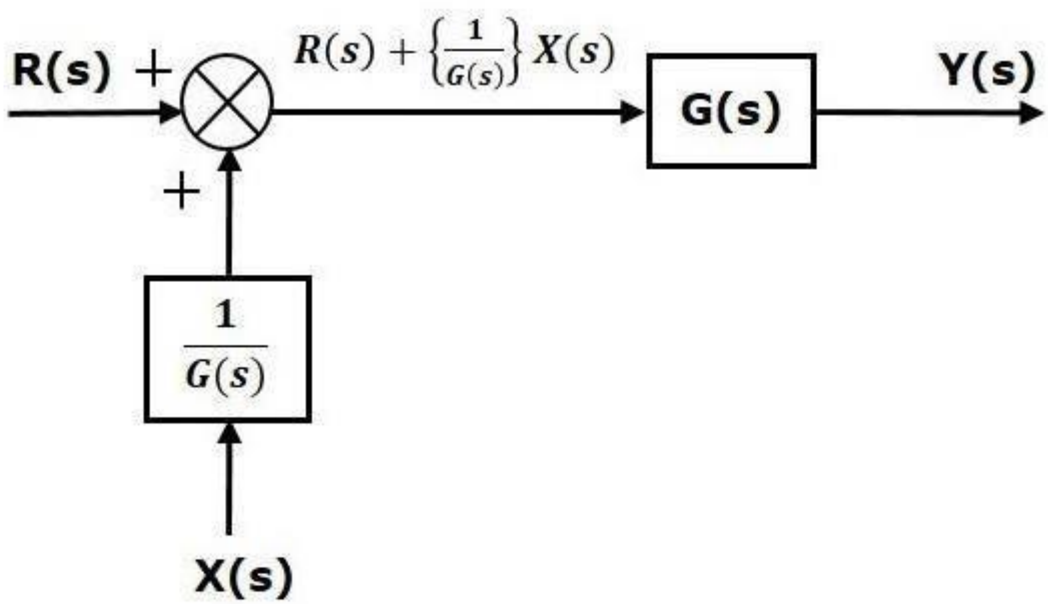


Output of this block diagram is -

$$Y(S) = G(s)R(s) + G(s)X(s) \qquad \text{(Equation 4)}$$

CompareEquation3andEquation4,

The first term 'G(s)R(s)′ is same in both equations. But, there is difference in the second term. In order to get the second term also same, we require one more block 1/G(s). It is having the input X(s) and the output of this block is given as input to summing point instead of X(s). This block diagram is shown in the following figure.



BlockDiagramAlgebraforTake-offPoints

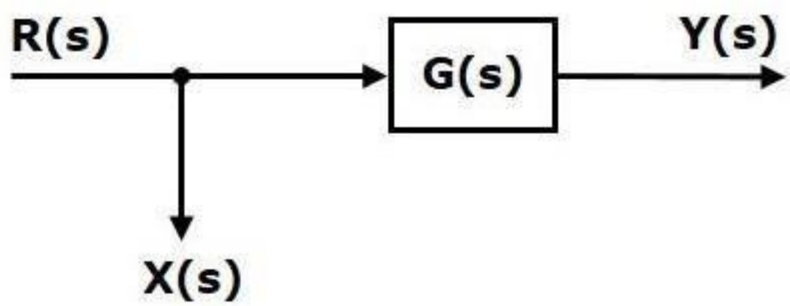
Therearetwopossibilitiesofshiftingthetake-offpointswithrespecttoblocks−

- Shiftingtake-offpointaftertheblock
- Shiftingtake-offpointbeforetheblock

Letusnow seewhatkind ofarrangementsistobedonein theabovetwo cases,onebyone.

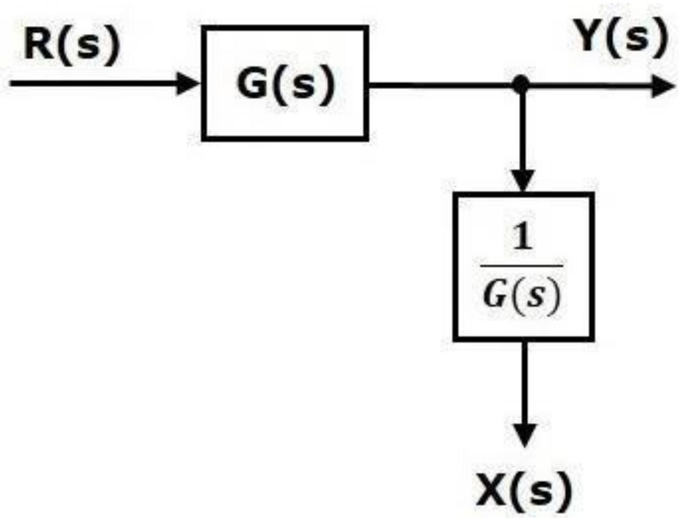
Shifting a Take-off Point form a Position before a Block to a position after the Block

Consider the block diagram shown in the following figure. In this case, the take-off point is present before the block.



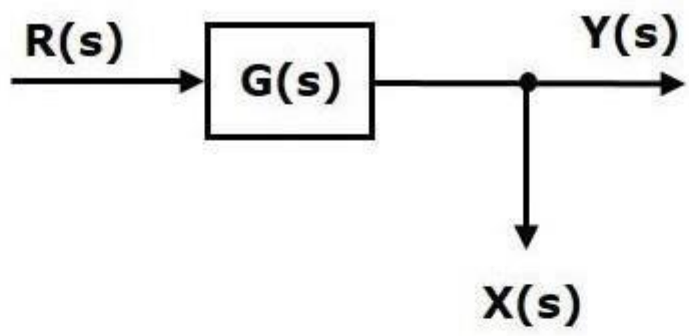
Here, $X(s) = R(s)$ and $Y(s) = G(s)R(s)$

When youshift thetake-off point after theblock, theoutputY(s) willbesame. But, thereis difference in X(s) value. So, in order togetthesame X(s)value,werequireonemore block 1/G(s). It is having the input Y(s) and the output is X(s) this block diagram is shown in the following figure.



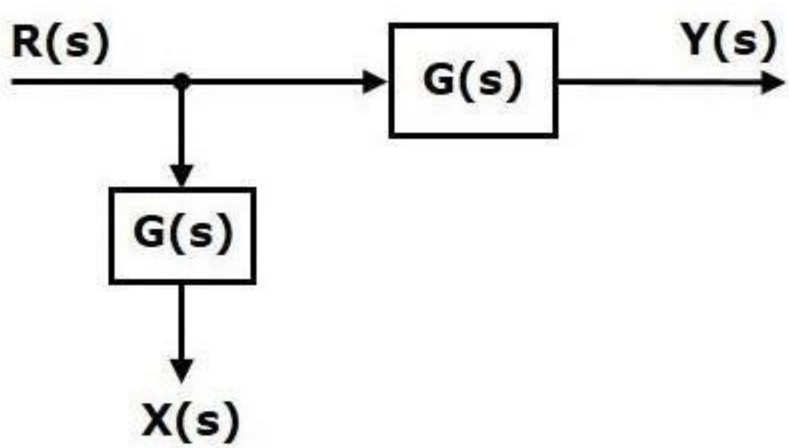
Shifting Take-off Point from a Position after a Block to a position before the Block

Considertheblockdiagramshowninthefollowingfigure.Here,thetake-offpointis present after the block.



Here, $X(s) = Y(s) = G(s)R(s)$

When you shift the take-off point before the block, the output Y(s) will be same. But, there is difference in X(s) value. So, in order to get same X(s) value, we require one more block G(s) It is having the input R(s) and the output is X(s). This block diagram is shown in the following figure.



Theconceptsdiscussedinthepreviouschapterarehelpfulforreducing(simplifying)theblock diagrams.

BlockDiagramReductionRules

Followtheserulesforsimplifying(reducing)theblockdiagram,whichishavingmany blocks, summing points and take-off points.

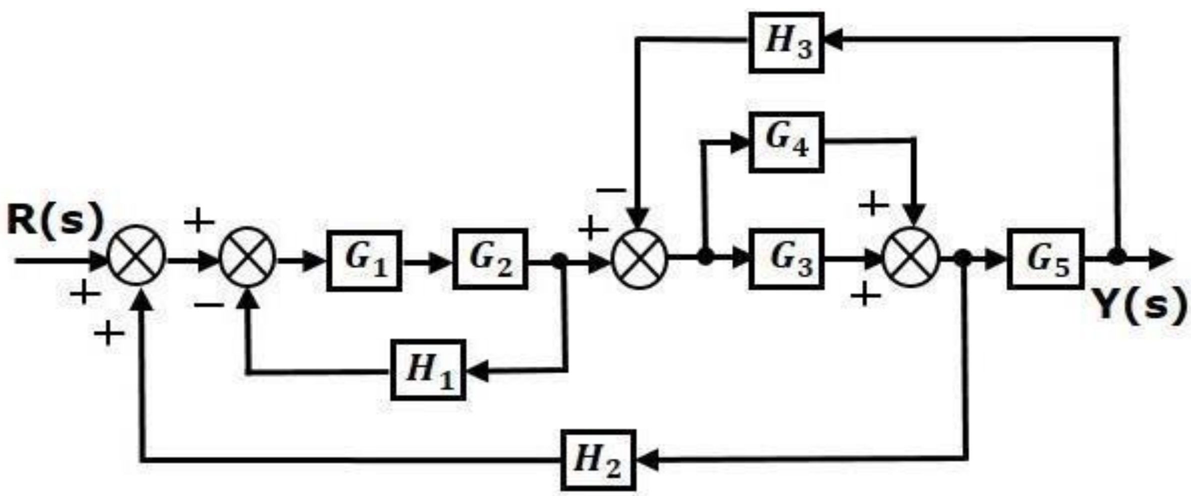
- **Rule1** –Checkfortheblocksconnectedinseriesandsimplify.
- **Rule2** –Checkfortheblocksconnectedinparallelandsimplify.

- **Rule3**–Checkfortheblocksconnectedinfeedbackloopandsimplify.
- **Rule4**– Ifthereisdifficultywithtake-offpointwhilesimplifying,shiftittowardsright.
- **Rule5**–Ifthereisdifficultywithsummingpointwhilesimplifying,shiftittowardsleft.
- **Rule6** – Repeattheabovestepstillyougetthesimplifiedform,i.e.,singleblock.

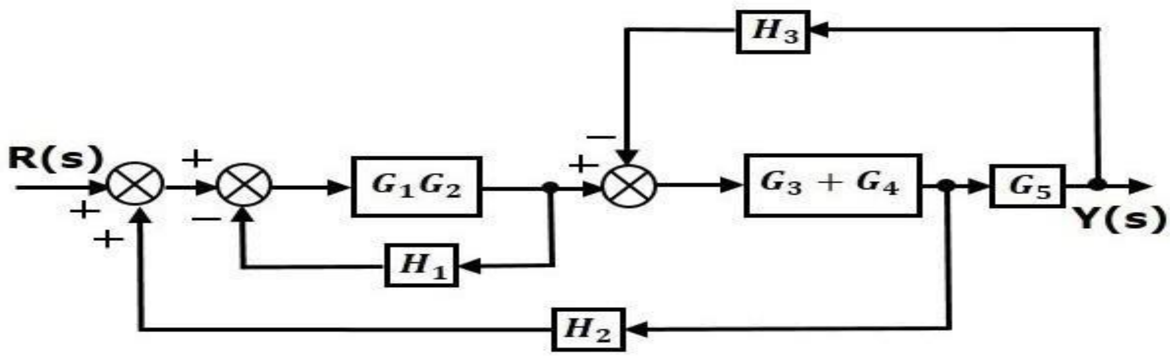
Note –Thetransferfunctionpresentinthissingleblockis thetransferfunctionofthe overall block diagram.

Example

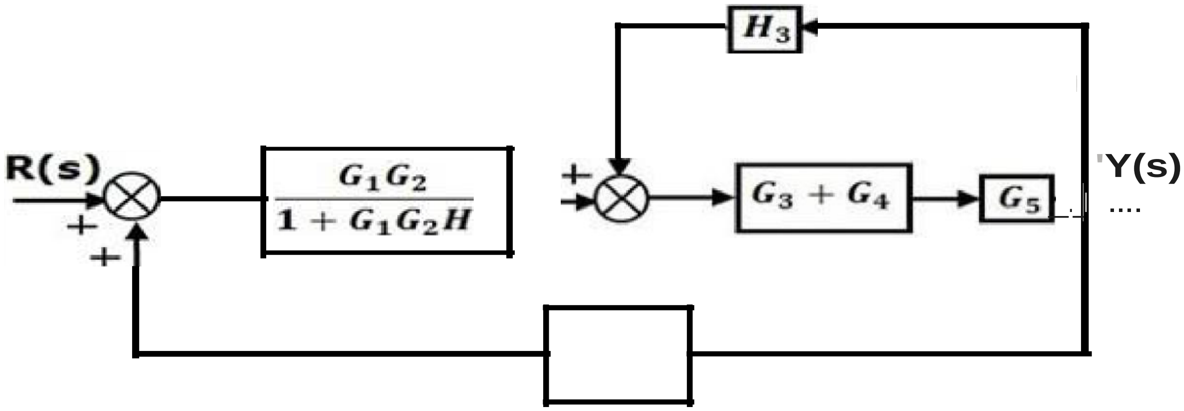
Considertheblockdiagramshowninthefollowingfigure.Letussimplify(reduce)this block diagram using the block diagram reduction rules.



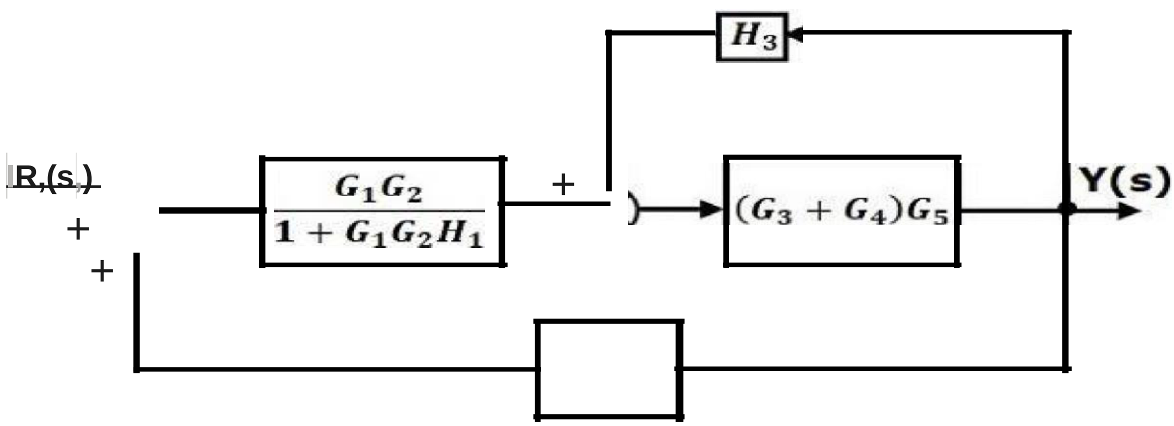
Step 1 – Use Rule 1 for blocks G_1 and G_2. Use Rule 2 for blocks G_3 and G_4. The modified block diagram is shown in the following figure.



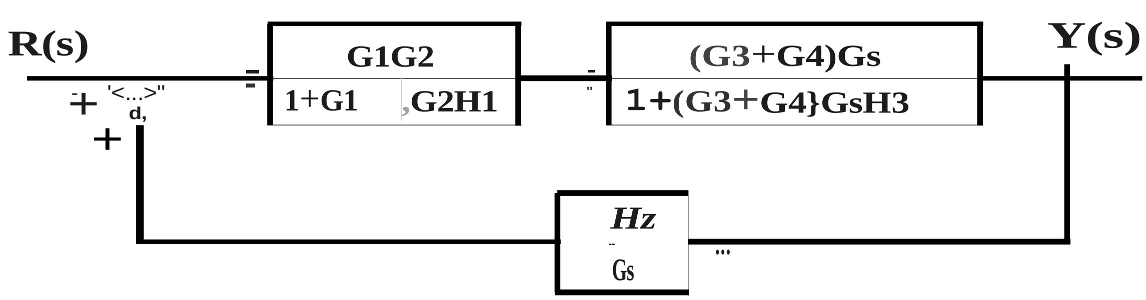
Step2-UseRule3furblocksG1,G2arid*Hi*.UseRule4forshi ir,gteke-off pointafterthe block*G* ,hemodifiedblockdiagramisshowninthefollowing figur ,



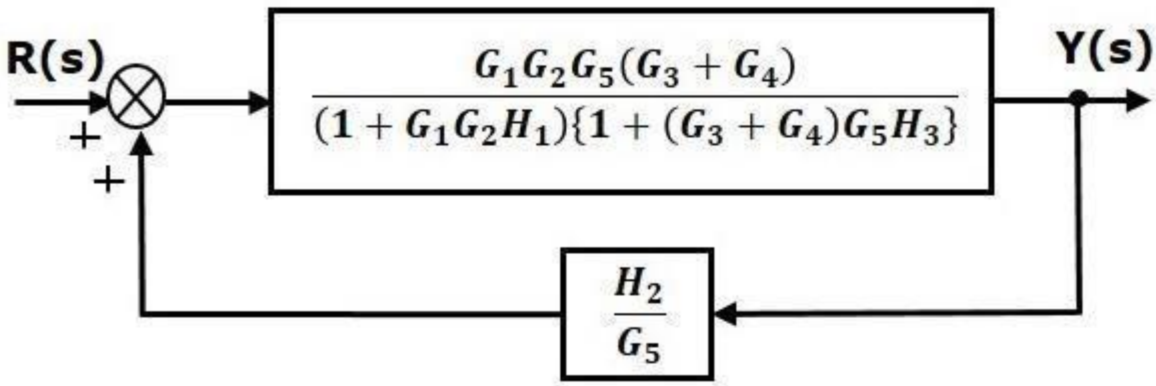
Step3 -UseRule1forblocks***(Ga,*** +G4)andGfl,Themodified blockdiagram isshowninthefollowingfigure,



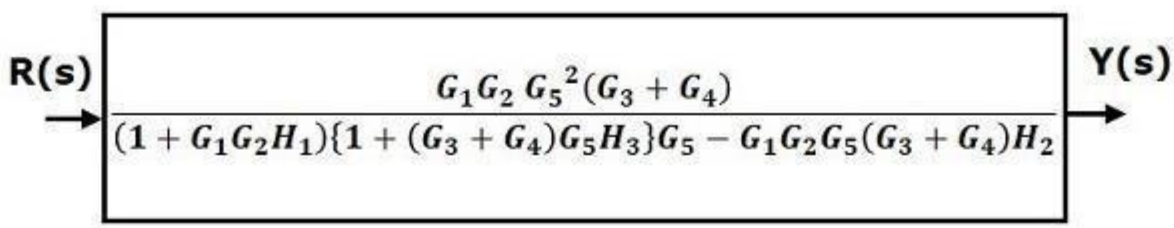
Step 4 -UseRule3forblocks*(Ga.*+*G4)G5*and*Ha*.Themodifiedblock diagramis shown in the following figure.



StepS -UseRule1forblocksconnectedin series.Themodifiedblockdiagram is shown in the followingfigure.



Step 6 – Use Rule 3 for blocks connected in feedback loop. The modified block diagram is shown in the following figure. This is the simplified block diagram.



Therefore, the transfer function of the system is

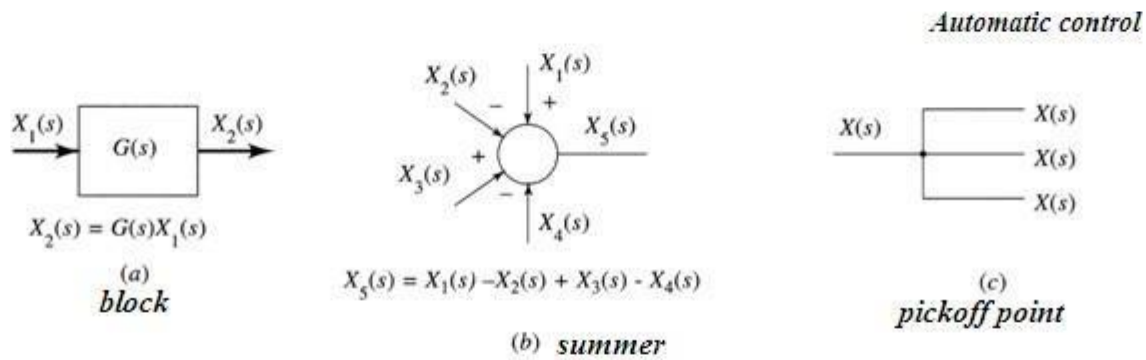
$$\frac{Y(s)}{R(s)}=\frac{G_1G_2G_5^2(G_3+G_4)}{(1+G_1G_2H_1)\{1+(G_3+G_4)G_5H_3\}G_5-G_1G_2G_5(G_3+G_4)H_2}$$

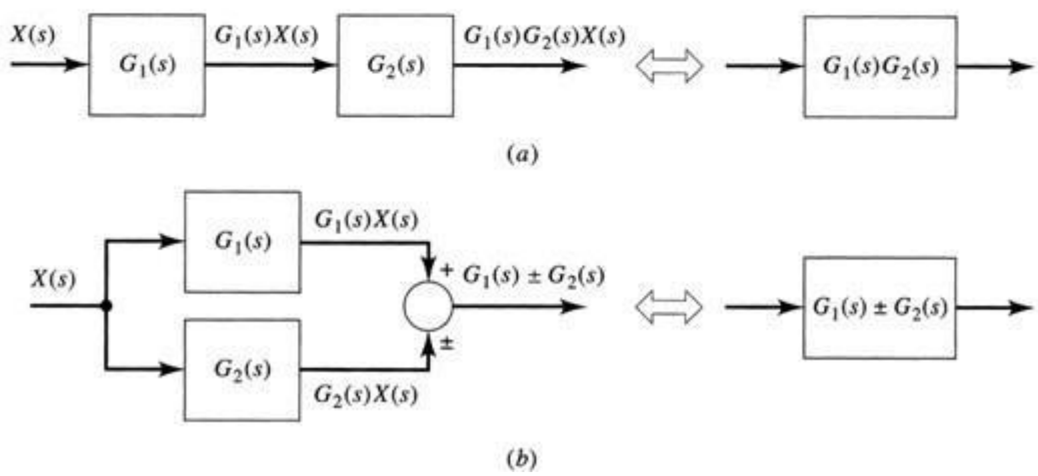
Note–Followthesestepsinordertocalculatethetransferfunctionoftheblockdiagram having multiple inputs.

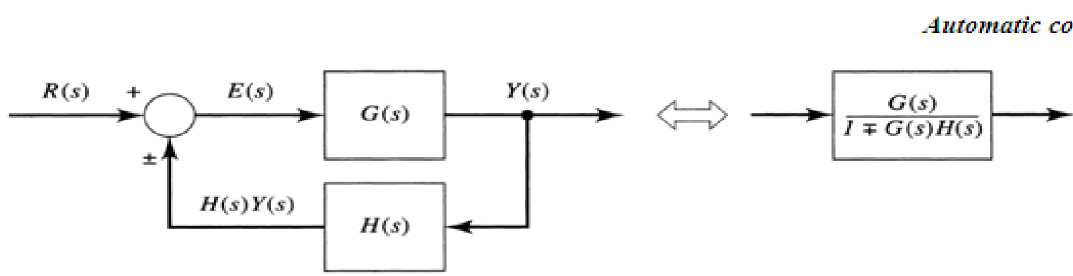
- **Step1**–Findthetransferfunctionofblockdiagrambyconsideringoneinputata time and make the remaining inputs as zero.
- **Step2** –Repeatstep1forremaininginputs.
- **Step3** –Gettheoveralltransferfunctionbyaddingallthosetransferfunctions.

The block diagram reduction process takes more time for complicated systems because; we have to draw the (partially simplified) block diagram after each step. So, to overcome this drawback, use signal flow graphs (representation).

