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ASPIRE TO EXCEL

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DEPARTMENT OF MECHANICAL ENGINEERING

THIRD SEMESTER

SUBJECT

MET-32 MECHANCIS OF FLUIDS

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UNIT I

INTRODUCTION TO FLUID

- ✓ Fluid may be defined as a substance which is capable of flowing. It has no definite shape of its own, but confirms to the shape of the containing vessel. A small amount of shear force exerted on a fluid will cause it to undergo a deformation. It continues as long as the force is to be applied.
- ✓ The study of fluids in motion, where pressure forces are not considered, is called fluid kinematics and if the pressure forces are also considered for the fluids in motion, that branch of science is called fluid dynamics.

Generally, matter exists in three states. They are

- ✓ Solid
- ✓ Liquid and
- ✓ Gas.

Although different in many respects, liquids and gases have a common characteristic in which they differ from solids. The liquid and gas together are called by the common term 'fluids'.

A fluid is a substance which deforms continuously under the action of shear stress. In addition, it has the following properties:

- It is unable to retain any unsupported shape.
- It flows under its own weight and takes the shape of any solid body with which it contained.
- A fluid in equilibrium cannot sustain any shear.
- It cannot regain its original shape on the removal of the shear force.
- Shear stresses occur in fluids only when they are in motion.
- Rate of strain is directly proportional to the applied stress.

- ✓ Fluid mechanics is a physical science related with the behavior of fluid at rest and in continuous motion. It consists of two approaches such as empirical hydraulics and classical hydrodynamics. Empirical hydraulics deals with the motion of water but classical hydrodynamics deals with the flow analysis based on concept of ideal fluid.

DIFFERENCE BETWEEN SOLID AND FLUID

- ✓ In nature, all matter exists in any one of three forms of states such as solid, liquid or gas, or in a mixture of these forms. The liquid and gaseous form are usually combined and given a common name of fluid, because of the common characteristics exhibited by liquids and gases.
- ✓ A solid is generally concerned as a substance that has its own shape and undergoes an infinitesimal change in volume under pure compressive load. It offers resistance to change in shape without a change in volume under the application of tangential forces. This force may cause some displacement of one layer over another in the direction of the applied force, but the material will not continue to

deform indefinitely. When this force is removed, the deformation will disappear provided a critical limit has not been exceeded.

- ✓ The spacing and latitude of motion of molecules are very small in solids, large in a liquid and extremely large in gas. Accordingly, the intermolecular bonds are very strong in solids, weak in liquids and very weak in gases. It is due to that the solid is very compact and rigid. The common materials classified as solids are bricks, steel, diamond, wood, rubber, plastics etc.
- ✓ The fluids do not have their own shape and they do not offer any resistance to change in shape when a deforming tangential force is applied. They continuously deform under the action of such forces, however small the force may be.
- ✓ The continuous deformation under the action of tangential force causes liquids and gases to flow rather than to remain as solid. The common examples of fluids are water, kerosene, milk, gasoline, air, steam etc.

CLASSIFICATION OF FLUIDS

The fluids may be classified into the following five types:

1. Ideal fluid
2. Real fluid
3. Newtonian fluid
4. Non-Newtonian fluid
5. Ideal plastic fluid

Fluids are classified as follows:

- ❖ Ideal fluids and Real or Practical fluids.
- ❖ Newtonian fluids and Non-Newtonian fluids.

1. Ideal Fluids

Ideal fluids have the following properties.

- ✓ It is incompressible.
- ✓ It has zero viscosity.
- ✓ Shear force is zero when the fluid is in motion i.e. No resistance is offered to the motion of any fluid particles.

2. Real or practical Fluids

- ✓ It is compressible.
- ✓ They are viscous in nature.
- ✓ Some resistance is always offered by the fluid when it is in motion.
- ✓ Shear stress always exists in such fluids.

3. Newtonian Fluids

In Newtonian fluids, a linear relationship exists between the magnitudes of shear stress (τ) and the resulting rate of deformation (du /dy). i.e. the constant of proportionality μ , does not change with the rate of deformation.

$$\tau = \mu \frac{du}{dy}$$

Example: Water, Kerosene.

The viscosity at any given temperature and pressure is constant for a Newtonian fluid and is independent of the rate of deformation.

4. Non-Newtonian Fluids

- ✓ In Non-Newtonian fluids, there is a non-linear relation between the magnitude of the applied shear stress and the rate of deformation. The viscosity will vary with variation in rate of deformation. They do not obey Newton's law of viscosity.
- ✓ The Non-Newtonian fluids can be further classified into five groups. They are simple Non-Newtonian, ideal plastic, shear thinning, shear thickening and real plastic fluids. Simple Non-Newtonian has already explained.
- ✓ In plastics, there is no flow upto a certain value of shear stress. After this limit, it has a constant viscosity at any given temperature.
- ✓ In shear thinning materials, the viscosity will increase with rate of deformation (du/dy).
- ✓ In shear thickening materials, viscosity will decrease with rate of deformation (du/dy).
- ✓ Example: Non-Newtonian fluids are paint, toothpaste, and printer's ink.

5. Ideal Plastic Fluid: A fluid, in which shear stresses more than the yield value and shear stress is proportional to the rate of shear strain (or velocity gradient), is known as ideal plastic fluid

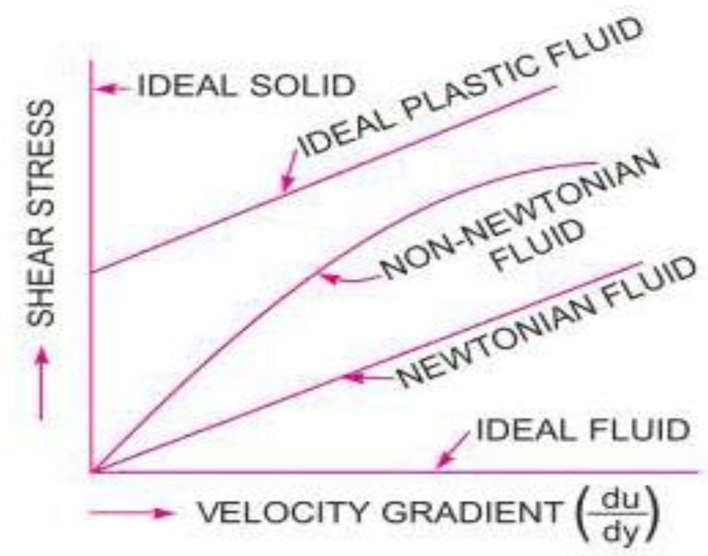


Figure. The relationship between shear stress and shear

FUNDAMENTAL DIMENSIONS AND UNITS

All physical quantities are measured in certain units. There are two types of units:

- ✓ Fundamental units
- ✓ Derived units

1. Fundamental units

All the physical quantities are expressed in terms of the following three fundamental units:

- Length (L)
- Mass (M)
- Time (T)

2. Derived units

Some units called derived units are expressed in terms of fundamental units, such as units of area, velocity, accelesration, pressure etc.

The following four systems of units are internationally accepted.

- C.G.S units
- F.P.S units
- M.K.S units
- S.I. units

PROPERTIES OF FLUIDS

Density or Mass Density: Density or mass density of a fluid is defined as the ratio of the mass of a fluid to its volume. Thus mass per unit volume of a fluid is called density. It is denoted the symbol (ρ rho). The unit of mass density in SI unit is kg per cubic metre, i.e. kg/m^3.

- ✓ The density of liquids may he considered as constant while that of gases changes with the variation of pressure and temperature.

Mathematically, mass density is written as.

$$\text{Density or mass density} = \frac{\text{mass of a fluid}}{\text{volume}}$$

The value of density of water is 1 gm/cm^3 or 1000 kg/m^3.

Specific Weight or Weight Density: Specific weight or weight density of a fluid is the ratio between the weight of a fluid to its volume. The weight per unit volume of a fluid is called weight density and it is denoted by the symbol ω

Weight of fluid (Mass of fluid) x Acceleration due to gravity

$$= \frac{(\text{mass of a fluid x g})}{\text{volume of fluid}}$$

The value of specific weight or weight density (ω) for water is 9.81 X 1000 Newton/m^3 in SI units.

Specific Volume: Specific volume of a fluid is defined as the volume of a fluid occupied by a unit mass or volume per unit mass of a fluid is called specific volume. Mathematically, it is expressed as

Specific volume = Volume of fluid /mass of fluid

$$= \frac{1}{(\text{mass /volume})} = \frac{1}{\rho}$$

Specific Gravity: Specific gravity is defined as the ratio of the weight density for density) of a fluid to the weight density (or density) of a standard fluid. For liquids, the standard fluid is taken water and for gases, the standard fluid is taken air. Specific gravity is also called as relative density. It is dimensionless quantity and is denoted by the symbol S.

Mathematically, S (for liquids) = $\frac{\text{Weight density (density)of liquid}}{\text{weight density (density)of water}}$

S (for gases) = $\frac{\text{Weight density (density)of gas}}{\text{weight density (density)of air}}$

Thus weight density of a liquid = S x Weight density of water

= S x 1000 x 9.81 N/m^3

The density of a liquid = S x Density of water

= S x 1000-kg/m^3.

- ✓ If the specific gravity of a fluid is known, then the density of the fluid will be equal to specific gravity

of fluid multiplied by the density of water. For example the specific gravity of mercury is 13.6, hence

density of mercury = 13.6 x 1000 = 13600 kg/m^3.

VISCOSITY

- ✓ Viscosity is defined as the property of a fluid which offers resistance to the movement of one layer of fluid over another adjacent layer of the fluid.
- ✓ When two layers of a fluid, a distance 'dy' apart, move one over the other at different velocities, say *u* and *u* + *du* as shown in Fig., the viscosity together with relative velocity causes a shear stress acting between the fluid layers.
- ✓ The top layer causes a shear stress on the adjacent lower layer while the lower layer causes a shear stress on the adjacent top layer.

- ✓ This shear stress is proportional to the rate of change of velocity with respect to y. It is denoted by symbol τ called Tau.

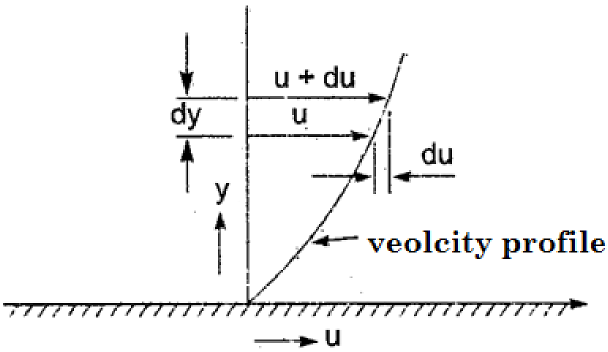


Fig. Velocity variation near a solid boundary

Mathematically,
$$\tau \propto \frac{du}{dy}$$

$$\tau = \mu \frac{du}{dy}$$

where μ (called mu) is the constant of proportionality and is known as the co-efficient of dynamic viscosity or only viscosity. $\frac{du}{dy}$ represents the rate of shear strain or rate of shear deformation or velocity gradient.

From equation, we have

Thus viscosity is also defined as the shear stress required to produce unit rate of shear strain.

Kinematic Viscosity. It is defined as the ratio between the dynamic viscosity and density of fluid. It is denoted by the Greek symbol (v) called 'nu'. Thus, mathematically.

$$v = \frac{\text{viscosity}}{\text{density}} = \frac{\mu}{\rho}$$

In MKS and SI, the unit of kinematic viscosity is *metre^2/sec or* m^2*/sec*, while in CGS units it is written as cm^2/s. In CGS units, kinematic viscosity is also known as stoke.

$$\text{one stoke} = cm^2/s = \left(\frac{1}{100}\right)^2 m^2/s = 10^{-4}\ m^2/s$$

$$\text{centistoke} = \frac{1}{100}\ \text{stoke}$$

Newton's Law of Viscosity. It states that the shear stress **(x)** on a fluid element layer is directly proportional to the rate of shear strain. The constant of proportionality is called the coefficient of viscosity. Mathematically, it is expressed as given by equation (1.2) or as

$$\tau = \mu \frac{du}{dy}$$

Fluids which obey the above relation are known as *Newtonian fluids* and the fluids which do not obey the above relation are known as *Non- Newtonian fluids*

Calculate the specific weight, density and specific gravity of one litre of a liquid which weighs 7 N.

Given :

Volume = l litre = $\frac{1}{1000}$ m^3 (or)1 litre = 1000 cm^3

Weight = 7 N

Solution:

(i)Specific weight (w) = $\frac{weight}{volume} = \frac{7N}{\frac{1}{1000}m^3}$ = **7000 N/m3. Ans.**

(ii) Density (ρ) = $\frac{w}{g} = \frac{7000}{9.81}$ kg/m^3 = **713.5 kg/m3. Ans.**

(in) Specific gravity = $\frac{density\ of\ liquid}{density\ of\ water} = \frac{7000}{1000}$ (Density of water = 1000 kg/m^3)

= 0.7135 Ans.

Calculate the density, specific weight and weight of one litre of petrol of specific gravity = 0.7

Given:

Volume = litre = x 1000 cm^3 = $\frac{1000}{10^6}$ m^3 **= 0.001 m3**

Sp. gravity S , = 0.7

(i) Density (p) = S x 1000 kg/m^3 = 0.7 x 1000

= 700 kg/m3

(ii) Specific weight (w)

w = ρ x g = 700 x 9.81 N/m^3

= 6867 N/m^3. Ans.

(iii) Weight (W)

We know that, specific weight = $\frac{weight}{volume} = \frac{W}{0.001}$ or 6867

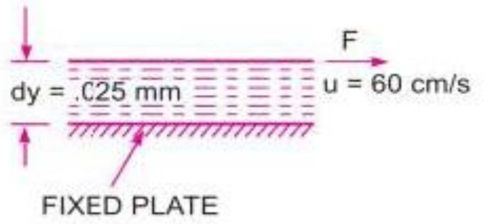
W = 6867 X 0.001 **= 6.867 N. Ans.**

A plate 0.025 mm distant from a fixed plate, moves at 60 cm/s and requires a force of 2 N per unit area i.e., 2 N/m^2 to maintain this speed. Determine the fluid viscosity between the plates.

Given :

Distance between plates, dy = .025 mm

= .025 x 10^{-3} m



Velocity of upper plate, u = 60 cm/s = 0.6 m/s

Force on upper plate, F = 2.0 N/m^2

This is the value of shear stress i.e., τ

Solution:

Let the fluid viscosity between the plates is μ,

we know that, $\tau = \mu \frac{du}{dy}$

where, du = change of velocity

dy = change of distance = 0.025 x 10^{-3} m

= force per unit area = 2.0 N/m^2

$$2.0 = \mu \frac{0.60}{0.025 * 10^{-3}}$$

$$\mu = \frac{2.0 \ x\ 0.025\ x\ 10^{-3}}{0.60} = 8.33 \times 10^{-5} \text{ Ns/m}^2$$

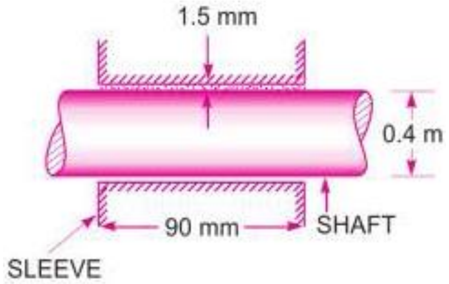
= 8.33 x 10^{-5} x 10 poise

μ = 8.33 x 10^{-4} poise

The dynamic viscosity of an oil, used for lubrication between a shaft and sleeve is 6 poise. The shaft is of diameter 0.4 m and rotates at 190 r.p.m. Calculate the power lost in the Bearing for a sleeve length of 90 mm. The thickness of the oil film is 1.5 mm.

Given

Viscosity μ = 6 poise = $\frac{6}{10}$ Ns/m^2



Diameter of shaft, D = 0.4

Speed of shaft, N = 190 r.p.m

Sleeve length L = 90 mm = 90 x 10^{-3} m

Thickness of oil film, t = 1.5 mm = 1.5 x 10^{-3} m

Tangential velocity of shaft, u = $\frac{\pi D N}{60} = \frac{\pi \text{ x } 0.4 \text{ x } 190}{60}$

u = 3.98 m/s

Solution:

using this relation,

$$\tau = \mu \frac{du}{dy}$$

where du = Change of velocity = u - 0 = u = 3.98 m/s

dy = Change of distance = t = 1.5 x 10 $^{-3}$ m

$$\tau = 10 \text{ x } \frac{3.98}{1.5 \text{ x } 10^{-3}} = \mathbf{15922 \ N/m^2}$$

This is shear stress on shaft,

Shear force on the shaft, F = shear stress x Area

= 1592 x π D L = 1592 x π x 0.4 x 90x10^{-3}

F = 180.05 N

Torque on the shaft, **T =** Force x $\frac{D}{2}$

= 180.05 x $\frac{0.4}{2}$

T = 36.01 Nm

Power lost, = $\frac{2 \pi N T}{60} = \frac{2 \pi \ 190 \text{ x } 36.01}{60}$

Power lost = 716.48 W

If the velocity profile of a fluid over a plate is parabolic with the vertex 20 cm from the plate, where the velocity is 120 cm/sec. Calculate the velocity gradients and shear stresses at a distance of 0, 10 and 20 cm from the plate, if the viscosity of the fluid is 8.5 poise.

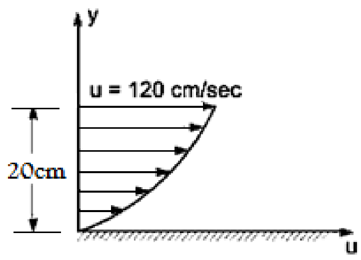
Given:

Distance of vertex from plate = 20 cm

Velocity at vertex, u = 120 cm/sec

Viscosity, μ = 8.5 poise

= 8.5 / 10 Ns/m^2 = 0.85



The velocity profile is given parabolic and equation of velocity profile is

u = ay^2 + by + c .... (i)

where a, b and c are constants. Their values are determined from boundary conditions as:

(a) at y = 0, u = 0

(b) at y = 20 cm, u = 120 cm/sec

(c) at y = 20 cm, (du/dy) = 0

Solution:

substituting boundary condition (a) in equation (i), we get

$$c = 0$$

boundary condition (b) on substitution in (i) gives

$$120 = a\,(20)^2 + b(20) = 400a + 20b \qquad \text{........(ii)}$$

boundary condition (c) on substitution in equation (i) gives

$$du / dy = 2ay+b$$

$$0 = 2 \times a \times 20 + b = 40a + b$$

Solving equations (ii) and (ii) for a and b,

From equation (iii), $b = - 40a$

Substituting this value in equation (ii), we get

$$120 = 400a + 20 \times (- 40a) = 400a - 800a = -400a$$

$$120 = -400a$$

$$a = 120 / (-400) = -3/10 = \mathbf{- 0.3}$$

$$b = - 40 \times (-0.3) = \mathbf{12.0}$$

Substituting the values of a, b and c in equation (i),

$$u = - 0.3y^2 + 12y$$

Velocity Gradient:

$$(du/dy) = -0.3 \times 2y +12 = -0.6y +12$$

at y= 0, Velocity Gradient, $(du/dy)_{y=0}$ = -0.6 x 0+12 = **12/s Ans.**

at y= 10 cm, $(du/dy)_{y=10}$ = -0.6 x 10+12 = - 6+12 = **6/s Ans.**

at y= 20 cm, , $(du/dy)_{y= 20}$ = -0.6 x 20+12 = -12+12 = **12/s Ans.**

Shear Stresses:

Shear stress is given by, $\tau = \mu \dfrac{du}{dy}$

Shear stress at y= o, $\tau = \mu \left(\frac{du}{dy}\right)_{y= 0}$ = 0.85 x 12.0 = **10.2 N/m^2 Ans**

Shear stress at y= o, $\tau = \mu \left(\frac{du}{dy}\right)_{y= 10}$ = 0.85 x 6.0 = **5.1 N/m^2 Ans**

Shear stress at y= o, $\tau = \mu \left(\frac{du}{dy}\right)_{y= 20}$ = 0.85 x 0 = **0 Ans**

Calculate the dynamic viscosity of an oil, which is used for lubrication between a square plate of size 0.8 m x 0.8 m and an inclined plane with angle of inclination 30° as shown in Fig. The weight of the square plate is 300 N and it slides down the inclined plane with a uniform velocity of 0.3 m/s. The thickness of oil film is 1.5 mm.

Given:

Area of plate, $A = 0.8 \times 0.8 = 0.64$ m

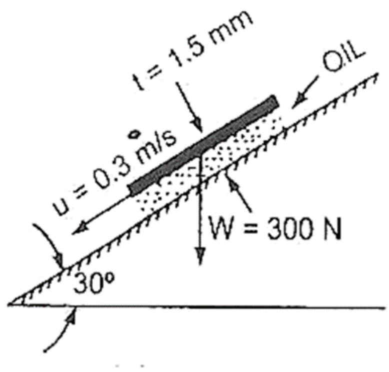
Angle of plane, $\theta = 30°$

Weight of plate, $W = 300$ N

Velocity of plate, u = 0.3 m/s

Thickness of oil film, $t = dy = 1.5$ mm

$= 1.5 \times 10^{3}$ m



Solution:

Let the viscosity of fluid between plate and inclined plane is μ.

Component of weight W, along the plane = Wcos 60° = 300 cos 60° = 150 N

Thus the shear force, *F*, on the bottom surface of the plate = 150 N

shear stress, $\tau = \frac{\text{Force}}{\text{Area}}$ = 150 / 0.64 N/m^2

Now using equation, we have

$$\tau = \mu \quad (du/dy)$$

where du = change of velocity = u - 0 = u = 0.3 m/s

dy = t = 1.5×10^{-3} m

$150/0.64 = \mu\ (0.3 / 1.5 \times 10^{-3})$

$\mu = (150 \times 1.5 \times 10^{-3}) / 0.64 \times 0.3$

μ = 17 N s/m^2 = 1.17 x 10

μ = 11.7 poise Ans

Two large plane surfaces are 2.4 cm apart. The space between the surfaces is filled with glycerine. What force is required to drag a very thin plate of surface area 0.5 square metre between the two large plane surfaces at a speed of 0.6 m/s, if:

(i) the thin plate is in the middle of the two plane surfaces, and

(ii) the thin plate is at a distance of 0.8 cm from one of the plane surfaces ? Take the dynamic viscosity of glycerine = 8. 10 x 10^{-1} N s/m^2.

Given:

Distance between two large surfaces = 2.4 cm

Area of thin plate, $A = 0.5\ m^2$

Velocity of thin plate, $u = 0.6$ m/s

Viscosity of glycerine, $\mu = 8.10 \times 10^{-1}$ N s/m^2

Solution:

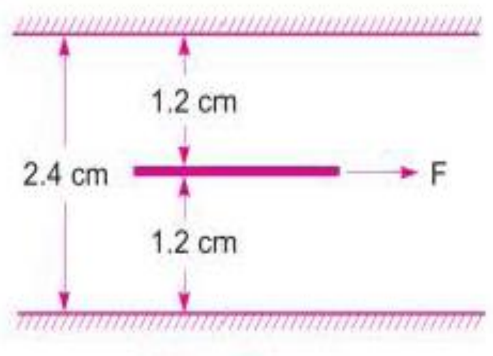
Case I: When the thin plate is in the middle of the two plane surfaces [Refer to Fig.]