BlockDiagramReduction-Summary





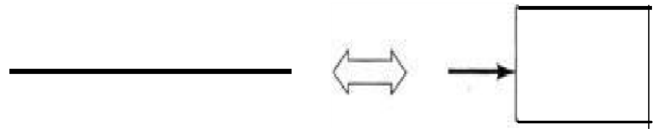


$$Y(s) = G(s)E(s)$$

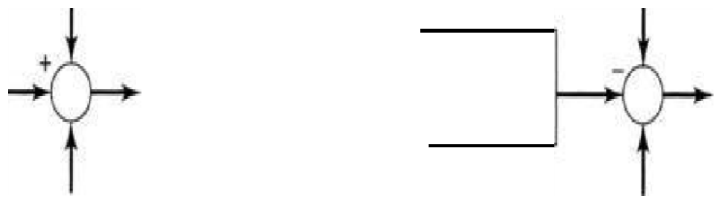
$$E(s) = R(s) \pm H(s)Y(s)$$

$$Y(s) = G(s)[R(s) \pm H(s)Y(s)] = G(s)R(s) \pm G(s)H(s)Y(s)$$

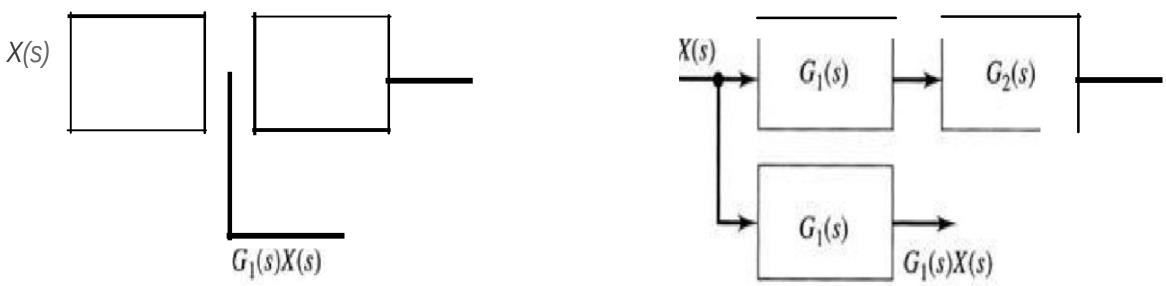
$$T(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1 \mp G(s)H(s)}$$



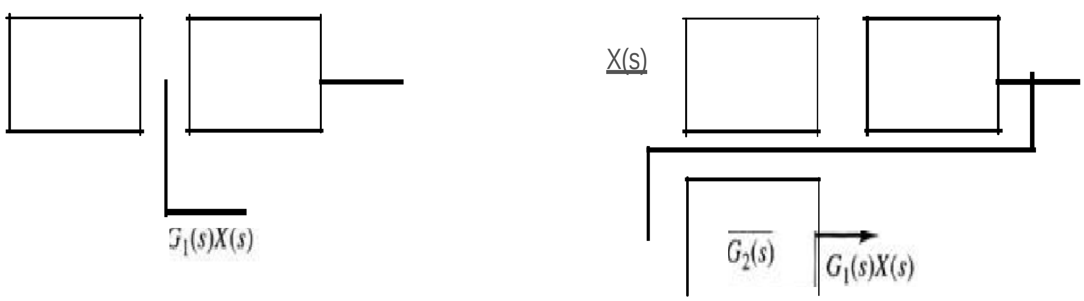
a)lns.ertionorremovalofunitygain



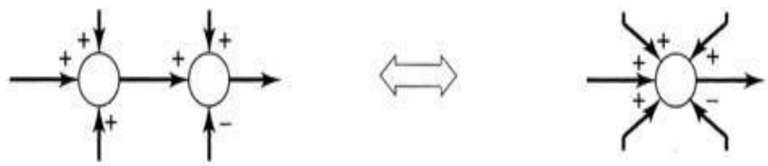
(b),Changingaummerign



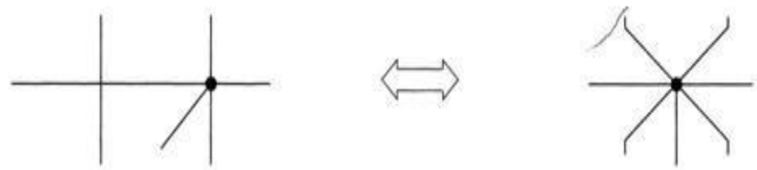
(.c)Movingapickff·pointback



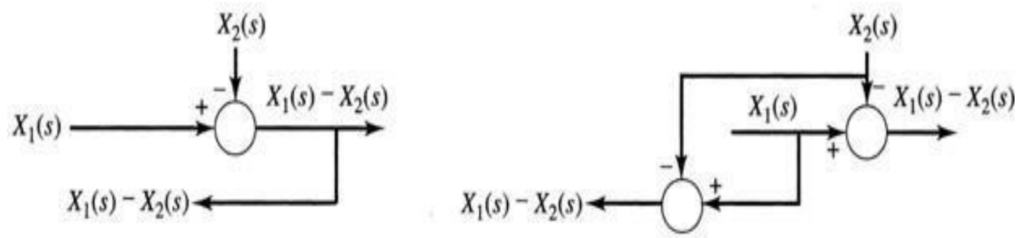
d)Movingapickoffpointforward



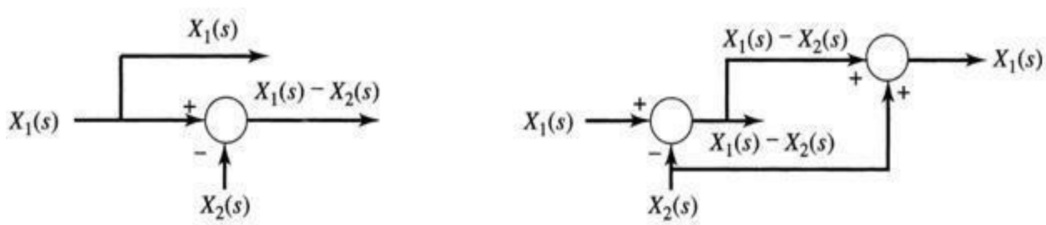
(*e*) Combining or expanding summations



(*f*) Combining or expanding junctions

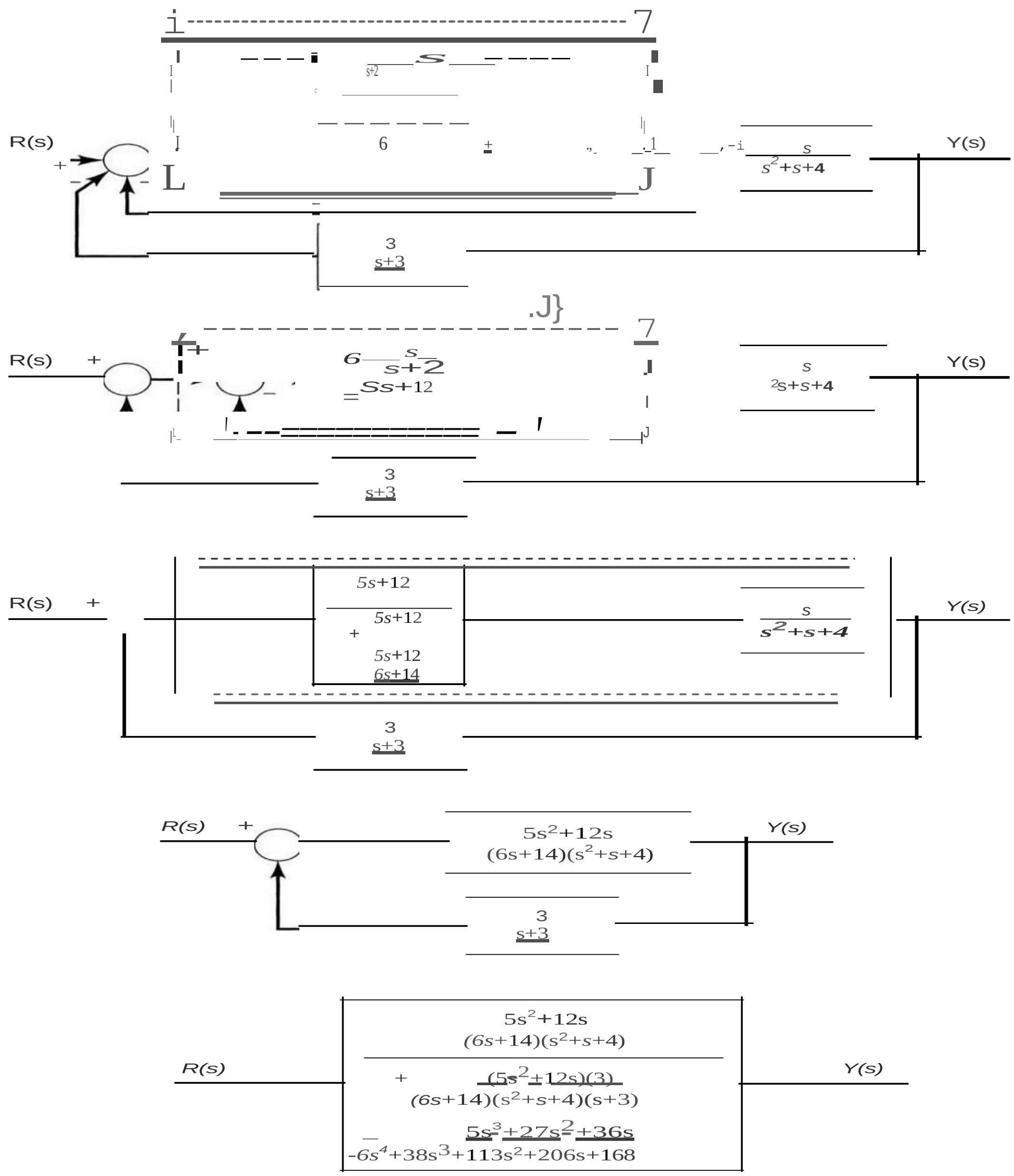


(*g*) Moving a pickoff point behind a summation

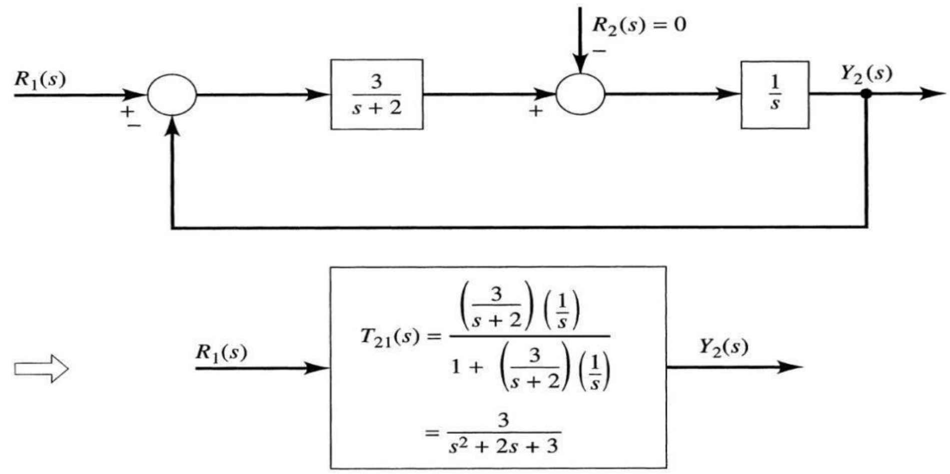


(*h*) Moving a pickoff point forward of a summation

Example-1:

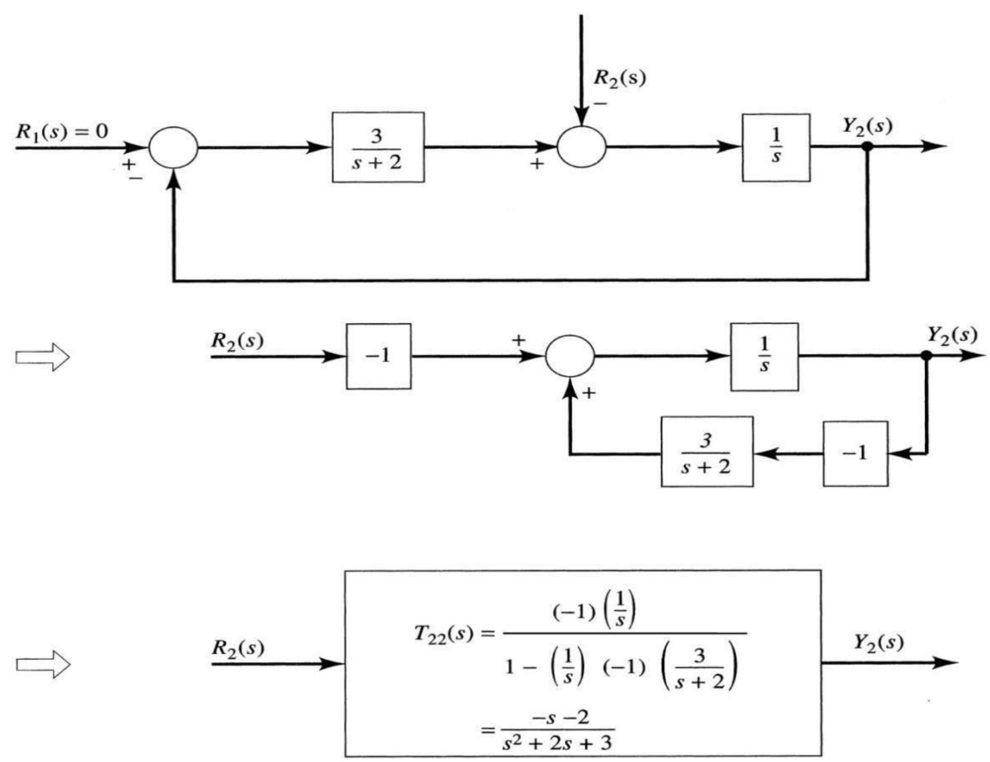

R(s)
Y(s)
6
s+2
s
s²+s+4
3
s+3
5s+12
6s+14
5s²+12s
(6s+14)(s²+s+4)
(5s²+12s)(3)
(6s+14)(s²+s+4)(s+3)
5s³+27s²+36s
6s⁴+38s³+113s²+206s+168

Example-2:



$R_2(s) = 0$
$R_1(s)$
$\frac{3}{s+2}$
$\frac{1}{s}$
$Y_2(s)$
$R_1(s)$
$T_{21}(s) = \dfrac{\left(\frac{3}{s+2}\right)\left(\frac{1}{s}\right)}{1 + \left(\frac{3}{s+2}\right)\left(\frac{1}{s}\right)} = \dfrac{3}{s^2 + 2s + 3}$
$Y_2(s)$

(a)



$R_2(s)$
$R_1(s) = 0$
$\frac{3}{s+2}$
$\frac{1}{s}$
$Y_2(s)$
$R_2(s)$
-1
$\frac{1}{s}$
$Y_2(s)$
$\frac{3}{s+2}$
-1
$R_2(s)$
$T_{22}(s) = \dfrac{(-1)\left(\frac{1}{s}\right)}{1 - \left(\frac{1}{s}\right)(-1)\left(\frac{3}{s+2}\right)} = \dfrac{-s-2}{s^2 + 2s + 3}$
$Y_2(s)$

(b)

Signal flow graph is a graphical representation of algebraic equations. In this chapter, let us discuss the basic concepts related signal flow graph and also learn how to draw signal flow graphs.

BasicElementsofSignalFlowGraph

Nodesandbranchesarethebasicelementsofsignalflow

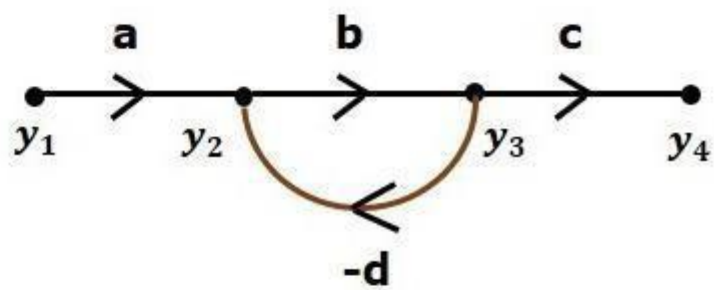
graph.Node

Nodeisapointwhich representseither avariableor asignal.Therearethreetypesof nodes —inputnode,outputnodeandmixednode.

- **InputNode–** Itisanode,which hasonly outgoingbranches.
- **OutputNode–** Itisanode,which hasonlyincomingbranches.
- **MixedNode–** Itisanode,whichhasboth incoming and outgoingbranches.

Example

Letusconsiderthefollowingsignalflowgraphtoidentifythesenodes.



- The **nodes** present in this signal flow graph are y_1, y_2, y_3 and y_4.
- y_1 and y_4 are the **input node** and **output node** respectively.
- y_2 and y_3 are **mixed nodes**.

Branch

Branch is a line segment which joins two nodes. It has both **gain** and **direction**. For example, there arefour branches in the above signal flow graph.Thesebranches have **gains** of **a, b,c** and **-d**.

ConstructionofSignalFlow Graph

Letusconstructasignalflowgraphbyconsideringthefollowingalgebraicequations−

$$y_2 = a_{12}y_1 + a_{42}y_4$$

$$y_3 = a_{23}y_2 + a_{53}y_5$$

$$y_4 = a_{34}y_3$$

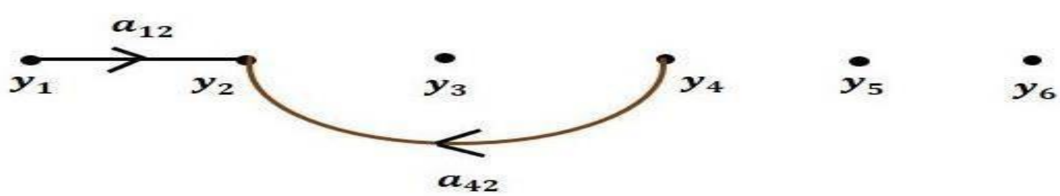
$$y_5 = a_{45}y_4 + a_{35}y_3$$

$$y_6 = a_{56}y_5$$

There will be six **nodes** (y_1, y_2, y_3, y_4, y_5 and y_6) and eight **branches** in this signal flow graph. The gains of the branches are a_{12}, a_{23}, a_{34}, a_{45}, a_{56}, a_{42}, a_{53} and a_{35}.

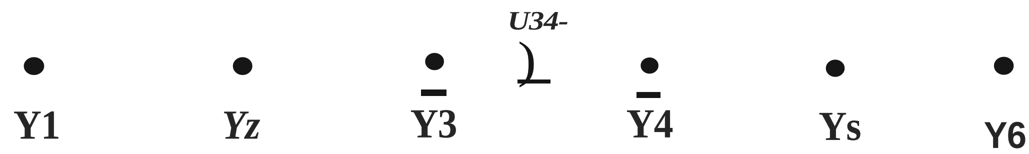
To get the overall signal flow graph, draw the signal flow graph for each equation, then combine all these signal flow graphs and then follow the steps given below −

Step 1 − Signal flow graph for $y_2 = a_{13}y_1 + a_{42}y_4$ is shown in the following figure.

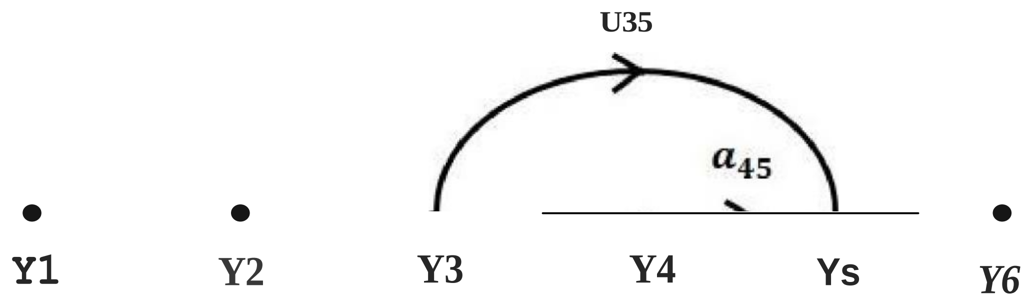


Step 2 − Signal flow graph for $y_3 = a_{23}y_2 + a_{53}y_5$ is shown in the following figure.

Step 3 − Signal flow graph for $y_4 = a_{34}y_3$ is shown in the following figure.

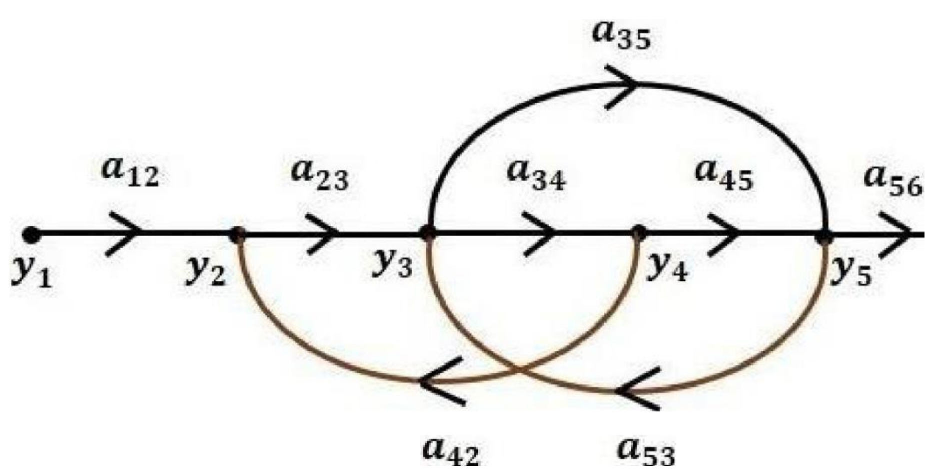


Step4-Signalflowgraphfor***y5***=**a415'Y4**+**aa6Ya**1sshowninthefollowing figure.



Step5-SignalflowgraphforYri=a155y5isshowninthefollowingfigure.

Step6-Siqnalflowqraphofoverallsystemisshowninthefollowinqfiqure.



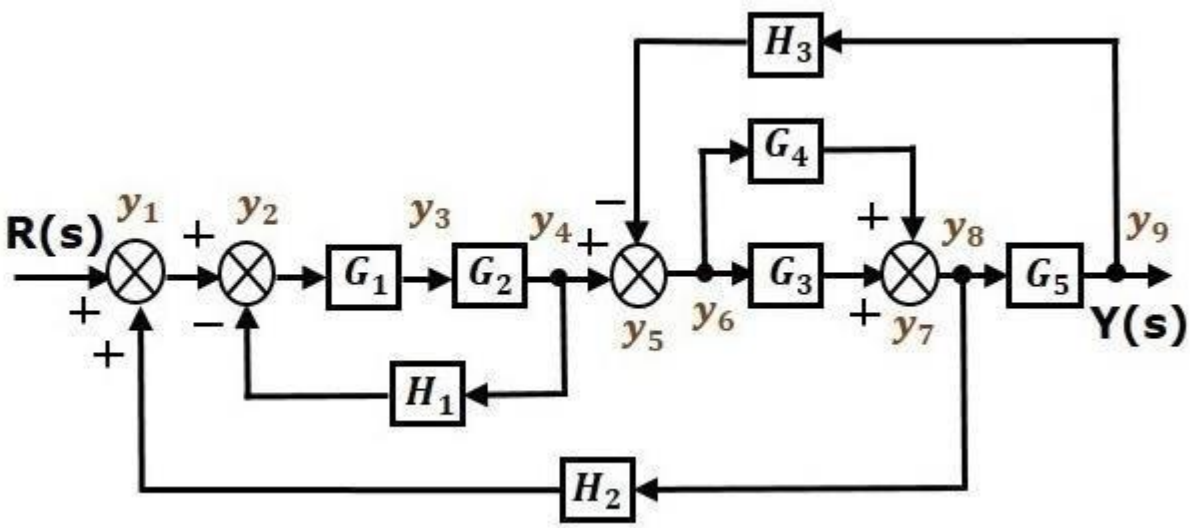
ConversionofBlockDiagramsintoSignalFlowGraphs

Followthesestepsforconvertingablockdiagramintoitsequivalentsignalflowgraph.

- Represent all the signals, variables, summing points and take-off points of block diagram as **nodes** in signal flow graph.
- Representtheblocksofblockdiagram as **branches** insignal flowgraph.
- Represent the transfer functions inside the blocks of block diagram as**gains** of the branches in signal flow graph.
- Connect the nodes as per the block diagram. If there is connection between two nodes (but there is no block in between), then represent the gain of the branch as one. **For example**, between summing points, between summing point and takeoff point, between input and summing point, between take-off point andoutput.

Example

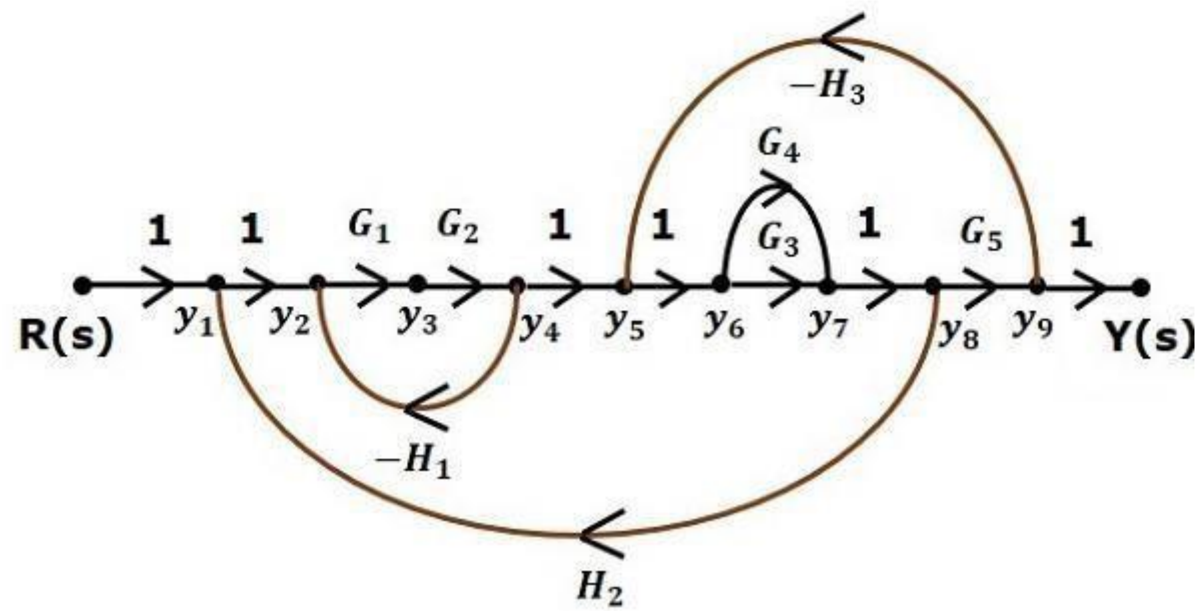
Letusconvertthefollowingblockdiagramintoitsequivalentsignalflowgraph.



Represent theinput signal R(s) and output signal C(s) of block diagram asinput nodeR(s) and output node C(s) of signal flow graph.

Just for reference, the remaining nodes (y_1 to y_9) are labelled in the block diagram. There are nine nodes other than input and output nodes. That is four nodes for four summing points, four nodes for four take-off points and one node for the variable between blocks $G_1$and G_2.

Thefollowingfigureshowstheequivalentsignalflowgraph.



Let us now discuss the Mason's Gain Formula. Suppose there are 'N' forward paths in a signal flow graph. The gain between the input and the output nodes of a signal flow graph is nothing but the **transfer function** of the system. It can be calculated by using Mason's gain formula.

Mason'sgainformulais

$$T = \frac{C(s)}{R(s)} = \frac{\Sigma_{i=1}^{N} P_i \Delta_i}{\Delta}$$

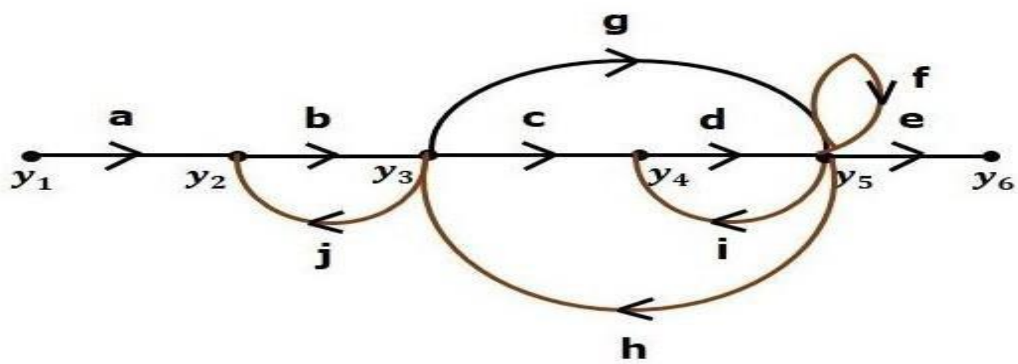
Where,

- **C(s)** is theoutputnode
- **R(s)** is theinputnode
- **T**isthetransferfunctionorgainbetweenR(s)andC(s)
- **Pi**is theithforwardpathgain

Δ=1−(sumofallindividualloopgains)+(sumofgainproductsofallpossibletwo nontouchingloops)−(sumofgainproductsofallpossiblethreenontouchingloops) +....

$\Delta_i$isobtainedfromΔbyremoving theloopswhicharetouchingtheithforwardpath.

Considerthefollowingsignalflow graphinordertounderstandthebasicterminology involved here.



Path

It is a traversal of branches from one node to any other node in the direction of branch arrows. It should not traverse any node more than once.

Examples – $y_2 \rightarrow y_3 \rightarrow y_4 \rightarrow y_5$ and $y_5 \rightarrow y_3 \rightarrow y_2$

Forward Path

The path that exists from the input node to the output node is known as **forward path**.

Examples – $y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow y_4 \rightarrow y_5 \rightarrow y_6$ and $y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow y_5 \rightarrow y_6$.

Forward Path Gain

It is obtained by calculating the product of all branch gains of the forward path.

Examples – $abcde$ is the forward path gain of $y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow y_4 \rightarrow y_5 \rightarrow y_6$ and abge is the forward path gain of $y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow y_5 \rightarrow y_6$.

Loop

Thepath thatstartsfromonenodeandendsatthesamenodeisknownasa **loop**.Hence, it is a closed path.

Examples – $y_2 \rightarrow y_3 \rightarrow y_2$ and $y_3 \rightarrow y_5 \rightarrow y_3$.

Loop Gain

It is obtained by calculating the product of all branch gains of a loop.

Examples – b_j is the loop gain of $y_2 \rightarrow y_3 \rightarrow y_2$ and g_h is the loop gain of $y_3 \rightarrow y_5 \rightarrow y_3$.

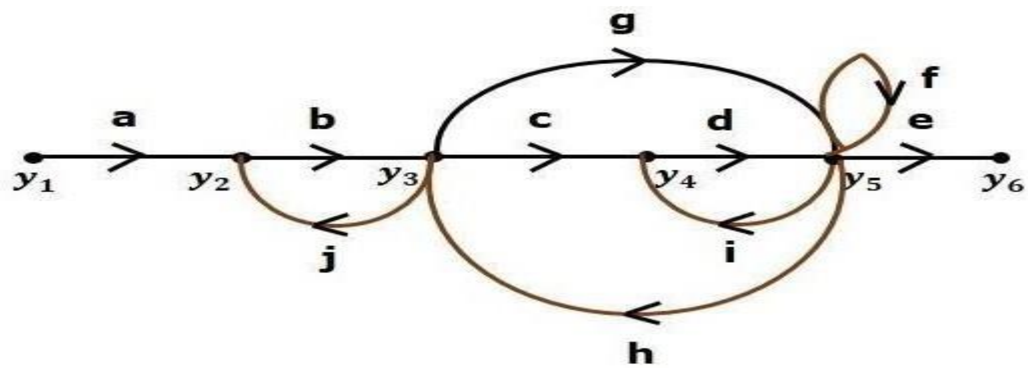
Non-touching Loops

These are the loops, which should not have any common node.

Examples – The loops, $y_2 \rightarrow y_3 \rightarrow y_2$ and $y_4 \rightarrow y_5 \rightarrow y_4$ are non-touching.

CalculationofTransferFunctionusingMason'sGain Formula

Letusconsiderthesamesignalflowgraphforfindingtransferfunction.



- Numberofforwardpaths,N=2.
- Firstforwardpathis-$y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow y_4 \rightarrow y_5 \rightarrow y_6$.

First forward path gain, $p_1 = abcde$

- Secondforwardpathis -$y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow y_5 \rightarrow y_6$
- Secondforwardpathgain,$p_2 = abge$
- Numberofindividualloops,L =5.

  Loops are - $y_2 \rightarrow y_3 \rightarrow y_2$, $y_3 \rightarrow y_5 \rightarrow y_3$, $y_3 \rightarrow y_4 \rightarrow y_5 \rightarrow y_3$, $y_4 \rightarrow y_5 \rightarrow y_4$ and $y_5 \rightarrow y_5$.

  Loop gains are - $l_1 = bj$, $l_2 = gh$, $l_3 = cdh$, $l_4 = di$ and $l_5 = f$.

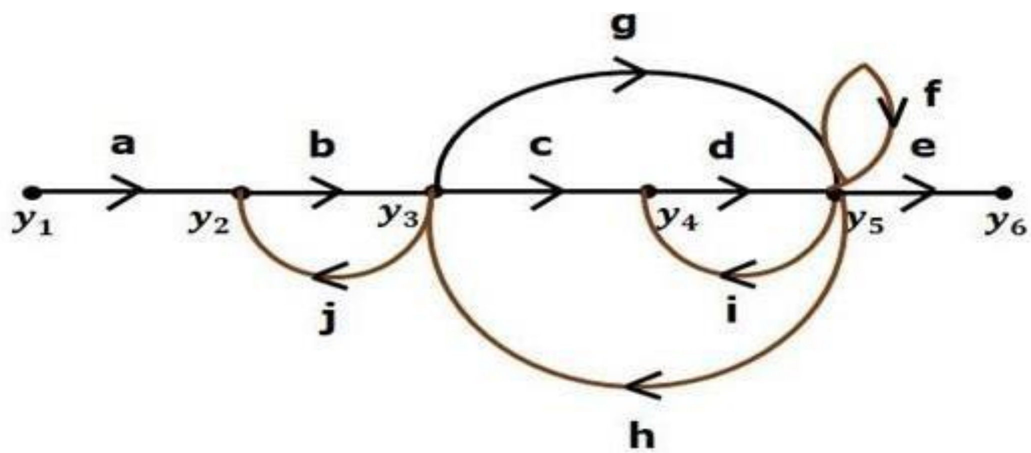
- Numberoftwonon-touching loops=2.

- Firstnon-touchingloopspairis-$y_2 \rightarrow y_3 \rightarrow y_2, y_4 \rightarrow y_5 \rightarrow y_4$.

Gain product of first non-touching loops pair $l_1 l_4 = bjdi$

- Second non-touching loops pair is - $y_2 \rightarrow y_3 \rightarrow y_2, y_5 \rightarrow y_5$.
- Gainproductofsecondnon-touchingloopspairis $l_1 l_5 = bjf$

Highernumberof(morethantwo)non-touchingloopsarenotpresentinthissignalflow graph.We know,



- Number of forward paths, N = 2.
- First forward path is - $y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow y_4 \rightarrow y_5 \rightarrow y_6$.
- First forward path gain, $p_1 = abcde$.
- Second forward path is - $y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow y_5 \rightarrow y_6$.
- Second forward path gain, $p_2 = abge$.
- Number of individual loops, L = 5.
- Loops are - $y_2 \rightarrow y_3 \rightarrow y_2$, $y_3 \rightarrow y_5 \rightarrow y_3$, $y_3 \rightarrow y_4 \rightarrow y_5 \rightarrow y_3$,

$y_4 \rightarrow y_5 \rightarrow y_4$ and $y_5 \rightarrow y_5$.

Loop gains are - $l_1 = bj$, $l_2 = gh$, $l_3 = cdh$, $l_4 = di$ and $l_5 = f$.

Numberoftwonon-touchingloops=2.

a Firstnon-touchingloopspairis-'!/2 '!/3 '!/2,'!/4 '!/15 '!/4,

aGainproductoffirstnon-touchingloopspair, l1*l4*=*bjdi*

aSecondnon-touchingloopspairis-*Y2* y3 *Y2,Yo* *Yo.*

Gainproductofsecondnon-touchingloopspairis-*lilts*=*bjf*

Highernumberof(morethantwo)non-touchingloopsarenotpresentinthis signal flow graph.

Weknow,

8 =1- (sttrn*ofallindividtialloopgains)*

+(sttrn*ofgainprodttctsofaUpossibletwonontmtchingloops)*

-(*sttm ofgainprodttctsofaUpossiblethreenontmtchingloops)*+...

Substitutethevaluesintheaboveequation,

8 =1- *(bj*+*g-li*+*cdh* +*di*+f)+*(bjdi*+*bj*f)-(O)

t:..=1-(bj+gh+cdh+di+f)+bjdi+bjf

Thereisnoloopwhichisnon-touchingtothefirstforwardpath.

So,D..1=1.

Similarly,D..2=1. Since,noloopwhichisnon-touchingtothesecondforward path.

Substitute,N=2inMason'sgainformula

$$T=\frac{C(s)}{R(s)}=\frac{l=1 J\{t:..i}{t:..}$$

$$T=\frac{C(s)}{R(s)}=\frac{P1D..1+P2D..2}{t:..}$$

Substitutealltheneccessaryvaluesintheaboveequation.

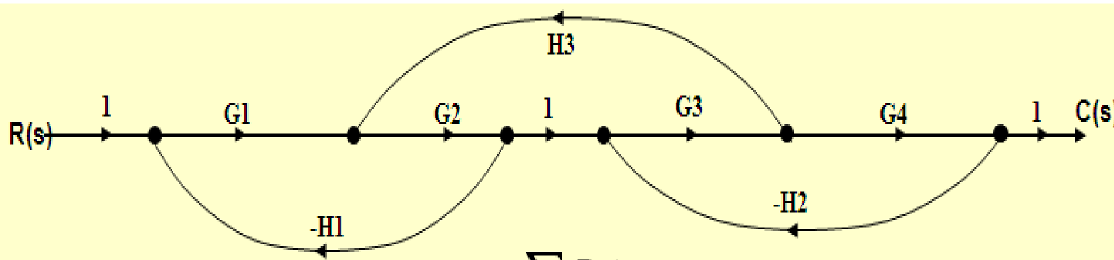
$$T=\frac{C(s)}{R(s)}=\frac{(abcde)l+(abge)l}{1-(bj+gh+cdh+di+f)+bjdi+bjf}$$

$$T=\frac{C(s)}{R(s)}=\frac{(abcde)+(abge)}{1-(bj+gh+cdh+di+f)+bjdi+bjf}$$

Therefore,thetransferfunctionis-

$$T=\frac{C(s)}{R(s)}=\frac{(abcde)+(abge)}{1-(bj+gh+cdh+di+f)+bjdi+bjf}$$

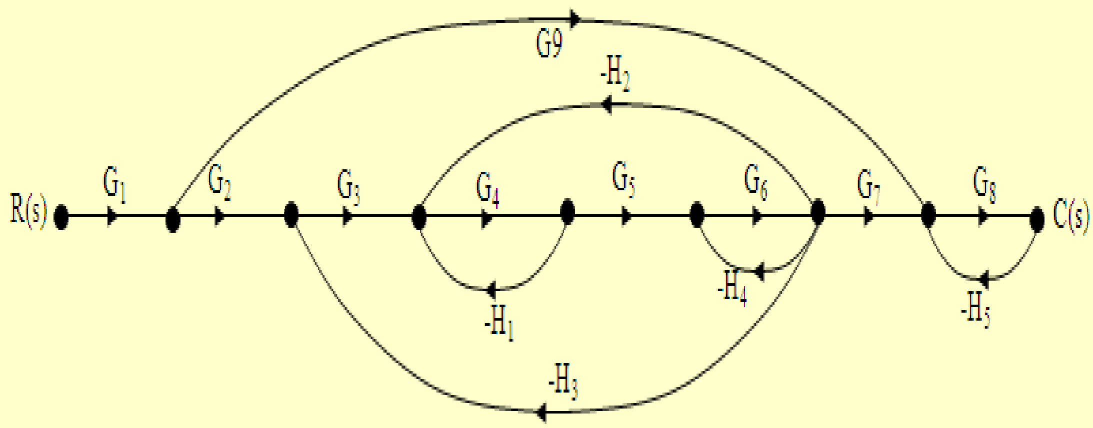
Example-1:



$$T(s) = \frac{\sum P_k \Delta_k}{\Delta}$$

- $P_1 = G_1 G_2 G_3 G_4$ $\Delta_1 = 1$ There is no P_2 or Δ_2 or more.
- $\sum L_1 = -G_1 G_2 H_1 + G_2 G_3 H_3 - G_3 G_4 H_2$
- $\sum L_2 = G_1 G_2 G_3 G_4 H_1 H_2$
- $\Delta = 1 - \sum L_1 + \sum L_2 = 1 + G_1 G_2 H_1 - G_2 G_3 H_3 + G_3 G_4 H_2 + G_1 G_2 G_3 G_4 H_1 H_2$
- $\mathrm{T(s)} = \frac{\sum \mathrm{P}_1 \Delta_1}{\Delta} = \frac{G_1 G_2 G_3 G_4}{1 + G_1 G_2 H_1 - G_2 G_3 H_3 + G_3 G_4 H_2 + G_1 G_2 G_3 G_4 H_1 H_2}$

Example-2:



$$M_1 = G_1G_2G_3G_4G_5G_6G_7G_8 \qquad \Delta_1 = 1$$

$$M_2 = G_1G_9G_8 \qquad \Delta_2 = 1-[-G_4H_1 - G_6H_4 - G_3G_4G_5G_6H_3 - G_4G_5G_6H_2] + G_4H_1G_6H_4$$

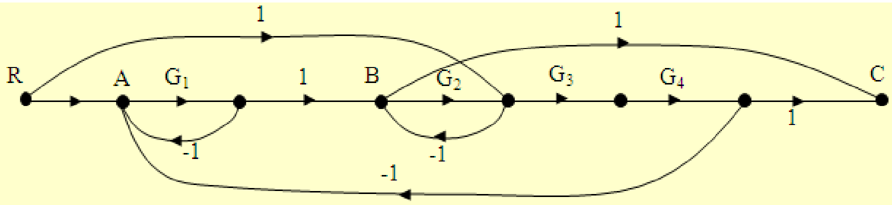
$$= 1 + G_4H_1 + G_6H_4 + G_3G_4G_5G_6H_3 + G_4G_5G_6H_2 + G_4H_1G_6H_4$$

$$\Delta = 1-[-G_4H_1 - G_6H_4 - G_3G_4G_5G_6H_3 - G_4G_5G_6H_2 - G_8H_5]$$
$$+\left[G_4H_1G_6H_4 + G_4H_1G_8H_5 + G_6H_4G_8H_5 + G_8H_5G_4G_5G_6H_2 + G_8H_5G_3G_4G_5G_6H_3\right]$$
$$+G_4H_1G_6H_4G_8H_5$$

$$\Delta = 1 + G_4H_1 + G_6H_4 + G_3G_4G_5G_6H_3 + G_4G_5G_6H_2 + G_8H_5$$
$$+G_4H_1G_6H_4 + G_4H_1G_8H_5 + G_6H_4G_8H_5 + G_8H_5G_4G_5G_6H_2 + G_8H_5G_3G_4G_5G_6H_3$$
$$+G_4H_1G_6H_4G_8H_5$$

$$T(s) = \frac{C(s)}{R(s)} = \frac{M_1\Delta_1 + M_2\Delta_2}{\Delta} = \frac{G_1G_2G_3G_4G_5G_6G_7G_8 + G_1G_9G_8[1 + G_4H_1 + G_6H_4 + G_3G_4G_5G_6H_3 + G_4G_5G_6H_2 + G_4H_1G_6H_4]}{\Delta}$$

Example-3:



$$\mathbf{M_1} = G_1G_2G_3G_4 \qquad \Delta_1 = 1$$

$$\mathbf{M_2} = G_1 \qquad \Delta_2 = 1$$

$$\mathbf{M_3} = G_3G_4 \qquad \Delta_3 = 1 + G_1$$

$$\mathbf{M_4} = -1 \qquad \Delta_4 = 1 + G_1$$

$$\mathbf{M_5} = -G_3G_4G_1 \qquad \Delta_5 = 1$$

$$\Delta = 1 - (-G_1 - G_2 - G_1G_2G_3G_4) + G_1G_2 \; = \; 1 + G_1 + G_2 + G_1G_2G_3G_4 + G_1G_2$$

a) $$\frac{C}{M} = \frac{G_1G_2G_3G_4 + G_1 + G_3G_4(1+G_1) - (1)(1+G_1) - G_1G_3G_4}{1 - (-G_1 - G_2 - G_1G_2G_3G_4) + G_1G_2}$$

STATESPACEANALYSISOFCONTINUOUS SYSTEMS

The**statespacemodel**ofLinearTime-Invariant(LTI)systemcan berepresentedas,

$$\dot{X}=AX+B$$

$$U$$

$$Y=CX+D$$

$$U$$

Thefirstandthesecondequationsareknownasstateequationandoutputequation respectively.

Where,

- XandX˙arethestatevector and thedifferential statevectorrespectively.
- Uand Yareinputvector and outputvectorrespectively.
- Aisthesystemmatrix.
- B andCaretheinputandtheoutputmatrices.
- Disthefeed-forward matrix.

BasicConceptsofStateSpaceModel

Thefollowingbasicterminologyinvolvedinthischapter.

State

Itisagroupofvariables,whichsummarizesthehistoryofthesysteminorderto predict the future values (outputs).

State Variable

Thenumberofthestatevariablesrequiredisequalto thenumberofthestorageelements present in the system.

Examples–currentflowingthroughinductor,voltageacross

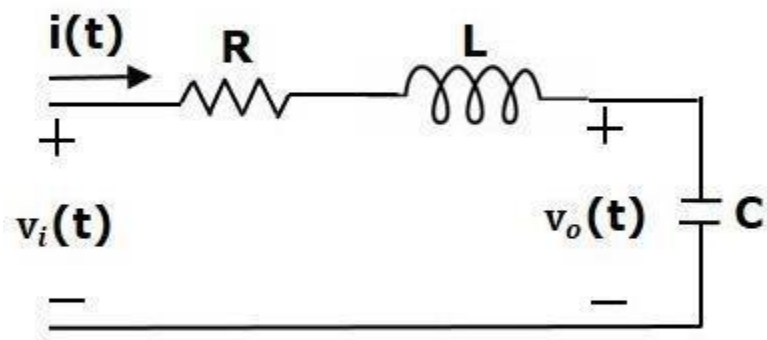
capacitorStateVector

Itisavector,which containsthestatevariablesaselements.

In the earlier chapters, we have discussed two mathematical models of the control systems. Those are the differential equation model and the transfer function model. The state space model can be obtained from any one of these two mathematical models. Let us now discuss these two methods one by one.

StateSpaceModelfromDifferentialEquation

Consider the following series of the RLC circuit. It is having an input voltage, vi(t) and the current flowing through the circuit is i(t).



There are two storage elements (inductor and capacitor) in this circuit. So, the number of the state variables is equal to two and these state variables are the current flowing through the inductor, i(t) and the voltage across capacitor, v_c(t).

Fromthecircuit,theoutputvoltage,v_0(t)is equal to thevoltageacrosscapacitor,v_c(t).

$$v_0(t) = v_c(t)$$

Apply KVL around the loop.

$$v_i(t) = Ri(t) + L\frac{\mathrm{d}i(t)}{\mathrm{d}t} + v_c(t)$$

$$\Rightarrow \frac{\mathrm{d}i(t)}{\mathrm{d}t} = -\frac{Ri(t)}{L} - \frac{v_c(t)}{L} + \frac{v_i(t)}{L}$$

The voltage across the capacitor is -

$$v_c(t) = \frac{1}{C}\int i(t)dt$$

Differentiate the above equation with respect to time.

$$\frac{\mathrm{d}v_c(t)}{\mathrm{d}t} = \frac{i(t)}{C}$$

State vector, $X = \begin{bmatrix} i(t) \\ v_c(t) \end{bmatrix}$

Differential state vector, $\dot{X} = \begin{bmatrix} \frac{\mathrm{d}i(t)}{\mathrm{d}t} \\ \frac{\mathrm{d}v_c(t)}{\mathrm{d}t} \end{bmatrix}$

We can arrange the differential equations and output equation into the standard form of state space model as,

$$\dot{X} = \begin{bmatrix} \frac{\mathrm{d}i(t)}{\mathrm{d}t} \\ \frac{\mathrm{d}v_c(t)}{\mathrm{d}t} \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} i(t) \\ v_c(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} \begin{bmatrix} v_i(t) \end{bmatrix}$$

$$Y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} i(t) \\ v_c(t) \end{bmatrix}$$

Where,

$$A = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix},\ B = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix},\ C = \begin{bmatrix} 0 & 1 \end{bmatrix}\ and\ D = \begin{bmatrix} 0 \end{bmatrix}$$

StateSpaceModelfromTransferFunction

Considerthetwotypesoftransferfunctionsbasedonthetypeoftermspresentinthe numerator.

- TransferfunctionhavingconstanttermінNumerator.
- Transfer function having polynomial function of ‘s’ in

Numerator.TransferfunctionhavingconstantterminNumerator

Consider the following transfer function of a system

$$\frac{Y(s)}{U(s)} = \frac{b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0}$$

Rearrange, the above equation as

$$(s^n + a_{n-1}s^{n-1} + \ldots + a_0)Y(s) = b_0 U(s)$$

Apply inverse Laplace transform on both sides.

$$\frac{\mathrm{d}^n y(t)}{\mathrm{d}t^n} + a_{n-1}\frac{\mathrm{d}^{n-1} y(t)}{\mathrm{d}t^{n-1}} + \ldots + a_1\frac{\mathrm{d}y(t)}{\mathrm{d}t} + a_0 y(t) = b_0 u(t)$$

Let

$$y(t) = x_1$$

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = x_2 = \dot{x}_1$$

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} = x_3 = \dot{x}_2$$

.

.

.

$$\frac{\mathrm{d}^{n-1} y(t)}{\mathrm{d}t^{n-1}} = x_n = \dot{x}_{n-1}$$

$$\frac{\mathrm{d}^n y(t)}{\mathrm{d}t^n} = \dot{x}_n$$

andu(t)=u

Then,

$$\dot{x}_n + a_{n-1}x_n + \ldots + a_1x_2 + a_0x_1 = b_0u$$

From the above equation, we can write the following state equation.

$$\dot{x}_n = -a_0x_1 - a_1x_2 - \ldots - a_{n-1}x_n + b_0u$$

The output equation is -

$$y(t) = y = x_1$$

The state space model is -

$$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-2} & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix} [u]$$

$$Y = \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

Here,D=[0].

Example

Findthestatespacemodelforthesystemhavingtransferfunction.

$$\frac{Y(s)}{U(s)} = \frac{1}{s^2 + s + 1}$$

Rearrange, the above equation as,

$$(s^2 + s + 1)Y(s) = U(s)$$

Apply inverse Laplace transform on both the sides.

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} + \frac{\mathrm{d}y(t)}{\mathrm{d}t} + y(t) = u(t)$$

Let

$$y(t) = x_1$$

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = x_2 = \dot{x}_1$$

and $u(t) = u$

Then, the state equation is

$$\dot{x}_2 = -x_1 - x_2 + u$$

The output equation is

$$y(t) = y = x_1$$

The state space model is

$$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [u]$$

$$Y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Transfer function having polynomial function of ‘s’ in NumeratorConsiderthefollowingtransferfunctionofa system

$$\frac{Y(s)}{U(s)} = \frac{b_n s^n + b_{n-1} s^{n-1} + \ldots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}$$

$$\Rightarrow \frac{Y(s)}{U(s)} = \left(\frac{1}{s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}\right)(b_n s^n + b_{n-1} s^{n-1} + \ldots + b_1 s + b_0)$$

The above equation is in the form of product of transfer functions of two blocks, which are cascaded.

$$\frac{Y(s)}{U(s)} = \left(\frac{V(s)}{U(s)}\right)\left(\frac{Y(s)}{V(s)}\right)$$

Here,

$$\frac{V(s)}{U(s)} = \frac{1}{s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}$$

Rearrange,theaboveequationas

$$(s^n + a_{n-1}s^{n-1} + \ldots + a_0)V(s) = U(s)$$

Apply inverse Laplace transform on both the sides.

$$\frac{\mathrm{d}^n v(t)}{\mathrm{d}t^n} + a_{n-1}\frac{\mathrm{d}^{n-1} v(t)}{\mathrm{d}t^{n-1}} + \ldots + a_1\frac{\mathrm{d}v(t)}{\mathrm{d}t} + a_0 v(t) = u(t)$$

Let

$$v(t) = x_1$$

$$\frac{\mathrm{d}v((t)}{\mathrm{d}t} = x_2 = \dot{x}_1$$

$$\frac{\mathrm{d}^2 v(t)}{\mathrm{d}t^2} = x_3 = \dot{x}_2$$

.

.

.

$$\frac{\mathrm{d}^{n-1} v(t)}{\mathrm{d}t^{n-1}} = x_n = \dot{x}_{n-1}$$

$$\frac{\mathrm{d}^n v(t)}{\mathrm{d}t^n} = \dot{x}_n$$

andu(t)=u

Then,thestateequationis

$$\dot{x}_n = -a_0x_1 - a_1x_2 - \ldots - a_{n-1}x_n + u$$

Consider,

$$\frac{Y(s)}{V(s)} = b_n s^n + b_{n-1}s^{n-1} + \ldots + b_1 s + b_0$$

Rearrange, the above equation as

$$Y(s) = (b_n s^n + b_{n-1}s^{n-1} + \ldots + b_1 s + b_0)V(s)$$

Apply inverse Laplace transform on both the sides.

$$y(t) = b_n\frac{\mathrm{d}^n v(t)}{\mathrm{d}t^n} + b_{n-1}\frac{\mathrm{d}^{n-1} v(t)}{\mathrm{d}t^{n-1}} + \ldots + b_1\frac{\mathrm{d}v(t)}{\mathrm{d}t} + b_0 v(t)$$

By substituting the state variables and $y(t) = y$ in the above equation, will get the output equation as,

$$y = b_n\dot{x}_n + b_{n-1}x_n + \ldots + b_1x_2 + b_0x_1$$

Substitute, $\dot{x}_n$ value in the above equation.

$$y = b_n(-a_0x_1 - a_1x_2 - \dots - a_{n-1}x_n + u) + b_{n-1}x_n + \dots + b_1x_2 + b_0x_1$$

$$y = (b_0 - b_na_0)x_1 + (b_1 - b_na_1)x_2 + \dots + (b_{n-1} - b_na_{n-1})x_n + b_nu$$

The state space model is

$$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-2} & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix} [u]$$

$$Y = \begin{bmatrix} b_0 - b_na_0 & b_1 - b_na_1 & \dots & b_{n-2} - b_na_{n-2} & b_{n-1} - b_na_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

If $b_n = 0$, then,

$$Y = \begin{bmatrix} b_0 & b_1 & \dots & b_{n-2} & b_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

TransferFunctionfromStateSpace Model

WeknowthestatespacemodelofaLinearTime-Invariant(LTI)systemis-

$$\dot{X}=AX+B$$

$$U$$

$$Y=CX+D$$

$$U$$

ApplyLaplaceTransform onbothsidesofthestateequation.

$$sX(s)=AX(s)+BU(s)$$

$$\Rightarrow(sI-A)X(s)=BU(s)$$

$$\Rightarrow X(s)=(sI-A)^{-1}BU(s)$$

ApplyLaplaceTransform onboth sidesoftheoutput equation.

$$Y(s)=CX(s)+DU(s)$$

Substitute,X(s)valueintheabove equation.

$$\Rightarrow Y(s)=C(sI-A)^{-1}BU(s)+DU(s)$$

$$\Rightarrow Y(s)=[C(sI-A)^{-1}B+D]U(s)$$

$$\Rightarrow Y(s)U(s)=C(sI-A)^{-1}B+D$$

Theaboveequationrepresentsthetransferfunctionofthesystem.So,wecancalculatethe transfer function of the system by using this formula for the system represented in the state space model.

Note – WhenD=[0],thetransfer functionwill be

$$\frac{Y(s)}{U(s)}=C(sI-A)^{-1}B$$

Example

Letuscalculatethetransferfunctionofthesystemrepresentedinthestatespacemodel as,

$$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} [u]$$

$$Y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Here,

$$A = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix} \quad and \quad D = [0]$$

The formula for the transfer function when $D = [0]$ is -

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B$$

Substitute, A, B & C matrices in the above equation.

$$\frac{Y(s)}{U(s)} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s+1 & 1 \\ -1 & s \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \frac{Y(s)}{U(s)} = \begin{bmatrix} 0 & 1 \end{bmatrix} \frac{\begin{bmatrix} s & -1 \\ 1 & s+1 \end{bmatrix}}{(s+1)s - 1(-1)} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \frac{Y(s)}{U(s)} = \frac{\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s \\ 1 \end{bmatrix}}{s^2 + s + 1} = \frac{1}{s^2 + s + 1}$$

Therefore,thetransferfunctionofthesystemfor thegivenstatespacemodel is

$$\frac{Y(s)}{U(s)} = \frac{1}{s^2 + s + 1}$$

StateTransitionMatrixanditsProperties

If the system is having initial conditions, then it will produce an output. Since, this outputis present even in the absence of input, it is called **zero input response** $x_{ZIR}(t)$. Mathematically, we can write it as,

$$x_{ZIR}(t) = e^{At}X(0) = L^{-1}\left\{[sI-A]^{-1}X(0)\right\}$$

Fromtheaboverelation,wecanwritethestatetransitionmatrixϕ(t)as

$$\phi(t) = e^{At} = L^{-1}[sI-A]^{-1}$$

So,thezeroinputresponsecanbeobtainedbymultiplyingthestatetransition matrix ϕ(t) with the initial conditions matrix.