Let F_1 = Shear force on the upper side of the thin plate Fig. 1.7 (a)

F_2 = Shear force on the lower side of the thin plate

F = Total force required to drag the plate

Then, $F = F_1 + F_2$



The shear stress (τ_1) on the upper side of the thin plate is given by equation.

$$\tau_1 = \mu \left(\frac{du}{dy}\right)_1$$

where, du = Relative velocity between thin plate and upper large plane surface

= 0.6 m/sec

dy = Distance between thin plate and upper large plane surface

= 1.2 cm = 0.012 m (plate is a thin one and hence thickness of plate is neglected)

$$\tau_1 = 8.10 \times 10\text{-}1 \times \left(\frac{0.6}{0.012}\right) = 40.5\ N/m^2$$

Now, shear force, F_1 = shear stress x Area

$= \tau_1 \times A = 40.5 \times 0.5 = 20.25$ N

F_1 = 20.25 N

Similarly shear stress (τ_2) on the lower side of the thin plate is given by

$$\tau_2 = \mu \left(\frac{du}{dy}\right)_2$$

τ_2 = 8.10 x 10-1 x ($\frac{0.6}{0.012}$) = 40.5 N/m^2

Now , shear force , F_2 = shear stress (τ_2) x Area

= τ_2 x A = 40.5 x 0.5 = 20.25 N

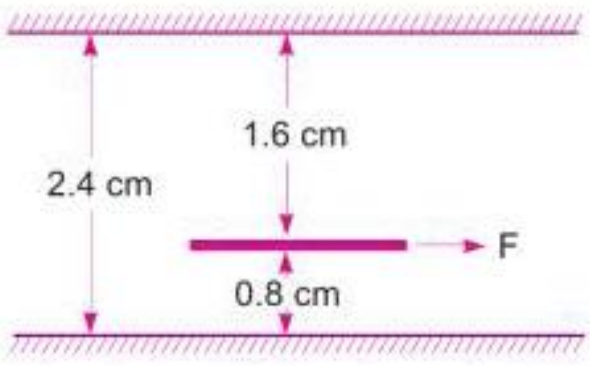
F_2 = 20.25 N

Then , Total force F = F_1 + F_2 = 20.25 + 20.25

F = 40.5 N

Case II. When the thin plate is at a distance of 0.8 cm from one of the plane surfaces [Refer to Fig].

Let the thin plate is at a distance 0.8 cm from the lower plane surface.



Then distance of the plate from the upper plane surface,

2.4 cm – 0.8 = 1.6 cm = .016 m

(Neglecting thickness of the plate)

The shear force on the upper side of the thin plate,

F_1 = Shear stress (τ_1) x Area = τ_1 x A

$$= \mu \left(\frac{du}{dy}\right)_1 \times A = 8.10 \times 10^{-1} \times \left(\frac{0.6}{0.016}\right) \times 0.5$$

F_1 = 15.18 N

The shear force on the lower side of the thin plate,

F_2 = Shear stress (τ_2) x Area = τ_2 x A

$$= \mu \left(\frac{du}{dy}\right)_2 \times A = 8.10 \times 10^{-1} \times \left(\frac{0.6}{0.8/100}\right) \times 0.5$$

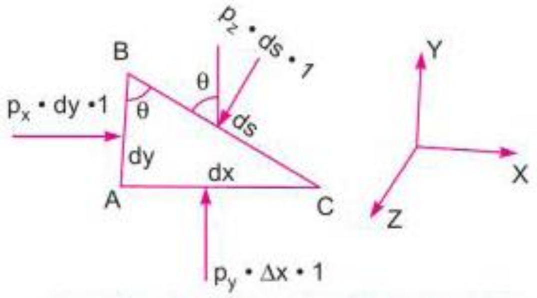
F_2 = 30.36 N

Total force required, F = F_1 + F_2 = 15.18 + 30.36 **F = 45.54 N**

A vertical gap 2.2 cm wide of infinite extent contains a fluid of viscosity 2.0 Ns/m^2 and specific gravity 0.9. A metallic plate 1.2 m X 1.2 m X0.2 cm is to be lifted up with a constant velocity of 0.15 m/sec, through the gap. If the plate is in the middle of the gap, find the force required. The weight of the plate is 40 N.

PASCAL LAW

It states that the pressure or intensity of pressure at a point in a fluid at rest is equal in all directions.



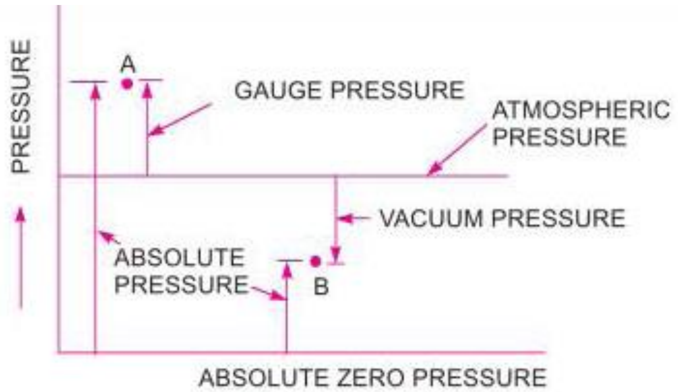
The fluid element is of very small dimensions ie, dx, dy and ds. Consider an arbitrary fluid element of wedge shape in a fluid mass at rest as shown. Let the width of the element perpendicular to the plane of paper is unity and p_x, p_y and p_z are the pressures or intensity of pressure acting on the face AB, AC and BC respectively. Let ABC = θ. Then the forces acting on the element are:

- ✓ Pressure forces normal to the surfaces.
- ✓ Weight of element in the vertical direction.

$$p_x = p_y = p_z$$

ABSOLUTE GAUGE, ATMOSPHERIC AND VACUUM PRESSURES

The pressure on a fluid is measured in two different systems. In one system, it is measured above the absolute zero or complete vacuum and it is called the absolute pressure and in other system, pressure is measured above the atmospheric pressure and it is called gauge pressure. Thus:



1. Absolute pressure is defined as the pressure which is measured with reference to absolute vacuum pressure.

2. Gauge pressure is defined as the pressure which is measured with the help of a pressure measuring instrument, in which the atmospheric pressure is taken as datum. The atmospheric pressure on the scale is marked as zero.

3. Vacuum pressure is defined as the pressures below the atmospheric pressure.

The relationship between the absolute pressure, gauge pressure and vacuum pressure are shown in Fig.

Mathematically :

(i) *Absolute pressure* = Atmospheric pressure + Gauge pressure

$$P_{ab} = P_{atm} + P_{gauge}$$

(ii) *Vacuum pressure* = Atmospheric pressure - Absolute pressure

The atmosphcric pressure at sea level at 15°C is 101.3 KN/m^2 or 10.13 KN/ Cm^2 in SI unit. In case of MKS units, it is equal to 1.033 kgf/cm^2.

MEASUREMENT OF PRESSURE

The pressure of a fluid is measured by the following devices.

1. Manometers
2. Mechanical gauges

Manometers:

i) Simple Manometers

ii) Differential manometers

a) U - tube differential Manometers

b) Inverted U - tube differential Manometers

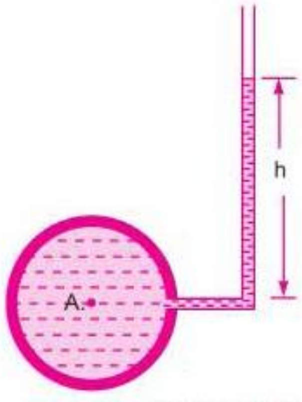
SIMPLE MANOMETER

A simple manometer consists of a glass tube having one of its ends connected to a point where pressure is to be measured and other end remains open to atmosphere. Common types of simple manometers are:

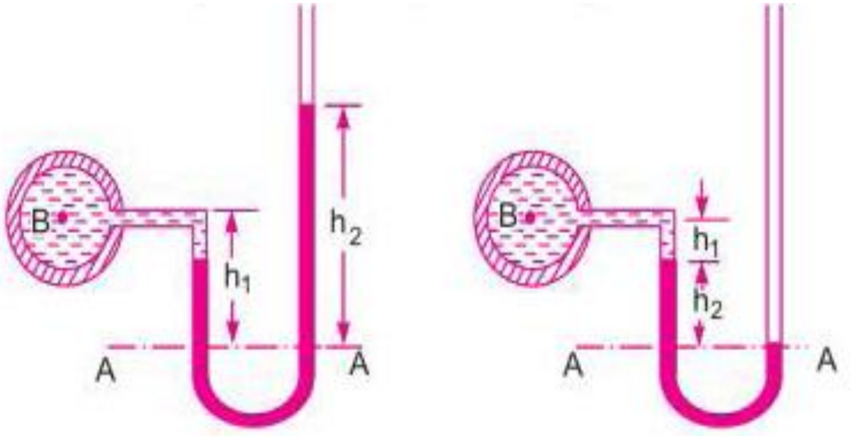
1. Piezometer
2. U-tube Manometer
3. Single Column Manometer

Piezometer: It is the simplest form of manometer used for measuring gauge pressures. One end of this manometer is connected to the point where pressure is to be measured and other end is open to the atmosphere as shown in Fig. The rise of liquid gives the pressure head at that point. If at a point A, the height of liquid say water is h in piezometer tube, then pressure at A

$$= \rho \times g \times h \ (N/m^2)$$



U-tube Manometer: It consists of glass tube bent in U-shape, one end of which is connected to a point at which pressure is to be measured and other end remains open to the atmosphere as shown in Fig.



(a) For gauge pressure (b) For vacuum pressure

The tube generally contains mercury or any other liquid whose specific gravity is greater than the specific gravity of the liquid whose pressure is to be measured.

For gauge pressure, $p = (\rho_2 g h_2 - \rho_1 g h_1)$

For vacuum pressure, $p = -(\rho_2 g h_2 + \rho_1 g h_1)$

A simple U-tube manometer containing mercury is connected to a pipe in which a fluid of sp. gr. 0.8 and having vacuum pressure is flowing. The other end of the manometer is open to atmosphere. Find the vacuum pressure in pipe, if the difference of mercury level in the two limbs is 40 cm and the height of fluid in the left from the centre of pipe is 15 cm below.

Given:

Sp. gr. of fluid, $S_1 = 0.8$

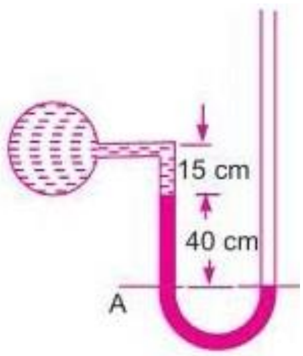
Sp. gr. of mercury, $S_2 = 13.6$

Density of fluid, $\rho_1 = 800$

Density of mercury, $\rho_2 = 13.6 \times 1000$

Difference of mercury level, $h_2 = 40$ cm $= 0.4$ m

Height of liquid in left limb, $h_1 = 15$ cm $= 0.15$ m



Solution.

Let the pressure in pipe = p

Equating pressure above datum line A-A, we get

$$\rho_2 \, g \, h_2 + \rho_1 \, g \, h_1 + P = 0$$

$$p = -(\rho_2 \, g \, h_2 + \rho_1 \, g \, h_1)$$

$$p = -[13.6 \times 1000 \times 9.81 \times 0.4 + 800 \times 9.81 \times 0.15]$$

$$= -[53366.4 + 1177.2] = -54543.6 \text{ N/m}^2$$

$$\mathbf{p = -5.454 \text{ N/cm}^2}$$

The right limb of a simple U-tube manometer containing mercury is open to the atmosphere while the left limb is connected to a pipe in which a fluid of sp. gr. 0.9 is flowing. The centre of the pipe is 12 cm below the level of mercury in the right limb. Find the pressure of fluid in the pipe if the difference of mercury level in the two limbs is 20 cm.

SINGLE COLUMN MANOMETER

Single column manometer is a modified form of a U-tube manometer in which a reservoir, having a large cross-sectional area (about 100 times) as compared to the area of the tube is connected to one of the limbs (say left limb) of the manometer as shown in Fig.

Due to large cross-sectional area of the reservoir, for any variation in pressure, the change in the liquid level in the reservoir will be very small which may be neglected and hence the pressure is given by the height of liquid in the other limb. The other limb may be vertical or inclined. Thus there are two types of single column manometer as:

- ✓ Vertical Single Column Manometer.
- ✓ Inclined Single Column Manometer.

Vertical Single Column Manometer

Fig. shows the vertical single column manometer. Let *X-X* be the datum line in the reservoir and in the right limb of the manometer, when it is not connected to the pipe. When the manometer is connected to the pipe, due to high pressure at A, the heavy liquid in the reservoir will be pushed downward and will rise in the right limb.

Let ,

Δh = Fall of heavy liquid in reservoir

h_2 = Rise of heavy liquid in right limb

h_1 = Height of centre of pipe above X-X

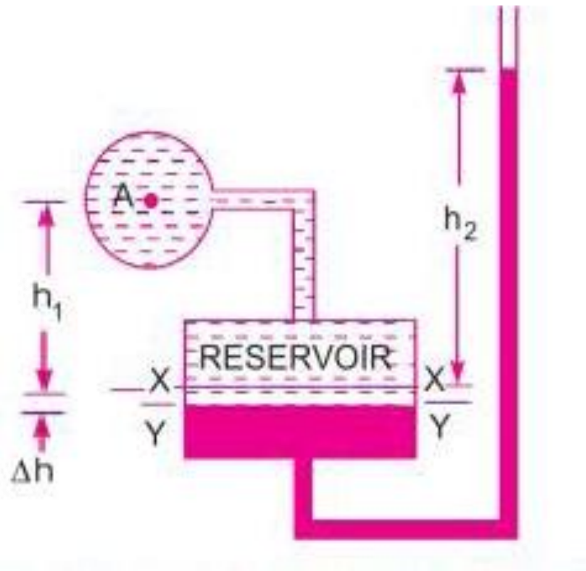
p_A = Pressure at A, which is to be measured

A = Cross-sectional area of the reservoir

a = Cross-sectional area of the right limb

S_1= Sp. gr. of liquid in pipe

S_2= Sp. gr. of heavy liquid in reservoir and right limb



ρ_1 = Density of liquid in pipe

ρ_2 = Density of liquid in reservoir

Fall of heavy liquid in reservoir will cause a rise of heavy liquid level in the right limb.

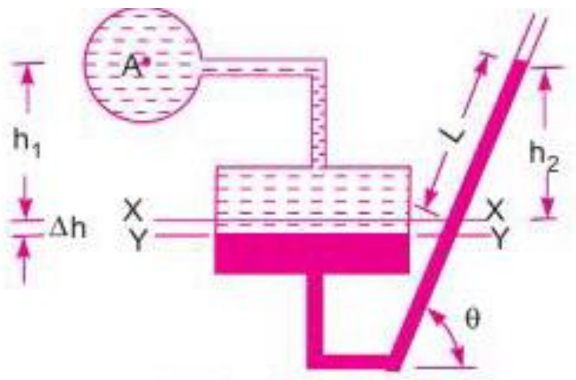
$$A \times \Delta h = a \times h_2$$

$$\Delta h = a \times h_2 / A$$

finally, $$\mathbf{P_A = \rho_2\, g\, h_2 - \rho_1\, g\, h_1}$$

Inclined Single Column Manometer

Fig. shows the inclined single column manometer. This manometer is more sensitive. Due to inclination the distance moved by the heavy liquid in the right limb will be more.



Let L = Length of heavy liquid moved in right limb from X-X

θ = Inclination of right limb with horizontal

h_2 = Vertical rise of heavy liquid in right limb from X-X = L x sin θ

From equation, the pressure at A is

$P_A = \rho_2\, g\, h_2 - \rho_1\, g\, h_1$

Substituting the value of h_2, we get

$$\mathbf{P_A = \sin\theta \times \rho_2\, g - \rho_1\, g\, h_1}$$

DIFFERENTIAL MANOMETERS

Differential manometers are the devices used for measuring the difference of pressures between two points in a pipe or in two different pipes. A differential manometer consists of a U-tube, containing a heavy liquid, whose two ends arc connected to the points, whose difference of pressure is to be measured. Most commonly types of differential manometers are:

1. U-tube differential manometer
2. Inverted U-tube differential manometer

U-Tube Differential Manometer

Fig. shows the differential manometers of U-tube type.

In Fig. the two points A and B are at different level and also contains liquids of different sp. gr. These points are connected to the U-tube differential manometer. Let the pressure at A and B are p_A and p_B.

Let h = Difference of mercury level in the U-tube.

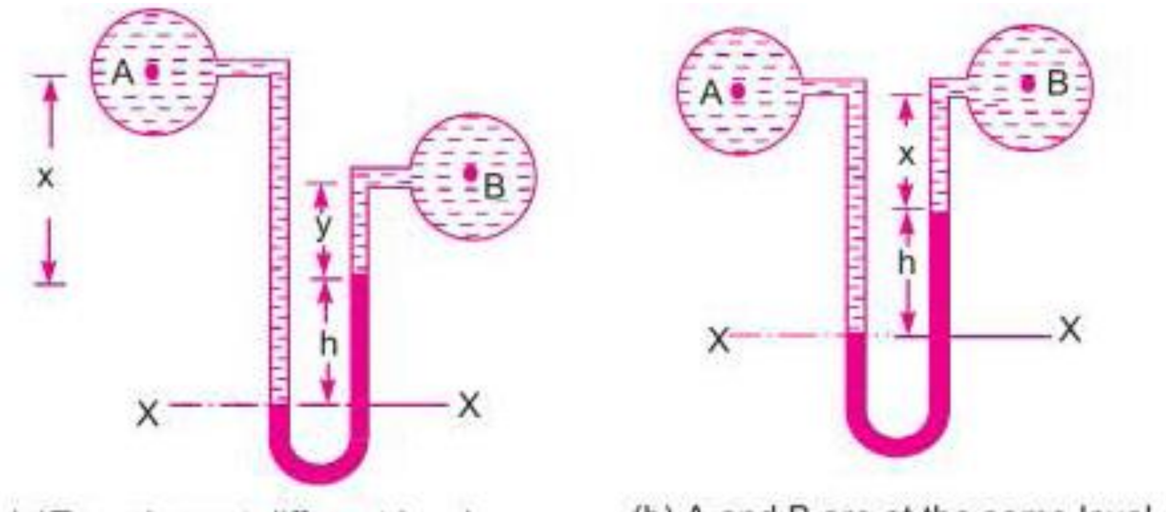
y = Distance of the centre of B, from the mercury level in the right limb.

x = Distance of the centre of A, from the mercury level in the right limb,

ρ_1 = Density of liquid at A.

ρ_2 = Density of liquid at B.

ρ_g = Density of heavy liquid or mercury.



(a)Two pipes at different levels (b) A and B are at the same level

Taking datum line at X-X.

Pressure above X-X in the left limb = $\rho_1 g(h + x) + p_A$

where p_A = pressure at A.

Pressure above X-X in the right limb = $\rho_g \times g \times h + \rho_2 \times g \times y + pB$

where p_B = Pressure at B.

Equating the two pressure, we have

$\rho_1 g(h + x) + p_A = = \rho_g \times g \times h + \rho_2 \times g \times y + pB$

$PA-P_B = \rho_g gh + \rho_2 g y - \rho_1 g(h + x)$

$= h\ g\ (\rho_g - \rho_1) + \rho_2\ g\ y - \rho_1 gx$

Difference of pressure at A and B = $h \times g(\rho_g - \rho_1) + \rho_2 gy - \rho_1 gx$

The two points A and B are at the same level and contains the same liquid of density ρ,

Then

$$PA-P_B = gh(\rho_g - \rho_1)$$

A Differential manometer is connected at the two points A and B of two pipes as shown in Fig. The pipe A contains a liquid of sp. gr. = 1.5 while pipe B contains a liquid of sp. gr. = 0.9. The pressures at A and B are 1 kgf/cm² and 1.80 kgf/cm² respectively. Find the difference in mercury level in the differential manometer.

Given:

Sp. gr. of liquid at A , $S_1 = 1.5$ [$\rho_1 = 1500$ kg/m³]

Sp. gr. of liquid at B , S2 = 0.9 [ρ1 = 900 kg/m3]

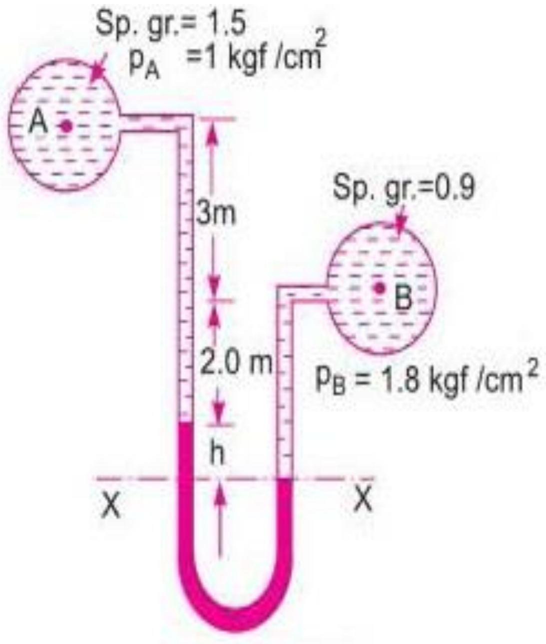
Pressure at A, $= P_A = 1$ kgf/cm² $= 1 \times 10^4$ kgf/m²

$= 10^4 \times 9.81$ N/m² (1 kgf = 9.81 N)

Pressure at B, $= P_A = 1.8$ kgf/cm² $= 1.8 \times 10^4$ kgf/m²

$= 1.8 \times 10^4 \times 9.81$ N/m²

Density of mercury $= 13.6 \times 1000$ kg/m³



Solution:

Taking X-X as datum line,

Pressure above X-X in the left limb

$= 13.6 \times 1000 \times 9.81 \times h + 1500 \times 9.81 \times (2 + 3) + P_A$

= 13.6 x 1000 X 9.81 x h + 7500 x 9.81 + 9.81 x 10^4

Pressure above X-X in the right limb

= 900 x 9.81 x (h + 2) + P_B

= 900 x 9.81 x (h + 2) + 1.8 x $I0^4$ x 9.81

Equating the two pressure, we get

13.6 x 1000 x 9.81h + 7500 x 9.81 + 9.81 x 10^4 = 900 x 9.81 x (h + 2) + 1.8 x 10^4 x 9.81

Dividing by 1000 x 9.81, we get

13.6 h + 7.5 + 10 = (h + 2.0) x 0.9 + 18

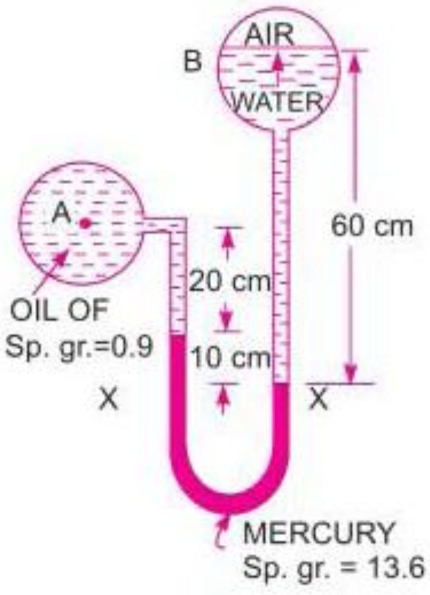
13.6 h+ 17.5 = 0.9 h+ 1.8+ 18 = 0.9 h+ 19.8

(13.6 - 0.9) h = 19.8 - 17.5 (or) 12.7/ h = 2.3

h = 2.3 / 12.7 = 0.181 m

h = 18.1 cm Ans

A differential manometer is connected at the two points A and B as shown in Fig. At B air pressure is 9.81 N/cm^2 (abs), find the absolute pressure at A.