Propertiesofthestate transitionmatrix

- Ift=0,thenstatetransitionmatrixwillbeequalto anIdentitymatrix.

ϕ(0)=I

- Inverseofstatetransitionmatrixwillbesameasthatofstatetransitionmatrixjust by replacing 't' by '-t'.

$$\phi^{-1}(t) = \phi(-t)$$

- If t=t1+t2 , then the corresponding state transition matrix is equal to the multiplicationofthetwostatetransitionmatricesatt=t1t=t1andt=t2t=t2.

ϕ(t1+t2)=ϕ(t1)ϕ(t2)

Controllabilityand Observability

Letusnowdiscusscontrollabilityandobservabilityofcontrolsystemoneby one.

Controllability

Acontrolsystemissaidto be**controllable**iftheinitialstatesofthecontrolsystemare transferred(changed)tosomeotherdesiredstatesbyacontrolledinputinfiniteduration of time.

Wecancheckthecontrollabilityofacontrolsystembyusing**Kalman's test**.

- WritethematrixQcinthefollowingform.

$$Q_c = [B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B]$$

- FindthedeterminantofmatrixQcQcandifitisnotequalto zero,thenthecontrol system is controllable.

Observability

Acontrolsystemissaidto be**observable** ifitisableto determinetheinitialstatesofthe control system by observing the outputs in finite duration of time.

Wecanchecktheobservabilityofacontrolsystembyusing**Kalman’s test**.

- WritethematrixQoinfollowingform.

$$Q_o = [C^T \quad A^TC^T \quad (A^T)^2C^T \quad \dots \quad (A^T)^{n-1}C^T]$$

- FindthedeterminantofmatrixQoQoandifitisnotequalto zero,thenthecontrol system is observable.

Example

Letusverifythecontrollabilityandobservabilityofacontrolsystemwhichisrepresented in the state space model as,

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} [u]$$

$$Y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Here,

$$A = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \end{bmatrix}, D = [0] \quad and \quad n = 2$$

For $n = 2$, the matrix Q_c will be

$$Q_c = [B \quad AB]$$

We will get the product of matrices A and B as,

$$AB = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow Q_c = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$|Q_c| = 1 \neq 0$$

SincethedeterminantofmatrixQcisnotequaltozero,thegivencontrolsystemis controllable.
Forn=2,thematrixQo will be –

$$Q_o = [C^T \quad A^T C^T]$$

Here,

$$A^T = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \quad and \quad C^T = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We will get the product of matrices A^T and C^T as

$$A^T C^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow Q_o = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\Rightarrow |Q_o| = -1 \quad \neq 0$$

Since,thedeterminant ofmatrixQo isnotequalto zero,thegivencontrol system is observable.Therefore,thegivencontrolsystemisbothcontrollableandobservable.

UNIT-III

TIMERESPONSE ANALYSIS

We can analyze the response of the control systems in both the time domain and the frequency domain. Wewill discussfrequency responseanalysisof control systemsin later chapters. Let us now discuss about the time response analysis of control systems.

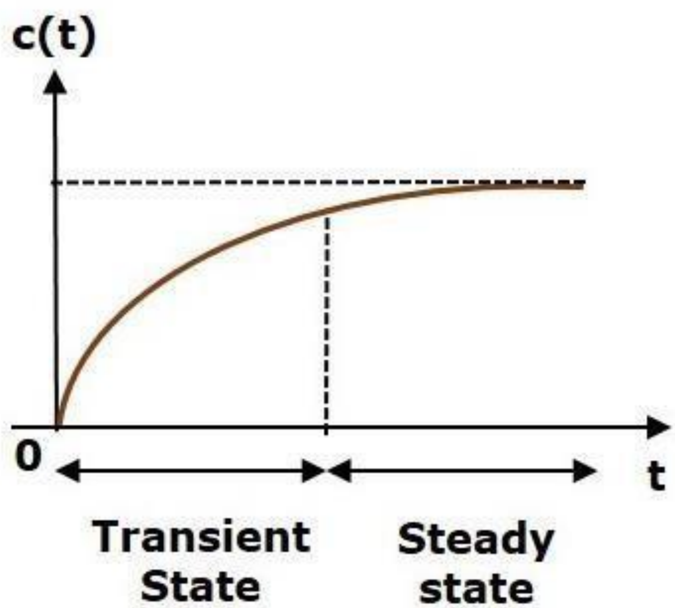
WhatisTimeResponse?

If the output of control system for an inputvarieswithrespecttotime,thenitis called the **time response** of the control system. The time response consists of twoparts.

 Transient response

Steadystateresponse

Theresponseofcontrolsystem intimedomainisshowninthefollowing figure.



Here, both the transient and the steady states are indicated in the figure. The responses corresponding to these states are known as transient and steady state responses.

Mathematically, we can write the time response c(t) as

$$c(t) = c_{tr}(t) + c_{ss}(t)$$

Where,

- $c_{tr}(t)$ is the transient response
- $c_{ss}(t)$isthesteadystateresponse

TransientResponse

After applying input to thecontrol system, output takescertain timeto reach steady state. So, the output will be in transient state till it goes to a steady state. Therefore, theresponseofthecontrolsystemduringthetransientstateisknownas**transientresponse**.

The transient response will be zero for large values of 't'. Ideally, this value of 't' is infinity and practically, it is five times constant.

Mathematically,wecanwriteitas

$$\lim_{t\to\infty} c_{tr}(t) = 0$$

SteadystateResponse

The part of the time response that remains even after the transient response has zero value for large values of 't' is known as **steady state response**. This means, the transient response will be zero even during the steady state.

Example

Letusfindthetransientandsteadystatetermsofthetimeresponseofthecontrolsystem $c(t) = 10 + 5e^{-t}$

Here, thesecond term $5e^{-t}$ will be zero as **t** denotes infinity. So, this is the **transient term**.Andthefirstterm10remainsevenas **t**approachesinfinity.So,thisisthe**steady stateterm**.

StandardTestSignals

Thestandardtestsignalsareimpulse,step,rampandparabolic.Thesesignalsareusedto know the performance of the control systems using time response of theoutput.

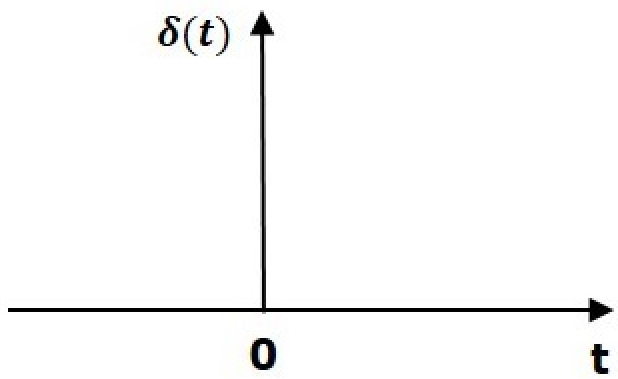
UnitImpulseSignal

Aunitimpulsesignal,δ(t)isdefinedas

$$\delta(t) = 0 \text{ for } t \neq 0$$

$$\text{and } \int_{0^-}^{0^+} \delta(t) dt = 1$$

The following figure shows unit impulse signal.



So, the unit impulse signal exists only at‘t’ is equal to zero. The area of this signal under small interval of time around‘t’ is equal to zero is one. The value of unit impulse signal is zero for all other values of‘t’.

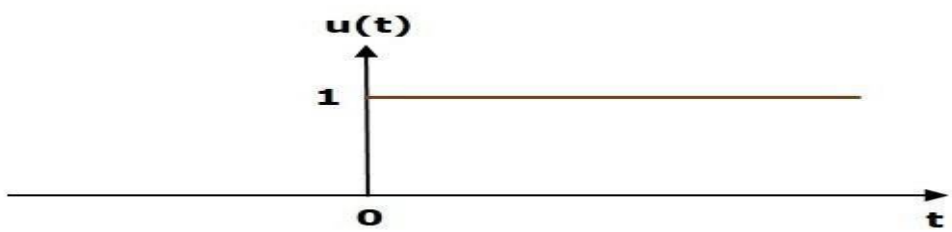
UnitStep Signal

Aunitstepsignal,u(t)isdefinedas

$$u(t) = 1; t \geq 0$$

$$= 0; t < 0$$

Followingfigureshowsunitstep signal.



So,theunitstepsignalexistsforallpositivevaluesof‘t’includingzero.Anditsvalueisone during this interval. The value of the unit step signal is zero for all negative values of‘t’.

UnitRamp Signal

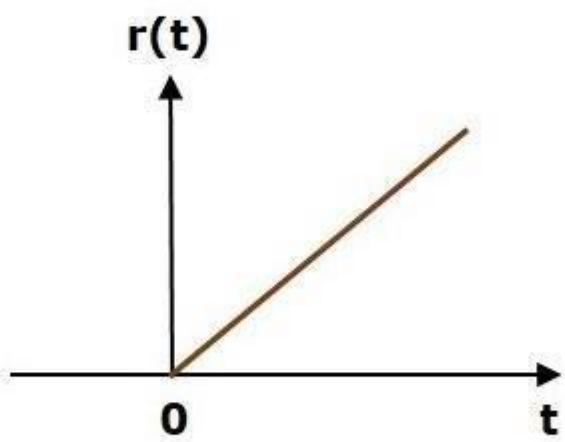
Aunitrampsignal,r (t)isdefined as

$$r(t) = t; t \geq 0$$
$$= 0; t < 0$$

We can write unit ramp signal, $r(t)$ in terms of unit step signal, $u(t)$ as

$$r(t) = tu(t)$$

Following figure shows unit ramp signal.



So, the unit ramp signal exists for all positive values of‘t’ including zero. And its value increases linearly with respect to‘t’ during this interval. The value of unit ramp signal is zero for all negative values of‘t’.

UnitParabolicSignal

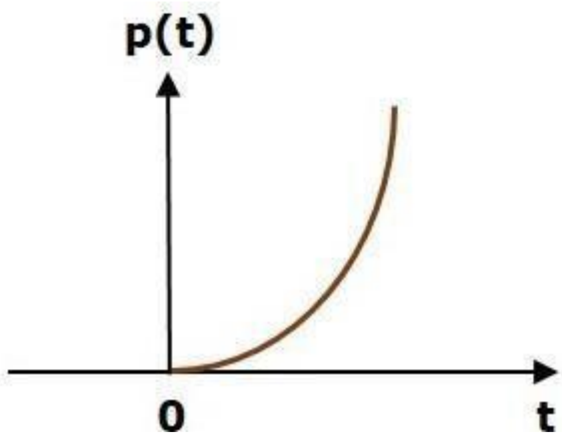
Aunitparabolicsignal,p(t)isdefinedas,

$$p(t) = \frac{t^2}{2}; t \geq 0$$
$$= 0; t < 0$$

We can write unit parabolic signal, $p(t)$ in terms of the unit step signal, $u(t)$ as,

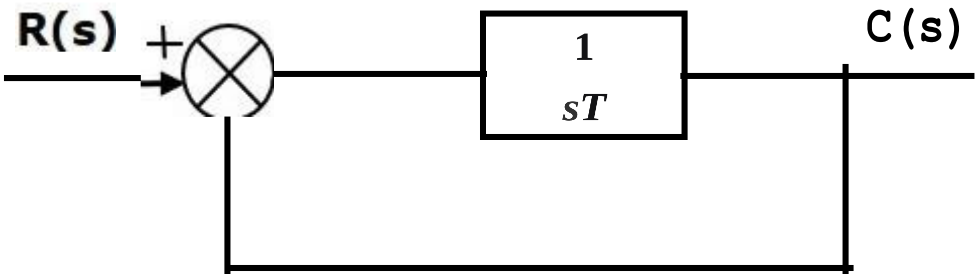
$$p(t) = \frac{t^2}{2}u(t)$$

The following figure shows the unit parabolic signal.



So, the unit parabolic signal exists for all the positive values of'‘t’ including zero. And its value increases non-linearly with respect to‘t’ during this interval. The value of the unit parabolic signal is zero for all the negative values of‘t’.

In this chapter, let us discuss the time response of the first order system. Consider the following block diagram of the closed loop control system. Here, an open loop transfer function, 1/sT is connected with a unity negative feedback.



We know that the transfer function of the closed loop control system has unity negative feedback as,

$$\frac{C(s)}{R(s)}=\frac{G(s)}{1+G(s)}$$

Substitute, $G(s)=\frac{1}{sT}$ in the above equation.

$$\frac{C(s)}{R(s)}=\frac{\frac{1}{sT}}{1+\frac{1}{sT}}=\frac{1}{sT+1}$$

The power of s is one in the denominator term. Hence, the above transfer function is of the first order and the system is said to be the **first order system.**

We can re-write the above equation as

$$C(s) = \left(\frac{1}{sT+1}\right) R(s)$$

Where,

- **C(s)** is the Laplace transform of the output signal c(t),
- **R(s)** is the Laplace transform of the input signal r(t), and
- **T** is the time constant.

Follow these steps to get the response (output) of the first order system in the time domain.

- Take the Laplace transform of the input signal $r(t)$.
- Consider the equation, $C(s) = \left(\frac{1}{sT+1}\right) R(s)$
- Substitute $R(s)$ value in the above equation.
- Do partial fractions of $C(s)$ if required.
- Apply inverse Laplace transform to $C(s)$.

Impulse Response ofFirstOrderSystem

Considerthe**unitimpulse signal** asaninput tothefirstordersystem.

So,r(t)=δ(t)

ApplyLaplacetransformonboththe sides.

R(s) =1

Consider the equation, $C(s) = \left(\frac{1}{sT+1}\right) R(s)$

Substitute, $R(s) = 1$ in the above equation.

$$C(s) = \left(\frac{1}{sT+1}\right)(1) = \frac{1}{sT+1}$$

RearrangetheaboveequationinoneofthestandardformsofLaplacetransforms.

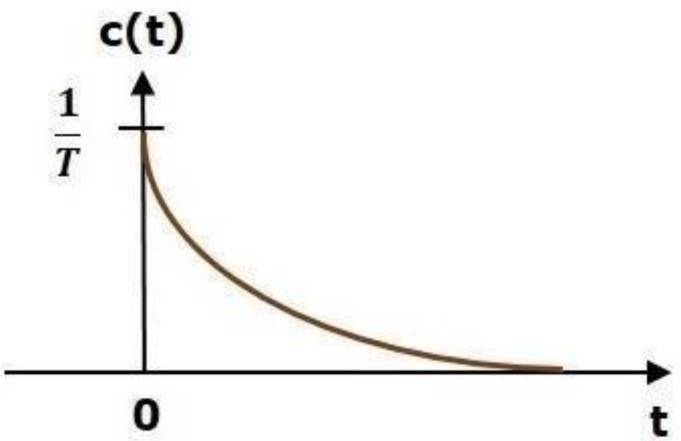
$$C(s) = \frac{1}{T\left(\ s + \frac{1}{T}\right)} \Rightarrow C(s) = \frac{1}{T}\left(\frac{1}{s + \frac{1}{T}}\right)$$

ApplyingInverseLaplaceTransformonboththesides,

$$c(t) = \frac{1}{T}e^{\left(-\frac{t}{T}\right)}u(t)$$

$$c(t) = \frac{1}{T}e^{\left(-\frac{t}{T}\right)}u(t)$$

Theunitimpulseresponseisshowninthefollowingfigure.



The**unitimpulseresponse**,c(t)isanexponentialdecayingsignalforpositivevaluesof‘t’ and it is zero for negative values of ‘t’.

StepResponseofFirstOrderSystem

Considerthe**unitstepsignal**asaninputtofirstorder

system. So, r(t)=u(t)

$$R(s) = \frac{1}{s}$$

Consider the equation, $C(s) = \left(\frac{1}{sT+1}\right) R(s)$

Substitute, $R(s) = \frac{1}{s}$ in the above equation.

$$C(s) = \left(\frac{1}{sT+1}\right)\left(\frac{1}{s}\right) = \frac{1}{s\,(sT+1)}$$

Do partial fractions of C(s).

$$C(s) = \frac{1}{s\,(sT+1)} = \frac{A}{s} + \frac{B}{sT+1}$$

$$\Rightarrow \frac{1}{s\,(sT+1)} = \frac{A\,(sT+1) + Bs}{s\,(sT+1)}$$

Onboththesides,thedenominatortermisthesame.So,theywillgetcancelledbyeach other. Hence, equate the numerator terms.

$$1=A(sT+1)+Bs$$

By equating the constant terms on both the sides, you will get A = 1.

Substitute,A=1andequatethecoefficientofthe **s**termsonboth the sides.

$$0=T+B$$

$$\Rightarrow B=-T$$

Substitute,A=1andB= −Tinpartialfractionexpansionof C(s)

$$C(s) = \frac{1}{s} - \frac{T}{sT+1} = \frac{1}{s} - \frac{T}{T\left(s+\frac{1}{T}\right)}$$

$$\Rightarrow C(s) = \frac{1}{s} - \frac{1}{s+\frac{1}{T}}$$

ApplyinverseLaplacetransform onboththe sides.

$$c(t) = \left(1 - e^{-\left(\frac{t}{T}\right)}\right) u(t)$$

The**unitstepresponse**,c(t)hasboth thetransientandthesteadystateterms.

The transient term in the unit step response is -

$$c_{tr}(t) = -e^{-\left(\frac{t}{T}\right)}u(t)$$

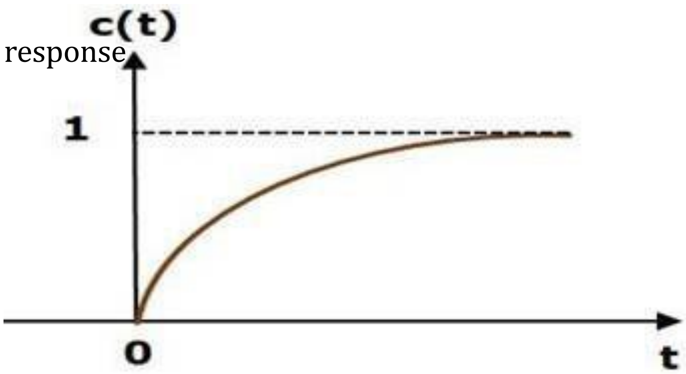
The steady state term in the unit step

$$c_{ss}(t) = u(t)$$

responseis–

Thefollowingfigureshowstheunitstep

response



The value of the **unit step response, c(t)** is zero at t = 0 and for all negative values of t. It is gradually increasing from zero value and finally reaches to one in steady state. So, the steady state value depends on the magnitude of the input.

RampResponse ofFirstOrderSystem

Considerthe**unitrampsignal**asaninputto thefirstorder system.

So,r(t)=tu(t)

ApplyLaplacetransform onboth thesides.

$$R(s) = \frac{1}{s^2}$$

Consider the equation, $C(s) = \left(\frac{1}{sT+1}\right)R(s)$

Substitute, $R(s) = \frac{1}{s^2}$ in the above equation.

$$C(s) = \left(\frac{1}{sT+1}\right)\left(\frac{1}{s^2}\right) = \frac{1}{s^2(sT+1)}$$

Onboththesides,thedenominatortermisthesame.So,theywillgetcancelledbyeach other. Hence, equate the numerator terms.

$$1 = A(sT+1) + Bs(sT+1) + Cs^2$$

By equating the constant terms on both the sides, you will get A = 1.

Substitute,A=1andequatethecoefficientofthestermsonboth the sides.

$$0=T+B \Rightarrow B=-T$$

Similarly,substituteB=−Tandequatethecoefficientofs^{2}termsonboth thesides.You will get $C=T^2$

SubstituteA= 1,B = −T and $C=T^2$inthepartial fractionexpansionof C(s).

$$C(s) = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{sT+1} = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{T\left(s+\frac{1}{T}\right)}$$

$$\Rightarrow C(s) = \frac{1}{s^2} - \frac{T}{s} + \frac{T}{s+\frac{1}{T}}$$

ApplyinverseLaplacetransformonboththe sides.

$$c(t) = \left(t - T + Te^{-\left(\frac{t}{T}\right)}\right)u(t)$$

The**unitrampresponse**,c(t)hasboththetransientandthesteadystate terms. The

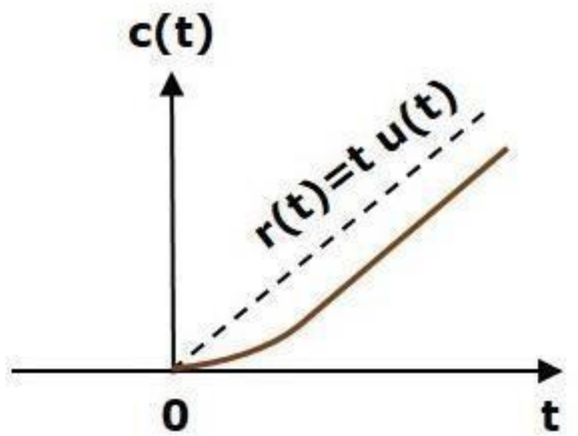
transient term in the unit ramp response is

$$c_{tr}(t) = Te^{-\left(\frac{t}{T}\right)}u(t)$$

Thesteadystateterm intheunitrampresponseis–

$$c_{ss}(t) = (t - T)u(t)$$

Thefigurebelowistheunit rampresponse:



The**unitrampresponse**,c(t)followstheunitrampinputsignalforallpositivevaluesoft. But, there is a deviation of T units from the input signal.

ParabolicResponse ofFirstOrderSystem

Considerthe**unitparabolicsignal**asaninputto thefirstorder system.

$$\text{So, } r(t) = \frac{t^2}{2}u(t)$$

Apply Laplace transform on both the sides.

$$R(s) = \frac{1}{s^3}$$

Consider the equation, $C(s) = \left(\frac{1}{sT+1}\right) R(s)$

Substitute $R(s) = \frac{1}{s^3}$ in the above equation.

$$C(s) = \left(\frac{1}{sT+1}\right)\left(\frac{1}{s^3}\right) = \frac{1}{s^3(sT+1)}$$ \

Do partial fractions of $C(s)$.

$$C(s) = \frac{1}{s^3(sT+1)} = \frac{A}{s^3} + \frac{B}{s^2} + \frac{C}{s} + \frac{D}{sT+1}$$

After simplifying, you will get the values of A, B, C and D as 1, $-T,\ T^2\ and\ -T^3$ respectively. Substitute these values in the above partial fraction expansion of C(s).

$$C(s) = \frac{1}{s^3} - \frac{T}{s^2} + \frac{T^2}{s} - \frac{T^3}{sT+1} \Rightarrow C(s) = \frac{1}{s^3} - \frac{T}{s^2} + \frac{T^2}{s} - \frac{T^2}{s+\frac{1}{T}}$$

ApplyinverseLaplacetransformonboththesides.

$$c(t) = \left(\frac{t^2}{2} - Tt + T^2 - T^2 e^{-\left(\frac{t}{T}\right)}\right)u(t)$$

The**unitparabolicresponse**,c(t)hasboth thetransientandthesteadystateterms. The

transient term in the unit parabolic response is

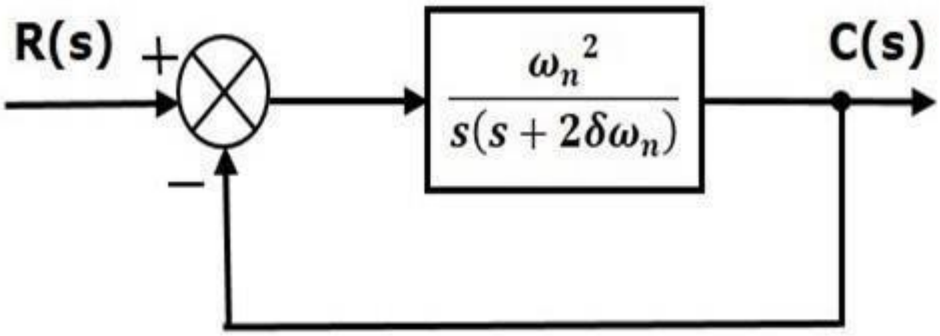
$$C_{tr}(t) = -T^2 e^{-\left(\frac{t}{T}\right)} u(t)$$

Thesteadystatetermintheunitparabolicresponse is

$$C_{ss}(t) = \left(\frac{t^2}{2} - Tt + T^2\right)u(t)$$

From these responses, we can conclude that the first order control systems are not stable with the ramp and parabolic inputs because these responses go on increasing even at infinite amount of time. The first order control systems are stable with impulse and step inputs because these responses have bounded output. But, the impulse response doesn't have steady state term. So, the step signal is widely used in the time domain for analyzing the control systems from their responses.

In this chapter, let us discuss the time response of second order system. Consider the following block diagram of closed loop control system. Here, an open loop trans$_n$fer function, ω ^{2} / s(s+2δωn) is connected with a unity negative feedback.



We know that the transfer function of the closed loop control system having unity negative feedback as

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

Substitute, $G(s) = \frac{\omega_n^2}{s(s+2\delta\omega_n)}$ in the above equation.

$$\frac{C(s)}{R(s)} = \frac{\left(\frac{\omega_n^2}{s(s+2\delta\omega_n)}\right)}{1+\left(\frac{\omega_n^2}{s(s+2\delta\omega_n)}\right)} = \frac{\omega_n^2}{s^2+2\delta\omega_n s+\omega_n^2}$$

The power of ‘s’ is two in the denominator term. Hence, the above transfer function is of the second order and the system is said to be the **second order system**.

Thecharacteristicequationis-

$$s^2+2\delta\omega_n s+\omega_n^2=0$$

The roots of characteristic equation are -

$$s = \frac{-2\omega\delta_n \pm \sqrt{(2\delta\omega_n)^2 - 4\omega_n^2}}{2} = \frac{-2(\delta\omega_n \pm \omega_n\sqrt{\delta^2 - 1})}{2}$$

$$\Rightarrow s = -\delta\omega_n \pm \omega_n\sqrt{\delta^2 - 1}$$

b

- The two roots are imaginary when δ = 0.
- The two roots are real and equal when δ =1.
- The two roots are real but not equal when δ > 1.

Thetworootsarecomplexconjugatewhen0<δ<
1.WecanwriteC(s)equationas,

$$C(s) = \left(\frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}\right) R(s)$$

Where,

- **C(s)**istheLaplacetransformoftheoutputsignal,c(t)

R(s) is the Laplace transform of the input signal,r(t) **ω_n**

is the naturalfrequency

- **δ** is thedampingratio.

Followthesestepsto gettheresponse(output)ofthesecondordersysteminthetime domain.

Take Laplace transform of the input signal, $r(t)$.

Consider the equation, $C(s) = \left(\frac{\omega_n^2}{s^2+2\delta\omega_n s+\omega_n^2}\right) R(s)$

Substitute $R(s)$ value in the above equation.

Do partial fractions of $C(s)$ if required.

Apply inverse Laplace transform to $C(s)$.

StepResponse ofSecondOrder System

Considertheunitstepsignalasaninputtothesecondordersystem.Laplacetransformof the unit step signal is,

$$R(s) = \frac{1}{s}$$

We know the transfer function of the second order closed loop control system is,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}$$

Case 1: δ = 0

Substitute, $\delta = 0$ in the transfer function.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + \omega_n^2}$$

$$\Rightarrow C(s) = \left(\frac{\omega_n^2}{s^2 + \omega_n^2}\right) R(s)$$

Substitute, $R(s) = \frac{1}{s}$ in the above equation.

$$C(s) = \left(\frac{\omega_n^2}{s^2 + \omega_n^2}\right)\left(\frac{1}{s}\right) = \frac{\omega_n^2}{s(s^2 + \omega_n^2)}$$

Apply inverse Laplace transform on both the sides.

$$c(t) = (1 - \cos(\omega_n t))\, u(t)$$

So, the unit step response of the second order system when $/delta = 0$ will be a continuous time signal with constant amplitude and frequency.

Case 2: δ = 1

Substitute, $/delta = 1$ in the transfer function.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2}$$

$$\Rightarrow C(s) = \left(\frac{\omega_n^2}{(s + \omega_n)^2}\right) R(s)$$

Substitute, $R(s) = \frac{1}{s}$ in the above equation.

$$C(s) = \left(\frac{\omega_n^2}{(s + \omega_n)^2}\right)\left(\frac{1}{s}\right) = \frac{\omega_n^2}{s(s + \omega_n)^2}$$

Do partial fractions of $C(s)$.

$$C(s) = \frac{\omega_n^2}{s(s+\omega_n)^2} = \frac{A}{s} + \frac{B}{s+\omega_n} + \frac{C}{(s+\omega_n)^2}$$

After simplifying, you will get the values of A, B and C as $1, -1\, and\ -\omega_n$ respectively. Substitute these values in the above partial fraction expansion of $C(s)$.

$$C(s) = \frac{1}{s} - \frac{1}{s+\omega_n} - \frac{\omega_n}{(s+\omega_n)^2}$$

Apply inverse Laplace transform on both the sides.

$$c(t) = (1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t})u(t)$$

So, the unit step response of the second order system will try to reach the step input in steady state.

Case 3: 0 < δ < 1

We can modify the denominator term of the transfer function as follows −

$$s^2 + 2\delta\omega_n s + \omega_n^2 = \left\{s^2 + 2(s)(\delta\omega_n) + (\delta\omega_n)^2\right\} + \omega_n^2 - (\delta\omega_n)^2$$

$$= (s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)$$

The transfer function becomes,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)}$$

$$\Rightarrow C(s) = \left(\frac{\omega_n^2}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)}\right) R(s)$$

Substitute, $R(s) = \frac{1}{s}$ in the above equation.

$$C(s) = \left(\frac{\omega_n^2}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)}\right)\left(\frac{1}{s}\right) = \frac{\omega_n^2}{s\left((s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)\right)}$$

Do partial fractions of $C(s)$.

$$C(s) = \frac{\omega_n^2}{s\left((s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)\right)} = \frac{A}{s} + \frac{Bs+C}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)}$$

After simplifying, you will get the values of A, B and C as $1,\ -1\, and\ -2\delta\omega_n$ respectively. Substitute these values in the above partial fraction expansion of C(s).

$$C(s) = \frac{1}{s} - \frac{s+2\delta\omega_n}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)}$$

$$C(s) = \frac{1}{s} - \frac{s+\delta\omega_n}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)} - \frac{\delta\omega_n}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)}$$

$$C(s) = \frac{1}{s} - \frac{(s+\delta\omega_n)}{(s+\delta\omega_n)^2 + (\omega_n\sqrt{1-\delta^2})^2} - \frac{\delta}{\sqrt{1-\delta^2}}\left(\frac{\omega_n\sqrt{1-\delta^2}}{(s+\delta\omega_n)^2 + (\omega_n\sqrt{1-\delta^2})^2}\right)$$

Substitute, $\omega_n\sqrt{1-\delta^2}$ as ω_d in the above equation.

$$C(s) = \frac{1}{s} - \frac{(s+\delta\omega_n)}{(s+\delta\omega_n)^2 + \omega_d^2} - \frac{\delta}{\sqrt{1-\delta^2}}\left(\frac{\omega_d}{(s+\delta\omega_n)^2 + \omega_d^2}\right)$$

Apply inverse Laplace transform on both the sides.

$$c(t) = \left(1 - e^{-\delta\omega_n t}\cos(\omega_d t) - \frac{\delta}{\sqrt{1-\delta^2}} e^{-\delta\omega_n t}\sin(\omega_d t)\right) u(t)$$

$$c(t) = \left(1 - \frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\left((\sqrt{1-\delta^2})\cos(\omega_d t) + \delta\sin(\omega_d t)\right)\right) u(t)$$

If $\sqrt{1-\delta^2} = \sin(\theta)$, then 'δ' will be cos(θ). Substitute these values in the above equation.

$$c(t) = \left(1 - \frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}(\sin(\theta)\cos(\omega_d t) + \cos(\theta)\sin(\omega_d t))\right) u(t)$$

$$\Rightarrow c(t) = \left(1 - \left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t + \theta)\right) u(t)$$

So,theunitstepresponseofthesecondordersystemishavingdampedoscillations (decreasing amplitude) when ‘δ’ lies between zero and one.

Case4:δ>1

Wecanmodifythedenominator termofthetransfer functionasfollows–

$$s^2+2\delta\omega_n s+\omega_n^2=\left\{s^2+2(s)(\delta\omega_n)+(\delta\omega_n)^2\right\}+\omega_n^2-(\delta\omega_n)^2$$

$$=(s+\delta\omega_n)^2-\omega_n^2\left(\delta^2-1\right)$$

The transfer function becomes,

$$\frac{C(s)}{R(s)}=\frac{\omega_n^2}{(s+\delta\omega_n)^2-\omega_n^2(\delta^2-1)}$$

$$\Rightarrow C(s)=\left(\frac{\omega_n^2}{(s+\delta\omega_n)^2-\omega_n^2(\delta^2-1)}\right)R(s)$$

Substitute, $R(s)=\frac{1}{s}$ in the above equation.

$$C(s)=\left(\frac{\omega_n^2}{(s+\delta\omega_n)^2-(\omega_n\sqrt{\delta^2-1})^2}\right)\left(\frac{1}{s}\right)=\frac{\omega_n^2}{s(s+\delta\omega_n+\omega_n\sqrt{\delta^2-1})(s+\delta\omega_n-\omega_n\sqrt{\delta^2-1})}$$

Do partial fractions of $C(s)$.

$$C(s)=\frac{\omega_n^2}{s(s+\delta\omega_n+\omega_n\sqrt{\delta^2-1})(s+\delta\omega_n-\omega_n\sqrt{\delta^2-1})}$$

$$=\frac{A}{s}+\frac{B}{s+\delta\omega_n+\omega_n\sqrt{\delta^2-1}}+\frac{C}{s+\delta\omega_n-\omega_n\sqrt{\delta^2-1}}$$

After simplifying, you will get the values of A, B and C as 1, $\frac{1}{2(\delta+\sqrt{\delta^2-1})(\sqrt{\delta^2-1})}$

.

and $\frac{-1}{2(\delta-\sqrt{\delta^2-1})(\sqrt{\delta^2-1})}$ respectively. Substitute these values in above partial fraction expansion of $C(s)$.

$$C(s) = \frac{1}{s} + \frac{1}{2(\delta + \sqrt{\delta^2 - 1})(\sqrt{\delta^2 - 1})} \left(\frac{1}{s + \delta\omega_n + \omega_n\sqrt{\delta^2 - 1}} \right) - \left(\frac{1}{2(\delta - \sqrt{\delta^2 - 1})(\sqrt{\delta^2 - 1})} \right) \left(\frac{1}{s + \delta\omega_n - \omega_n\sqrt{\delta^2 - 1}} \right)$$

Apply inverse Laplace transform on both the sides.

$$c(t) = \left(1 + \left(\frac{1}{2(\delta+\sqrt{\delta^2-1})(\sqrt{\delta^2-1})} \right) e^{-(\delta\omega_n + \omega_n\sqrt{\delta^2-1})t} - \left(\frac{1}{2(\delta-\sqrt{\delta^2-1})(\sqrt{\delta^2-1})} \right) e^{-(\delta\omega_n - \omega_n\sqrt{\delta^2-1})t} \right) u(t)$$

Sinceitisoverdamped,theunitstepresponseofthesecondordersystemwhenδ>1will never reach step input in the steady state.

ImpulseResponseofSecondOrderSystem

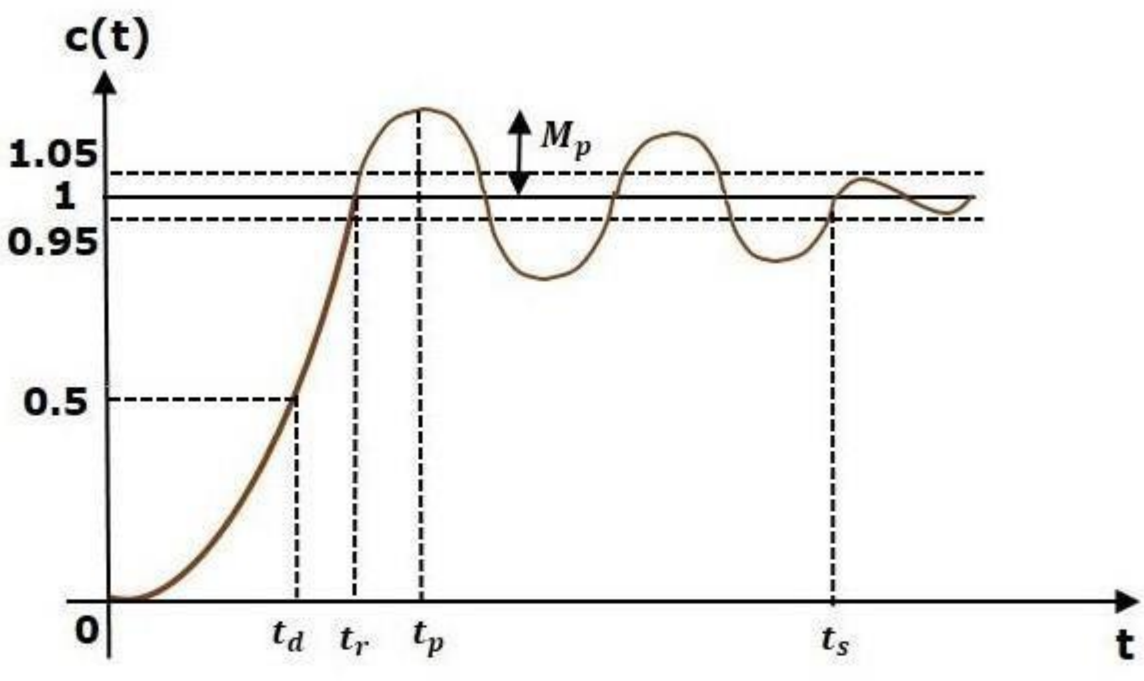
The**impulseresponse**ofthesecondordersystemcanbeobtainedbyusinganyoneof these two methods.

- Followtheprocedureinvolvedwhilederivingstepresponsebyconsideringthe valueof R(s) as 1 instead of 1/s.
- Do thedifferentiationofthestepresponse.

Thefollowingtableshowstheimpulseresponseofthesecondordersystemfor4casesof the damping ratio.

Condition of Damping ratio	Impulse response for t ≥ 0
δ = 0	$\omega_n \sin(\omega_n t)$
δ = 1	$\omega_n^2 t e^{-\omega_n t}$
0 < δ < 1	$\left(\frac{\omega_n e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t)$
δ > 1	$\left(\frac{\omega_n}{2\sqrt{\delta^2-1}}\right)\left(e^{-(\delta\omega_n-\omega_n\sqrt{\delta^2-1})t} - e^{-(\delta\omega_n+\omega_n\sqrt{\delta^2-1})t}\right)$

In this chapter, let us discuss the time domain specifications of the second order system. The step response of the second order system for the underdamped case is shown in the following figure.



All the time domain specifications are represented in this figure. The response up to the settling time is known as transient response and the response after the settling time is known as steady state response.

Delay Time

It is the time required for the response to reach **half of its final value** from the zero instant. It is denoted by tdtd.

Considerthestepresponseofthesecondordersystemfort≥0,when'δ'liesbetweenzero and one.

$$c(t) = 1 - \left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t + \theta)$$

The final value of the step response is one.

Therefore, at $t = t_d$, the value of the step response will be 0.5. Substitute, these values in the above equation.

$$c(t_d) = 0.5 = 1 - \left(\frac{e^{-\delta\omega_n t_d}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t_d + \theta)$$

$$\Rightarrow \left(\frac{e^{-\delta\omega_n t_d}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t_d + \theta) = 0.5$$

By using linear approximation, you will get the **delay time t_d** as

$$t_d = \frac{1 + 0.7\delta}{\omega_n}$$

RiseTime

It is the time requiredfor the response to rise from **0% to 100%of its final value**.This is applicable for the **under-damped systems**. For the over-damped systems, consider the duration from 10% to 90% of the final value. Rise time is denoted by $\mathbf{t_r}$.

Att $=t_1=0$,c(t)=0.

We know that the final value of the step response is one.Therefore, at t=t2, the value of step response is one. Substitute, these values in the following equation.

$$c(t) = 1 - \left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t + \theta)$$

$$c(t_2) = 1 = 1 - \left(\frac{e^{-\delta\omega_n t_2}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t_2 + \theta)$$

$$\Rightarrow \left(\frac{e^{-\delta\omega_n t_2}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t_2 + \theta) = 0$$

$$\Rightarrow \sin(\omega_d t_2 + \theta) = 0$$

$$\Rightarrow \omega_d t_2 + \theta = \pi$$

$$\Rightarrow t_2 = \frac{\pi - \theta}{\omega_d}$$

Substitute t_1 and t_2 values in the following equation of **rise time**,

$$t_r = t_2 - t_1$$

$$\therefore\ t_r = \frac{\pi - \theta}{\omega_d}$$

Fromaboveequation,weconcludethattherisetime$t_r$andthedampedfrequency$\omega_d$are inversely proportional to each other.

PeakTime

Itisthetimerequiredfortheresponseto reachthe **peakvalue** forthefirsttime.Itis denoted by t_p. At $t=t_p$ the first derivate of the response is zero.

Weknowthestepresponseofsecondordersystemforunder-dampedcaseis

$$c(t) = 1 - \left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t + \theta)$$

Differentiate $c(t)$ with respect to 't'.

$$\frac{\mathrm{d}c(t)}{\mathrm{d}t} = -\left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\omega_d \cos(\omega_d t + \theta) - \left(\frac{-\delta\omega_n e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t + \theta)$$

$$c(t) = 1 - \left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}} \right) \sin(\omega_d t + \theta)$$

Differentiate $c(t)$ with respect to 't'.

$$\frac{\mathrm{d}c(t)}{\mathrm{d}t} = -\left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}} \right) \omega_d \cos(\omega_d t + \theta) - \left(\frac{-\delta\omega_n e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}} \right) \sin(\omega_d t + \theta)$$

Substitute, $t = t_p$ and $\frac{\mathrm{d}c(t)}{\mathrm{d}t} = 0$ in the above equation.

$$0 = -\left(\frac{e^{-\delta\omega_n t_p}}{\sqrt{1-\delta^2}} \right) [\omega_d \cos(\omega_d t_p + \theta) - \delta\omega_n \sin(\omega_d t_p + \theta)]$$

$$\Rightarrow \omega_n \sqrt{1-\delta^2} \cos(\omega_d t_p + \theta) - \delta\omega_n \sin(\omega_d t_p + \theta) = 0$$

$$\Rightarrow \sqrt{1-\delta^2} \cos(\omega_d t_p + \theta) - \delta \sin(\omega_d t_p + \theta) = 0$$

$$\Rightarrow \sin(\theta) \cos(\omega_d t_p + \theta) - \cos(\theta) \sin(\omega_d t_p + \theta) = 0$$

$$\Rightarrow \sin(\theta - \omega_d t_p - \theta) = 0$$

$$\Rightarrow sin(-\omega_d t_p) = 0 \Rightarrow -\sin(\omega_d t_p) = 0 \Rightarrow sin(\omega_d t_p) = 0$$

$$\Rightarrow \omega_d t_p = \pi$$

$$\Rightarrow t_p = \frac{\pi}{\omega_d}$$

Fromtheaboveequation,wecanconcludethatthepeaktime $t_p$andthedamped frequency ω_d are inversely proportional to eachother.

Peak Overshoot

Peakovershoot $\mathbf{M_p}$isdefinedasthedeviationoftheresponseatpeaktimefromthefinal value of response. It is also called the **maximum overshoot**.

Mathematically,wecanwriteit as

$$Mp=c(t_p)-c(\infty)$$

Where,$c(t_p)$isthepeakvalueoftheresponse,$c(\infty)$isthefinal(steady state)valueofthe response.

Att=tp,theresponsec(t)is -

$$c(t_p) = 1 - \left(\frac{e^{-\delta\omega_n t_p}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t_p + \theta)$$

Substitute, $t_p = \frac{\pi}{\omega_d}$ in the right hand side of the above equation.

$$c(t_P) = 1 - \left(\frac{e^{-\delta\omega_n\left(\frac{\pi}{\omega_d}\right)}}{\sqrt{1-\delta^2}}\right)\sin\left(\omega_d\left(\frac{\pi}{\omega_d}\right) + \theta\right)$$

$$\Rightarrow c(t_p) = 1 - \left(\frac{e^{-\left(\frac{\delta\pi}{\sqrt{1-\delta^2}}\right)}}{\sqrt{1-\delta^2}}\right)(-\sin(\theta))$$

We know that

$$\sin(\theta) = \sqrt{1-\delta^2}$$

So, we will get $c(t_p)$ as

$$c(t_p) = 1 + e^{-\left(\frac{\delta\pi}{\sqrt{1-\delta^2}}\right)}$$

Substitute the values of $c(t_p)$ and $c(\infty)$ in the peak overshoot equation.

$$M_p = 1 + e^{-\left(\frac{\delta\pi}{\sqrt{1-\delta^2}}\right)} - 1$$

$$\Rightarrow M_p = e^{-\left(\frac{\delta\pi}{\sqrt{1-\delta^2}}\right)}$$

Percentage of peak overshoot % M_p can be calculated by using this formula.

$$\%M_p = \frac{M_p}{c(\infty)} \times 100\%$$

From the above equation, we can conclude that the percentage of peak overshoot %Mp will decrease if the damping ratio δ increases.

Settlingtime

It is the time required for the response to reach the steady state and stay within the specified tolerance bands around the final value. In general, the tolerance bands are 2% and 5%. The settling time is denoted by ts.

Thesettling timefor5%tolerancebandis–

$$t_s = \frac{3}{\delta\omega_n} = 3\tau$$

Thesettling timefor2%tolerancebandis–

$$t_s = \frac{4}{\delta\omega_n} = 4\tau$$

Where,τisthetimeconstantandisequal to$1/\delta\omega_n$.

- Both the settling time ts and the time constant τ are inversely proportional to the damping ratio δ.
- Both thesettlingtimetsandthetimeconstantτ areindependentofthesystemgain. That means even the system gain changes, the settling time ts and time constant τ will never change.

Example

Letusnowfindthetimedomain specificationsofacontrolsystemhavingtheclosedloop transfer function when the unit step signal is applied as an input to this control system.

Weknowthatthestandardformofthetransferfunctionofthesecondorderclosedloop control system as

$$\frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}$$

Byequatingthesetwotransferfunctions,wewillgettheun-dampednaturalfrequency$\omega_n$as 2 rad/sec and the damping ratio δ as 0.5.

Weknowtheformulafordamped frequency $\omega_d$as

$$\omega_d = \omega_n\sqrt{1-\delta^2}$$

$$\omega_d = \omega_n \sqrt{1-\delta^2}$$

Substitute, ω_n and δ values in the above formula.

$$\Rightarrow \omega_d = 2\sqrt{1-(0.5)^2}$$

$$\Rightarrow \omega_d = 1.732\, rad/sec$$

Substitute, δ value in following relation

$$\theta = \cos^{-1}\delta$$

$$\Rightarrow \theta = \cos^{-1}(0.5) = \frac{\pi}{3}\, rad$$

Substitutetheabovenecessaryvaluesintheformulaofeachtimedomainspecification and simplify in order to get the values of time domain specifications for given transfer function.

Thefollowingtableshowstheformulaeoftimedomainspecifications,substitutionof necessary values and the final values

Time domain specification	Formula	Substitution of values in Formula	Final value
Delay time	$t_d = \frac{1+0.7\delta}{\omega_n}$	$t_d = \frac{1+0.7(0.5)}{2}$	t_d =0.675 sec
Rise time	$t_r = \frac{\pi-\theta}{\omega_d}$	$t_r = \frac{\pi-(\frac{\pi}{3})}{1.732}$	t_r =1.207 sec
Peak time	$t_p = \frac{\pi}{\omega_d}$	$t_p = \frac{\pi}{1.732}$	t_p =1.813 sec
% Peak overshoot	$\%M_p = \left(e^{-\left(\frac{\delta\pi}{\sqrt{1-\delta^2}}\right)}\right) \times 100\%$	$\%M_p = \left(e^{-\left(\frac{0.5\pi}{\sqrt{1-(0.5)^2}}\right)}\right) \times 100\%$	$\% M_p$ =16.32%
Settling time for 2% tolerance band	$t_s = \frac{4}{\delta\omega_n}$	$t_S = \frac{4}{(0.5)(2)}$	t_s =4 sec

Thedeviation oftheoutput of control system from desired responseduring steady state is known as **steady stateerror**. It isrepresentedas e_{ss}. Wecan find steady stateerror using the final value theorem as follows.

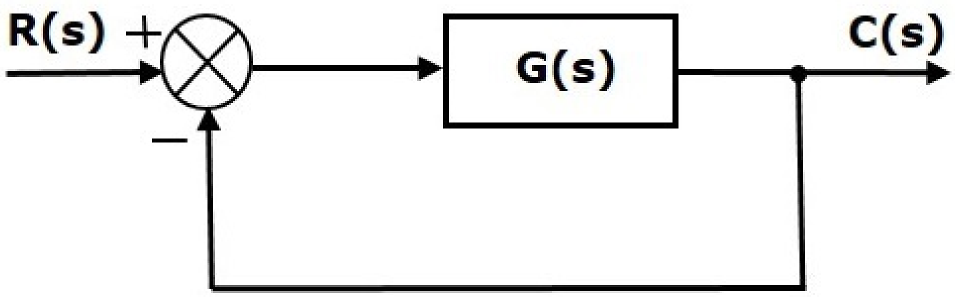
$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} E(s)$$

Where,

E(s)istheLaplacetransformoftheerror signal,e(t)
Let us discuss how to find steady state errors for unity feedback and non-unity feedback control systems one by one.

SteadyStateErrorsforUnityFeedbackSystems

Considerthefollowingblockdiagramofclosedloopcontrolsystem,whichishavingunity negative feedback.



Where,

- R(s) is the Laplace transform of the reference Input signal $r(t)$
- C(s) is the Laplace transform of the output signal $c(t)$

We know the transfer function of the unity negative feedback closed loop control system as

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

$$\Rightarrow C(s) = \frac{R(s)G(s)}{1+G(s)}$$

The output of the summing point is -

$$E(s) = R(s) - C(s)$$

Substitute $C(s)$ value in the above equation.

$$E(s) = R(s) - \frac{R(s)G(s)}{1+G(s)}$$

$$\Rightarrow E(s) = \frac{R(s) + R(s)G(s) - R(s)G(s)}{1+G(s)}$$

$$\Rightarrow E(s) = \frac{R(s)}{1+G(s)}$$

Substitute $E(s)$ value in the steady state error formula

$$e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1+G(s)}$$

Thefollowingtableshowsthesteadystateerrorsandtheerrorconstantsforstandard input signals like unit step, unit ramp & unit parabolic signals.

Input signal	Steady state error e_{ss}	Error constant
unit step signal	$\frac{1}{1+k_p}$	$K_p = \lim_{s \to 0} G(s)$
unit ramp signal	$\frac{1}{K_v}$	$K_v = \lim_{s \to 0} sG(s)$
unit parabolic signal	$\frac{1}{K_a}$	$K_a = \lim_{s \to 0} s^2G(s)$

Where,Kp,KvandKaarepositionerrorconstant,velocityerrorconstantandacceleration error constant respectively.

Note – If any of the above input signals has the amplitude other than unity, then multiply corresponding steady state error with that amplitude.

Note – We can't define the steady state error for the unit impulse signal because, it exists only at origin. So, we can't compare the impulse response with the unit impulse input as **t** denotes infinity

Example

Let us find the steady state error for an input signal $r(t) = \left(5 + 2t + \frac{t^2}{2}\right) u(t)$ of unity negative feedback control system with $G(s) = \frac{5(s+4)}{s^2(s+1)(s+20)}$

The given input signal is a combination of three signals step, ramp and parabolic. The following table shows the error constants and steady state error values for these three signals.

Input signal	Error constant	Steady state error
$r_1(t) = 5u(t)$	$K_p = \lim_{s\to 0} G(s) = \infty$	$e_{ss1} = \frac{5}{1+k_p} = 0$
$r_2(t) = 2tu(t)$	$K_v = \lim_{s\to 0} sG(s) = \infty$	$e_{ss2} = \frac{2}{K_v} = 0$
$r_3(t) = \frac{t^2}{2}u(t)$	$K_a = \lim_{s\to 0} s^2G(s) = 1$	$e_{ss3} = \frac{1}{k_a} = 1$

Wewill gettheoverallsteadystateerror,byadding theabovethreesteadystateerrors.

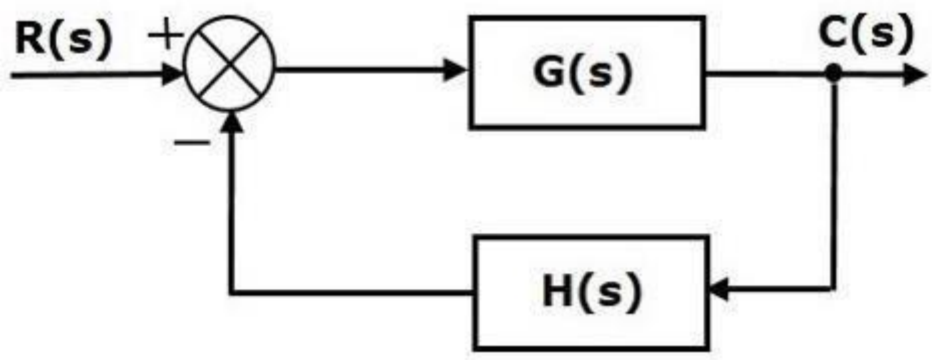
$$e_{ss} = e_{ss1} + e_{ss2} + e_{ss3}$$

$$\Rightarrow e_{ss} = 0+0+1 = 1 \Rightarrow e_{ss} = 0+0+1 = 1$$

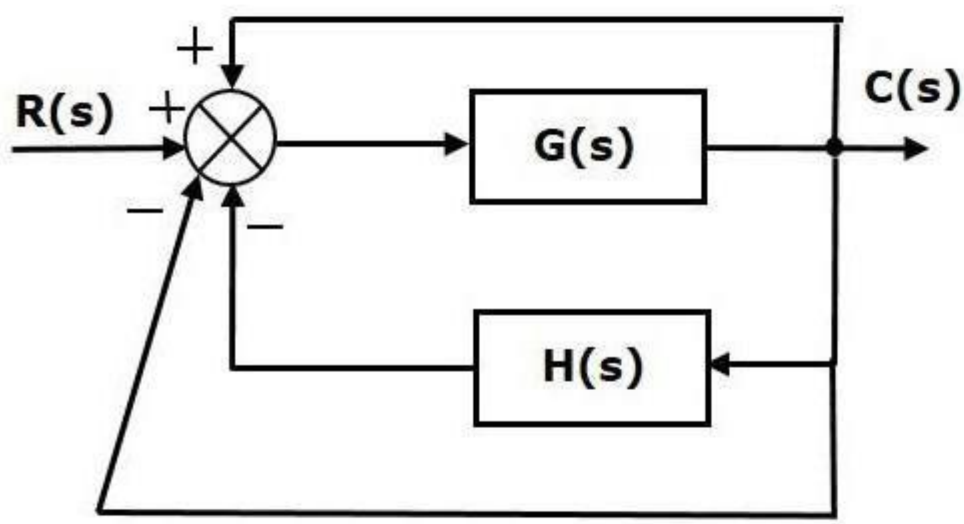
Therefore,wegotthesteadystateerrore_{ss} as**1** forthisexample.