Inverted u-tube differential manometer:

It consists of an inverted U-tube, containing a light liquid. The two ends of the tube are connected to the points whose difference of pressure is to be measured. It is used for measuring difference of low pressures. Fig. shows an inverted U-tube differential manometer connected to the two points A and B. Let the pressure at A is more than the pressure at B.

Let

h_1 = Height of liquid in left limb below the datum line X-X

h_2 = Height of liquid in right limb

h = Difference of light liquid

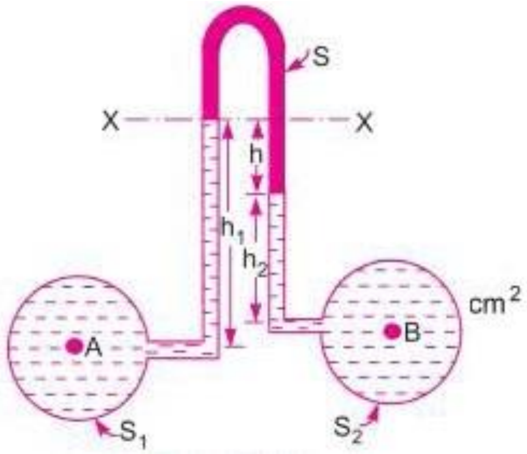
ρ_1 = Density of liquid at A

ρ_2 = Density of liquid at B

ρ_s = Density of light liquid

p_A = Pressure at A

p_B = Pressure at B.



Taking X-X as datum line. Then pressure in the left limb below X-X

$= P_A - \rho_1 g h_1$

Pressure in the right limb below X-X

$= P_B - \rho_2 g h_2 - \rho_s g h$

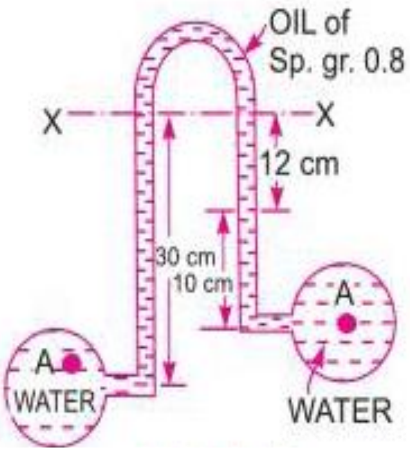
Equating the two pressure

$P_A - \rho_1 g h_1 = P_B - \rho_2 g h_2 - \rho_s g h$

$P_A - P_B = \rho_1 g h_1 - \rho_2 g h_2 - \rho_s g h$

Water is flowing through two different pipes to which an inverted differential manometer having an oil of sp. gr. 0.8 is connected. The pressure head in the pipe A is 2 m of water, find the pressure in the pipe B for the manometer readings as shown in Fig.

Given:



Solution:

Pressure head at A = $\frac{P_A}{\rho g}$ = 2 m of water

P_A = p x g x 2 = 1000 x 9.81 x 2

= 1000 x 9.81 x 2

P_A = 19620 N/m^2

Fig. shows the arrangement. Taking X-X as datum line.

Pressure below X-X in the left limb = P_A - ρ_1 g h_1

= 19620 - 1000 x 9.81 x 0.3

= 16677 N/m^2

Pressure below X-X in the right limb

= P_B - 1000 x 9.81 x 0.1 - 800 x 9.81 x0.12

= P_B - 981 - 941.76 = P_B - 1922.76

Equating the two pressure, we get

16677 = P_B - 1922.76

P_B = 16677 + 1922.76 = 18599.76 N/m^2

P_B = 1.8599 N/cm^2 Ans.

In Fig. an inverted differential manometer is connected to two pipes A and B which convey water. The fluid in manometer is oil of sp. gr. 0.8. For the manometer readings shown in the figure, find the pressure difference between A and B.

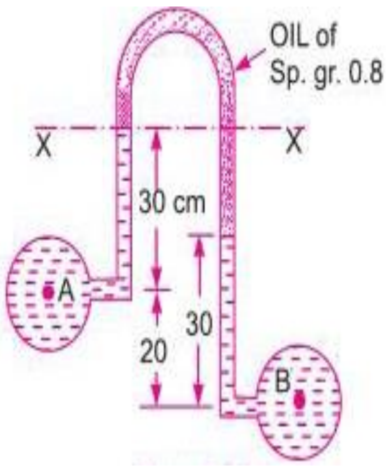
Given :

Sp. gr. of oil S_0 = 0.8

Density of oil ρ_0 = (S_0 x 1000) = 800 kg/m^3

Difference of oil in the two limbs,

= (30 + 20) - 30 **= 20 cm**



Solution:

Taking datum line at X-X

Pressure in the left limb below X-X

$= P_A - 1000 \times 9.81 \times 0$

$\mathbf{= P_A - 2943}$

Pressure in the right limb below X-X

$= P_B - 1000 \times 9.81 \times 0.3 - 800 \times 9.81 \times 0.2$

$= P_B - 2943 - 1569.6$

$\mathbf{= P_B - 4512.6}$

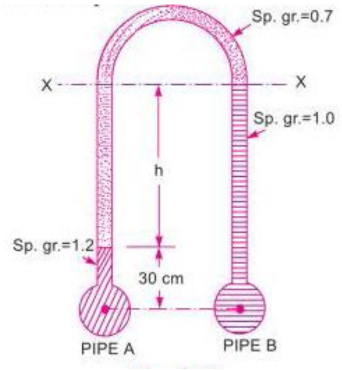
Equating the two pressure,

$P_A - 2943 = P_B - 4512.6$

$P_B - P_A = 4512.6 - 2943$

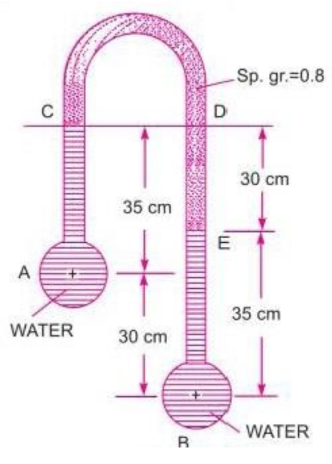
$\mathbf{P_B - P_A = 1569.6\ N/m^2}$**. Ans**

Find out the differential reading 'h' of an inverted U-tube manometer containing oil of specific gravity 0.7 as the manometric fluid when connected across pipes A and B as shown in Fig. below, conveying liquids of specific gravities 1.2 and 1.0 and immiscible with manometric fluid. Pipes A and B are located at the same level and assume the pressures at A and B to be equal.



An inverted U-tube manometer is connected to two horizontal pipes A and B through which water is flowing. The vertical distance between the axes of these pipes is 30 cm. When an oil of specific gravity 0.8 is used as a gauge fluid, the vertical heights of water columns in the two limbs of the inverted manometer (when measured from the respective centre lines of the pipes) are found to be same and equal to 35 cm. Determine the difference of pressure between the pipes.

Given:



Specific gravity of measuring liquid = 0.8

The arrangement is shown in Fig.

Let P_A = pressure at A

P_B = pressure at H.

The points C and D lie on the same horizontal line.

Hence pressure at C should be equal to pressure at D.

Solution:

But pressure at C $= P_A - \rho g h$

$= P_A - 1000 \times 9.81 \times (0.35)$

But pressure at D $= P_B - \rho_1 g h_1 - \rho_2 g h_2$

$= P_B - 1000 \times 9.81 \times (0.35) - 800 \times 9.81 \times 0.3$

But pressure at C = pressure at D

$$P_A - 1000 \times 9.81 \times .35 = P_B - 1000 \times 9.81 \times 0.35 - 800 \times 9.81 \times 0.3$$

$$800 \times 9.81 \times 0.3 = P_B - P_A$$

$$P_B - P_A = 800 \times 9.81 \times 0.3$$

$$\mathbf{P_B - P_A = 2354.4\ N/m^2}$$ **Ans.**

MECHANICAL GAUGES

Mechanical gauges are used to measure high fluid pressure and where high precision is not required. Some of common types of mechanical gauges are given below:

(a) ***Diaphragm Pressure Gauge***

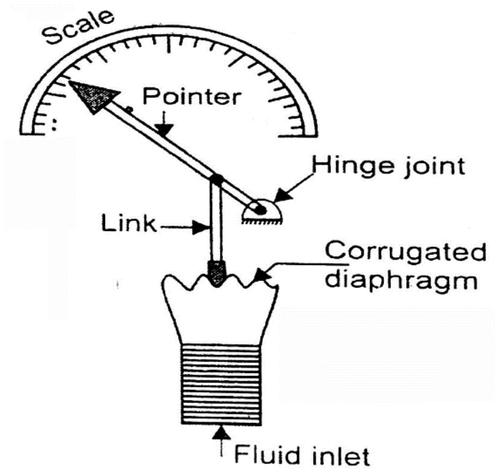
(b) Bourdon tube pressure gauge

(c) Bellows pressure gauge

(d) Dead-weight pressure gauge

Diaphragm Pressure Gauge

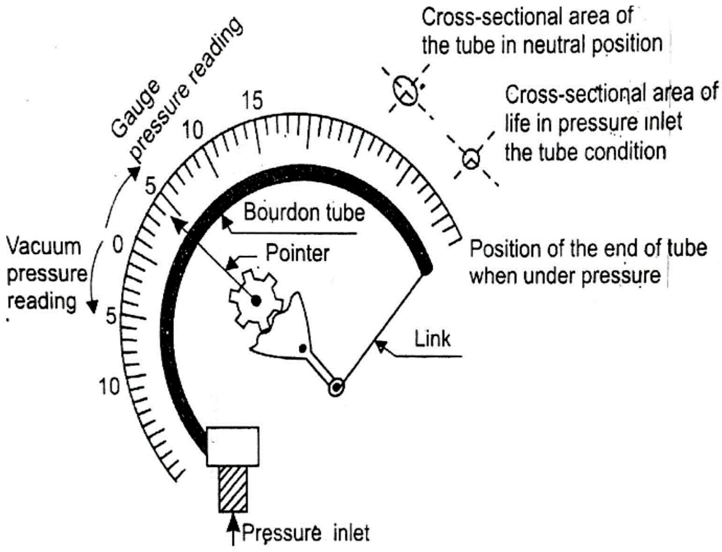
- ✓ The pressure responsive element in this gauge is an elastic sheet corrugated diaphragm. The diaphragm gets deflection being towards the low pressure side.



- ✓ When the fluid enters into the diaphragm, it causes its elastic deformation under pressure to be transmitted to a pointer through link and hinge joints as shown in fig.
- ✓ A pointer is moved on a graduated circular dial in pressure units. However, this pressure gauge is used to measure relatively low pressure. The Aneroid barometer operates on a similar principle.

Bourdon Tube Pressure Gauge:

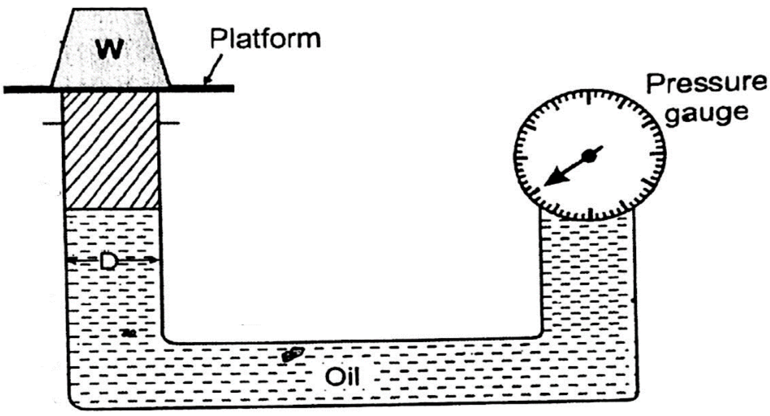
- ✓ The pressure responsive element in Bourdon tube pressure gauge is a tube of steel or bronze which is elliptical cross-section and curved into a-circular arc, called Bourdon tube.
- ✓ The outer end of the tube is closed and free to move. The other end of the tube, through which the fluid enters, is rigidly fixed to the frame as shown in Fig.
- ✓ The pressure gauge is connected to the vessel containing fluid under pressure. Due to increase in internal pressure, the elliptical cross-section of the tube tends to become circular, thus causing the tube to straighten out slightly.



Bourdon tube pressure gauge

- ✓ The outward movement of the free end of the tube is transmitted, through a link, quadrant and pinion, to a pointer which moving clockwise on the graduated circular dial indicates the pressure intensity of the fluid.
- ✓ When a gauge is connected to a partial vacuum, the Bourdon tube tends to close, thereby moving the pointer in anti-clockwise duration, indicating the negative or vacuum pressure.
- ✓ The movement of the free end of the Bourdon tube in directly proportional to the difference between the external atmospheric pressure and internal fluid pressure.
- ✓ Hence the Bourdon pressure gauge records (a) the gauge pressure; which is the difference between fluid pressure and outside atmospheric pressure, and (b) the negative or vacuum pressure which is difference between outside atmospheric pressure and fluid pressure.

Dead-weight Pressure Gauge:



Dead weight pressure gauge

- ✓ It consists of placing a dead weight on the top of a plunger fitted in a vertical cylinder. Oil is used as the working fluid in dead weight pressure gauge. We know that intensity of pressure for any load "W" on the plunger is given by p= W/A

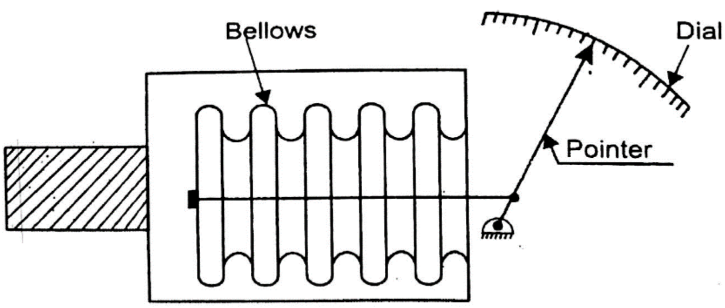
  where, A = cross-sectional area of the plunger, ($\frac{\pi}{4}$ D^2)

  D = diameter of the plunger

- ✓ According to Pascal's law: the intensity of pressure (p) at any point in a fluid at rest is same in all direction. So same pressure (p) is transmitted to the pressure gauge to be calibrated and the pointer of the pressure gauge moves and takes up a steady position on the dial.
- ✓ That position is marked as 'p'. In this way, by loading the plunger by loads of other magnitudes, other intensities of pressures are marked on the dial.

Bellows Pressure Gauge:

- ✓ In this pressure gauge the pressure responsive element is made of a thin metallic tube having deep circumferential corrugations.
- ✓ In response to the pressure changes this elastic element expands or contracts, thereby moving the pointer on a graduated circular dial as shown in Fig.

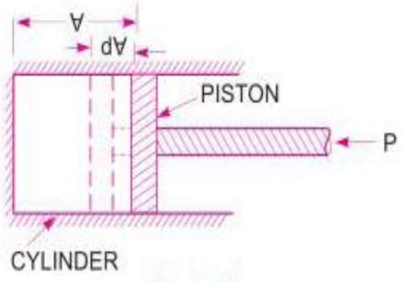


Bellows Pressure Gauge

COMPRESSIBILITY AND BULK MODULUS

Compressibility is the reciprocal of the bulk modulus of elasticity, K which is defined as the ratio of compressive stress to volumetric strain.

Consider a cylinder fitted with a piston as shown in Fig.



Let,,

V = Volume of a gas enclosed in the cylinder

p = Pressure of gas when volume is V

Let,

The pressure is increased to p + dp,

The volume of gas decreases from V to V - dV,

Then increase in pressure = dp kgf / m^2

Decrease in volume = dV

Volumetric strain $= \frac{dV}{V}$

[- ve sign means the volume decreases with increase of pressure]

$$\text{Bulk modulus } K = \frac{\text{Increase of pressure}}{\text{Volumetric strain}}$$

$$= \frac{dp}{\frac{dV}{V}} = \frac{-dp}{V} V$$

$$\text{Compressibility,} \quad = \frac{1}{K}$$

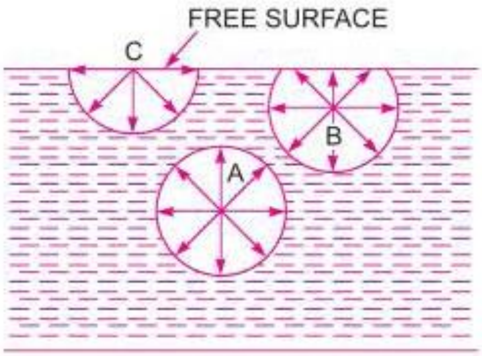
VAPOUR PRESSURE AND CAVITATION

- ✓ A change from the liquid state to the gaseous state is known as vaporization. The vaporization (which depends upon the prevailing pressure and temperature condition) occurs because of continuous escaping of the molecules through the free liquid surface.
- ✓ Consider a liquid (say water) which is confined in a closed vessel. Let the temperature of liquid is 20°C and pressure is atmospheric. This liquid will vaporise at 100°C. When vaporization takes place, the molecules escapes from the free surface of the liquid.
- ✓ These vapour molecules get accumulated in the space between the free liquid surface and top of the vessel. These accumulated vapours exert a pressure on the liquid surface. This pressure is known as vapour pressure of the liquid or this is the pressure at which the liquid is converted into vapours.
- ✓ Again consider the same liquid at 20°C at atmospheric pressure in the closed vessel. If the pressure above the liquid surface is reduced by some means, the boiling temperature will also reduce. If the pressure is reduced to such an extent that it becomes equal to or less than the vapour pressure, the boiling of the liquid will start, though the temperature of the liquid is 20°C.
- ✓ Thus a liquid may boil even at ordinary temperature, if the pressure above the liquid surface is reduced so as to be equal or less than the vapour pressure of the liquid at that temperature.

- Now consider a flowing liquid in a system. If the pressure at any point in this flowing liquid becomes equal to or less than the vapour pressure, the vaporization of the liquid starts. The bubbles of these vapours are carried by the flowing liquid into the region of high pressure where they collapse, giving rise to high impact pressure.
- The pressure developed by the collapsing bubbles is so high that the material from the adjoining boundaries gets eroded and cavities are formed on them. This phenomenon is known as cavitation.
- Hence the cavitation is the phenomenon of formation of vapour bubbles of a flowing liquid in a region where the pressure of the liquid falls below the vapour pressure and sudden collapsing of these vapour bubbles in a region of higher pressure.
- When the vapour bubbles collapse, a very high pressure is created. The metallic surfaces, above which the liquid is flowing, is subjected to these high pressures, which cause pitting action on the surface. Thus cavities are formed on the metallic surface and hence the name is cavitation.

SURFACE TENSION AND CAPILLARITY

Surface tension is defined as the tensile force acting on the surface of a liquid in contact with a gas or on the surface between two immiscible liquids such that the contact surface behaves like a membrane under tension. The magnitude of this force per unit length of the free surface will have the same value as the surface energy per unit area. It is denoted by Greek letter σ (called sigma). In MKS units, it is expressed as kgf /m while in SI units as N/m.



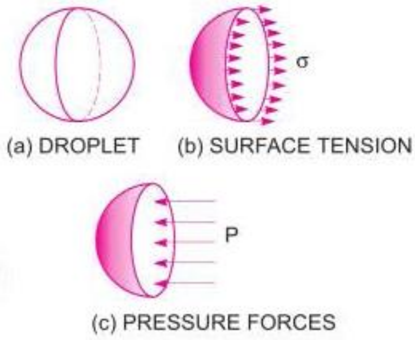
The phenomenon of surface tension is explained by Fig. Consider three molecules A, B, C of a liquid in a mass of liquid. The molecule A is attracted in all directions equally by the surrounding molecules of the liquid. Thus the resultant force acting on the molecule A is zero. But the molecule B, which is situated near the free surface, is acted upon by upward and downward forces which are unbalanced. Thus a net resultant force on molecule B is acting in the downward direction. The molecule C, situated on the free surface of liquid, does experience a resultant downward force. All the molecules on the free surface experience a downward force. Thus the free surface of the liquid acts like a very thin film under tension of the surface of the liquid act as though it is an elastic membrane under tension.

Surface Tension on Liquid Droplet: Consider a small spherical droplet of a liquid of radius V. On the entire surface of the droplet, the tensile force due to surface tension will be acting.

Let G = Surface tension of the liquid

p = Pressure intensity inside the droplet (in excess of the outside pressure intensity)

d = Dia. of droplet.



Let the droplet is cut into two halves. The forces acting on one half (say left half) will be

(i) tensile force due to surface tension acting around the circumference of the cut portion as shown in Fig. (b) and this is equal to, $= \sigma \times$ Circumference

$= \sigma \times \pi d$

(ii) pressure force on the area, $\frac{\pi}{4} d^2 = p \frac{\pi}{4} d^2$

Equating the above equations, w get

$$p = \frac{4\sigma}{d}$$

Surface Tension on a Hollow Bubble: A hollow bubble like a soap bubble in air has two surfaces in contact with air, one inside and other outside. Thus two surfaces are subjected to surface tension. In such case, we have

Surface Tension on a Liquid Jet:

Consider a liquid jet of diameter 7f and length 'L' as shown in Fig.

Let p = Pressure intensity inside the liquid jet above the outside pressure

a = Surface tension of the liquid. A

Consider the equilibrium of the semi jet, we have Force due to pressure = p x area of semi jet

= pxLxd

Force due to surface tension = a x 2L. Equating the forces, we have

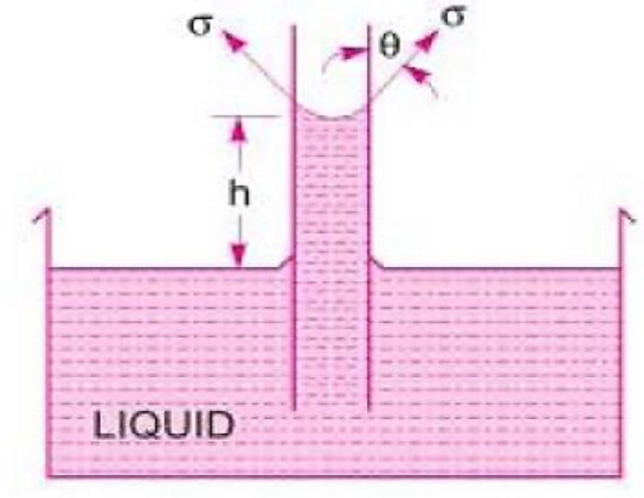
p x Lx d = G x2L

CAPILLARITY

Capillarity is defined as a phenomenon of rise or fall of a liquid surface in a small tube relative to the adjacent general level of liquid when the tube is held vertically in the liquid. The rise of liquid surface is known as capillary rise while the fall of the liquid surface is known as capillary depression. It is expressed in terms of cm or mm of liquid. Its value depends upon the specific weight of the liquid, diameter of the tube and surface tension of the liquid.