SteadyStateErrorsforNon-UnityFeedbackSystems

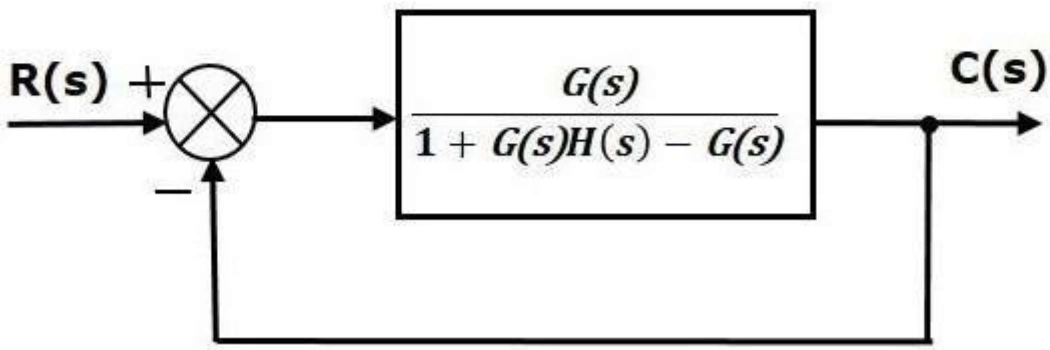
Considerthefollowingblockdiagramofclosedloopcontrolsystem,whichishavingnon unity negative feedback.



We can find the steady state errors only for the unity feedback systems. So, we have to convert the non-unity feedback system into unity feedback system. For this, include one unity positive feedback path and one unity negative feedback path in the above block diagram. The new block diagram looks like as shown below.



Simplify the above block diagram by keeping the unity negative feedback as it is. The following is the simplified block diagram



This block diagram resembles the block diagram of the unity negative feedback closedloop control system. Here, the single block is having the transferfunctionG(s)/[1+G(s)H(s)−G(s)] instead of G(s).You can now calculate the steady state errors by using steady state error formula given for the unity negative feedback systems.

Note − It is meaningless to find the steady state errors for unstable closed loop systems. So, we have to calculate the steady state errors only for closed loop stable systems. This means we need to check whether the control system is stable or not before finding the steady state errors. In the next chapter, we will discuss the concepts-relatedstability.

The various types of controllers are used to improve the performance of control systems. In this chapter, we will discuss the basic controllers such as the proportional, the derivative and the integral controllers.

ProportionalController

Theproportionalcontrollerproducesanoutput,whichisproportionaltoerror signal.

$$u(t) \propto e(t)$$

$$\Rightarrow u(t) = K_P e(t)$$

Apply Laplace transform on both the sides -

$$U(s) = K_P E(s)$$

$$\frac{U(s)}{E(s)} = K_P$$

Therefore,thetransferfunctionoftheproportionalcontrollerisKPKP.

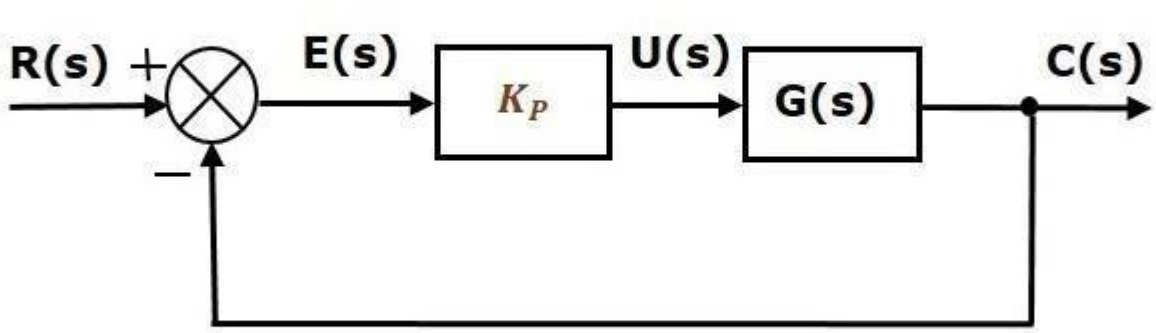
Where,

U(s)istheLaplacetransformoftheactuatingsignalu(t) E(s) is

the Laplace transform of the error signal e(t)

$K_P$istheproportionality constant

Theblockdiagramoftheunitynegativefeedbackclosedloopcontrolsystemalongwiththe proportional controller is shown in the following figure.



DerivativeController

Thederivativecontroller producesanoutput,whichisderivativeoftheerror signal.

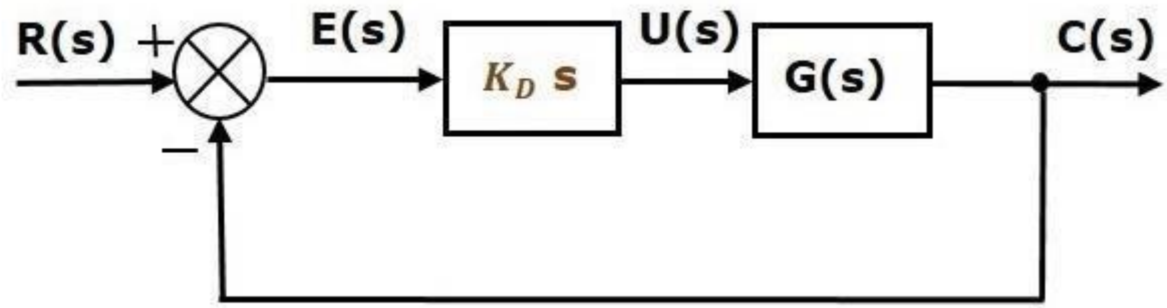
$$u(t) = K_D \frac{\mathrm{d}e(t)}{\mathrm{d}t}$$

Apply Laplace transform on both sides.

$$U(s) = K_D s E(s)$$

$$\frac{U(s)}{E(s)} = K_D s$$

Therefore,thetransferfunctionofthederivativecontrolleris $K_D s$.
Where, KD is the derivative constant.
Theblockdiagramoftheunitynegativefeedbackclosedloopcontrolsystemalongwith the derivative controller is shown in the following figure.



Thederivativecontrollerisusedto maketheunstablecontrolsysteminto astable one.

IntegralController

Theintegralcontrollerproducesanoutput,whichisintegraloftheerrorsignal.

$$u(t) = K_I \int e(t)dt$$

Apply Laplace transform on both the sides -

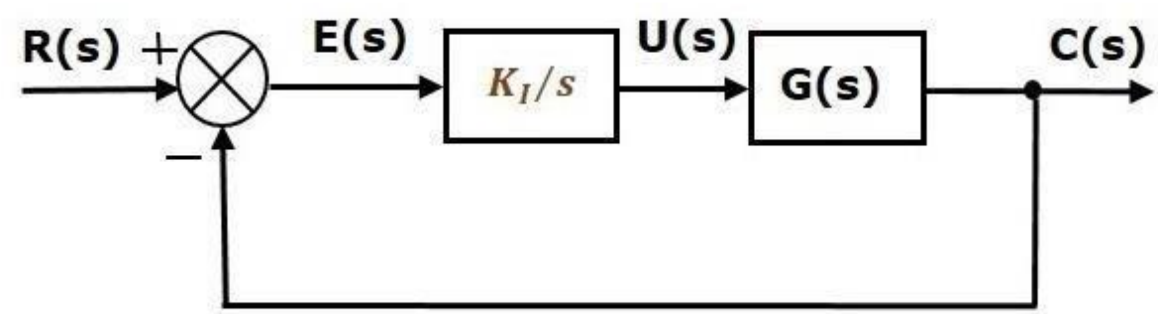
$$U(s) = \frac{K_I E(s)}{s}$$

$$\frac{U(s)}{E(s)} = \frac{K_I}{s}$$

Therefore, the transfer function of the integral controller is $\frac{K_I}{s}$.

Where,KIKIistheintegral constant.

Theblockdiagramoftheunitynegativefeedbackclosedloopcontrolsystemalongwith the integral controller is shown in the following figure.



Theintegralcontrollerisusedtodecreasethesteadystate error.

Let us now discuss about the combination of basic controllers.

ProportionalDerivative(PD)Controller

Theproportionalderivativecontroller producesanoutput,whichisthecombinationof the outputs of proportional and derivative controllers.

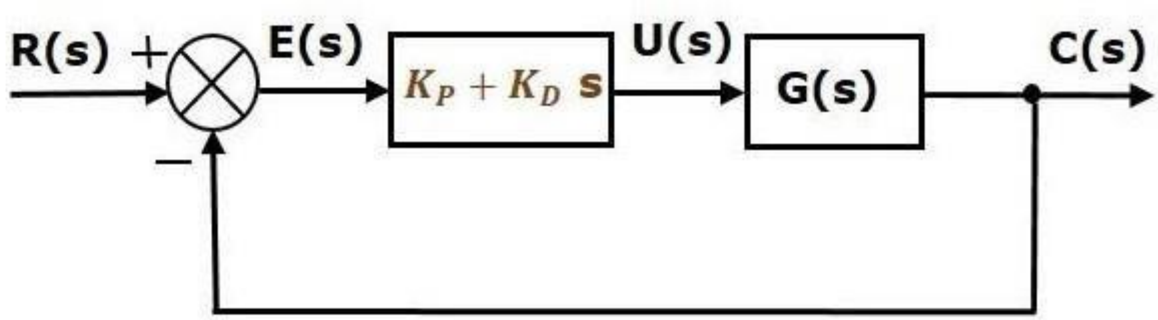
$$u(t) = K_P e(t) + K_D \frac{\mathrm{d}e(t)}{\mathrm{d}t}$$

Apply Laplace transform on both sides -

$$U(s) = (K_P + K_D s)E(s)$$

$$\frac{U(s)}{E(s)} = K_P + K_D s$$

Therefore,thetransferfunctionoftheproportionalderivativecontrollerísK_P+K_Ds. Theblockdiagramoftheunitynegativefeedbackclosedloop controlsystemalongwiththe proportional derivative controller is shown in the following figure.



Theproportionalderivativecontrollerisusedtoimprovethestabilityofcontrolsystem without affecting the steady state error.

ProportionalIntegral(PI)Controller

Theproportionalintegralcontrollerproducesanoutput,whichisthecombinationof outputs of the proportional and integral controllers.

$$u(t) = K_P e(t) + K_I \int e(t) dt$$

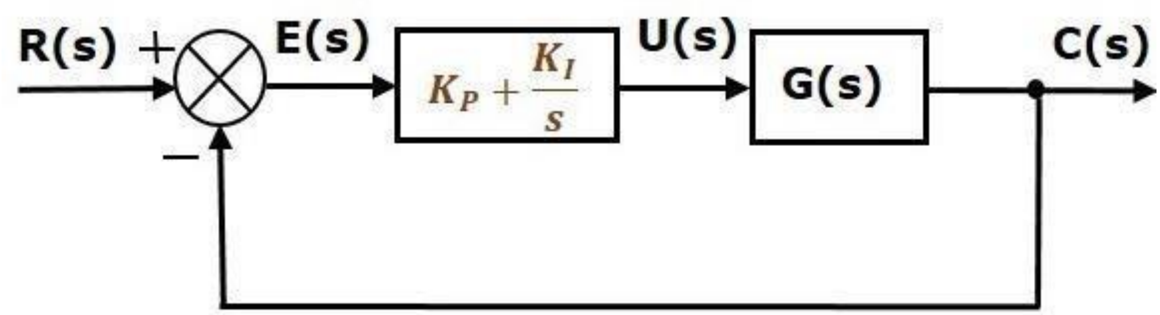
Apply Laplace transform on both sides -

$$U(s) = \left(K_P + \frac{K_I}{s}\right) E(s)$$

$$\frac{U(s)}{E(s)} = K_P + \frac{K_I}{s}$$

Therefore, the transfer function of proportional integral controller is $K_P + \frac{K_I}{s}$.

Theblockdiagramoftheunitynegativefeedbackclosedloopcontrolsystem alongwiththe proportional integral controller is shown in the following figure.



Theproportionalintegralcontrollerisusedtodecreasethesteadystateerrorwithout affecting the stability of the control system.

ProportionalIntegralDerivative(PID)Controller

Theproportionalintegralderivativecontrollerproducesanoutput,whichisthe combination of the outputs of proportional, integral and derivative controllers.

$$u(t) = K_P e(t) + K_I \int e(t)dt + K_D \frac{\mathrm{d}e(t)}{\mathrm{d}t}$$

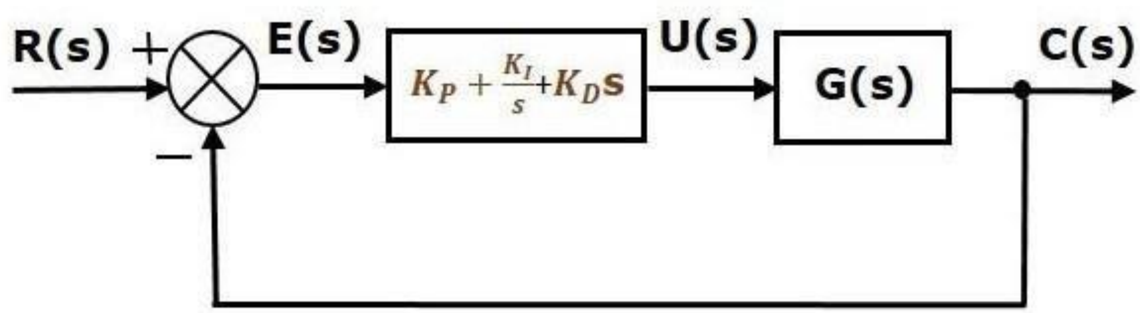
Apply Laplace transform on both sides -

$$U(s) = \left(K_P + \frac{K_I}{s} + K_D s\right) E(s)$$

$$\frac{U(s)}{E(s)} = K_P + \frac{K_I}{s} + K_D s$$

Therefore, the transfer function of the proportional integral derivative controller is $K_P + \frac{K_I}{s} + K_D s$.

Theblockdiagramoftheunitynegativefeedbackclosedloop controlsystemalongwiththe proportional integral derivative controller is shown in the following figure.



UNIT-IV

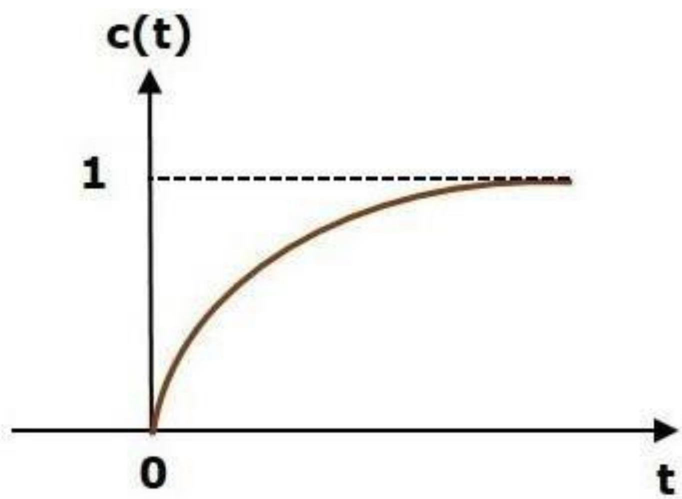
STABILITYANALYSISINS-DOMAIN

Stabilityisanimportantconcept.Inthischapter,letusdiscussthestabilityof system and types of systems based on stability.

WhatisStability?

Asystemissaidtobestable,ifitsoutputisundercontrol.Otherwise,itissaidtobe unstable. A **stable system** produces a bounded output for a given bounded input.

Thefollowingfigureshowstheresponseofastablesystem.



This is the response of firstorder control system forunit step input. This response has the valuesbetween0 and 1.So,itis boundedoutput.Weknow thattheunitstepsignal has the value of one for all positive values of **t** including zero. So, it is bounded input. Therefore, the first order control system is stable since both the input and the output are bounded.

TypesofSystemsbasedon Stability

Wecanclassifythesystems basedonstabilityasfollows.

- Absolutely stable system
- Conditionallystablesystem
- Marginally stable system

AbsolutelyStableSystem

Ifthesystem isstablefor all therangeof system component values,thenitisknownas the **absolutely stable system**. The open loop control system is absolutely stable if all the poles of the open loop transfer function present in left half of **'s' plane**. Similarly, the closed loop control system is absolutely stable if all the poles of the closed loop transfer function present in the left half of the 's' plane.

ConditionallyStableSystem

If the system is stable for acertainrange ofsystem component values,thenitis known as **conditionally stable system**.

MarginallyStableSystem

Ifthe system is stable byproducing anoutputsignal with constantamplitudeand constant frequency of oscillations for bounded input, then it is known as **marginally stablesystem**. The open loop control system is marginally stable if any two poles of the open loop transfer function is present on the imaginary axis. Similarly, the closed loop control system ismarginally stable if any two poles of the closed loop transfer function is present on the imaginary axis.

nthischapter,letusdiscussthestabilityanalysisinthe **'s'** domainusing theRouthHurwitz stability criterion. In this criterion, we require the characteristic equation to find the stability of the closed loop control systems.

Routh-HurwitzStabilityCriterion

Routh-Hurwitz stability criterion is having one necessary condition and one sufficient condition for stability. If any control system doesn't satisfy the necessary condition, then we can say that the control system is unstable. But, if the control system satisfies the necessarycondition,thenitmayor maynotbe stable. So,thesufficientconditionishelpful for knowing whether the control system is stable or not.

NecessaryConditionforRouth-HurwitzStability

The necessary condition is that the coefficients of the characteristic polynomial should be positive. This implies that all the roots of the characteristic equation should have negative real parts.

Considerthecharacteristicequationoftheorder'n'is-

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1} s^1 + a_n s^0 = 0$$

Note that, there should not be any term missing in the **nth** order characteristic equation. This means that the **nth** order characteristic equation should not have any coefficient that is of zero value.

SufficientConditionforRouth-HurwitzStability

The sufficient condition is that all the elements of the first column of the Routh array should have the same sign. This means that all the elements of the first column of the Routh array should be either positive or negative.

RouthArray Method

If all the roots of the characteristic equation exist to the left half of the 's' plane, then the controlsystemisstable. Ifatleastonerootofthecharacteristicequationexists to theright half ofthe 's' plane, thenthe control system is unstable. So, we have to findtheroots ofthe characteristic equation to know whether the control system is stable or unstable. But, it is difficult to find the roots of the characteristic equation as orderincreases.

So, to overcome this problem there we have the **Routh array method**. In this method, there is no need to calculate the roots of the characteristic equation. First formulate the Routh table and find the number of the sign changes in the first columnof the Routh table. The number of sign changes in the first column of the Routh table gives the number of roots of characteristic equation that exist in the right half of the 's' plane and the control system is unstable.

FollowthisprocedureforformingtheRouth table.

- Fill the first two rows of the Routh array with the coefficients of the characteristic polynomial as mentioned in the table below. Start with the coefficient of sn and continue up to the coefficient of s0.
- Fill the remaining rows of the Routh array with the elements as mentioned in the table below. Continue this process till you get the first column element of **row** s0s0 is an. Here, an is the coefficient of s0 in the characteristic polynomial.

Note – If any row elements of the Routh table have some common factor, then you can divide the row elements with that factor for the simplification will be easy.

ThefollowingtableshowstheRouth arrayofthenthordercharacteristic polynomial.

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1} s^1 + a_n s^0$$

s^n	a_0	a_2	a_4	a_6	$\ldots$	$\ldots$
s^{n-1}	a_1	a_3	a_5	a_7	$\ldots$	$\ldots$
s^{n-2}	$b_1 = \frac{a_1 a_2 - a_3 a_0}{a_1}$	$b_2 = \frac{a_1 a_4 - a_5 a_0}{a_1}$	$b_3 = \frac{a_1 a_6 - a_7 a_0}{a_1}$	$\ldots$	$\ldots$	$\ldots$
s^{n-3}	$c_1 = \frac{b_1 a_3 - b_2 a_1}{b_1}$	$c_2 = \frac{b_1 a_5 5 - b_3 a_1}{b_1}$	$\vdots$			
$\vdots$	$\vdots$	$\vdots$	$\vdots$			
s^1	$\vdots$	$\vdots$				
s^0	a_n					

Example

Letusfindthestabilityofthecontrolsystemhavingcharacteristicequation,

$$s^4 + 3s^3 + 3s^2 + 2s + 1 = 0$$

Step1–VerifythenecessaryconditionfortheRouth-Hurwitz

stability. All the coefficients of the characteristic polynomial, $s^4 + 3s^3 + 3s^2 + 2s + 1$ arepositive.So,thecontrolsystemsatisfiesthenecessary condition.

Step2–Form theRoutharrayforthegivencharacteristicpolynomial.

s^4	1	3	1
s^3	3	2	
s^2	$\frac{(3\times3)-(2\times1)}{3}=\frac{7}{3}$	$\frac{(3\times1)-(0\times1)}{3}=\frac{3}{3}$ $=1$	
s^1	$\frac{\left(\frac{7}{3}\times2\right)-(1\times3)}{\frac{7}{3}}$ $=\frac{5}{7}$		
s^0	1		

Step3–VerifythesufficientconditionfortheRouth-Hurwitz stability.

AlltheelementsofthefirstcolumnoftheRoutharrayarepositive.Thereisno sign change in the first column of the Routh array. So, the control system is stable.

SpecialCasesofRouthArray

Wemaycomeacrosstwotypesofsituations,whileformingtheRouthtable.Itisdifficult to complete the Routh table from these two situations.

Thetwospecialcasesare–

- The first element of any row of the Routh'sarray iszero.
- AlltheelementsofanyrowoftheRouth'sarrayarezero.

Letusnow discusshowto overcomethedifficultyinthesetwocases,onebyone.

FirstElementofanyrowof theRouth'sarrayiszero

If any row of the Routh's array contains only the first element as zero and at least one of the remaining elements have non-zero value, then replace the first element with a small positive integer, ϵ. And then continue the process of completing the Routh's table. Now, find the number of sign changes in the first column of the Routh's table by substituting ϵϵ tends to zero.

Example

Letusfindthestabilityofthecontrolsystemhavingcharacteristicequation,

$$s^4+2s^3+s^2+2s+1=0$$

Step1−VerifythenecessaryconditionfortheRouth-Hurwitz

stability. All the coefficients of the characteristic polynomial,

$$s^4+2s^3+s^2+2s+1=0$$

arepositive.So,thecontrolsystemsatisfiedthe necessarycondition.

Step2−Form theRoutharrayforthegivencharacteristicpolynomial.

s^4	1	1	1
s^3	$\cancel{2}\ 1$	$\cancel{2}\ 1$	
s^2	$\frac{(1\times1)-(1\times1)}{1}=0$	$\frac{(1\times1)-(0\times1)}{1}=1$	
s^1			
s^0			

Therows^{3}elementshave2asthecommonfactor.So,alltheseelementsare dividedby2.
Specialcase (i) − Only thefirst element of rows^{2}iszero.So,replaceit by ϵ and continue the process of completing the Routh table.

s^4	1	1	1
s^3	1	1	
s^2	ϵ	1	
s^1	$\frac{(\epsilon\times1)-(1\times1)}{\epsilon}=\frac{\epsilon-1}{\epsilon}$		
s^0	1		

Step3–Verifythesufficientconditionforthe Routh-Hurwitz

stability.Asϵtendstozero,theRouth tablebecomeslikethis.

s^4	1	1	1
s^3	1	1	
s^2	0	1	
s^1	$-\infty$		
s^0	1		

Therearetwosignchangesinthefirstcolumnof Routh table.Hence,thecontrolsystemis unstable.

AlltheElementsofanyrowoftheRouth'sarrayarezero

Inthiscase,followthesetwosteps–

- Writetheauxilaryequation,A(s)oftherow,which isjustabovetherow ofzeros.
- Differentiatetheauxiliaryequation,A(s)withrespectto s.Filltherowofzeroswith these coefficients.

Example

Letusfindthestabilityofthecontrolsystemhavingcharacteristicequation,

$$s^5 + 3s^4 + s^3 + 3s^2 + s + 3 = 0$$

Step1–Verifythenecessaryconditionfor theRouth-Hurwitz stability.

Allthecoefficientsofthegivencharacteristicpolynomialarepositive.So,thecontrol system satisfied the necessary condition.

Step2–Form theRoutharrayforthegivencharacteristicpolynomial.

s^5	1	1	1
s^4	$\cancel{3}$ 1	$\cancel{3}$ 1	$\cancel{3}$ 1
s^3	$\frac{(1\times1)-(1\times1)}{1} = 0$	$\frac{(1\times1)-(1\times1)}{1} = 0$	
s^2			
s^1			
s^0			

The row s^4 elements have the common factor of 3. So, all these elements are divided by 3.

Special case (ii) – All the elements of row s^3 are zero. So, write the auxiliary equation, A(s) of the row s^4.

$$A(s) = s^4 + s^2 + 1$$

$$\frac{\mathrm{d}A(s)}{\mathrm{d}s} = 4s^3 + 2s$$

Place these coefficients in row s^3.

s^5	1	1	1
s^4	1	1	1
s^3	$\cancel{4}$ 2	$\cancel{2}$ 1	
s^2	$\frac{(2\times1)-(1\times1)}{2} = 0.5$	$\frac{(2\times1)-(0\times1)}{2} = 1$	
s^1	$\frac{(0.5\times1)-(1\times2)}{0.5} = \frac{-1.5}{0.5} = -3$		
s^0	1		

Step3–VerifythesufficientconditionfortheRouth-Hurwitz stability.

There are two sign changes in the first columnof Routh table. Hence, the control system is unstable.

In theRouth-Hurwitz stability criterion, wecanknowwhetherthe closed looppolesarein on left half of the ‘s’ plane or on the right half of the ‘s’ plane or on an imaginary axis. So,we can’t find the nature of the control system. To overcome this limitation, there is a technique known as the root locus.