Expression for Capillary Rise. Consider a glass tube of small diameter V/' opened at both ends and is inserted in a liquid, say water. The liquid will rise in the tube above the level of the liquid.

Let h = height of the liquid in the tube. Under a state of equilibrium, the weight of liquid of height h is balanced by the force at the surface of the liquid in the tube. But the force at the surface of the liquid in the tube is due to surface tension.



Capillary Rise

Let a = Surface tension of liquid

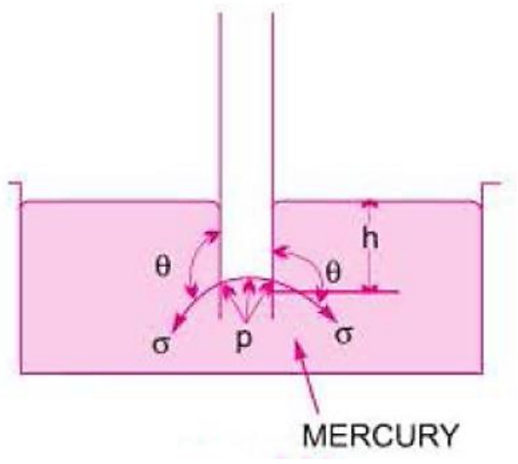
8 = Angle of contact between liquid and glass tube.

The weight of liquid of height h in the tube = (Area of tube

Expression for Capillary Fall: If the glass tube is dipped in mercury, the level of mercury in the tube will be lower than the general level of the outside liquid as shown in Fig.

Let h = Height of depression in tube.

Then in equilibrium, two forces are acting on the mercury inside the tube. First one is due to surface tension acting in the downward direction and is equal to a x nd x cos 0.



Second force is due to hydrostatic force acting upward and is equal to intensity of pressure at a depth

Equating the two, we get

Value of θ for mercury and glass tube is 128°.

Calculate the capillary effect in millimetres in a glass tube of 4 mm diameter, when immersed in (i) water and (ii) mercury. The temperature of the liquid is 20°C and the values of the surface tension of wafer and mercury at 20°C in contact with air are 0.073575 N/m and 0.51 N/m respectively. The angle of contact for water is zero and that for mercury is 130°. Take density of water at 20°C as equal to 998 kg/m^3

Solution. Given :

Dia. of tube, d = 4mm = 4 x 10^{-3} m

The capillary effect (ie. capillary rise or depression) is given by equation as,

$$h = \frac{4\sigma \cos\theta}{\rho\, g\, d}$$

where, σ = surface tension in N/m

θ = angle of contact

ρ = density

(i) Capillary effect for water

σ = 0.073575 N/m

θ = 0°

ρ = 998 kg/m^3 at 20°C

$$h = \frac{4 \text{ x} .073575 \text{ x} \cos 0°}{998 \text{ x } 9.81 \text{ x } 4\text{x}10^{-3}}$$

h = 7.51x 10^{-3} m

h = 7.51 mm

(ii) Capillary effect for mercury

σ = 0.51 N/m,

θ = 130°

ρ = sp. gr. x 1000 = 13.6 x 1000 = 13600 kg/m^3

$$h = \frac{4 \text{ x } 0.51 \text{ x} \cos 130°}{13600 \text{ x } 9.81 \text{ x } 4\text{x}10^{-3}}$$

= - 2.46 x 10^{-3} m

h = - 2.46 mm

[The negative sign indicates the capillary depression]

An oil of viscosity 5 poise is used for lubrication between a shaft and sleeve. The diameter of the shaft is 0.5 m and it rotates at 200 r.p.m. Calculate the power lost in oil for a sleeve length of 100 mm. The thickness of oil film is 1.0 mm.

Solution. Given :

Viscosity μ = 5 poise = $\frac{5}{10}$ Ns/m^2 = 0.5 Ns/m^2

Diameter of shaft, D = 0.5

Speed of shaft, N = 200 r.p.m

Sleeve length L = 100 mm = 100 x 10^{-3} m = 0.1 m

Thickness of oil film, t = 1.0 mm = 1.0 x 10^{-3} m

Tangential velocity of shaft, u = $\frac{\pi D N}{60} = \frac{\pi \text{ x } 0.5 \text{ x } 200}{60}$

u = 5.235 m/s

using this relation,

$$\tau = \mu \frac{du}{dy}$$

where du = Change of velocity = u - 0 = u = 5.235 m/s

dy = Change of distance = t = 1.0 x 10^{-3} m

$$\tau = \frac{0.5 \text{ x } 5.235}{1 \text{ x } 10^{-3}} = \mathbf{2617.5\ N/m^2}$$

This is shear stress on shaft,

Shear force on the shaft, F = shear stress x Area

= 2617.5 x π D L = 2617.5 x π x 0.5 x 0.1

F = 410.95 N

Torque on the shaft, **T =** Force x $\frac{D}{2}$

= 410.95 x $\frac{0.5}{2}$

T = 102.74 Nm

Power lost, =T x ω (Watts) = $\frac{2\pi\ N T}{60}$ (W) = $\frac{2\pi\ 200 \text{ x } 102.74}{60}$

Power lost = 2150 W = 2.15 kW

Calculate the capillary rise in a glass tube of 2.5 mm diameter when immersed vertically in (a) water and fb) mercury. Take surface tensions σ = 0.0725 N/m for water and σ = 0.52 N/m for mercury in contact with air. The specific gravity for mercury is given as 13.6 and angle of contact = 130°.

The capillary rise in the glass tube is not to exceed 0.2 mm of water. Determine its minimum size, given that surface tension for water in contact with air = 0.0725 N/m.

HYDROSTATIC FORCES ON SURFACES

This chapter deals with the fluids (i.e., liquids and gases) at rest. This means that there will be no relative motion between adjacent or neighboring fluid layers. The velocity gradient, which is equal to the change of velocity between two adjacent fluid layers divided by the distance between the layers, will be zero or $\frac{du}{dy} = 0$. The shear stress which is equal to $\mu \frac{du}{dy}$ will also be zero. Then the forces acting on the fluid particles will be :

- ✓ due to pressure of fluid normal to the surface
- ✓ due to gravity (or self-weight of fluid particles)

TOTAL PRESSURE AND CENTRE OF PRESSURE

Total pressure is defined as the force exerted by a static fluid on a surface either plane or curved when the fluid comes in contact with the surfaces. This force always acts normal to the surface.

Centre of pressure is defined as the point of application of the total pressure on the surface. There are four cases of submerged surfaces on which the total pressure force and centre of pressure is to be determined. The submerged surfaces may be:

- ✓ Vertical plane surface
- ✓ Horizontal plane surface
- ✓ Inclined plane surface
- ✓ Curved surface

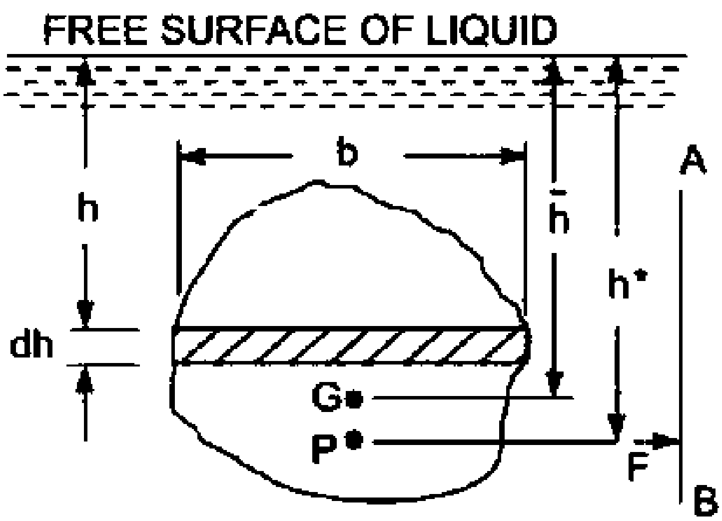
1. VERTICAL PLANE SURFACE SUBMERGED IN LIQUID

Consider a plane vertical surface of arbitrary shape immersed in a liquid as shown in Fig. Let A = Total area of the surface

h = Distance of C.G. of the area from free surface of liquid

G = Centre of gravity of plane surface P = Centre of pressure

h* = Distance of centre of pressure from free surface of liquid.



(a) Total Pressure (F): The total pressure on the surface may be determined by dividing the entire surface into a number of small parallel strips. The force on small strip is then calculated and the total pressure force on the whole area is calculated by integrating the force on small strip.
Consider a strip of thickness dh and width b at a depth of h from free surface of liquid as shown in Fig.

$$F=pgAh$$

Center of pressure (h*):

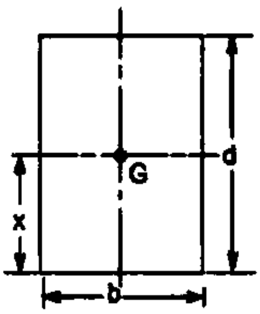
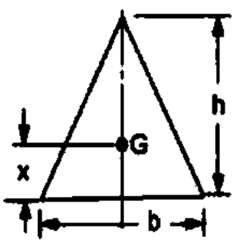
(b) Centre of Pressure (h*): Centre of pressure is calculated by using the "Principle of Moments", which states that the moment of the resultant force about an axis is equal to the sum of moments of the components about the same axis.
The resultant force F is acting at P, at a distance h* from free surface of the liquid as shown in Fig

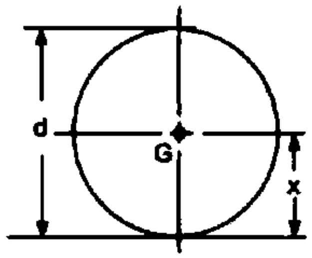
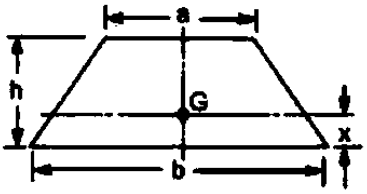
$$h^* = \frac{I_G + A\bar{h}^2}{A\bar{h}} = \frac{I_G}{A\bar{h}} + \bar{h}$$

In the above equation equation, h is the distance of C.G. of the area of the vertical surface from free surface of the liquid. Hence from equation, it is clear that:

- ✓ Centre of pressure (i.e., h*) lies below the centre of gravity of the vertical surface.
- ✓ The distance of centre of pressure from free surface of liquid is independent of the density of the liquid.

The moments of inertia and other geometric properties of some important plane surfaces

Plane surface	*C.G. from the base*	*Area*	*Moment of inertia about an axis passing through C.G. and parallel to base (I_G)*	*Moment of inertia about base (I_0)*
1. Rectangle 	$x = \frac{d}{2}$	bd	$\frac{bd^3}{12}$	$\frac{bd^3}{3}$
2. Triangle 	$x = \frac{h}{3}$	$\frac{bh}{2}$	$\frac{bh^3}{36}$	$\frac{bh^3}{12}$

Plane surface	*C.G. from the base*	*Area*	*Moment of inertia about an axis passing through C.G. and parallel to base (I_G)*	*Moment of inertia about base (I_0)*
3. Circle 	$x = \frac{d}{2}$	$\frac{\pi d^2}{4}$	$\frac{\pi d^4}{64}$	—
4. Trapezium 	$x = \left(\frac{2a+b}{a+b}\right)\frac{h}{3}$	$\frac{(a+b)}{2} \times h$	$\left(\frac{a^2+4ab+b^2}{36(a+b)}\right) \times h^3$	—

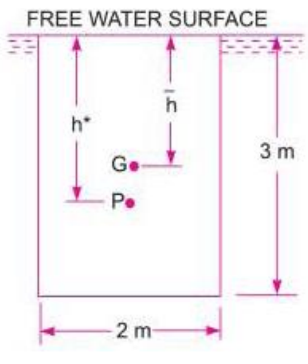
A rectangular plane surface is 2 m wide and 3 m deep. It lies in vertical plane in water. Determine the total pressure and position of centre of pressure on the plane surface when its upper edge is horizontal and (a) coincides with water surface, (b) 2.5 m below the free water surface.

Given :

Width of plane surface, b = 2 m

Depth of plane surface, d = 3 m

$A = 3 \times 2 = 6 \text{ m}^2$,



Solution:

(a)Upper edge coincides with water surface

Total pressure is given by equation as ,

$$F = \rho g A \bar{h}$$

where, $\rho = 1000 \text{ kg/ m}^3$

$g = 9.81 \text{ m/s}^2$

$\bar{h} = \frac{1}{2}(3) = 1.5 \text{ m}$

F= 1000 x 9.81 x 6 x 1.5 = **88290 N. Ans.**

Depth of centre of pressure is given by equation as,

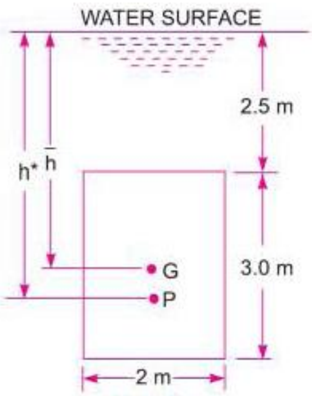
$$h^* = \frac{I_G}{A\bar{h}} + \bar{h}$$

where, I_G = M.O.I, about C.G. of the area of surface

$$\frac{bd^3}{12} = \frac{2 \times 3^3}{12} = 4.5\ m^4$$

$$\frac{4.5}{6 \times 1.5} + 1.5 = 0.5 + 1.5 = \mathbf{2.0\ m}$$

(b) Upper edge is 2.5 m below water surface (Fig.)



Total pressure (F) is given by

$F = \rho\, g\, A\, \bar{h}$

where h = Distance of C.G. from free surface of water

$$2.5 + \frac{3}{2} = 4.0\ \text{m}$$

F = 1000 x9.81 x 6 x 4.0

= 235440 N. Ans.

Centre of pressure is given by, **h* =** $\frac{I_G}{A\,h} + \bar{h}$

where, I_G = 4.5, A = 6.0, $\bar{h}$ = 4.0

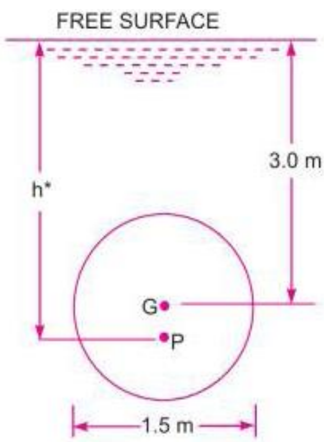
h* = $\frac{4.5}{6.0 \times 4.0} + 4.0$

= 0.1875 + 4.0

h* = 4.1875 m Ans.

Determine the total pressure on a circular plate of diameter 1.5 m which is placed vertically in water in such a way that the centre of the plate is 3 m below the free surface of water. Find the position of centre of pressure also.

Given:



Dia. of plate, d = 1.5 m

A = $\frac{\pi}{4}$ $(1.5)^2$ = 1.767 m^2

$\bar{h}$ = 3.0 m

Solution:

Total pressure is given by equation,

$F = \rho\, g\, A\, \bar{h}$

= 1000 x 9.81 x 1.767 x 3.0 N

F = 52002.81 N Ans.

Position of centre of pressure (h*) is given by equation,

$$h^* = \frac{I_G}{A\,h} + \bar{h}$$

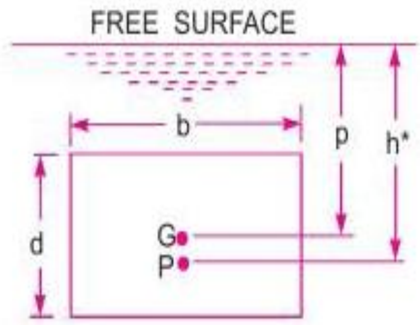
where, $I_G = \frac{\pi d^4}{64} = \frac{\pi \times 1.5^4}{64}$

$$h^* = \frac{0.2485}{1.767 \times 3.0} + 3.0$$

$$= 0.0468 + 3.0$$

h* = 3.0468 m Ans.

A rectangular sluice gate is situated on the vertical wall of a lock. The vertical side of the sluice is 'd' metres in length and depth of centroid of the area is 'p' m below the water surface.



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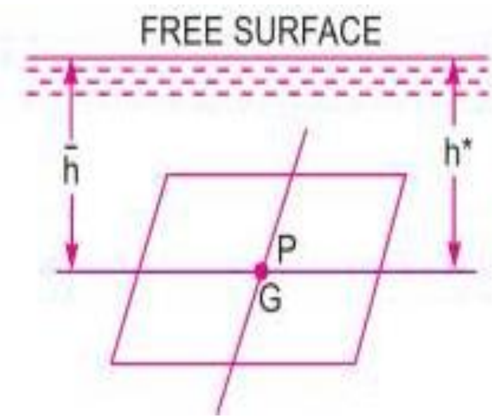
2. HORIZONTAL PLANE SURFACE SUBMERGED IN LIQUID

Consider a plane horizontal surface immersed in a static fluid. As every point of the surface is at the same depth from the free surface of the liquid, the pressure intensity will be equal on the entire surface and equal to, $p = \rho g h$, where h is depth of surface.



Let A = Total area of surface

Then total force, F, on the surface

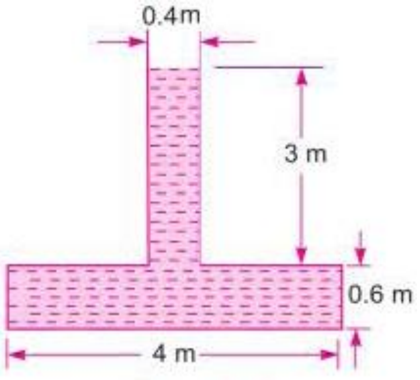
$$= p \times \text{Area} = \rho g h \times A = \rho g A \bar{h}$$

where $\bar{h}$ = Depth of C.G. from free surface of liquid = h

h* = Depth of centre of pressure from free surface = h

Fig. Shows a tank full of water. Find:

1. **Total pressure on the bottom of tank.**
2. **Weight of water in the tank.**
3. **Hydrostatic paradox between the results of (i) and (ii).**
4. **Width of tank is 2 m.**



Given:

Depth of water on bottom of tank, $h_1 = 3 + 0.6 = 3.6$ m

Width of tank = 2 m

Length of tank at bottom Area at the bottom, = 4 m

Area at the bottom = 4 x 2 = 8 m^2

(i)Total pressure F, on the bottom is

$F = \rho\ g\ A\ \bar{h}$ = 1000 x 9.81 x 8 x 3.6

= 282528 N Ans.

(ii) Weight of water in tank = ρ g x Volume of tank

= 1000 x 9.81 x [3 x 0.4 x 2 + 4 x 0.6 x 2]

= 1000 x 9.81 [2.4 + 4.8]

= 70632 N Ans.

(iii)From the results of (i) and (ii),

It is observed that the total weight of water in the tank is much less than the total pressure at the bottom of the tank. This is known as Hydrostatic paradox.

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3. INCLINED PLANE SURFACE SUBMERGED IN LIQUID

Consider a plane surface of arbitrary shape immersed in a liquid in such a way that the plane of the surface makes an angle 9 with the free surface of the liquid as shown in Fig.

Let

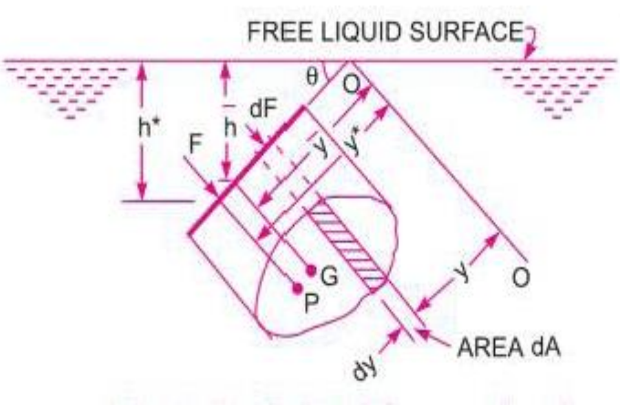
A = Total area of inclined surface

$\bar{h}$ = Depth of C.G. of inclined area from free surface

h* = Distance of centre of pressure from free surface of liquid

θ = Angle made by the plane of the surface with free liquid surface

Let the plane of the surface, if produced meet the free liquid surface at O. Then 0-0 is the axis perpendicular to the plane of the surface.



Let $\bar{y}$ = distance of the C.G. of the inclined surface from 0-0

y* = distance of the centre of pressure from O-O.

F = ρ g sin θ $\bar{y}$ x A ($\bar{h} = \bar{y}$ sin θ)

F = ρ g A $\bar{h}$

Centre of Pressure (h*)

$$h^* = \frac{I_G \ \sin^2\theta}{A\,\bar{h}} + \bar{h}$$

If θ = 90°. the above equation, becomes same as equation which is applicable to vertically plane submerged surfaces.

In above equation,, IG = M.O.I, of inclined surfaces about an axis passing through G and parallel to 0-0.

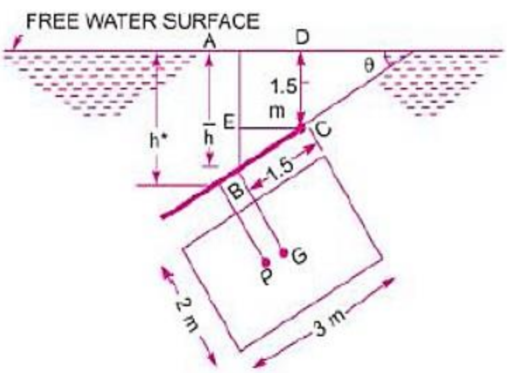
A rectangular plane surface 2 m wide and 3 m deep lies in water in such a way that its plane makes an angle of 30° with the free surface of water. Determine the total pressure and position of centre of pressure when the upper edge is 1.5 m below the free water surface.

Given :

Width of plane surface, b = 2 m

Depth, d = 3 m Angle, θ = 30°

Distance of upper edge from free water surface = 1.5 m



Solution:

(i) Total pressure force is given by equation as

$$F = \rho g A \bar{h}$$

where ρ = 1000 kg/m^3

A = b x d= 3 x 2 = 6 m^2

$\bar{h}$ = Depth of C.G. from free water surface

= 1.5+ 1.5 sin 30°

{ $\bar{h}$ = AE + EB = 1.5 + BC sin 30° = 1.5+ 1.5 sin 30° }

= 1.5+ 1.5 x $\frac{1}{2}$ = 2.25 m

F = 1000 x 9.81 x 6 x 2.25

F = 132435 N. Ans.