RootlocusTechnique

In the root locus diagram, we can observe the path of the closed loop poles. Hence, we can identify the nature of the control system. In this technique, we will use an open loop transfer function to know the stability of the closed loop control system.

BasicsofRootLocus

TheRootlocusisthelocusoftheroots ofthecharacteristicequationbyvaryingsystem gain K from zero to infinity.

Weknowthat,thecharacteristicequationoftheclosedloopcontrolsystemis

$$1 + G(s)H(s) = 0$$

We can represent $G(s)H(s)$ as

$$G(s)H(s) = K\frac{N(s)}{D(s)}$$

Where,

- K represents the multiplying factor
- N(s) represents the numerator term having (factored) n^{th} order polynomial of 's'.
- D(s) represents the denominator term having (factored) m^{th} order polynomial of 's'.

Substitute, $G(s)H(s)$ value in the characteristic equation.

$$1 + k\frac{N(s)}{D(s)} = 0$$

$$\Rightarrow D(s) + KN(s) = 0$$

Case 1 − K = 0

If $K = 0$, then $D(s) = 0$.

That means, the closed loop poles are equal to open loop poles when K is zero.

Case 2 − K = ∞

Re-write the above characteristic equation as

$$K\left(\frac{1}{K} + \frac{N(s)}{D(s)}\right) = 0 \Rightarrow \frac{1}{K} + \frac{N(s)}{D(s)} = 0$$

Substitute, $K = \infty$ in the above equation.

$$\frac{1}{\infty} + \frac{N(s)}{D(s)} = 0 \Rightarrow \frac{N(s)}{D(s)} = 0 \Rightarrow N(s) = 0$$

If $K = \infty$, then $N(s) = 0$. It means the closed loop poles are equal to the open loop zeros when K is infinity.

From above two cases, we can conclude that the root locus branches start at open loop poles and end at open loop zeros.

AngleConditionandMagnitude Condition

Thepointson theroot locusbranchessatisfy theanglecondition. So,theanglecondition is used to know whether thepointexist on rootlocus branch or not. We can findthe valueof K for the points on the root locus branches by using magnitude condition. So, we can use the magnitude condition for the points, and this satisfies the angle condition.

Characteristicequationofclosedloopcontrolsystemis

$$1 + G(s)H(s) = 0$$
$$\Rightarrow G(s)H(s) = -1 + j0$$

The **phase angle** of $G(s)H(s)$ is

$$\angle G(s)H(s) = \tan^{-1}\left(\frac{0}{-1}\right) = (2n+1)\pi$$

The**anglecondition** isthepointatwhichtheangleoftheopenlooptransferfunction isan odd multiple of 180^0.

MagnitudeofG(s)H(s)G(s)H(s)is–

$$|G(s)H(s)| = \sqrt{(-1)^2 + 0^2} = 1$$

The magnitude condition is that the point (which satisfied the angle condition) at which the magnitude of the open loop transfer function is one.

The **root locus** is a graphical representation in s-domain and it is symmetrical about the real axis. Because the open loop poles and zeros exist in the s-domain having the values either asreal or ascomplexconjugatepairs.Inthischapter,letusdiscusshow to construct (draw) the root locus.

Rulesfor ConstructionofRootLocus

Followtheserulesforconstructingarootlocus.

Rule1–Locatetheopenlooppolesandzerosinthe‘s’ plane.

Rule 2– Findthenumber ofrootlocusbranches.

We know that the root locus branches start at the open loop poles and end at open loop zeros. So, the number of root locus branches **N** is equal to the number of finite open loop poles **P** or the number of finite open loop zeros **Z**, whichever is greater.

Mathematically,wecanwritethenumber ofrootlocusbranches**N**as

N=PifP≥Z N=Z
if P<Z
Rule3 –Identifyanddrawthe**realaxisrootlocusbranches**.

If the angle of the open loop transfer function at a point is an odd multiple of 180^0, then that point is on the root locus. If odd number of the open loop poles and zeros exist to the left side of a point on the real axis, then that point is on the root locus branch. Therefore, the branch of points which satisfies this condition is the real axis of the root locus branch.

Rule4–Findthecentroid andtheangleofasymptotes.

- If P=Z, then all the root locus branches start at finite open loop poles and end at finite open loop zeros.
- If P>Z, then Z number of root locus branches start at finite open loop poles and end at finite open loopzerosandP−Z number ofroot locusbranchesstart at finite open loop poles and end at infinite open loop zeros.

- If P<Z , then P number ofroot locus branches start at finite open looppoles and end at finite open loop zeros and Z−P number of root locus branches start at infinite open loop poles and end at finite open loop zeros.

So, some of the root locus branches approach infinity, when P≠Z. Asymptotes give the direction of these root locus branches. The intersection point of asymptotes on the real axis is known as **centroid**.

Wecancalculatethe**centroidα** byusing thisformula,

$$\alpha = \frac{\sum Real\ part\ of\ finite\ open\ loop\ poles - \sum Real\ part\ of\ finite\ open\ loop\ zeros}{P-Z}$$

The formula for the angle of **asymptotes θ** is

$$\theta = \frac{(2q+1)180^0}{P-Z}$$

Where,

$$q = 0, 1, 2, \ldots, (P-Z)-1$$

Rule5–Findtheintersectionpointsofrootlocusbrancheswithanimaginary axis.

Wecancalculatethepointatwhichtherootlocusbranchintersectstheimaginaryaxisandthe value of **K** at that point by using the Routh array method and special **case (ii)**.

- Ifall elementsofanyrow oftheRouth arrayarezero,thentherootlocusbranch intersects the imaginary axis and vice-versa.
- Identifytherowinsuchawaythatifwemakethefirstelementaszero,thenthe elements of the entire row are zero. Find the value of **K** for thiscombination.
- Substitutethis**K**valueintheauxiliaryequation.Youwillgettheintersectionpoint of the root locus branch with an imaginary axis.

Rule6 –FindBreak-awayandBreak-inpoints.

- Iftheexistsarealaxisrootlocusbranchbetweentwoopenlooppoles,thenthere will be a **break-away point** in between these two open loop poles.

- Ifthereexistsarealaxisrootlocusbranchbetweentwoopenloopzeros,thenthere will be a **break-in point** in between these two open loopzeros.

Note−Break-awayandbreak-inpointsexistonlyontherealaxisrootlocusbranches. Follow

these steps to find break-away and break-in points.

- WriteKin termsofsfrom thecharacteristicequation1+G(s)H(s)=0.
- Differentiate Kwithrespectto sandmakeitequaltozero.Substitutethese values of ss in the above equation.
- Thevaluesofssfor which theKvalueispositivearethe**breakpoints**.

Rule7− Findtheangleofdepartureand theangleofarrival.

TheAngleofdepartureandtheangleofarrivalcanbecalculatedatcomplexconjugate open loop poles and complex conjugate open loop zeros respectively.

Theformulaforthe**angleofdeparture**$\phi_d$is

$$\phi_d = 180^0 - \phi$$

The formula for the **angle of arrival** ϕ_a is

$$\phi_a = 180^0 + \phi$$

Where,

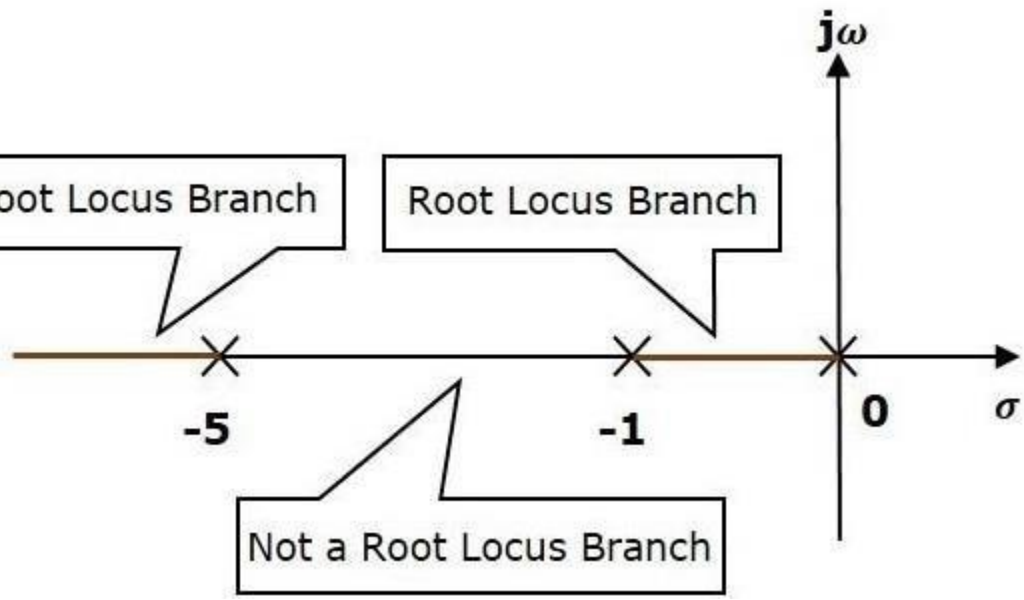
$$\phi = \sum \phi_P - \sum \phi_Z$$

Example

Letusnowdrawtherootlocusofthecontrolsystemhavingopenlooptransfer function, $G(s)H(s) = \frac{K}{s(s+1)(s+5)}$

Step1−Thegivenopenlooptransferfunctionhasthreepolesats=0,
s= -1,s= -5.Itdoesn'thaveanyzero.Therefore,thenumber ofrootlocusbranchesis equal to the number of poles of the open loop transfer function.

N=P=3



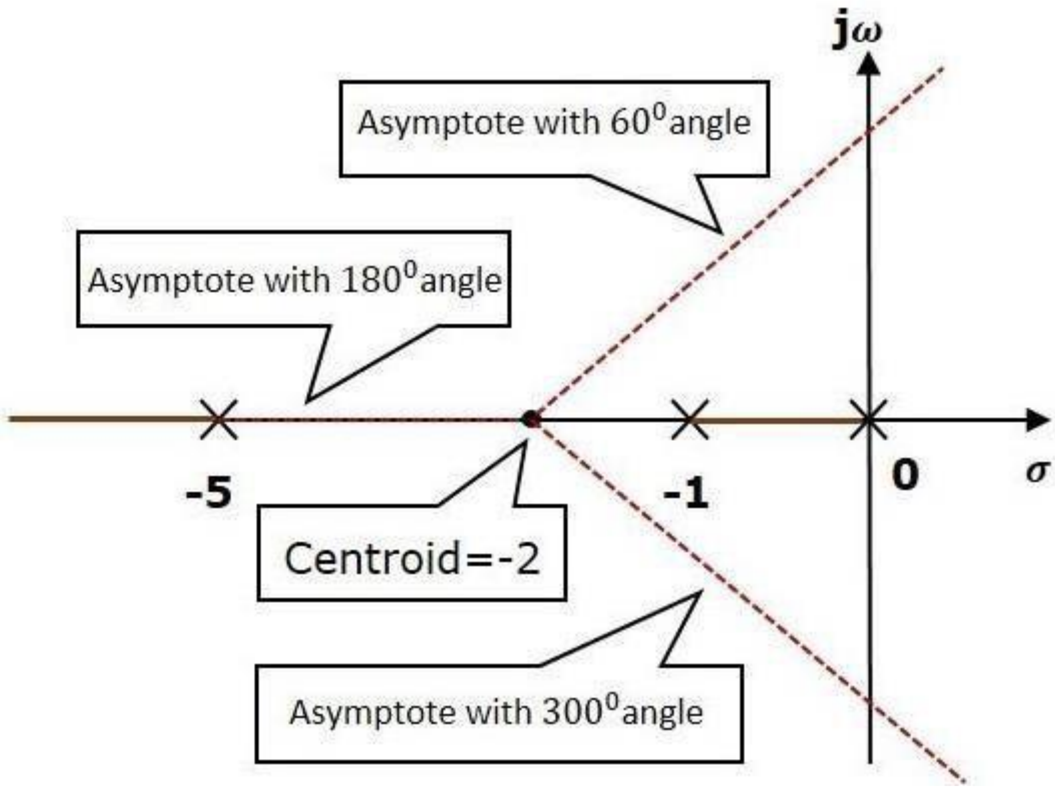
Thethreepolesarelocatedare shownintheabovefigure.Thelinesegmentbetweens=−1, and s=0 is one branch of root locus on real axis. And the other branch of the root locus on the real axis is the line segment to the left of s=−5.

Step 2 – We will get the values of the centroid and the angle of asymptotes by using the given formulae.

Centroid

Theangleofasymptotesare $\theta = 60^0, 180^0$ and 300^0.

Thecentroidandthreeasymptotesareshowninthefollowing figure.

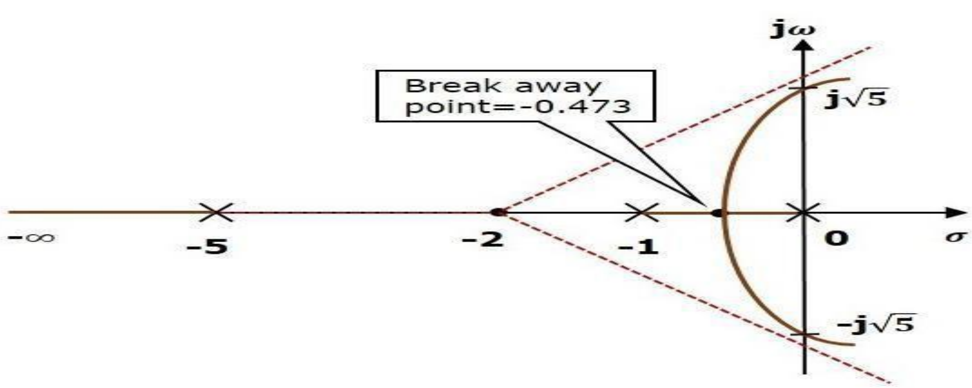


Step3 −Sincetwoasymptoteshavetheanglesof 600600and30003000,tworootlocus branches intersect the imaginary axis. By using the Routh array method and special case(ii), $j\sqrt{5}$ $-j\sqrt{5}$.

therootlocusbranchesintersectstheimaginaryaxisat and

Therewill beonebreak-awaypointonthereal axisrootlocusbranchbetweenthepoless =−1ands=0.Byfollowingtheproceduregivenforthecalculationofbreak-awaypoint,we will get it as s =−0.473.

Therootlocusdiagram forthegivencontrolsystem isshowninthefollowing figure.



Inthisway,youcandrawtherootlocusdiagramofanycontrolsystemandobservethe movement of poles of the closed loop transfer function.

Fromtherootlocusdiagrams,wecanknowtherangeofKvaluesfordifferenttypesof damping.

EffectsofAddingOpenLoopPolesand ZerosonRootLocus

Therootlocuscanbeshiftedin **‘s’plane** byaddingtheopenlooppolesandtheopenloop zeros.

- If we include a pole in the open loop transfer function, then some of root locus branches will move towards right half of ‘s’ plane.Becauseofthis,thedamping ratio δ decreases. Which implies, damped frequency ωd increases and the time domainspecificationslikedelaytimetd,risetimetrandpeaktimetpdecrease. But, it effects the system stability.
- If we include a zero in the open loop transfer function, then some of root locus branches will move towards left half of ‘s’ plane. So, it will increase the control system stability. In this case, the damping ratio δ increases.Whichimplies,dampedfrequencyωd decreases andthetimedomainspecificationslikedelaytime td,rise time tr and peak time tp increase.

So, based on the requirement, we can include (add) the open loop poles or zeros to the transfer function.

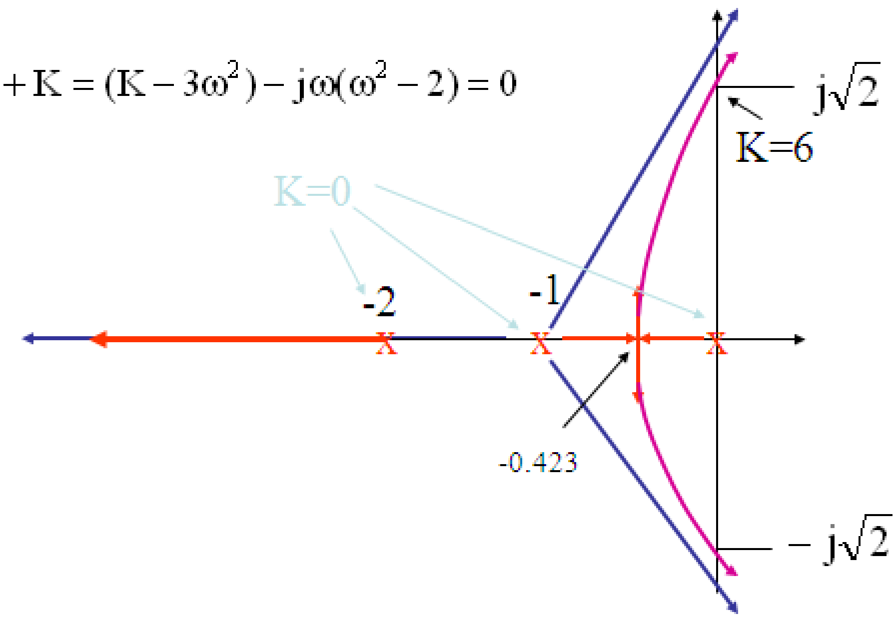
Example

$$\boxed{s^3 + 3s^2 + 2s + K = 0}$$

$$s = j\omega$$

$$-j\omega^3 - 3\omega^2 + 2j\omega + K = (K - 3\omega^2) - j\omega(\omega^2 - 2) = 0$$

$$\boxed{\begin{aligned} \omega^2 &= 2 \\ K &= 3\omega^2 = 6 \end{aligned}}$$



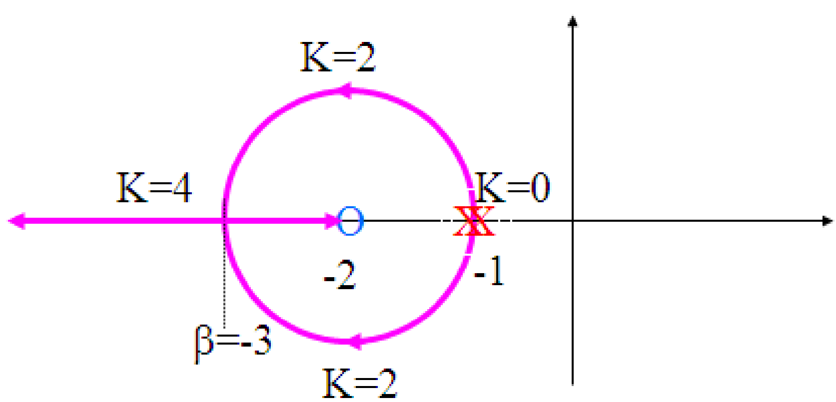
Example

$$\boxed{GH(s) = \frac{K(s+2)}{(s+1)^2}}$$

Two poles at –1
One zero at –2
One asymptote at 180°
Break-in point at -3

$$\frac{2}{\beta+1} = \frac{1}{\beta+2}$$

$$2\beta + 4 = \beta + 1$$



Example Why a circle ?

Characteristic equation $s^2 + s(2+K) + 2K + 1 = 0$

For K<4

$$s_{1,2} = \frac{-(2+K) \pm j\sqrt{K(4-K)}}{2}$$

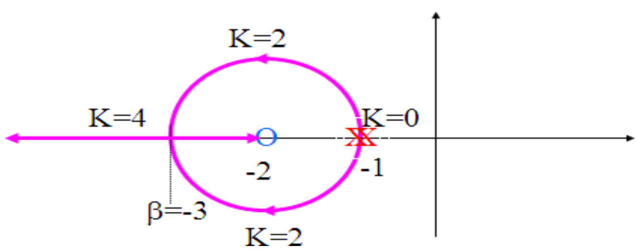
For K>4

$$s_{1,2} = \frac{-(2+K) \pm \sqrt{K(K-4)}}{2}$$

Change of origin $s_{1,2} + 2 = \frac{-(-2+K) \pm j\sqrt{K(4-K)}}{2}$

$$4m = (K-2)^2 + K(4-K) = K^2 - 4K + 4 + 4K - K^2$$

$$m = 1$$



UNIT-V

FREQUENCYRESPONSEANALYSIS

WhatisFrequencyResponse?

The response of a system can be partitioned into both the transient response and the steady state response. We can find the transient response by using Fourier integrals. The steadystateresponseofasystem for an input sinusoidal signalisknown asthe **frequency response**. In this chapter, we will focus only on the steady state response.

If a sinusoidal signal is applied as an input to a Linear Time-Invariant (LTI) system, then it produces the steady state output, which is also a sinusoidal signal. The input and output sinusoidal signalshavethesamefrequency, but different amplitudesand phaseangles. Let the input signal be

$$r(t) = A\sin(\omega_0 t)$$

The open loop transfer function will be −

$$G(s) = G(j\omega)$$

We can represent $G(j\omega)$ in terms of magnitude and phase as shown below.

$$G(j\omega) = |G(j\omega)|\angle G(j\omega)$$

Substitute, $\omega = \omega_0$ in the above equation.

$$G(j\omega_0) = |G(j\omega_0)|\angle G(j\omega_0)$$

The output signal is

$$c(t) = A|G(j\omega_0)|\sin(\omega_0 t + \angle G(j\omega_0))$$

- The **amplitude** of the output sinusoidal signal is obtained by multiplying the amplitude of the input sinusoidal signal and the magnitude of $G(j\omega)$ at $\omega = \omega_0$.
- The **phase** of the output sinusoidal signal is obtained by adding the phase of the input sinusoidal signal and the phase of $G(j\omega)$ at $\omega = \omega_0$.

Where,

- **A** istheamplitudeoftheinputsinusoidalsignal.
- **ω_0**isangularfrequencyoftheinputsinusoidal

signal. We can write, angular frequency ω_0 as shown below.

$$\omega_0=2\pi f_0$$

Here,$f_0$isthefrequencyoftheinputsinusoidalsignal.Similarly,youcanfollowthesame procedure for closed loop control system.

FrequencyDomainSpecifications

Thefrequencydomainspecifications are

- **Resonant peak**
- **Resonantfrequency**
- **Bandwidth.**

Considerthetransferfunctionofthesecondorderclosedcontrolsystemas

$$T(s)=\frac{C(s)}{R(s)}=\frac{\omega_n^2}{s^2+2\delta\omega_n s+\omega_n^2}$$

Substitute, $s=j\omega$ in the above equation.

$$T(j\omega)=\frac{\omega_n^2}{(j\omega)^2+2\delta\omega_n(j\omega)+\omega_n^2}$$

$$\Rightarrow T(j\omega)=\frac{\omega_n^2}{-\omega^2+2j\delta\omega\omega_n+\omega_n^2}=\frac{\omega_n^2}{\omega_n^2\left(1-\frac{\omega^2}{\omega_n^2}+\frac{2j\delta\omega}{\omega_n}\right)}$$

$$\Rightarrow T(j\omega)=\frac{1}{\left(1-\frac{\omega^2}{\omega_n^2}\right)+j\left(\frac{2\delta\omega}{\omega_n}\right)}$$

Let, $\frac{\omega}{\omega_n}=u$ Substitute this value in the above equation.

$$T(j\omega)=\frac{1}{(1-u^2)+j(2\delta u)}$$

Magnitude of $T(j\omega)$ is -

$$M=|T(j\omega)|=\frac{1}{\sqrt{(1-u^2)^2+(2\delta u)^2}}$$

Phase of $T(j\omega)$ is -

$$\angle T(j\omega) = -tan^{-1}\left(\frac{2\delta u}{1-u^2}\right)$$

Resonant Frequency

It is the frequency at which the magnitude of the frequency response has peak value for the first time. It is denoted by ω_r. At $\omega = \omega_r$, the first derivate of the magnitude of $T(j\omega)$ is zero.

Differentiate M with respect to u.

$$\frac{\mathrm{d}M}{\mathrm{d}u} = -\frac{1}{2}\left[(1-u^2)^2 + (2\delta u)^2\right]^{\frac{-3}{2}}\left[2(1-u^2)(-2u) + 2(2\delta u)(2\delta)\right]$$

$$\Rightarrow \frac{\mathrm{d}M}{\mathrm{d}u} = -\frac{1}{2}\left[(1-u^2)^2 + (2\delta u)^2\right]^{\frac{-3}{2}}\left[4u(u^2 - 1 + 2\delta^2)\right]$$

Substitute, $u = u_r$ and $\frac{\mathrm{d}M}{\mathrm{d}u} == 0$ in the above equation.

$$0 = -\frac{1}{2}\left[(1-u_r^2)^2 + (2\delta u_r)^2\right]^{-\frac{3}{2}}\left[4u_r(u_r^2 - 1 + 2\delta^2)\right]$$

$$\Rightarrow 4u_r(u_r^2 - 1 + 2\delta^2) = 0$$

$$\Rightarrow u_r^2 - 1 + 2\delta^2 = 0$$

$$\Rightarrow u_r^2 = 1 - 2\delta^2$$

$$\Rightarrow u_r = \sqrt{1-2\delta^2}$$

Substitute, $u_r = \frac{\omega_r}{\omega_n}$ in the above equation.

$$\frac{\omega_r}{\omega_n} = \sqrt{1-2\delta^2}$$

$$\Rightarrow \omega_r = \omega_n\sqrt{1-2\delta^2}$$

ResonantPeak

Itisthepeak(maximum)valueofthemagnitudeof T(jω).Itisdenotedby M_r. At $u = u_r$, the Magnitude of T(jω) is -

$$M_r = \frac{1}{\sqrt{(1-u_r^2)^2 + (2\delta u_r)^2}}$$

Substitute, $u_r = \sqrt{1-2\delta^2}$ and $1 - u_r^2 = 2\delta^2$ in the above equation.

$$M_r = \frac{1}{\sqrt{(2\delta^2)^2 + (2\delta\sqrt{1-2\delta^2})^2}}$$

$$\Rightarrow M_r = \frac{1}{2\delta\sqrt{1-\delta^2}}$$

Resonant peak in frequency response corresponds to the peak overshoot in the time domain transient response for certain values of damping ratio δδ. So, the resonant peak and peak overshoot are correlated to each other.

Bandwidth

Itistherangeoffrequenciesover which,themagnitudeofT(jω)dropsto 70.7%from itszero frequency value.

Atω=0,thevalueofuwillbezero. Substitute,
u=0 in M.

$$M = \frac{1}{\sqrt{(1-0^2)^2 + (2\delta(0))^2}} = 1$$

Therefore,themagnitudeofT(jω)isoneatω=0

At3-dB frequency,themagnitudeofT(jω)willbe70.7%ofmagnitudeofT(jω))atω=0

i.e., at $\omega = \omega_B, M = 0.707(1) = \frac{1}{\sqrt{2}}$

$$\Rightarrow M = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{(1-u_b^2)^2 + (2\delta u_b)^2}}$$

$$\Rightarrow 2 = (1-u_b^2)^2 + (2\delta)^2 u_b^2$$

Let, $u_b^2 = x$

$$\Rightarrow 2 = (1-x)^2 + (2\delta)^2 x$$

$$\Rightarrow x^2 + (4\delta^2 - 2)x - 1 = 0$$

$$\Rightarrow x = \frac{-(4\delta^2 - 2) \pm \sqrt{(4\delta^2 - 2)^2 + 4}}{2}$$

Consider only the positive value of x.

$$x = 1 - 2\delta^2 + \sqrt{(2\delta^2 - 1)^2 + 1}$$

$$\Rightarrow x = 1 - 2\delta^2 + \sqrt{(2 - 4\delta^2 + 4\delta^4)}$$

Substitute, $x = u_b^2 = \frac{\omega_b^2}{\omega_n^2}$

$$\frac{\omega_b^2}{\omega_n^2} = 1 - 2\delta^2 + \sqrt{(2 - 4\delta^2 + 4\delta^4)}$$

$$\Rightarrow \omega_b = \omega_n \sqrt{1 - 2\delta^2 + \sqrt{(2 - 4\delta^2 + 4\delta^4)}}$$

Bandwidthωbinthefrequencyresponseisinverselyproportionaltotherisetimetrin the time domain transient response.