(II) Centre of pressure (h*) Using equation , we have

$$h^* = \frac{I_G \sin^2\theta}{A\bar{h}} + \bar{h}$$

where, $I_G = \frac{bd^3}{12} = \frac{2 \times 3^3}{12}$

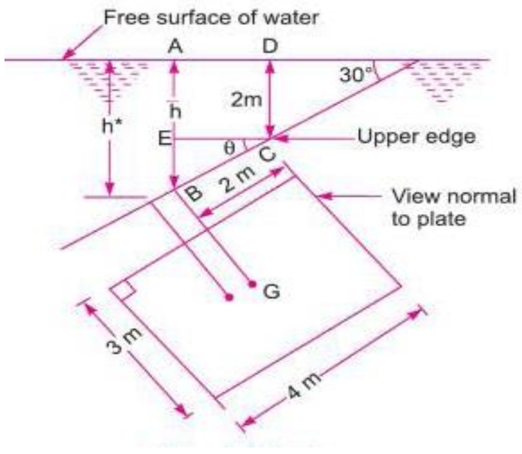
I_G = **4.5 m^4**

$h^* = \frac{4.5 \times \sin^2 30}{6 \times 2.25} + 2.25$

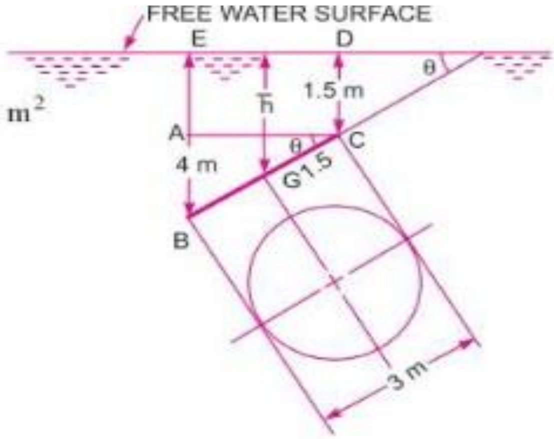
h* = 0.0833 + 2.25

h* = 2.3333 m

A rectangular plane surface 3 m wide and 4 m deep lies in water in such a way that its plane makes an angle of 30° with the free surface of water. Determine the total pressure force and position of centre of pressure, when the upper edge is 2 m below the free surface.



A circular plate 3.0 m diameter is immersed in water in such a way that its greatest and least depth below the free surface are 4 m and 1.5 m respectively. Determine the total pressure on one face of the plate and position of the centre of pressure.



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4. CURVED SURFACE SUB-MERGED IN LIQUID

Consider a curved surface AB, sub-merged in a static fluid as shown in Fig. Let dA is the area of a small strip at a depth of It from water surface.

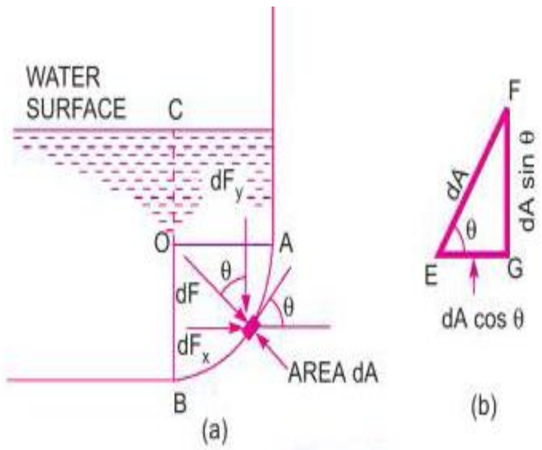
Then pressure intensity on the area dA is = pgh

Pressure force, dF = p x Area = ρ g h x dA

This force d_F acts normal to the surface.

Hence total pressure force on the curved surface should be

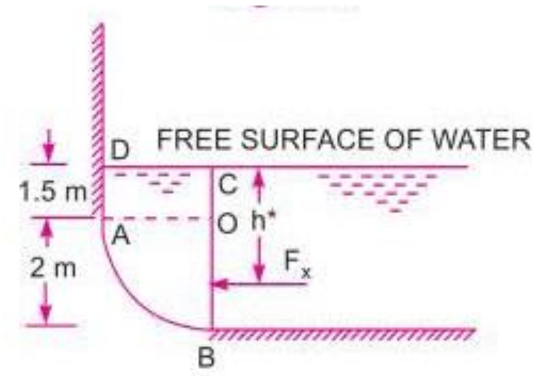
$$F = \int \rho\, g\, h\, dA$$



But here as the direction of the forces on the small areas are not in the same direction, but varies from point to point. Hence integration of equation for curved surface is impossible.

Compute the horizontal and vertical components of the total force acting on a curved surface AB, which is in the form of a quadrant of a circle of radius 2 m as shown in Fig. Take the width of the gate as unity.

Given :



Width of gate = 1.0 m,

Radius of the gate = 2.0 m

Distance , AO = OB = 2 m

Solution:

Horizontal force, F_x exerted by water on gate is given by equation as

F_x = Total pressure force on the projected area of curved surface AB on vertical plane

= Total pressure force on OB

{projected area of curved surface on vertical plane = OB x 1}

$F = p\ g\ A\ \bar{h}$

$= 1000 \times 9.81 \times 2 \times 1 \times (1.5 + \frac{2}{2})$

[area of OB = A = BO x 1 = 2 x 1 = 2]

$\bar{h}$ = Depth of C.G. of OB from free surface = $1.5 + \frac{2}{2}$

Fx = 9.81 x 2000 x 2.5 **= 49050 N**

The point of application of F_x is given by , **h* =** $\frac{I_G}{A\,\bar{h}} + \bar{h}$

where, I_G = M.O.I, about C.G. of the area of surface

$$\frac{bd^3}{12} = \frac{1 \times 2^3}{12} = \frac{2}{3}\ m^4$$

h* = $\frac{\frac{2}{3}}{2 \times 2.5} + 2.5 = \frac{1}{7.5} + 2.5$ m

= 0.1333 + 2.5

h* = 2.633 m from free surface

Vertical force Fy exerted by water is given by equation,

Fy = Weight of water supported by AB upto free surface

= Weight of portion DABOC

= Weight of DAOC + Weight of water A OB

= ρ g [Volume of DAOC + Volume of AOB]

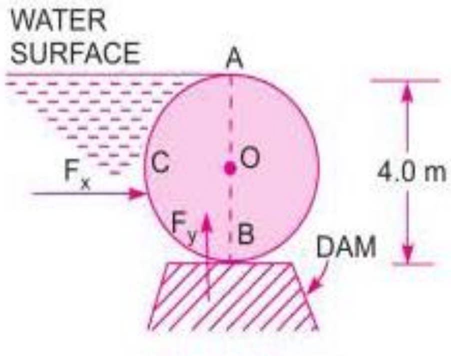
$= 1000 \times 9.81 \, [\, AD \times AO \times 1 + \frac{\pi}{4} \times (OA)^2 \times 1 \,]$

$= 1000 \times 9.81 \, [\, 1.5 \times 2.0 \times 1 + \frac{\pi}{4} \times (2)^2 \times 1 \,]$

$= 1000 \times 9.81 \, [\, 3.0 + \pi \,]$ N

Fy = 60249.1 N

Find the magnitude and direction of the resultant force due to water acting on a roller gate of cylindrical form of 4.0 m diameter, when the gate is placed on the dam in such a way that water is just going to spill. Take the length of the gate as 8 m



BUOYANCY

When a body is immersed in a fluid, an upward force is exerted by the fluid on the body. This upward force is equal to the weight of the fluid displaced by the body and is called the force of buoyancy or simply buoyancy.

CENTRE OF BUOYANCY

It is defined as the point, through which the force of buoyancy is supposed to act. As the force of buoyancy is a vertical force and is equal to the weight of the fluid displaced by the body, the centre of buoyancy will be the centre of gravity of the fluid displaced.

Find the volume of the water displaced and position of centre of buoyancy for a wooden block of width 2.5 m and of depth 1.5 m, when it floats horizontally in water. The density of wooden block is 650 kg/m3 and its length 6.0 m

Given:

Width = 2.5m

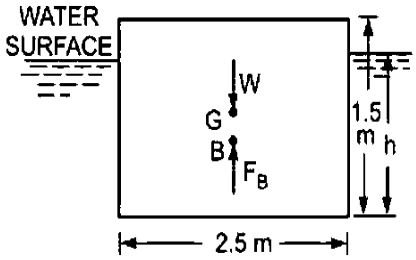
Depth = 1.5m

Length = 6.0m

Volume of the block = 2.5 x 1.5x6.0

= 22.50 m^3

Density of wood, ρ = 650 kg/m^3



Solution:

Weight of block = ρ x g x Volume

= 650 x 9.81 x 22.50 N

= 143471 N

For equilibrium the weight of water displaced = Weight of wooden block

= 143471 N

Volume of water displaced = $\frac{Weight\ of\ water\ displaced}{Weight\ density\ of\ water}$ = $\frac{143471}{1000 \times 9.81}$

(Weight density of water = 1000 x 9.81 N/ m^3)

Position of Centre of Buoyancy.

Volume of wooden block in water = Volume of water displaced

2.5 x h x 6.0 = 14.625 m^3, where h is depth of wooden block in water

$$h = \frac{14.625}{2.5 \times 6.2}$$

h = 0.975 m

Centre of Buoyancy = $\frac{0.975}{2}$ = **0.4875 m from base. Ans.**

A wooden log of 0.6 m diameter and 5m length is floating in river water. Find the depth of the wooden log in water when the sp. Gravity of the log is 0.7.

Given:

Dia. of log $D = 0.6$ m

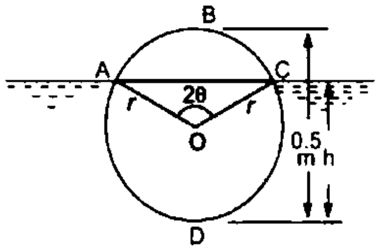
Length, $L = 5$ m

Sp. gr. $S = 0.7$

Density of log (ρ) = 0.7 x 1000 = 700 kg/m^3

Weight density of log, $w = \rho \times g$

= 700 x 9.81 N/m^3



Solution:

Find depth of immersion (or) h

Weight of wooden log = Weight density x Volume of log

$= 700 \times 9.81 \times \frac{\pi}{4}(D)^2 \times L$

$= 700 \times 9.81 \times \frac{\pi}{4}(.6)^2 \times 5$ N

= 989.6 x 9.91 N

For equilibrium,

Weight of wooden log = Weight of water displaced

= Weight density of water x Volume of water displaced

∴ Volume of water displaced $= \frac{989.6 \times 9.81}{1000 \times 9.81}$

= 0.9896 m^3

(∴Weight density of water = 1000x9.81 N/m^3)

Let h is the depth of immersion

∴ Volume of log inside water = Area of ADCA x Length

= Area of ADCA x 5.0

But volume of log inside water = Volume of water displaced = 0.9896 m^3

∴ 0.9896 = Area of ADCA x 5.0

∴ Area of ADCA $= \frac{0.9896}{5.0}$ = **0.1979 m^2**

But area of ADCA = Area of curved surface ADCOA + Area of ΔAOC

$= \pi r^2 \left[\frac{360^0 - 2\theta}{360^0}\right] + \frac{1}{2} r \cos\theta \times 2r \sin\theta$

$= \pi r^2 \left[1 - \frac{\theta}{180^0}\right] + r^2 \cos\theta \sin\theta$

∴ $0.1979 = \pi(.3)^2 \left[1 - \frac{\theta}{180^0}\right] + (.3)^2 \cos\theta \sin\theta$

$0.1979 = .2827 - .00157\,\theta + 0.9 \cos\theta \sin\theta$

or $.00157\,\theta - .09 \cos\theta \sin\theta = .2827 - .1979 = 0.0848$

$$\theta - \frac{.09}{.00157} \cos\theta \sin\theta = \frac{.0848}{.00157}$$

or $\theta - 57.32 \cos\theta \sin\theta =$ 54.01

or $\theta - 57.32 \cos\theta \sin\theta$ -54.01 = 0

For $\theta = 60^0$, 60-57.32 x 0.5 x .866 – 54.01 = 60 – 24.81 – 54.01 = -18.82

For $\theta = 70^0$, 70-57.32 x 0.342 x .0.9396 – 54.01 = 70 – 18.4 – 54.01 = -2.41

For $\theta = 72^0$, 72-57.32 x 0.309 x .951 – 54.01 = 72 – 16.84 – 54.01 = +1.14

For $\theta = 71^0$, 71-57.32 x 0.325x .9455 – 54.01 = 71 – 17.61 – 54.01 = -0.376

$\therefore \theta = 71.5^0$, 71.5-57.32 x 0.3173 x .948 – 54.01 = 71.5 – 17.24 – 54.01 = +.248

Then h = r + r cos 71.5^0

= 0.3 + 0.3 x 0.3173

h = 0.395 m.

META-CENTRE

- ✓ It is defined as the point about which a body starts oscillating when the body is tilted by a small angle. The meta-centre may also be defined as the point at which the line of action of the force of buoyancy will meet the normal axis of the body when the body is given a small angular displacement.
- ✓ Consider a body floating in a liquid as shown in Fig.(a). Let the body is in equilibrium and G is the centre of gravity and B the centre of buoyancy. For equilibrium, both the points lie on the normal axis, which is vertical.

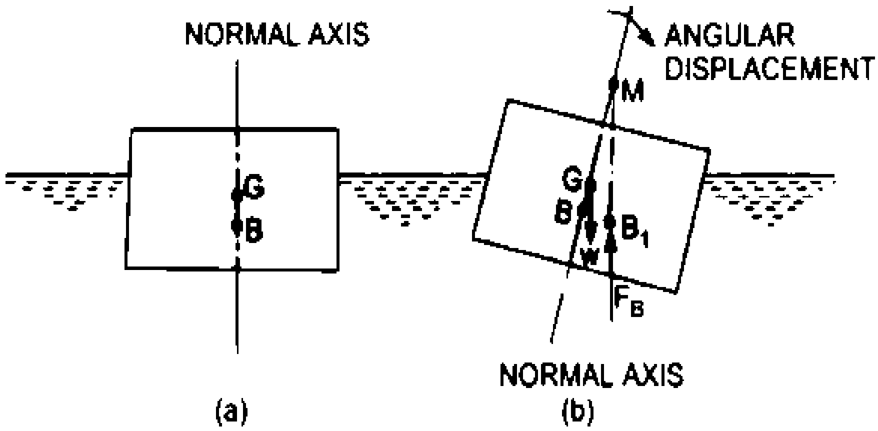


Fig. Meta – centre

- ✓ Let the body is given a small angular displacement in the clockwise direction as shown in Fig. (b). the centre of buoyancy, which is the centre of gravity of the displaced liquid or centre of gravity of the portion of the body sub-merged in liquid, will now be shifted towards right from the normal axis.
- ✓ Let it is at B_1 as shown in Fig.(b). The line of action of the force of buoyancy in this new position, will intersect the normal axis of the body at some point say M. This point M is called Meta-centre.

META-CENTRIC HEIGHT

The distance MG, i.e., the distance between the meta-centre of a floating body and the centre of gravity of the body is called meta-centric height.

Analytical Method For Meta-Centre Height:

- ✓ Fig (a) shows the position of a floating body in equilibrium. The location of centre of gravity and centre of buoyancy in this position is at G and B. The floating body is given a small angular displacement in the clockwise direction.
- ✓ This is shown in Fig (b). The new centre of buoyancy is at B_1. The vertical line through B_1 cuts the normal axis at M. Hence M is the meta-centre and GM is meta-centric height.

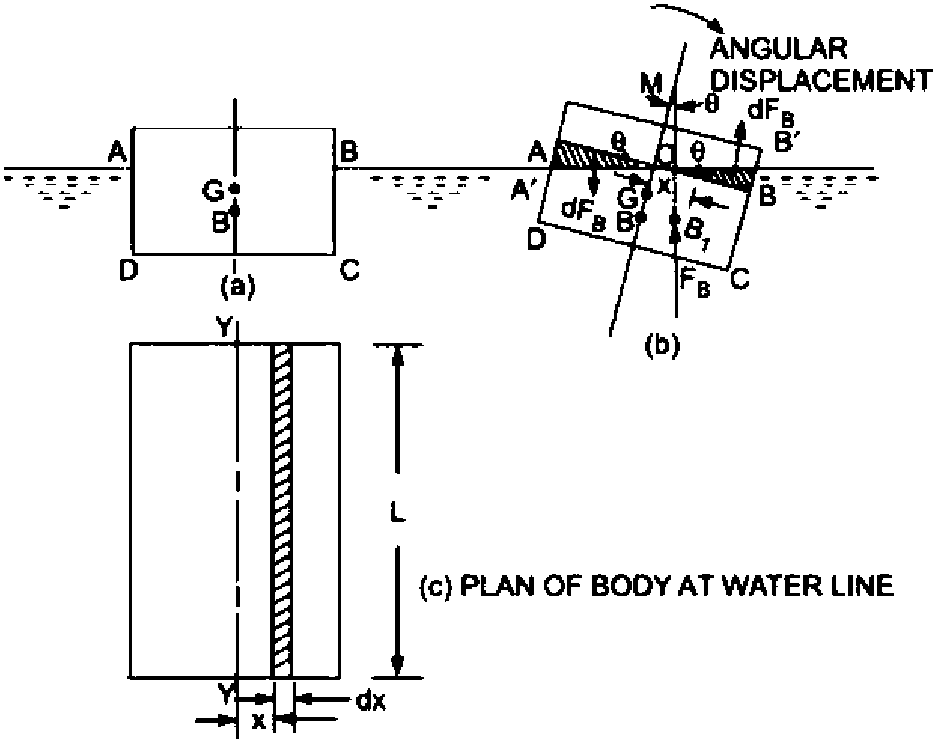


Fig Meta-centre height of floating body

- ✓ The angular displacement of the body in the clockwise direction causes the wedge-shaped prism BOB′ on the right of the axis to go inside the water while the identical wedge-shaped prism represented by AOA′ emerges out of the water on the left of the axis.
- ✓ These wedges represent a gain in buoyant force on the right side and a corresponding loss of buoyant force on the left side.
- ✓ The gain is represented by a vertical force dF_B acting through the C.G of the prism BOB′ while the loss is represented by an equal and opposite forece dF_B acting vertically downward through the centroid of AOA′. The couple due to these buoyant forces dF_B tends to rotate the ship in the counter clock wise direction.
- ✓ Also the moment caused by the displacement of the centre of buoyancy from B to B_1 is also in the counter clock wise direction. Thus these two couples must be equal.

Couple due to wedges. Consider towards the right of the axis a small strip of thickness dx at a distance x from O as shown in Fig.(b). The height of strip $x \times \angle BOB' = x \times \theta$.

$$[\because \angle BOB' = \angle AOA' = \angle BMB_1' = \theta]$$

$\therefore$ Area of strip = Height x Thickness = $x \times \theta \times dx$

If L is the length of the floating body, then

Volume of strip = Area x L

= $x \times \theta \times L \times dx$

$\therefore$ Weight of strip = ρg x Volume = $\rho g x\, \theta L\, dx$

Similarly, if a small strip of thickness dx at a distance x from O towards the left of the axis is considered, the weight of strip will be $\rho g x\, \theta L\, dx$. The two weights are acting in the opposite direction and hence constitute a couple.

Moment of this couple = Weight of each strip x Distance between these two weights

= $\rho g x\, \theta L\, dx\, [x + x]$

= $\rho g x\, \theta L\, dx \times 2x = \rho g x^2\, \theta L\, dx$

$\therefore$ Moment of the couple for the whole wedge

= $\int 2\, \rho g x^2\, \theta L\, dx$ ……. (1)

Moment of couple due to shifting of centre of buoyancy from B to B_1

= $F_B \times BB_1$

= $F_B \times BM \times \theta$ $\{\because BB_1 = BM x\theta \text{ if } \theta \text{ is very small}\}$

= $W \times BM \times \theta$ $\{\because F_B = W\}$….(2)

But these two couples are the same. Hence equating equations (4.1) and (4.2), we get

$W \times BM \times \theta = \int 2 \rho g x^2\, \theta\, L dx$

$W \times BM \times \theta = 2\rho g\theta \int x^2\, L dx$

$W \times BM = 2\rho g \int x^2\, L dx$

Now Ldx = Elemental area on the water line shown in Fig. (c) and = dA

$W \times BM = 2\rho g \int x^2\, dA$

But from Fig 4.5(c) it is clear that $\int x^2\, dA$ is the second moment of area of the plan of the body at water surface about the axis Y-Y. Therefore

$W \times BM = \rho g I$ *{Where* $I = 2 \int x^2\, dA$ *}*

$BM = \frac{\rho g I}{W}$

But W = Weight of the body

= Weight of the fluid displaced by the body

= ρg x Volume of the fluid displaced by the body

= ρg x Volume of the body sub-merged in water

= $\rho g \times \forall$

$\therefore$ $BM = \frac{\rho g \times I}{\rho g \times \forall} = \frac{I}{\forall}$ ………..(3)

$$GM = BM - BG = \frac{I}{\forall} - BG$$

$\therefore$ **Meta-centric height** = $$\mathbf{GM = \frac{I}{\forall} - BG} \qquad \text{……….(4)}$$

A rectangular pontoon is 5 m long, 3 m wide and 1.20 m high. The depth of immersion of the pontoon is 0.80 m in sea water. If the centre of gravity is 0.6 m above the bottom of the pontoon, determine the meat-centric height. The density for sea water = 1025 kg/m^3.

Given:

Dimension of pontoon = 5 m x 3 m x 1.20 m

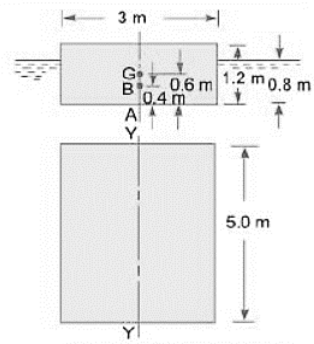
Depth of immersion = 0.8 m

Distance AG = 0.6 m

Distance AB = $\frac{1}{2}$ x Depth of immersion

= $\frac{1}{2}$ x .8 = 0.4 m

Density for sea water = 1025 kg/m^3



PLAN AT WATER SURFACE

Solution:

Meta-centre height GM, given by equation is

$$\mathbf{GM = \frac{I}{\forall} - BG}$$

Where I = M.O. Inertia of the plan of the pontoon about Y-Y axis

= $\frac{1}{2}$ x 5 x 3^3 m^4 = $\mathbf{\frac{45}{4}}$ **m^4**

$\forall$ = Volume of the body sub-merged in water

= 3 x 0.8 x 5.0 = **12.0 m^3**

BG = AG – AB

= 0.6 – 0.4

= 0.2 m

$\therefore$ GM = $\frac{45}{4}$ x $\frac{1}{12.0}$ - 0.2

= $\frac{45}{48}$ - 0.2 = 0.9375 – 0.2

GM = 0.7375 m.

A uniform body of size 3 m long x 2 m wide x 1 m deep floats in water. What is the weight of the body if depth of immersion is 0.8 m? Determine the meta-centric height also.

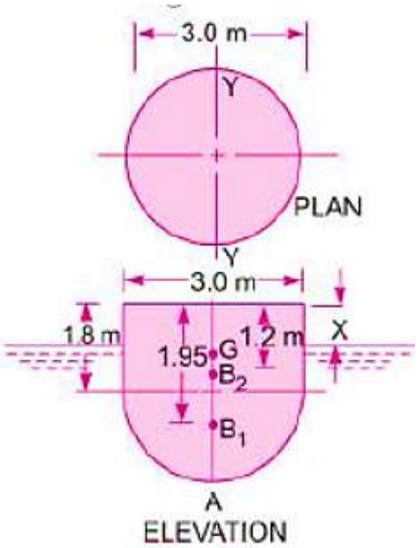
Given:

Dimension of body = 3 x 2 x 1

Depth of immersion = 0.8 m

Find (i) Weight of body, W

(ii) Meta-centric height, GM



Solution:

(i) Weight of Body , W

= Weight of water displaced

= ρg x Volume of water displaced

= 1000 x 9.81 x Volume of body in water

= 1000 x 9.81 x 3 x 2 x 0.8 N

W = 47088 N.