Bodeplots

TheBodeplotortheBodediagram consistsoftwo plots−

- Magnitudeplot
- Phase plot

In both the plots, x-axis represents angular frequency (logarithmic scale). Whereas, yaxis represents the magnitude (linear scale) of open loop transfer function in the magnitude plot and the phase angle (linear scale) of the open loop transfer function in the phase plot.

The**magnitude**oftheopenlooptransfer functionindBis-

$$M = 20\ \log|G(j\omega)H(j\omega)|$$

The**phaseangle**oftheopenloop transferfunctionindegreesis-

$$\phi = \angle G(j\omega)H(j\omega)$$

BasicofBodePlots

Thefollowingtableshowstheslope,magnitudeandthephaseanglevaluesoftheterms present in the open loop transfer function. This data is useful while drawing the Bode

Type of term	G(jω)H(jω)	Slope(dB/dec)	Magnitude (dB)	Phase angle(degrees)
Constant	K	0	$20 \log K$	0
Zero at origin	$j\omega$	20	$20 \log \omega$	90
'n' zeros at origin	$(j\omega)^n$	$20\ n$	$20\ n \log \omega$	$90\ n$
Pole at origin	$\frac{1}{j\omega}$	-20	$-20 \log \omega$	$-90\ or\ 270$
'n' poles at origin	$\frac{1}{(j\omega)^n}$	$-20\ n$	$-20\ n \log \omega$	$-90\ n\ or\ 270\ n$
Simple zero	$1 + j\omega r$	20	$0\ for\ \omega < \frac{1}{r}$ $20\ \log \omega r\ for\ \omega > \frac{1}{r}$	$0\ for\ \omega < \frac{1}{r}$ $90\ for\ \omega > \frac{1}{r}$

plots.

Simple pole	$\frac{1}{1+j\omega r}$	-20	$0 \; for \, \omega < \frac{1}{r}$ $-20 \; \log \omega r \; for \, \omega > \frac{1}{r}$	$0 \; for \, \omega < \frac{1}{r}$ $-90 \; or \; 270 \; for \, \omega > \frac{1}{r}$
Second order derivative term	$\omega_n^2 \left(1 - \frac{\omega^2}{\omega_n^2} + \frac{2j\delta\omega}{\omega_n}\right)$	40	$40 \; \log \; \omega_n \; for \, \omega < \omega_n$ $20 \; \log \; (2\delta\omega_n^2) \; for \; \omega = \omega_n$ $40 \; \log \; \omega \; for \, \omega > \omega_n$	$0 \; for \, \omega < \omega_n$ $90 \; for \, \omega = \omega_n$ $180 \; for \, \omega > \omega_n$
Second order integral term	$\frac{1}{\omega_n^2 \left(1 - \frac{\omega^2}{\omega_n^2} + \frac{2j\delta\omega}{\omega_n}\right)}$	-40	$-40 \; \log \; \omega_n \; for \, \omega < \omega_n$ $-20 \; \log \; (2\delta\omega_n^2) \; for \; \omega = \omega_n$ $-40 \; \log \; \omega \; for \, \omega > \omega_n$	$-0 \; for \, \omega < \omega_n$ $-90 \; for \, \omega = \omega_n$ $-180 \; for \, \omega > \omega_n$

Consider the open loop transfer function $G(s)H(s) = K$.

Magnitude $M = 20\ \log K$ dB

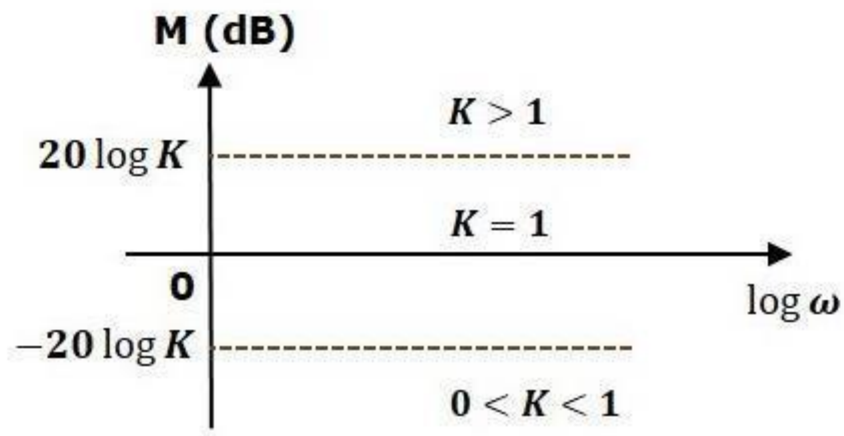
Phase angle $\phi = 0$ degrees

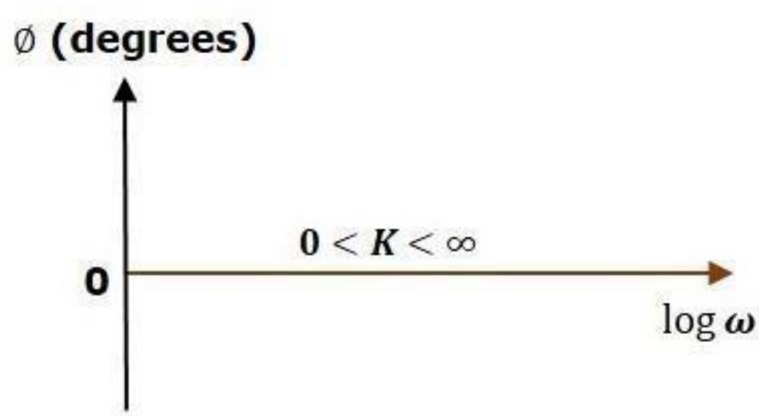
If $K = 1$, then magnitude is 0 dB.

If $K > 1$, then magnitude will be positive.

If $K < 1$, then magnitude will be negative.

The following figure shows the corresponding Bode plot.





The magnitude plot is a horizontal line, which is independent of frequency. The 0 dB line itself is the magnitude plot when the value of K is one. For the positive values of K, the horizontal line will shift 20logK dB above the 0 dB line. For the negative values of K, the horizontal line willshift20logKdBbelowthe0dBline.TheZerodegreeslineitselfisthephaseplotforall the positive values of K.

Considertheopenlooptransferfunction

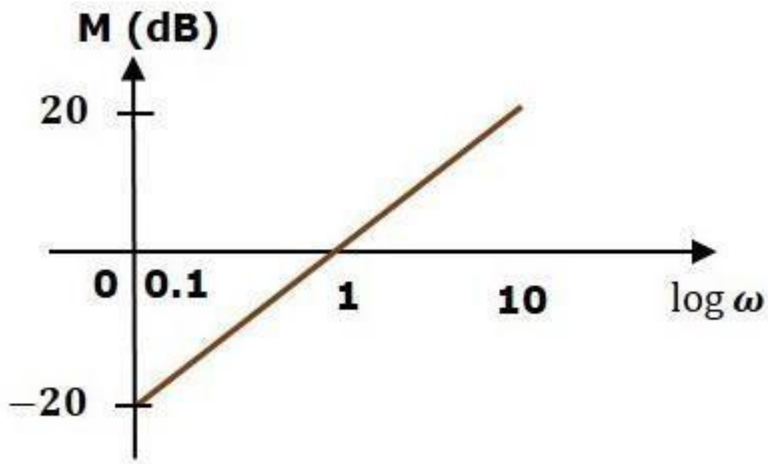
G(s)H(s)=s Magnitude M=20logω dB

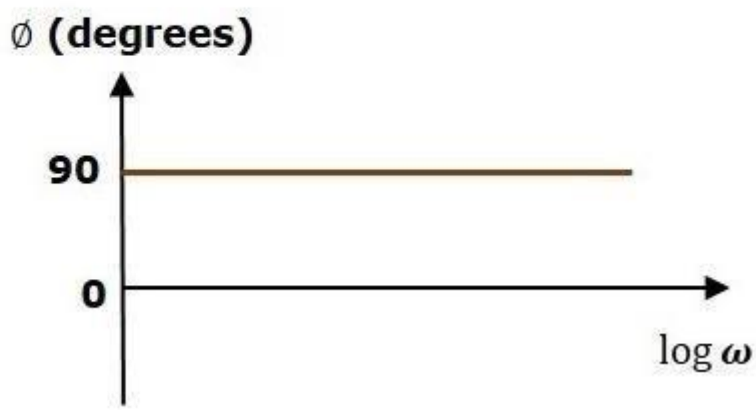
Phase angle ϕ=90^0

At ω=0.1rad/sec, the magnitude is -20dB.Atω=1rad/sec,themagnitudeis0dB.

Atω=10rad/sec,themagnitudeis20dB.

ThefollowingfigureshowsthecorrespondingBodeplot.





Themagnitudeplotisaline,whichishavingaslopeof20dB/dec.Thisline startedatω=0.1rad/sechavingamagnitudeof -20dBanditcontinuesonthesame slope. It is touching 0 dB line at ω=1 rad/sec. In this case, the phase plot is 90^0 line.

Consider the open loop transfer function

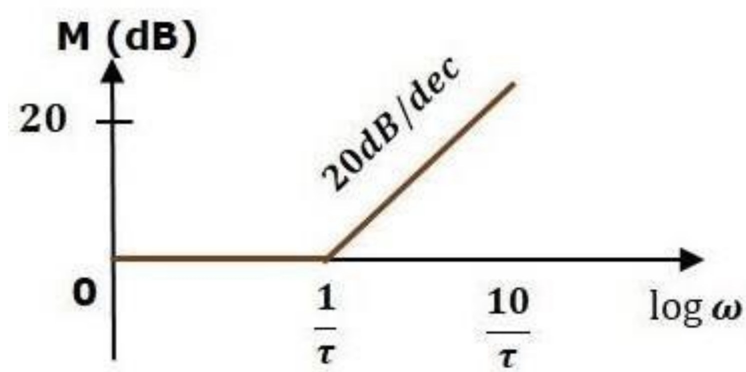
G(s)H(s)=1+sτ. Magnitude $M = 20\,log\sqrt{1+\omega^2\tau^2}$ dB

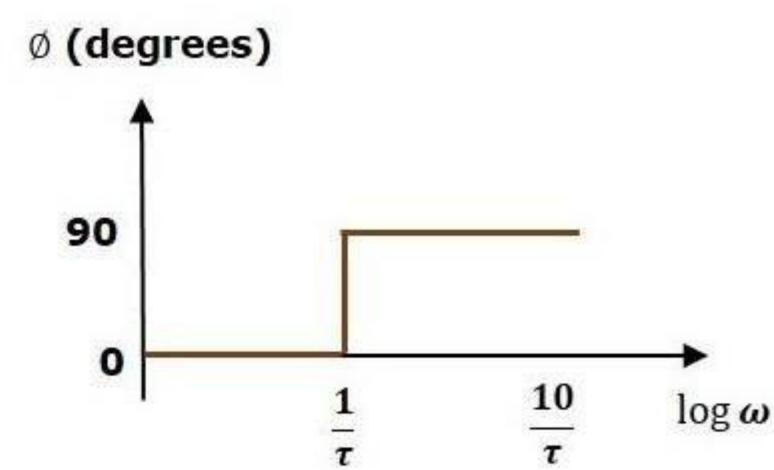
Phaseangle $\phi = \tan^{-1} \omega\tau$ degrees

For $\omega < \frac{1}{\tau}$,themagnitudeis0dB andphaseangleis0degrees.

For $\omega > \frac{1}{\tau}$,themagnitudeis20logωτdBandphaseangleis 90^0.

The following figure shows the corresponding Bodeplot





The magnitude plot is having magnitude of 0 dB upto ω=1τω=1τ rad/sec. From ω=1τ rad/sec, it is having a slope of 20 dB/dec. In this case, the phase plot is having phase angle of 0degrees up toω=1τ rad/sec and fromhere, it ishavingphase angle of90^0.ThisBode plot iscalledthe **asymptotic Bode plot**.

As the magnitude and the phase plots are represented with straight lines, the Exact Bode plots resemble the asymptotic Bode plots. The only difference is that the Exact Bode plots will have simple curves instead of straight lines.

Similarly,youcandrawtheBodeplotsforothertermsoftheopenlooptransferfunction which are given in the table.

RulesforConstructionofBodePlots

FollowtheseruleswhileconstructingaBodeplot.

- Representtheopenlooptransferfunctioninthestandardtimeconstantform.

Substitute, s=jωs=jω in the above equation.

- Findthecornerfrequenciesandarrangetheminascendingorder.
- Consider the starting frequency of the Bode plot as $1/10^{th}$ of the minimum corner frequency or 0.1 rad/sec whichever is smaller value and draw the Bode plot upto 10 times maximum corner frequency.
- Drawthemagnitudeplotsforeachtermandcombinetheseplotsproperly.

Draw the phase plots for each term and combine these plotsproperly.

Note –Thecornerfrequencyisthefrequencyat whichthereisachangeintheslopeof the magnitude plot.

Example

Considertheopenlooptransferfunctionofaclosedloopcontrolsyste

$$G(s)H(s) = \frac{10s}{(s+2)(s+5)}$$

Let us convert this open loop transfer function into standard time constant form.

$$G(s)H(s) = \frac{10s}{2\left(\frac{s}{2}+1\right)5\left(\frac{s}{5}+1\right)}$$

$$\Rightarrow G(s)H(s) = \frac{s}{\left(1+\frac{s}{2}\right)\left(1+\frac{s}{5}\right)}$$

So, we can draw the Bode plot in semi log sheet using the rules mentioned earlier.

StabilityAnalysisusingBodePlots

FromtheBodeplots, wecan say whether thecontrolsystem isstable, marginally stableor unstable based on the values of these parameters.

- Gaincrossoverfrequencyandphasecrossoverfrequency
- Gain margin and phase margin

PhaseCrossoverFrequency

The frequency at which the phase plot is having the phase of -180^0 is known as **phase cross over frequency**. It is denoted by ω_{pc}. The unit of phase cross over frequency is **rad/sec**.

GainCrossoverFrequency

The frequency at which the magnitudeplotishavingthemagnitudeof zerodB is knownas**gaincross over frequency**. It is denoted byω_{gc}. The unit ofgain cross over frequency is **rad/sec**.

The stability of the control system based on the relation between the phase cross over frequency and the gain cross over frequency is listed below.

- Ifthephasecrossoverfrequencyω_{pc}isgreater thanthegaincrossoverfrequency ωgc, then the control system is **stable**.
- Ifthephasecrossoverfrequencyω_{pc}isequal to thegaincrossoverfrequencyω_{gc}, then the control system is **marginally stable**.
- Ifthephasecrossoverfrequencyω_{pc}islessthanthegaincrossoverfrequencyω_{gc}, then the control system is **unstable**.

GainMargin

GainmarginGMGMisequal tonegativeofthemagnitudeindB atphasecrossover frequency.

$$GM=20\log(1M_{pc})=20\log M_{pc}$$

Where,MpcMpcisthemagnitudeatphasecrossoverfrequency.Theunitofgainmargin (GM) is **dB**.

PhaseMargin

Theformulafor phasemarginPMPMis

$$PM=180^0+\phi_{gc}$$

Where,ϕgcisthephaseangleatgaincrossoverfrequency.Theunitofphase marginis **degrees**.

NOTE:

Thestabilityofthecontrolsystembasedontherelationbetweengainmarginandphase margin is listed below.

- Ifboth thegainmarginGM andthephasemarginPMarepositive,thenthecontrol system is **stable**.
- Ifboth thegainmarginGMandthephasemarginPMareequalto zero,thenthe control system is **marginally stable**.

IfthegainmarginGMand/orthephasemarginPMare/isnegative, thenthe control system is **unstable**.

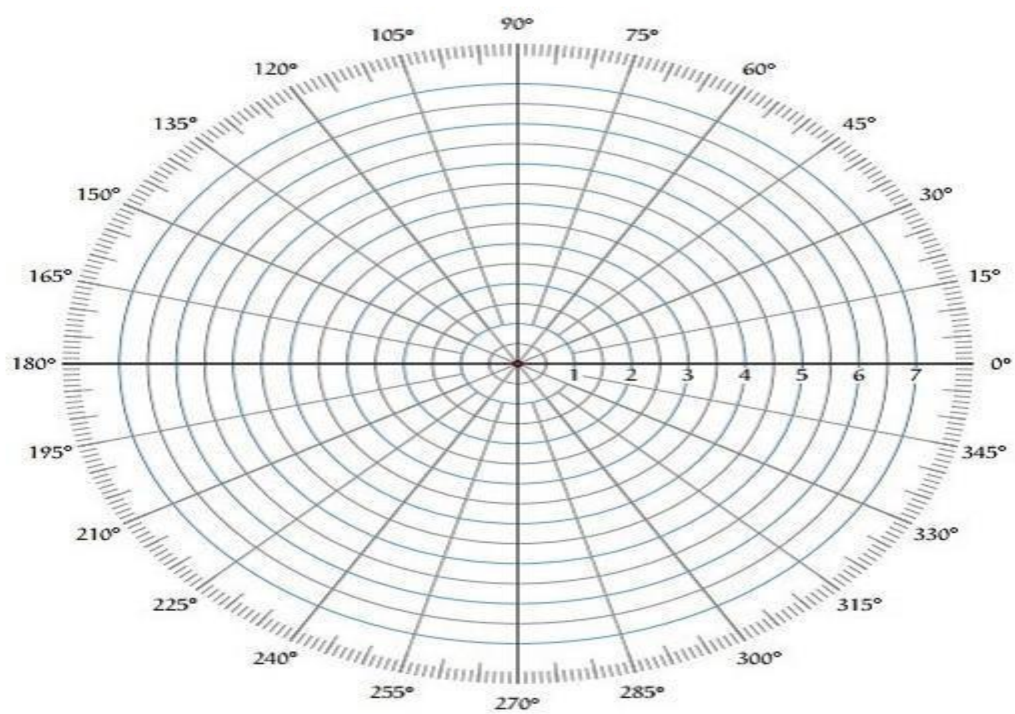
Polarplots

Polarplotisaplotwhichcanbedrawnbetweenmagnitudeandphase.Here,the magnitudes are represented by normal values only.

The polar form of $G(j\omega)H(j\omega)$ is

$$G(j\omega)H(j\omega) = |G(j\omega)H(j\omega)|\angle G(j\omega)H(j\omega)$$

The **Polar plot** is a plot, which can be drawn between the magnitude and the phase angle of $G(j\omega)H(j\omega)$ by varying ω from zero to ∞. The polar graph sheet is shown in the following figure.



This graph sheet consists of concentric circlesandradiallines.The **concentriccircles** and the **radial lines** represent the magnitudes and phase angles respectively. Theseangles are represented by positive values in anti-clock wise direction. Similarly, we can represent angles with negative values in clockwise direction. For example, the angle 270^0in anti-clock wise direction is equal to the angle -90^0 in clockwisedirection.

RulesforDrawingPolarPlots

Followtheserulesforplottingthepolarplots.

- Substitute,s=jω intheopenlooptransferfunction.
- Writetheexpressionsfor magnitudeand thephaseofG(jω)H(jω)
- FindthestartingmagnitudeandthephaseofG(jω)H(jω)bysubstitutingω=0.So, the polar plot starts with this magnitude and the phase angle.
- FindtheendingmagnitudeandthephaseofG(jω)H(jω)bysubstitutingω=∞So, the polar plot ends with this magnitude and the phase angle.
- Checkwhetherthepolarplotintersectstherealaxis,bymakingtheimaginary termof G(jω)H(jω) equal to zero and find the value(s) ofω.
- Checkwhetherthepolarplotintersectstheimaginaryaxis,bymakingreal termof G(jω)H(jω) equal to zero and find the value(s) ofω.
- Fordrawingpolarplotmoreclearly,findthemagnitudeandphaseofG(jω)H(jω)by considering the other value(s) of ω.

Example

Considertheopenlooptransferfunctionofaclosedloopcontrolsystem.

$$G(s)H(s) = \frac{5}{s(s+1)(s+2)}$$

Let us draw the polar plot for this control system using the above rules.

Step 1 − Substitute, $s = j\omega$ in the open loop transfer function.

$$G(j\omega)H(j\omega) = \frac{5}{j\omega(j\omega+1)(j\omega+2)}$$

The magnitude of the open loop transfer function is

$$M = \frac{5}{\omega(\sqrt{\omega^2+1})(\sqrt{\omega^2+4})}$$

The phase angle of the open loop transfer function is

$$\phi = -90^0 - \tan^{-1}\omega - \tan^{-1}\frac{\omega}{2}$$

Frequency (rad/sec)	Magnitude	Phase angle(degrees)
0	∞	-90 or 270
∞	0	-270 or 90

So, the polar plot starts at $(\infty,-90^0)$ and ends at $(0,-270^0)$. The first and the second terms within the brackets indicate the magnitude and phase angle respectively.

Step 3 – Based on the starting and the ending polar co-ordinates, this polar plot will intersectthenegativerealaxis.Thephaseanglecorrespondingtothenegativerealaxisis -180^0or180^0.So,byequatingthephaseangle ofthe openlooptransferfunctiontoeither -180^0or180^0,wewillgettheωvalueas$\sqrt{2}$.
By substituting $\omega=\sqrt{2}$ in the magnitude of the open loop transfer function, we will get M=0.83. Therefore, the polar plot intersects the negative real axis when $\omega=\sqrt{2}$ and the polar coordinate is $(0.83,-180^0)$.
So,wecandraw thepolarplotwith theaboveinformationonthepolar graph sheet.

NyquistPlots

Nyquist plots are the continuation of polar plots for finding the stability of the closed loop control systems by varying ω from $-\infty$ to ∞. That means, Nyquist plots are used to draw the complete frequency response of the open loop transfer function.

NyquistStability Criterion

The Nyquist stability criterion works on the **principle of argument**. It states that if there are P polesandZzerosareenclosedbythe‘s’planeclosedpath,thenthe corresponding G(s)H(s)G(s)H(s) plane must encircle the origin P−ZP−Z times. So, we can write the number of encirclements N as,

$$N=P-ZN=P-Z$$

- If the enclosed ‘s’ plane closed path contains only poles, then the direction of the encirclement in the G(s)H(s)G(s)H(s) plane will be opposite to the direction of the enclosed closed path in the ‘s’ plane.
- If the enclosed ‘s’ plane closed path contains only zeros, then the direction of the encirclement in the G(s)H(s)G(s)H(s) plane will be in the same direction as that of the enclosed closed path in the ‘s’ plane.

Let us now apply the principle of argument to the entire right half of the ‘s’ plane by selecting it as a closed path. This selected path is called the **Nyquist**contour.

We know that the closed loop control system is stable if all the poles of the closed loop transfer function are inthe lefthalf ofthe ‘s’ plane. So, the poles of the closed loop transfer function are nothing but the roots of the characteristic equation. As the order of the characteristic equation increases, it is difficult to find the roots. So, let us correlate these roots of the characteristic equation as follows.

- The Poles of the characteristic equation are same as that of the poles of the open loop transfer function.
- The zeros of the characteristic equation are same as that of the poles of the closed loop transfer function.

Weknow thattheopenloopcontrol system isstableifthereisno openlooppoleinthe the right half of the ‘s’ plane.

i.e.,$P=0 \Rightarrow N=-Z P=0 \Rightarrow N=-Z$

Weknowthattheclosedloopcontrolsystemisstableifthereisnoclosedlooppoleinthe right half of the ‘s’ plane.

i.e.,$Z=0 \Rightarrow N=P Z=0 \Rightarrow N=P$

Nyquist stability criterion states the number of encirclements about the critical point (1+j0) must beequal tothepolesof characteristic equation, whichisnothingbut thepoles of the open loop transfer function in the right half of the ‘s’ plane. The shift in origin to (1+j0) gives the characteristic equation plane.

RulesforDrawingNyquist Plots

FollowtheserulesforplottingtheNyquistplots.

- Locatethepolesand zerosofopenlooptransfer functionG(s)H(s)in‘s’plane.
- Draw the polar plot by varying ω from zero to infinity. If pole or zero present at s = 0, then varying ω from 0+ to infinity for drawing polarplot.
- Draw the mirror image of above polar plot for values of ω ranging from $-\infty$ to zero (0^- if any pole or zero present at s=0).
- The number of infinite radius half circles will be equal to the number of poles or zeros at origin. The infinite radius half circle will start at the point where themirror image of the polar plot ends. And this infinite radius half circle will end at the point where the polar plot starts.

After drawing the Nyquist plot, we can find the stability of the closed loop control system using the Nyquist stability criterion. If the critical point (-1+j0) lies outside the encirclement, then the closed loop control system is absolutelystable.

StabilityAnalysisusingNyquist Plots

From the Nyquist plots, we can identify whether the control system is stable, marginally stable or unstable based on the values of these parameters.

- Gaincrossoverfrequencyandphasecrossoverfrequency
- Gainmarginandphasemargin

PhaseCrossoverFrequency

ThefrequencyatwhichtheNyquistplotintersectsthenegativerealaxis(phaseangleis 180^0) is known as the **phase cross over frequency**. It is denoted byω_{pc}.

GainCrossover Frequency

ThefrequencyatwhichtheNyquistplotishaving themagnitudeofoneisknownasthe **gaincrossoverfrequency**.Itisdenotedbyωgc.
Thestabilityofthecontrolsystembasedontherelationbetweenphasecrossover frequency and gain cross over frequency is listed below.

- Ifthephasecrossoverfrequencyωpcisgreater thanthegaincrossoverfrequency ωgc, then the control system is **stable**.
- Ifthephasecrossoverfrequencyωpcisequal to thegaincrossoverfrequencyωgc, then the control system is **marginally stable**.
- Ifphasecrossoverfrequencyωpcislessthangaincrossoverfrequencyωgc,then the control system is **unstable**.

GainMargin

ThegainmarginGMisequal to thereciprocal ofthemagnitudeoftheNyquistplotatthe phase cross over frequency.

$$GM = \frac{1}{M_{pc}}$$

Where,Mpcisthemagnitudeinnormal scaleatthephasecrossover frequency.

PhaseMargin

ThephasemarginPMisequalto thesumof180^0andthephase angleatthegaincross over frequency.

$$PM = 180^0 + \phi_{gc}$$

Where, ϕ_{gc} isthephase angleatthegaincrossover frequency.

Thestabilityofthecontrolsystembasedontherelationbetweenthegainmarginandthe phase margin is listed below.

- IfthegainmarginGMisgreaterthanoneandthephasemarginPMispositive,then the control system is **stable**.
- IfthegainmarginGMsequaltooneandthephasemarginPMiszerodegrees,then the control system is **marginally stable**.
- IfthegainmarginGMislessthanoneand/orthephasemarginPMisnegative, then the control system is **unstable**.