(ii) Meta-centric Height, GM

Using equation (4) , we get

$$GM = \frac{I}{\forall} - BG$$

Where I = M.O.I about Y-Y axis of the plan of the body

$$= \frac{I}{12} \times 3 \times 2^3 \qquad = \frac{3 \times 2^3}{12}$$

= 2.0 m⁴

$\forall$ = Volume of body in water

= 3 x 2 x 0.8 = 4.8 m^3

$$BG = AG - AB = \frac{1.0}{2} - \frac{0.8}{2} = 0.5 - 0.4 = 0.1$$

$$\therefore \quad GM = \frac{2.0}{4.8} - 0.1 = 0.4167 - 0.1$$

GM = 0.3167 m.

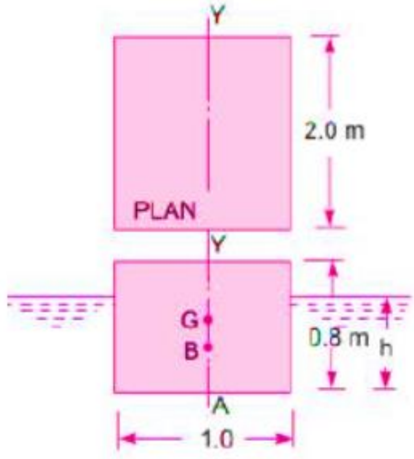
A block of wood of specific gravity 0.7 floats in water. Determine the meta-centric height of the block if its size is 2 m x 1 m x 0.8 m.

Given:

Dimension of block = 2 x 1 x 0.8

Let depth of immersion = *h* m

Sp. gr. of wood = 0.7



Solution:

Weight of wooden piece = Weight density of wood* x Volume

= 0.7 x 1000 x 9.81 x 2 x 1 x 0.8 N

Weight of water displaced = Weight density of water x Volume of the wood sub-merged in water

= 1000 x 9.81 x 2 x 1 x *h* N

For equilibrium,

Weight of wooden piece = Weight of water displaced

∴ 700 x 9.81 x 2 x 1 x 0.8 = 1000 x 9.81 x 2 x 1 x h

∴ $h = \frac{700 \text{ x } 9.81 \text{ x } 2 \text{ x } 1 \text{ x } 0.8}{1000 \text{ x } 9.81 \text{ x } 2 \text{ x } 1} = 0.7 \text{ x } 0.8 = 0.56$ m

∴ Distance of centre of Buoyancy from bottom, i.e.,

$$AB = \frac{h}{2} = \frac{0.56}{2} = 0.28 \text{ m}$$

and AG = 0.8 / 2.0 = 0.4 m

∴ BG = AG – AB = 0.4 – 0,28 = 0.12 m

The meta-centric height is given by equation (4) or

$$GM = \frac{I}{\forall} - BG$$

Where $I = \frac{I}{12} \text{ x } 2 \text{ x } 1.0^3$ $\mathbf{= \frac{I}{6} \text{ m}^4}$

∀ = Volume of wood in water

= 2 x 1 x h = 2 x 1 x .56 **= 1.12 m³**

∴ $GM = \frac{I}{6} \text{ x } \frac{I}{1.12} - 0.12$

= 0.1488 – 0.12

GM = 0.0288 m.

* Weight density of wood = ρx g, where ρ= density of wood

= 0.7 x 1000 = 700 kg/m³. Hence w for wood

= 700 x 9.81 N/m³.

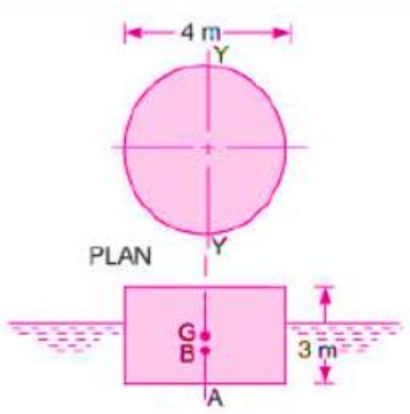
A solid cylinder of diameter 4.0 m has a height of 3 metres. Find the meta-centric height of the cylinder when it is floating in water with its axis vertical. The sp. gr. of the cylinder = 0.6

Given:

Dia. Of cylinder D = 4.0 m

Height of cylinder h = 3.0 m

Sp. gr. of cylinder = 0.6



Solution:

Depth of immersion of cylinder

$= 0.6 \times 3.0 = 1.8$ m

∴ AB $= \frac{1.8}{2} = 0.9$ m

and AG $= \frac{3}{2} = 1.5$ m

∴ BG = AG – AB

$= 1.5 - 0.9 = 0.6$ m

Now the meta –centric height GM is given by equation (4.4)

$$GM = \frac{I}{\forall} - BG$$

But I = M.O.I about Y-Y axis of the plan of the body

$$= \frac{\pi}{64} D^4 = \frac{\pi}{64}(4.0)^4$$

and ∀ = Volume of cylinder in water

$$= \frac{\pi}{4} D^4 \text{ Depth of immersion}$$

$$= \frac{\pi}{4} (4)^2 \times 1.8 \text{ m}^3$$

∴ GM $= \dfrac{\frac{\pi}{64} \times (4.0)^4}{\frac{\pi}{4} \times (4.0)^2 \times 1.8} - 0.6$

$= \frac{1}{16} \times \frac{4.0^2}{1.8} - 0.6 = \frac{1}{1.8} - 0.6 = 0.55 - 0.6 = -0.05$ m.

-ve sign means that meta-centre, (M) is below the centre of gravity (G).

CONDITIONS OF EQUILIBRIUM OF A FLOATING AND SUB-MERGED BODIES

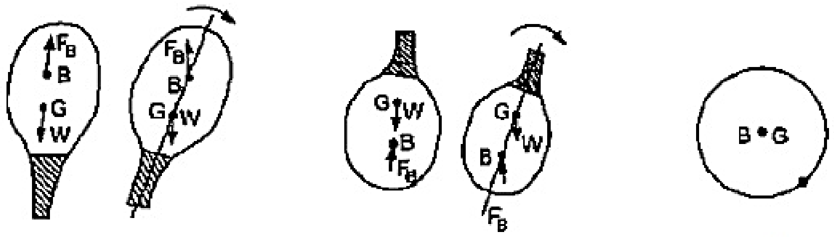
✓ A sub-merged or a floating body is said to be stable if it comes back to its original position after a slight disturbance. The relative position of the centre of gravity (G) and centre of buoyancy (B_1) of a body determines the stability of a sub-merged body.

Stability of a Sub-merged body

The position of centre of gravity and centre of buoyancy in case of completely sub-merged body are fixed. Consider a balloon, which is completely sub-merged in air. Let the lower position of the balloon contains heavier material, so that its centre of gravity is lower than its centre of buoyancy as shown in Fig. (a).

Let the weight of the balloon is W. The weight W is acting through G, vertically in the downward direction, while the buoyant force F_B is acting vertically up, through B. For the equilibrium of the balloon $W = F_B$.

If the balloon is given an angular displacement in the clockwise direction as shown in Fig.(a), then W and F_B constitute a couple acting in the anti-clock wise direction and brings the balloon in the original position. Thus the balloon in the position shown by Fig. (a) is in stable equilibrium.



(a) Stable Equilibrium (b) UnStable Equilibrium (c) Neutral Equilibrium

Stabilities of sub-merged bodies

(a) **Stable Equilibrium,** When $W = F_B$ and point B is above G, the body is said to be in stable equilibrium.

(b) **Unstable Equilibrium,** If $W = F_B$, but the centre of buoyancy (B) is below centre of gravity (G), the body is in unstable equilibrium as shown in Fig (b). A slight displacement of the body, in the clockwise direction, gives the couple due to W and F_B also in the clockwise direction. Thus the body does not return to its original position and hence the body is in unstable equilibrium.

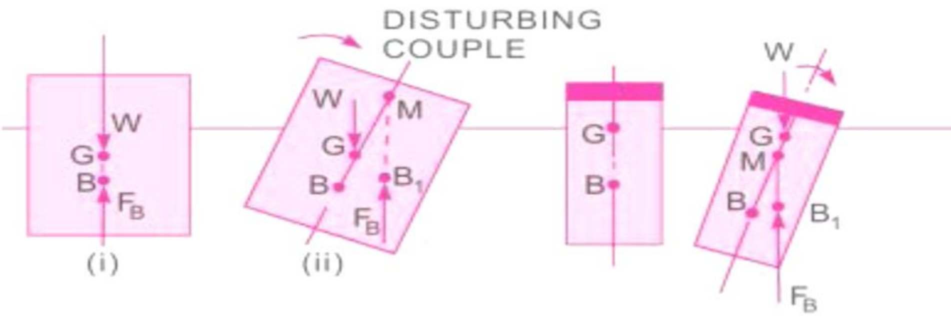
(c) **Neutral Equilibrium.** If F_B=W and B and G are at the same point as shown in Fig. (c), the body is said to be in neutral equilibrium.

STABILITY OF FLOATING BODY

The stability of a floating body is determined from the position of Meta-centre (M). In case of floating body, the weight of the body is equal to the weight of liquid displaced.

(a) Stable Equilibrium.

- ✓ If the point M is above G, the floating body will be in stable equilibrium shown in Fig.(a). If a slight angular displacement is given to the floating body in the clockwise direction, the centre of buoyancy shifts from B to B_1 such that the vertical line through B_1 cuts at M.
- ✓ Then the buoyant force F_B through B_1 and weight W through G constitute a couple acting in the anti clockwise direction and thus bringing the floating body in the original position.



Stability of floating bodies

(b) Unstable Equilibrium.

- ✓ If the point M is below G, the floating body will be in unstable equilibrium as shown in Fig 4.13 (b). The disturbing couple is acting in the clockwise direction. The couple due to buoyant force F_B and W is also acting in the clockwise direction and thus overturning the floating body.

(c)Neutral Equilibrium.

- ✓ If the point M is at the centre of gravity of the body, the floating body will be in neutral equilibrium.

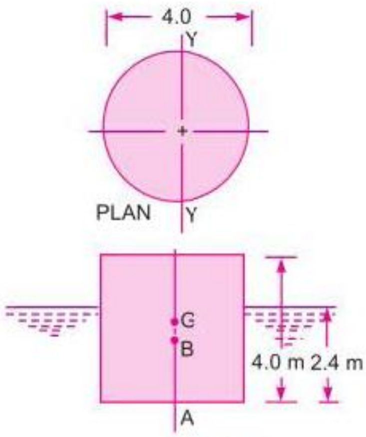
A solid cylinder of diameter 4.0 m has a height of 4.0 m. Find the meta-centric height of the cylinder if the specific gravity of the material of cylinder = 0.6 m and it is floating in water with its axis vertical. State whether the equilibrium is stable or unstable.

Given:

D = 4 m

Height, h = 4 m

Sp. gr. = 0.6



Solution:

Depth of cylinder in water = Sp. gr. x h

= 0.6 x 4.0 = 2.4 m

∴ Distance of centre of buoyancy (B) from A

$$AB = \frac{2.4}{2} = \mathbf{1.2\ m}$$

Distance of centre of gravity (G) from A

$$AG = \frac{4.0}{2} = \frac{4.0}{2} = \mathbf{2.0\ m}$$

∴ BG = AG – AB = 2.0 – 1.2 = 0.8 m

Now the meta-centric height GM is given by

$$GM = \frac{I}{\forall} - BG$$

Where I = M.O.I of the plan of the body about Y-Y axis

$$= \frac{\pi}{64} D^4 = \frac{\pi}{64}(4.0)^4$$

∀ = Volume of cylinder in water

$$= \frac{\pi}{4.0} D^4 \text{ Depth of cylinder in water} = \frac{\pi}{4} \times 4^2 \times 2.4\ m^3$$

$\therefore \quad \frac{I}{\forall} = \frac{\frac{\pi}{64} \text{ x } 4^4}{\frac{\pi}{4} \text{ x } 4^2 \text{ x } 2.4} = \frac{1}{16} \text{ x } \frac{4^2}{2.4} = \frac{1}{2.4}$ = 0.4167 m.

GM = $\frac{I}{\forall}$ - BG = 0.4167 – 0.8 = -0.3833 m.

-ve sign means that meta-centre, (M) is below the centre of gravity (G). Thus the cylinder is in unstable equilibrium.

A rectangular pontoon 10.0 m long , 7 m broad and 2.5 m deep weighs 686.7 kN. Its carmes on its upper deck an empty boiler of 5.0 m diameter weighing 588.6 kN. The centre of gravity of the boiler and the pontoon are at their respective centre along a vertical line. Find the meta-centric height. Weight density of sea water is 10.104 kN/m³.

Given:

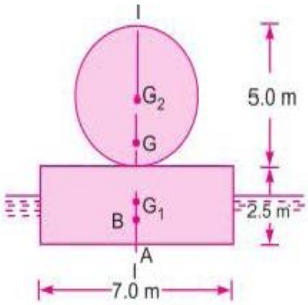
Dimension of pontoon = 10 x 7 x 2.5

Weight of pontoon W_1 = 686.7 kN

Dia. Of boiler D = 5.0 m

Weight of boiler W_2 = 588.6 kN

w for sea water = 10.104 kN/m³



Solution.

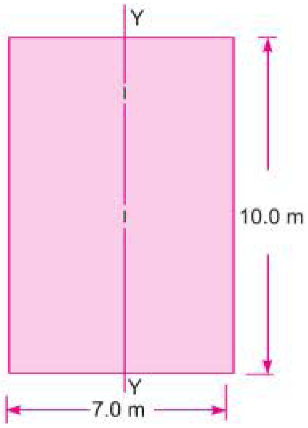
To find the meta-centric height, first determine the common centre of gravity G and common centre of buoyancy B of the boiler and pontoon. Let G_1 and G_2 are the centre of gravities of pontoon and boiler respectively.

Then

$AG_1 = \frac{25}{2} = 1.25$ m

$AG_2 = 2.5 + \frac{5.0}{2} = 2.5 + 2.5$

= 5.0 m



The distance of common centre of gravity G from A is given as

$$AG = \frac{W_1 \text{x } AG_1 + W_2 \text{x} AG_2}{W_1 + W_2}$$

$$\frac{686.7\text{x } 1.25 + 588.6\text{x}5.0}{(686.7 + 588.6)} = 2.98 \text{ m}$$

Let h is the depth of immersion. Then

Total weight of pontoon and boiler = Weight of sea water displaced

or $(686.7 + 588.6)$ = w x Volume of the pontoon in water

= 10.104 x L x b x Depth of immersion

1275.3 = 10.104 x 10 x 7 x h

$$h = \frac{1275.3}{10 \times 7 \times 10.104} = 1.803 \text{ m}$$

∴ The distance of the common centre of buoyancy B from A is

$$AB = \frac{h}{2} = \frac{1.803}{2} = \mathbf{.9015\ m}$$

∴ $BG = AG - AB$

$= 2.98 - .9015$

= 2.078 m

Meta – centric height is given by, $\mathbf{GM = \frac{I}{\forall} - BG}$

Where I = M.O.I of the plan of the body at the water level along Y-Y

$$= \frac{1}{12} \times 10.0 \times 7^3 = \frac{10 \times 49 \times 7}{12} \text{ m}^4$$

$\forall$ = Volume of the body in water

$= L \times b \times h = 10.0 \times 7 \times 1.857$

$$\therefore \quad \frac{I}{\forall} = \frac{10 \times 49 \times 7}{12 \times 10 \times 7 \times 1.857} = \frac{49}{12 \times 1.857} = \mathbf{2.198\ m.}$$

$$\therefore \quad GM = \frac{I}{\forall} - BG$$

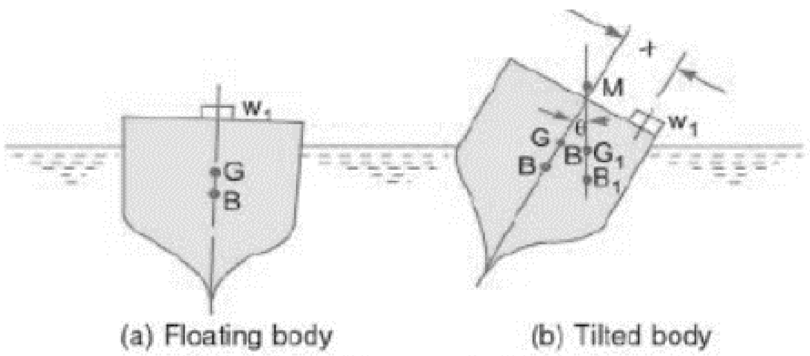
$= 2.198 - 2.078$

GM = 0.12 m.

∴ *Meta-centric height of both the pontoon and boiler = 0.12 m.*

EXPERIMENTAL METHOD OF DETERMINATION OF META-CENTRIC HEIGHT

The meta-centric height of a floating vessel can be determined, provided we know the centre of gravity of the floating vessel. Let w_1 is a known weight placed over the centre of the vessel as shown in Fig.(a) and the vessel is floating.



Meta-centric height

Let W = Weight of vessel including w_1

G = Centre of gravity of the vessel

B = Centre of buoyancy of the vessel

The weight w_1 is moved across the vessel towards right through a distance x as shown in Fig.(b). The vessel will be tilted. The angel of heel θ is measured by means of a plumbline and a protractor attached on the vessel. The new centre of gravity of the vessel will shift to G_1 as the weight w1 has been moved towards the right.

Also the centre of buoyancy will change to B_1 as the vessel has tilted. Under equilibrium, the moment caused by the movement of the load w_1 through a distance x must be equal to the moment caused by the shift of the centre of gravity from G to G_1. Thus

The moment due to change of $w_1 = w_1 \times x$

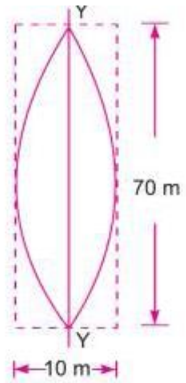
$\therefore$ $w_1 x = WGM \tan\theta$

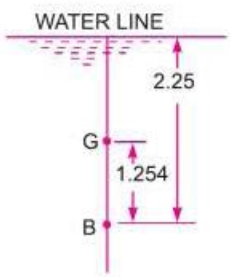
Hence $GM = \frac{w_1 x}{W \tan\theta}$ ……………(4.5)

A ship 70 m long and 10 m broad has a displacement of 19620 kN. A weight of 343.35 kN is moved across the deck through a distance of 6 m. The ship is tilted through 6^0. The moment of inertia of the ship at water-line about its fore and aft axis is 75% of M.O.I of the circumscribing rectangle. The centre of buoyancy is 2.25 m below water-line. Find the meat-centric height and position of centre of gravity of ship. Specific weight of sea water is 10104 N/m^3.

Given:

Length of ship,	L	=	70 m
Breadth of ship,	b	=	10 m
Displacement,	W	=	19620 kN
Angle of heel,	θ	=	6^0
M.O.I. of ship at water-line		=	75 % of M.O.I. of circumscribing rectangle
w for sea-water		=	10104 n/m^3 kN
Movable weight	w_1	=	343.35 kN
Distance moved by w_1	x	=	6m
Centre of buoyancy		=	2.25 m below water surface





Solution:

Find (i) Meta-centric height, GM

(ii) Position of centre of gravity, G

(i) Meta-centric height, GM is given by equation

$$\therefore \quad GM = \frac{w_1 x}{W \tan\theta} = \frac{343.35 kN \times 6.0}{19620\ kN \times \tan 6^0}$$

$$= \frac{343.35kN \times 6.0}{19620\ kN \times .1051}$$

= 0.999 m.

(ii) Position of centre of gravity, G

$$GM = \frac{I}{\forall} - BG$$

Where I = M.O.I of the ship at water-line about Y-Y

$$= \quad 75\% \text{ of } \frac{1}{12} \times 70 \times 10^3$$

$$= .75 \times \frac{1}{12} \times 10^3 = \mathbf{4375\ m^4}$$

and $\forall$ = Volume of ship in water = $\frac{\text{Weight of ship}}{\text{Weight density of water}}$

$$= \frac{19620}{10.104} = \mathbf{1941.74\ m^3}$$

∴ $\frac{I}{\forall} = \frac{4375}{1941.74}$ **= 2.253 m**

∴ GM = 2.253 – BG

.999 = 2.253 – BG

∴ BG = 2.253 - .999 **= 1.254 m**

From Fig., it is clear that the distance of G from free surface of the water = distance of B from water surface – BG

= 2.25 – 1.254

= 0.996 m.

A pontoon of 15696 kN displacement is floating in water. A weight of 245.25 kN is moved through a distance of 8 m across the deck of pontoon, which tilts the pontoon through an angle 4^0. Find meta-centric height of the pontoon.

Given:

Weight of pontoon = Displacement

W = 15696 kN

Movable weight w_1 = 245.25 kN

Distance moved by weight w_1, x=8 m

Angle of heel, $\theta = 4^0$

Solution:

The meta-centric height, GM is given by equation

$$GM = \frac{w_1 x}{W\ tan\theta}$$

$$= \frac{245.25kN \times 8}{15696\ kN \times \tan 4^0} = \frac{1962}{15696 \times 0.0699}$$

GM = 1.788 m

UNIT II

KINEMATICS OF MOTION

INTRODUCTION

Kinematics is defined as that branch of science which deals with motion of particles without considering the forces causing the motion. The velocity at any point in a flow field at any time is studied in this branch of fluid mechanics.

Once the velocity is known, then the pressure distribution and hence forces acting on the fluid can be determined, In this chapter, the methods of determining velocity and acceleration are discussed.

METHODS OF DESCRIBING FLUID MOTION

The fluid motion is described by two methods. They are (i) Lagraugian Method, and (ii) Eulerian Method.

- ✓ In the Lagrangian method, **a single fluid particle** is followed during its motion and its velocity * acceleration, density, etc.. are described.
- ✓ In case of Eulerian method, the velocity, acceleration, pressure, density etc., arc described **at a point** in Row field. The Eulerian method is commonly used in fluid mechanics.

TYPES OF FLUID FLOW

- ✓ The fluid flow is classified as:
- ✓ Steady and unsteady flows
- ✓ Uniform and (ton-uniform flows
- ✓ Laminar and turbulent flows
- ✓ Compressible and incompressible flows
- ✓ Rotational and irrotational flows
- ✓ One, two and three-dimensional flows.

Steady and Unsteady flow:

Steady flow

Fluid flow is said to be steady if at any point in the flowing fluid various characteristics such as velocity, density, pressure, etc. do not change with time.

$$\partial V/\partial t = 0 \qquad \partial p/\partial t = 0 \qquad \partial \rho/\partial t = 0$$

Unsteady flow

Fluid flow is said to be unsteady if at any point flowing fluid any one or all characteristics which describe the behaviour of the fluid in motion change with time.

$$\partial V/\partial t \neq 0 \qquad \partial p/\partial t \neq 0 \qquad \partial \rho/\partial t \neq 0$$

Uniform and Non-uniform flow.

Uniform flow

When the velocity of flow of fluid does not change both in direction and magnitude from point to point in the flowing fluid for any given instant of time, the flow is said to be uniform.

$$\partial V/\partial s = 0 \quad \partial p/\partial s = 0 \quad \partial \rho/\partial s = 0$$

Non-uniform flow

If the velocity of flow of fluid changes from point to point in the flowing fluid at any instant, the flow is said to be non-uniform flow.

$$\partial V/\partial s \neq 0 \quad \partial p/\partial s \neq 0 \quad \partial \rho/\partial s \neq 0$$

Laminar and Turbulent flow:

Laminar flow

A flow is said to be laminar if Reynolds number is less than 2000 for pipe flow. Laminar flow is possible only at low velocities and high viscous fluids. In laminar type of flow, fluid particles move in laminas or layers gliding smoothly over the adjacent layer.

Turbulent flow

In Turbulent flow, the flow is possible at both velocities and low viscous fluid. The flow is said to be turbulent if Reynolds number is greater than 4000 for pipe flow. In Turbulent type of flow fluid, particles move in a zig – zag manner.

Compressible and incompressible flow

Compressible flow

The compressible flow is that type of flow in which the density of the fluid changes from point to point i.e. the density is not constant for the fluid. It is expressed in kg/sec.

$\rho \neq$ constant

Incompressible flow

The incompressible flow is that type of flow in which the density is constant for the fluid flow. Liquids are generally incompressible. It is expressed in m^3/s.

$\rho =$ constant

Rotational and Irrotational flow.

Rotational flow

Rotational flow is that type of flow in which the fluid particles while flowing along stream lines and also rotate about their own axis.

Irrotational flow

If the fluid particles are flowing along stream lines and do not rotate about their own axis that type of flow is called as ir-rotational flow

One, Two and Three dimensional flow.

One dimensional flow

The flow parameter such as velocity is a function of time and one space co- ordinate only

$u = f(x), \quad v = 0 \quad \& \quad w = 0.$

Two dimensional flow

The velocity is a function of time and two rectangular space co-ordinates.

$u = f_1(x,y)$, $v = f_2(x,y)$ & $w = 0$.

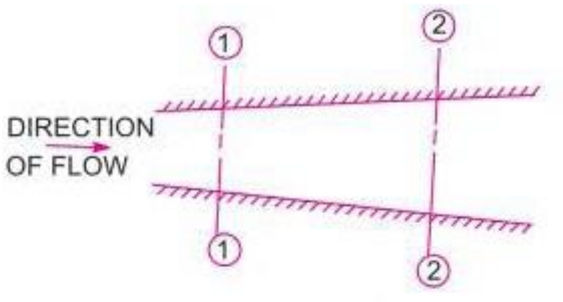
Three dimensional flow

The velocity is a function of time and three mutually perpendicular directions.

$u = f_1(x,y,z)$, $v = f_2(x,y,z)$ & $w = f_3(x,y,z)$.

CONTINUITY EQUATION

The equation based on the principle of conservation of mass is called continuity equation. Thus for a fluid flowing through the pipe at all the cross-section, the quantity of fluid per second is constant. Consider two cross-sections of a pipe as shown in Fig.



Let V_1 = Average velocity at cross-sect ion I-1

ρ_1 = Density at section 1-1

A_1 = Area of pipe at section I -1

and V_2, ρ_1, A_2 arc corresponding values at section. 2-2.

Then rate of flow at section 1-1 = $\rho_1 A_1 V_1$

Rate of flow at .section 2-2 = $\rho_2 A_2 V_2$

According to hlw of conservation of mass of flow

Rate of flow at section I - I = Rate of flow at section 2-2

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2$$

Equation is applicable to the compressible as well as incompressible fluids and is called Continuity Equation. If the fluid is in- compressible, then $\rho_1 = \rho_2$ and continuity equation reduces to

$$A_1 V_1 = A_2 V_2$$

RATE OF FLOW OR DISCHARGE (Q)

It is defined as the quantity of a fluid flowing per second through a section of a pipe or a channel. For an incompressible fluid (or liquid) the rate of flow or discharge is expressed as the volume or fluid flowing across the section per second. For compressible fluids, the rate of flow is usually expressed as the weight of fluid flowing across the section. Thus

(i) For liquids the units or Q are m^3/s or litres/s

(ii) For gases the units of Q is kgf/s or Newton/s

Consider a liquid flowing through a pipe in which

A = Cross-sectional area of pipe

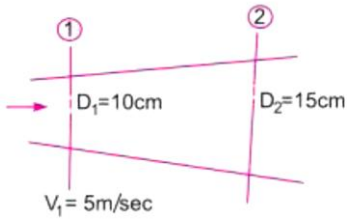
V = Average velocity of fluid across the section

Then discharge, **Q = A xV**

The diameters of a pipe at the sections 1 and 2 are 10 cm and 15 cm respectively. Find the discharge through the pipe if the velocity of water flowing through the pipe at section 1 m /s. Determine also the velocity at section 2.

Given:

At section 1, D_1 = 10cm = 0.1 m



$A_1 = \frac{\pi}{4}(D_1^2) = \frac{\pi}{4}(.1)^2$

$= 0.007854\ m^2$

V_1 = 5 m/s.

At section 2, D_2 = 15 cm = 0.15 m

$A_2 = \frac{\pi}{4}(.15)^2 = 0.01767\ m^2$

Solution:

(i) Discharge through pipe is given by equation

$Q = A_1 \times V_1$

$= 0.007854 \times 5$

$= 0.03927\ m^3/s.$

Using equation we have,

$A_1 V_1 = A_2 V_2$

(ii) $\therefore$ $V_2 = \frac{A_1V_1}{A_2} = \frac{0.007854}{0.01767} \times 5.0$

$V_2 = 2.22$ m/s.

A 30 cm diameter pipe, conveying water, branches into two pipes of diameters 20 cm and 15 cm respectively. If the average velocity in the 30 cm diameter pipe is 2.5 m/s, find the discharge in this pipe. Also determine the velocity in 15 cm pipe if the average velocity in 20 cm diameter pipe is 2 m/s.

Given :

D_1 = 30 cm = 0.30 m

$A_1 = \frac{\pi}{4} D_1^2 = \frac{\pi}{4}.3^2 = 0.007068\ m^2$

V_1 = 2.5 m/s

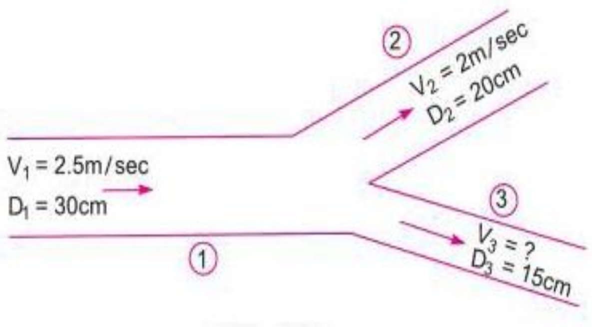
D_2 = 20 cm = 0.20 m

$A_2 = \frac{\pi}{4}(.2)^2 = \frac{\pi}{4} \times .4 = 0.0314\ m^2$

V_2 = 2 m/s

D_3 = 15 cm = 0.15 m

$A_3 = \frac{\pi}{4}(.15)^2 = \frac{\pi}{4} \times 0.225 = .01767\ m^2$



Solution:

Find (i) Discharge in pipe 1 or Q_1

(ii) Velocity in pipe of dia. 15 cm or V_3

Let Q_1, Q_2 and Q_3 are discharges in pipe 1 , 2 and 3 respectively.

Then according to continuity equation

$Q_1 = Q_2 + Q_3$ ………………(1)

(i) The discharge Q_1 in pipe 1 is given by

$Q_1 = A_1 V_1 = 0.07068 \times 2.5\ m^3/s =$ **$0.1767\ m^3/s$.**

(ii) Value of V_3

$Q_2 = A_2 V_2 = 0.0314 \times 2.0 =$ **$0.0628\ m^3/s$**.

Substituting the values Q_1 and Q_2 in equation (1)

$0.1767 = 0.0628 + Q_3$

∴ $Q_3 = 0.1767 = 0.0628 = 0.1139\ m^3/s$.

But $Q_3 = A_3 \times V_3 = 0.01767\ V_3$ or $0.1139 = 0.01767 \times V_3$

∴ $V_3 = \frac{0.1139}{0.01767} =$ **6.44 m/s.**

---

25 cm diameter pipe carries oil of sp.gr. 0.9 at a velocity of 3 m/s. At another section the diameter is 20cm. Find the velocity at this section and also mass rate of flow of oil.

Given.

at section 1, D_1 = 25 cm = 0.25 m

$A_1 = \frac{\pi}{4} D_1^2 = \frac{\pi}{4} \times 0.25^2 = 0.049\ m^3$

V_1 = 3 m/s

at section 2, D_2 = 20 cm = 0.2 m

$A_2 = \frac{\pi}{4}(0.2)^2 = 0.0314\ m^2$

V_2 = ?

Mass rate of flow of oil = ?

Solution:

Applying continuity equation at section 1 and 3

$A_1 V_1 = A_2 V_2$

or $0.049 \times 3.0 = 0.0314 \times V_2$

$\therefore \quad V_2 = \frac{0.049 \times 3.0}{0.0314}$

= 4.68 m/s.

Mass rate of flow of oil = Mass density x Q = $\rho \times A_1 \times V_1$

Sp.gr.of oil = $\frac{\text{Density of oil}}{\text{Density of water}}$

$\therefore$ Density of oil = Sp. gr. of oil x Density of water

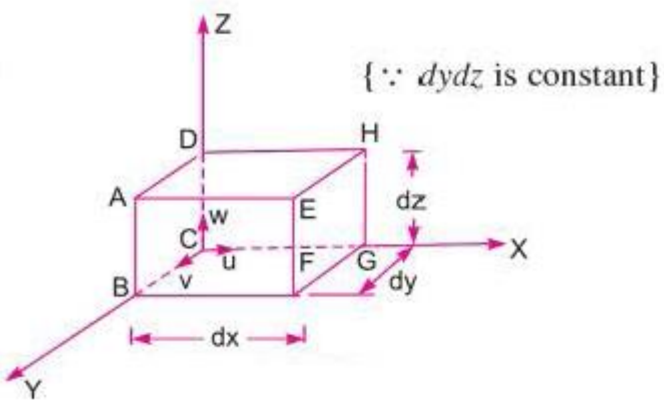
$= 0.9 \times 1000 \text{ kg/m}^3 = \frac{900\ kg}{m^3}$

$\therefore$ Mass rate of flow **= 900 x 0.049 x 3.0 kg/s**

= 132.23 kg/s.

CONTINUITY EQUATION IN THREE – DIMENSIONS

Consider a fluid element of lengths *dx* , *dy* and *dz* in the direction of *x*, *y* and *z*. Let *u*, *v* and *w* are the inlet velocity components in *x*, *y* and *z* direction respectively.



Mass of fluid entering the face ABCD per second

= ρ x Velocity in *x*-direction x Area of ABCD

= $\rho \times u \times (dy \times dz)$

Then mass of fluid leaving the face EFGH per second = $\rho u\ dydz + \frac{\partial}{\partial x}(\rho u\ dydz)\,dx$

$\therefore$ Gain of mass in *x*-direction

= Mass through ABCD – Mass through EFGH per second

$= \rho u\ dydz - \rho u\ dydz - \frac{\partial}{\partial x}(\rho u\ dydz)\,dx$

$= -\frac{\partial}{\partial x}(\rho u\ dydz)\,dx$

$= -\frac{\partial}{\partial x}(\rho u)\,dx\ dydz$

Similarly, *the net gain of mass in y-direction*

$= \quad -\frac{\partial}{\partial y}(\rho v)\ dxdydz$

and in z-direction $= \quad -\frac{\partial}{\partial z}(\rho w)\ dxdydz$

$\therefore$ *Net gain of masses* $= -\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right] dxdydz$

Since the mass is neither created nor destroyed in the fluid element, the net increase of mass per unit time in the fluid element must be equal to the rate of increase of mass of fluid in the element. But mass of fluid in the element is $\rho\ .\ dx\ .\ dy\ .\ dz$ and its rate of increase with time is $\frac{\partial}{\partial t}(\rho\ .\ dx\ .\ dy\ .\ dz)$ or $\frac{\partial \rho}{\partial t}\ .\ dx\ dy\ dz$

Equating the two expressions,

$$-\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right] dxdydz = \frac{\partial \rho}{\partial t}\ .\ dx\ dydz$$

$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$ [Cancelling $dx\ dy\ dz$ from both sides]….. (1)

Equation is the continuity equation in Cartesian co-ordinates in its most general form.

This equation is applicable to :

(i) Steady and unsteady flow

(ii) Uniform and non-uniform flow and

(iii) Compressible and incompressible fluids

For steady flow, $\frac{\partial \rho}{\partial t} = 0$ and hence equation (1) becomes as

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \qquad \text{……….(2)}$$

If the fluid is incompressible, then ρ is constant and the above equation becomes as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad \text{…………(3)}$$

Equation (3) is the continuity equation in three-dimensions. For a two-dimensional flow, the component w=0 and hence continuity equation becomes as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \text{…………(4)}$$

VELOCITY AND ACCELERATION

Let V is the resultant velocity at any point in a fluid flow. Let u, v and w are its component in *x*,y and z directions. The velocity components are function of space co-ordinates and time. Mathematically the velocity components are given as

$u = f_1(x, y, z, t)$

$v = f_2(x, y, z, t)$

$w = f_3(x, y, z, t)$

and Resultant velocity, $V = ui + vj + wk = \sqrt{u^2 + v^2 + w^2}$

Let a_x, a_y and a_z are the total acceleration in x , y and z direction respectively. Then by the chain rule of differentiation, we have

$$a_x = \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt} + \frac{du}{dt}$$

But $\frac{dx}{dt} = u$, $\frac{dy}{dt} = v$ and $\frac{dz}{dt} = w$

$\therefore$ $a_x = \frac{du}{dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{dt}$

Similarly $a_y = \frac{dv}{dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{dt}$ ....(5)

$a_z = \frac{dw}{dt} = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{dt}$

For steady flow, $\frac{\partial V}{\partial t} = 0$, where V is resultant velocity

or $\frac{\partial u}{\partial t} = 0$, $\frac{\partial V}{\partial t} = 0$ and $\frac{\partial w}{\partial t} = 0$

Hence acceleration in *x* , *y* and *z* direction becomes

$$\left.\begin{aligned} a_x &= \frac{du}{dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \\ a_y &= \frac{dv}{dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \\ a_z &= \frac{dw}{dt} = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \end{aligned}\right\}$$ ....(6)

Acceleration vector $A = a_x i + a_y j + a_z k$

$$= \sqrt{a_x^2 + a_y^2 + a_z^2}$$ ....(7)

Local Acceleration and Convective Acceleration.

Local acceleration is defined as the rate of increase of velocity with respect to time at a given point in a flow field. In the equation given by (5.6) the expression $\frac{\partial u}{\partial t}$, $\frac{\partial V}{\partial t}$ or $\frac{\partial w}{\partial t}$ is known as local acceleration

Convective acceleration: It is defined as the rate of change of velocity due to the change of position of fluid particles in a fluid flow. The expression other than $\frac{\partial u}{\partial t}$, $\frac{\partial V}{\partial t}$ or $\frac{\partial w}{\partial t}$ in equation (5.6) are known as convective acceleration.

The velocity vector in a fluid flow is given $V = 4x^3i - 10x^2yj + 2tk$. Find the velocity and acceleration of a fluid particle at (2, 1, 3) at time t = 1.

Solution:

The velocity components u, v and w are $u = 4x^3$, $y = -10x^2$, $w = 2t$

For the point (2, 1, 3) , we have $x = 2$, y = 1 and z = 3 at time t = 1

Hence velocity components at (2 ,1 , 3) are u = 4 x $(2)^3$ = 32 units

v = -10$(2)^2$ (t) = -40 units

w = 2 x 1 = 2 units

∴ Velocity vector V at (2, 1, 3) = 32i – 40j + 2k

Resultant velocity = $\sqrt{u^2 + v^2 + w^2}$

= $\sqrt{32^2 + (-40)^2 + 2^2}$

= $\sqrt{1024 + 1600 + 4}$

= 51.26 units.

Acceleration is given by equation,

$$a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

$$a_y = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}$$

$$a_z = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}$$

now from velocity components, we have

$$\frac{\partial u}{\partial x} = 12x^2, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial u}{\partial z} = 0 \quad and \quad \frac{\partial u}{\partial t} = 0$$

$$\frac{\partial v}{\partial x} = -20xy, \quad \frac{\partial v}{\partial y} = -10x^2, \quad \frac{\partial v}{\partial z} = 0 \quad \frac{\partial v}{\partial t} = 0$$

$$\frac{\partial w}{\partial x} = 0, \quad \frac{\partial w}{\partial y} = 0, \quad \frac{\partial w}{\partial z} = 0 \quad \frac{\partial w}{\partial t} = 2.1$$

Substituting the values, the acceleration components at (2,1,3) at time t = 1 are

a_x = $4x^3(12x^2)+(-10x^2y)$ (0)+2t x (0) + 0

= $48x^5$= 48 x $(2)^5$

= 48 x 32 **= 1536 units**

a_y = $4x^3$(-20xy)+($-10x^2y$) (-10x^2)+2t(0) + 0

= $-80x^4y + 100x^4y$

= -80 $(2)^4$(1) + 100 $(2)^4$ x 1

= -1280 + 1600 **= 320 units.**

a_z = $4x^3$(0)+($-10x^2y$) (0)+ (2t)(0) +2.1

= 2.0 units

∴ Acceleration is A = a_xi + a_yj + a_zk = 1536i + 320j + 2k.

Resultant A = $\sqrt{(1536)^2 + (320)^2 + (2)^2}$ units

= $\sqrt{2359296 + 102400 + 4}$ **= 1568.9 units.**

The following cases represent the two velocity components, determine the third component of velocity such that they satisfy the continuity equation:

(i) $u = x^2 + y^2 + z^2$; $v = xy^2 - yz^2 + xy$

(ii) $v = 2y^2$, $w = 2xyz$

Solution:

The continuity equation for incompressible fluid is given by equation as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Case I. $u = x^2 + y^2 + z^2$ $\therefore \frac{\partial u}{\partial x} = 2x$

$v = xy^2 - yz^2 + xy$ $\therefore \frac{\partial v}{\partial y} = 2xy - z^2 + x$

Substituting the values of $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$ in continuity equation

$$2x + 2xy - z^2 + x + \frac{\partial w}{\partial z} = 0$$

or $\frac{\partial w}{\partial z} = -3x - 2xy + z^2$ or $\partial w = (-3x - 2xy + z^2)\, \partial z$

Integration of both sides gives $\int dw = \int(-3x - 2xy + z^2)dz$

or $w = \left[-3xz - 2xyz + \frac{z^3}{3}\right]$ + Constant of integration

where constant of integration cannot be a function of *z*. But if can be a function of *x* and *y* that is *f(x,y)*.

$\therefore$ $w = \left[-3xz - 2xyz + \frac{z^3}{3}\right] + f(x,y)$.

Case II. $v = 2y^2$ $\therefore \frac{\partial v}{\partial y} = 4y$

$w = 2xyz$ $\therefore \frac{\partial w}{\partial z} = 2xy$

Substituting the values of $\frac{\partial v}{\partial y}$ and $\frac{\partial w}{\partial z}$ in continuity equation, we get

$$\frac{\partial u}{\partial x} + 4y + 2xy = 0$$

or $\frac{\partial u}{\partial x} = -4y - 2xy$ or $du = (-4y - 2xy)\, dx$

Integrating, we get $u = 4xy - 2y\frac{x^2}{2} + f(y,z) = -4xy - x^2y + f(y,z).$

A fluid flow field is given by V = x^2yi – y^2zj – (2xyz + yz²) k. Prove that it is a case of possible steady incompressible fluid flow. Calculate the velocity and acceleration at the point (2,1,3)

Solution: For the given fluid flow field

$u = x^2y$ $\therefore \frac{\partial u}{\partial x} = 2xy$

$v = y^2z$ $\therefore \frac{\partial v}{\partial y} = 2yz$

$w = -2xyz - y^2$ $\therefore \frac{\partial w}{\partial z} = -2xy - 2yz$

For a case of possible steady incompressible fluid flow, the continuity equation (5.4) should be satisfied.

i.e., $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

Substituting the values of $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}$ and $\frac{\partial w}{\partial z}$, we get

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 2xy + 2yz - 2xy - 2yz = 0$$

Hence the velocity field V = V = $x^2yi - y^2zj - (2xyz + yz^2)\ k$ is possible case of fluid flow.

Velocity at (2,1,3)

Substituting the values $x = 2$, $y = 1$ and $z = 3$ in velocity field, we get

$$V = x^2yi - y^2zj - (2xyz + yz^2)\ k$$

$$= 2^2 + 1i + 1^2 \text{ x } 3j - (2 \text{ x } 2 \text{ x } 1 \text{ x } 3 + 1 \text{ x } 3^2)k$$

$$= 4i + 3j - 21k.$$

and Resultant velocity $= \sqrt{4^2 + 3^4 + (-21)^2} = \sqrt{16 + 9 + 441} = 466 = 21.587$ units.

Acceleration at (2,1,3)

The acceleration components ax , ay and az for steady flow are

$$a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$a_y = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$a_z = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

$$u = x^2y,\ \frac{\partial u}{\partial x} = 2xy,\ \ \frac{\partial u}{\partial y} = x^2\ \ and\ \ \frac{\partial u}{\partial z} = 0$$

$$v = y^2z,\ \ \frac{\partial v}{\partial x} = 0,\ \ \frac{\partial v}{\partial y} = 2yz,\ \ \frac{\partial v}{\partial z} = y^2$$

$$w = -2xyz - yz^2,\ \frac{\partial w}{\partial x} = -2yz,\ \ \frac{\partial w}{\partial y} = -2xz - z^2,\ \ \frac{\partial w}{\partial z} = 2xy - 2yz$$

Substituting these values in acceleration components, we get acceleration at (2,1,3)

a_x = $x^2y\ (2xy) + y^2z(x^2) - (2xyz + yz^2)\ (0)$

= $2x^3y^2 + x^2y^2z$

= $2\ (2)^3 1^2 + 2^2 \text{ x } 1^2 \text{ x } 3 = 2 \text{ x } 8 + 12$ = 16 + 12 = 28 units.

a_y = $x^2y\ (0) + y^2z(2yz) - (2xyz + yz^2)\ (y^2)$

= $2y^3y^2 - 2xy^3z - y^3z^2$

= $2 \text{ x } 1^3 \text{ x } 3^2 - 2\text{x}2\text{x}1^3 \text{ x } 3 - 1^3 \text{ x } 3^2 = 18 - 12 - 9 = -3$ units

a_z = $x^2y\ (-2yz) + y^2z(2xz - z^2) - (2xyz + yz^2)\ (y^2)\ (-2xy - 2yz)$

= $-\ 2x^2y^2z - 2xy^2z - y^2z^3 + [4x^2y^2z + 2xy^2z^2 + 4xy^2z^2 + 2y^2z^3]$

= $-2 \text{ x } 2^2 \text{ x } 1^2 \text{ x } 3 - 2 \text{ x } 2 \text{ x } 1^2 \text{ x } 3^2 - 1^2 \text{ x } 3^3$

$+ [4 \text{ x } 2^2 \text{ x } 1^2 \text{ x } 3 + 2 \text{ x } 2 \text{ x } 1^2 \text{ x } 3^2 + 4 \text{ x } 2 \text{ x } 1^2 \text{ x } 3^2 + 2 \text{ x } 1^2 \text{ x } 3^3]$

= $-24-36-27+[48+36+72+54]$

= $-24-36-27+48+36+72+54 = 123$

∴Acceleration = $a_xi + a_yj + a_zk$ = 28i-3j+123k.

or Resultant acceleration = $\sqrt{28^2 + (-3)^2 + 123^2} = \sqrt{784 + 9 + 15129}$

= $\sqrt{15922}$ = **126.18 units.**

Find the convective acceleration at the middle of a pipe which converges uniformly from 0.4 m diameter to 0.2 m diameter over 2 m length. The rate of flow is 20 lit/s. If the rate of flow changes uniformly from 20 l/s to 40 l/s in 30 seconds. Find the total acceleration at the middle of the pipe at 15th second.

Given:

Diameter at section 1.

D_1 = 0.4 m ; D_2 = 0.2 m , L=2 m , Q=20 l/s = 0.02 m^3/s as one litre = 0.001m^3 = 1000 cm^3.

Find (i) Convective acceleration at middle i.e., at A when Q=20 l/s.

(ii) Total acceleration at A when Q changes from 20 l/s to 40 l/s in 30 seconds.

Solution:

Case I. In this case, the rate of flow is constant and equal to 0.02 m^3/s. The velocity of flow is in *x*-direction only. Hence this is one-dimensional flow and velocity components in y and z directions are zero or v=0 , z=0.

∴ Convective acceleration = $u\frac{\partial u}{\partial y}$ only …………..(i)

Let us find the value of u and $\frac{\partial u}{\partial x}$ at a distance *x* from inlet

The diameter (D_1) at a distance x from inlet or at section X-X is given by,

$$D_1 = 0.4 - \frac{0.4-.02}{2} \times x$$

$$= (0.4 - 0.1\,x) \text{ m}$$

The area of cross section (A_x) at section X-X is given by,

$$A_x = \frac{\pi}{4}D_x^2 = \frac{\pi}{4}(0.4-0.1\,x)^2$$

Velocity (u) at the section X-X in terms of Q (i.e., in terms of rate of flow)

$$u = \frac{Q}{Area} = \frac{Q}{A} = \frac{Q}{\frac{\pi}{4}D_x^2} = \frac{4Q}{\pi(0.4-0.1\,x)^2}$$

$$= \frac{1.273Q}{(0.4-0.1\,x)^2} = 1.273Q\,(0.4-0.1\,x)^{-2} \text{ m/s} \quad \text{.....(ii)}$$

To find $\frac{\partial u}{\partial x}$, we must differentiate equation (ii) with respect to *x*.

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}[1.273Q\,(0.4-0.1\,x)^{-2}]$$

$$= 1.273Q(-2)\,(0.4-0.1\,x)^{-1} \times (-0.1) \quad \text{[Here Q is constant].....(iii)}$$

$$= 0.2546Q\,(0.4-0.1\,x)^{-1}$$

Substituting the value of u an $\frac{\partial u}{\partial x}$ in equation (i), we get

Convective acceleration $= [1.273Q\,(0.4-0.1\,x)^{-2}] \times [0.2546Q\,(0.4-0.1\,x)]^{-1}$

$$= 1.273 \times 0.2546 \times Q^2 \times (0.4-0.1\,x)^{-3}$$

$$= 1.273 \times 0.2546 \times (0.02)^2 \times (0.4-0.1\,x)^{-3} \quad [\because Q = 0.02\,m^3/s]$$

∴ Convective acceleration at the middle (where *x* = 1 m)

$$= 1.273 \times 0.2546 \times (0.02)^2 \times (0.4-0.1 \times 1)^{-3} \text{m/s}^2$$

$$= 1.273 \times 0.2546 \times (0.02)^2 \times (0.3)^{-3} \text{m/s}^2$$

$$= 0.0048 \text{ m/s}^2.$$

Case II. When Q changes from 0.02 m^3/s to 0.04 m^3/s in 30 seconds, find the total acceleration at x=1 and t=15 seconds

Total acceleration = Convective acceleration + Local acceleration at t=15 seconds.

The rate of flow at t=15 seconds is given by

$$Q = Q_1 + \frac{Q_2 - Q_1}{30} \times 15 \text{ where } Q_1 = 0.04 \text{ m}^3\text{/s and } Q_1 = 0.02 \text{ m}^3\text{/s}$$

$$= 0.02 + \frac{(0.04 - 0.02)}{30} \times 15$$

$$\mathbf{= 0.03\ m^3/s}$$

The velocity (u) and gradient $\left(\frac{\partial u}{\partial x}\right)$in terms of Q are given by equations (ii) and (iii) respectively

$\therefore$ Convective acceleration = $u.\frac{\partial u}{\partial x}$

$$= [1.273Q\ (0.4 - 0.1\ x)^{-2}] \times [0.2546Q(0.4 - 0.1\ x)^{-1}]$$

$$= 1.273 \times 0.2546 \times Q^2 \times (0.4 - 0.1 \times 1)^{-3}$$

$$= 1.273 \times 0.2546 \times (0.03)^2 \times (0.3)^{-3} \text{m/s}^2$$

$$\mathbf{= 0.0108\ m/s^2.} \qquad \text{...........(iv)}$$

Local acceleration = $\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}$, $[1.273Q\ (0.4 - 0.1\ x)^{-2}]$

$[\because u\ from\ equation\ (ii) is\ u = 1.273Q(0.4 - 0.1\ x)^{-2}]$

$$= \quad 1.273Q(0.4 - 0.1\ x)^{-2} \times \frac{\partial Q}{\partial t}$$

$[\because$ Local acceleration is at a point wher x is constant but Q is changing$]$

Local acceleration (at x = 1 m)

$$= \quad 1.273 \times (0.4 - 0.1\ x)^{-2} \times \frac{\partial Q}{\partial t}$$

$$= \quad 1.273 \times (0.3)^{-2} \times \frac{0.02}{30} \qquad [\because \frac{\partial Q}{\partial t} = \frac{Q_2 - Q_1}{t} = \frac{0.04 - 0.02}{30} = \frac{0.02}{30}$$

$$= \quad \mathbf{0.00943\ m/s^2.} \qquad \text{............(v)}$$

Hence adding equations (iv) and (v), we get total acceleration.

$\therefore$ Total acceleration = Convective acceleration + Local acceleration

$$= 0.0108 + 0.00943$$

$$\mathbf{= 0.02023\ m/s^2.}$$

VELOCITY POTENTIAL FUNCTION AND STREAM FUNCTION

Velocity Potential Functions. It is defined as a scalar function of space and time such that its negative derivative with respect to any direction gives the fluid velocity in that direction. It is defined by ϕ (Phi). Mathematically, the velocity, potential is defined as $\phi = \int f(s, y, z)$ for steady flow such that

$$\left.\begin{aligned} u &= -\frac{\partial \phi}{\partial x} \\ v &= -\frac{\partial \phi}{\partial y} \\ w &= -\frac{\partial \phi}{\partial z} \end{aligned}\right\} \qquad \text{............(1)}$$

where u, v and w are the components of velocity in x,y and z directions respectively.

The velocity components in cylindrical polar co-ordinates in terms of velocity potential functions are given by

$$\left.\begin{aligned} u_r &= \frac{\partial\phi}{\partial r} \\ u_\theta &= \frac{1}{r}\frac{\partial\phi}{\partial\theta} \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots(1A)$$

where u_r = velocity component in radial direction (i.e., in *r* direction)

and u_θ = velocity component in tangential direction (i.e., in θ direction)

The continuity equation for an incompressible steady flow is $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

Substituting the values of u , v and w from equation (5.9) we get

$$\frac{\partial}{\partial x}\left(-\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial\phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(-\frac{\partial\phi}{\partial z}\right) = 0$$

or

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \qquad \ldots\ldots\ldots(2)$$

equation (2) is a Laplace equation.

For two-dimension case, equation (1) reduces to $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$ ……….(5.11)

If any value of ϕ that satisfies the Laplace equation, will correspond to some case of fluid flow.

Properties of the Potential Function. The rotational components* are given by

$$\omega_z = \frac{1}{2}\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right]$$

$$\omega_y = \frac{1}{2}\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right]$$

$$\omega_x = \frac{1}{2}\left[\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right]$$

Substituting the values of u, v and w from equation (1A) in the above rotational components, we get

$$\omega_z = \frac{1}{2}\left[\frac{\partial}{\partial x}\left(-\frac{\partial\phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial\phi}{\partial x}\right)\right] = \frac{1}{2}\left[-\frac{\partial^2\phi}{\partial x\partial y} + \frac{\partial^2\phi}{\partial y\partial x}\right]$$

$$\omega_y = \frac{1}{2}\left[\frac{\partial}{\partial z}\left(-\frac{\partial\phi}{\partial x}\right) - \frac{\partial}{\partial x}\left(-\frac{\partial\phi}{\partial z}\right)\right] = \frac{1}{2}\left[-\frac{\partial^2\phi}{\partial z\partial x} + \frac{\partial^2\phi}{\partial x\partial z}\right]$$

$$\omega_x = \frac{1}{2}\left[\frac{\partial}{\partial y}\left(-\frac{\partial\phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(-\frac{\partial\phi}{\partial y}\right)\right] = \frac{1}{2}\left[-\frac{\partial^2\phi}{\partial y\partial z} + \frac{\partial^2\phi}{\partial z\partial y}\right]$$

If ϕ is a continuous function, then $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$; $\frac{\partial^2\phi}{\partial z\partial x} = \frac{\partial^2\phi}{\partial x\partial z}$; etc.

$\therefore \quad \omega_z = \omega_y = \omega_x = 0$

When rotational components are zero, the flow is called ir-rotational. Hence the properties of the potential function are:

1. If velocity potential (ϕ) exists, the flow should be irrotational.
2. If velocity potential (ϕ) satisfies the Laplace equation, it represents the possible steady incompressible irrotational flow.

Stream Function. It is defined as the scalar function of space and time ; such that its partial derivative with respect to any direction gives the velocity component at right angles to that direction. It is denoted by ψ (Psi) and defined only for two-dimensional flow. Mathematically, for steady flow it is defined as $\psi = f(x,y)$ such that and $\frac{\partial\psi}{\partial x} = v$, $\frac{\partial\psi}{\partial y} = -u$ ………….(1)

The velocity components in cylindrical polar co-ordinates in terms of stream function are given as

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} \text{ and } u_\theta = -\frac{\partial\psi}{\partial r} \quad \text{……………(1A)}$$

where u_r = radial velocity and u_θ =tangential velocity

The continuity equation for two-dimensional flow is $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$.

Substituting the values of u and v from equation (5.12), we get

$$\frac{\partial}{\partial x}\left(-\frac{\partial\psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial x}\right) = \text{or } -\frac{\partial^2\psi}{\partial x\partial y} + \frac{\partial^2\psi}{\partial x\partial y} = 0$$

Hence existence of ψ means a possible case of fluid flow. The flow may be rotational or irrotational.

The rotational component ω_z is given by $\omega_z = \frac{1}{2}\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right]$

Substituting the values of u and v from equation (5.12) in the above rotational component, we get

$$\omega_z = \frac{1}{2}\left[\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial y}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial\psi}{\partial y}\right)\right] = \frac{1}{2}\left[\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right]$$

For irrotational flow, ω_z =0. Hence above equation becomes as $\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$

Which is Laplace equation for ψ .

The Properties of stream function (ψ) are:

1. If stream function (ψ) exists, it is a possible case of fluid which may be rotational or irrotational.
2. If stream function (ψ) satisfies the Laplace equation, it is possible case of an irrotational flow.

Equipotential Line. A line along which the velocity potential ϕ is constant, is called equipotential line.

For equipotential line ϕ = Constant

∴ $d\psi = 0$

But $\phi = f(x,y)$ for steady flow

∴ $d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy$

$= -udx - vdy$ $\quad \left\{\because \frac{\partial\phi}{\partial x} = -u, \frac{\partial\phi}{\partial y} - v\right\}$

$= -(udx + vdy)$

For equipotential line, $d\phi$ =0

$-(udx + vdy) = 0$ or $udx + vdy = 0$

∴ $\frac{dy}{dx} = -\frac{u}{v}$ ………..(2)

But $\frac{dy}{dx}$ = slope of equipotential line.

Line of Constant Stream Function

ψ = Constant

∴ $d\psi$ = 0

But $d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi\phi}{\partial y} dy + vdx - udy$ $\{\because \frac{\partial \psi}{\partial x} = v ; \frac{\partial \psi}{\partial y} = -u\}$

For a line of constant stream function

$= d\psi = 0$ or $vdx - udy = 0$

$\frac{dy}{dx} = \frac{v}{u}$ ....(3)

But $\frac{dy}{dx}$ is slope of stream line.

From equation (2) and (3) it is clear that the product of the slope of the equipotential line and the slope of the stream line at the point of intersection is equal to -1. Thus the equipotential lines are orthogonal to the stream lines at all points of intersection.

Flow Net. A grid obtained by drawing a series of equipotential lines and stream lines is called a flow net. The flow net is an important tool in analyzing two-dimensional irrotational flow problems.

Relational Between Stream Function and Velocity Potential Function

From equation (5.9),

We have $u = -\frac{\partial \phi}{\partial x}$ *and* $v = -\frac{\partial \phi}{\partial y}$

From equation (5.12), We have $u = -\frac{\partial \psi}{\partial y}$ *and* $v =- \frac{\partial \psi}{\partial x}$

Thus we have $u = -\frac{\partial \phi}{\partial x} =- \frac{\partial \psi}{\partial y}$ and $v = -\frac{\partial \phi}{\partial y} = \frac{\partial \psi}{\partial x}$

Hence $\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$

and $\frac{\partial \phi}{\partial y} = \frac{\partial \psi}{\partial x}$ ....(5.15)

The velocity potential function (ϕ) is given by an expression $\boldsymbol{\phi = -\frac{xy^3}{3} - x^2 + \frac{x^3y}{3} + y^2}$

I. **Find the velocity components in *x* and y direction.**
II. **Show that ϕ represents a possible case of flow.**

Solution: Given: $\phi = -\frac{xy^3}{3} - x^2 + \frac{x^3y}{3} + y^2$

The partial derivatives of ϕ w.r.t. *x* and y are

$\frac{\partial \phi}{\partial x} = -\frac{y^3}{3} - 2x + \frac{3x^2y}{3}$ ....(1)

and $\frac{\partial \phi}{\partial y} = -\frac{3xy^2}{3} + \frac{x^3}{3} + 2y$ ....(2)

(i) The velocity components u and v are given by equation (5.9)

$u = -\frac{\partial \phi}{\partial x} = -\left[-\frac{y^3}{3} - 2x + \frac{3x^2y}{3}\right] = \frac{y^3}{3} + 2x - x^2y$

∴ $u = \frac{y^3}{3} - 2x - x^2y$

∴ $v = -\frac{\partial \phi}{\partial y} = -\left[-\frac{3xy^2}{3} + \frac{x^3}{3} + 2y\right] = \frac{3xy^2}{3} - \frac{x^3}{3} - 2y = xy^2 - \frac{x^3}{3} - 2y$

(ii) The given value of ϕ,will represent a possible case of flow if it satisfies the Laplace equation, i.e.,

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

From equations (1) and (2), we have

Now $\frac{\partial\phi}{\partial x} = -\frac{y^3}{3} - 2x - x^2y$

$\therefore$ $\frac{\partial^2\phi}{\partial x^2} = -2 + 2xy$

and $\frac{\partial\phi}{\partial y} = xy^2 + \frac{x^3}{3} + 2y$

$\therefore$ $\frac{\partial^2\phi}{\partial y^2} = -2xy + 2$

$\therefore$ $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = (-2+2xy)+(-2xy+2)=0$

$\therefore$ Laplace equation is satisfied and hence ϕ represent a possible case of flow.

If for a two-dimensional potential flow, the velocity potential is given by $\phi = x$ (2y-1). determine the velocity at the point P(4,5). Determine also the value of stream function ψ at the point P.

Solution: Given: $\phi = x\,(2y-1)$

(i) The velocity components in the direction of x and y are

$$u = -\frac{\partial\phi}{\partial x} = -\frac{\partial}{\partial x}\,[x(2y-1)]] = -[2y-1] = 1-2y$$

$$v = -\frac{\partial\phi}{\partial y} = -\frac{\partial}{\partial y}\,[x(2y-1)]] = -[2x] = -2x$$

At the point P(4,5), i.e., at x=4 , y=5

u = 1-2x5=-9 units/sec

v = -2x4=-8 units/sec

$\therefore$ Velocity at P = -9i -8j

or Resultant velocity at P = $\sqrt{9^2+8^2} = \sqrt{81+64}$ 12.04 units/sec. 12.04 units/sec.

(ii) Value of Stream Function at P

We know that $\frac{\partial\psi}{\partial y} = -u = -(1-2y) = 2y-1$ ………..(i)

and $\frac{\partial\psi}{\partial x} = v = -2x$ ……….(ii)

Integrating equation (i) w.r.t. 'y', we get

$$\int d\psi = \int(2y-1)dy \text{ or } \psi = \frac{2y^2}{2} - y + \text{Constant of integration.}$$

The constant of integration is not a function of y but it can be a function of x. Let the value of constant of integration is k. Then

$\psi = y^2 - y + k$

Differentiating the above equation w.r.t. 'x', we get ………….(iii)

$$\frac{\partial\psi}{\partial x} = \frac{\partial k}{\partial x}$$

But from equation (ii) $\frac{\partial\psi}{\partial x} = -2x$

Equating the value of $\frac{\partial\psi}{\partial x}$, we get $\frac{\partial k}{\partial x} = -2x$

Integrating this equation, we get $k = \int -2x\,dx = -\frac{2x^2}{2} = -x^2$.

Substituting this value of k in equation (iii), we get $\psi = y^2 - y - x^2$.

∴ Stream function ψ at P(4,5) = $5^2 - 5 - 4^2 = 25 - 5 - 16 = 4$ units.

The stream function for a two-dimensional flow is given by ψ = 2xy. Calculate the velocity at the point P(2,3). Find the velocity potential function ϕ.

Solution. Given: $\psi = 2xy$

The velocity components u and v in terms of ψ are

$$u = -\frac{\partial\psi}{\partial y} = -\frac{\partial}{\partial y}(2xy) = -2x$$

$$v = -\frac{\partial\psi}{\partial x} = \frac{\partial}{\partial x}(2xy) = 2y$$

At the point P(2,3) we get u = -2x2=-4 units/sec.

v = 2 x 3 =6 units/sec.

∴ Resultant velocity at P = $\sqrt{u^2+v^2} = \sqrt{4^2+6^2} = \sqrt{16+36} = \sqrt{52}$ = 7.21 units/sec.

Velocity potential function ϕ

We know $\frac{\partial\phi}{\partial x} = -u = -(-2x) = 2x$ ....(i)

$\frac{\partial\phi}{\partial y} = -v = -2y$ ....(ii)

Integrating equation (i), we get

$$\int d\phi = \int 2x dx$$

or $\phi = \frac{2x^2}{2} = + C = x^2 + C$ ....(iii)

where C is a constant which is independent of x but can be a function of y.

Differentiating equation (iii) w.r.t. 'y', we get $\frac{\partial\phi}{\partial y} = \frac{\partial C}{\partial y}$

But from (ii), $\frac{\partial\phi}{\partial y} = -2y$

∴ $\frac{\partial C}{\partial y} = -2y$

Integrating this equation, we get $C = \int -2y\, dy = -\frac{2y^2}{2} = y^2$

Substituting this value of C in equation (iii), we get $\phi = x^2 - y^2$.

The velocity components in a two-dimensional flow field for an incompressible fluid are as follows: u = $\frac{y^3}{3}$ + 2x – x²y and v = xy² – 2y – x³/3 , obtain an expression for the stream function ψ.

Solution: Given: $u = y^3/3 + 2x - x^2y$

$v = xy^2 - 2y - x^3/3$

The velocity components in terms of stream function are

$\frac{\partial\psi}{\partial x} = v = xy^2 - 2y - x^3/3$ ....(i)

$\frac{\partial\psi}{\partial y} = -u = -y^{3/3} - 2x + x^2y$ ....(ii)

Integrating (i) w.r.t. x, we get $\psi = \int (xy^2 - 2y - x^3/3)$/dx or $\psi = \frac{x^2y^2}{2}$ - 2xy - $\frac{x^4}{4 \text{ x } 3} + k$ …………..(iii)

where k is a constant of integration which is independent of *x* but can be a function of y.

Differentiating equation (iii) w.r.t y, we get

$$\frac{\partial\psi}{\partial y} = \frac{2x^2y}{2} - 2x + \frac{\partial k}{\partial y} = x^2y - 2x + \frac{\partial k}{\partial y}$$

But from (ii), $\frac{\partial\psi}{\partial y} = \frac{y^3}{3} - 2x + x^2y$

Comparing the value of $\frac{\partial\psi}{\partial y}$, we get $x^2y - 2x + \frac{\partial k}{\partial y} = \frac{y^3}{3} - 2x + x^2y$

$$\frac{\partial k}{\partial y} = -\frac{y^3}{3}$$

Integrating, we get $k = \int(-y^3/3)dy = \frac{-y^4}{4 \text{ x } 3} = \frac{-y^4}{12}$

Substituting this value in (iii), we get

$$\psi = \frac{x^2y^2}{2} - 2xy - \frac{x^4}{12} - \frac{y^4}{12}$$

TYPES OF MOTION

A fluid particle while moving may undergo anyone or combination of following four types of displacements:

(i) Linear Translation or Pure Translation

(ii) Linear Deformation

(iii) Angular Deformation and

(iv) Rotation

Linear Translation. It is defined as the movement of a fluid element in such a way that it moves bodily from one position to another position and the two axes ab and cd represented in new positions by a′b′ and c′d′ are parallel as shown in Fig. (a)

Linear Deformation. It is defined as the deformation of a fluid element in linear direction when the element moves. The axes of the element in the deformed position and un-deformed position are parallel, but their lengths change as shown in Fig. (b).

Angular Deformation or Shear Deformation. It is defined as the average change in the angle contained by two adjacent sides. Let $\Delta\theta_1 \, and \, \Delta\theta_2$ is the change in angle between two adjacent sides of a fluid element as shown in Fig. (c) , then angular deformation or shear strain rate.

$$\frac{1}{2}[\Delta\theta_1 + \Delta\theta_2]$$

Now $\Delta\theta_1 = \frac{\partial v}{\partial x} \text{ x } \frac{\Delta x}{\Delta x} = \frac{\partial v}{\partial x}$ and $\Delta\theta_2 = \frac{\partial u}{\partial y} \cdot \frac{\Delta y}{\Delta y} = \frac{\partial u}{\partial y}$

∴ Angular deformation = $\frac{1}{2}[\Delta\theta_1 + \Delta\theta_2]$

or shear strain rate $= \frac{1}{2}[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}]$

Rotation. It is defined as the movement of a fluid element in such a way that both of its axes (horizontal as well as vertical) rotate in the same direction as shown in Fig.(d). It is equal to $\frac{1}{2}[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}]$ for a two-dimensional element in *x*-y plane. The rotational components are

$$\omega_z = \frac{1}{2}\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right]$$

$$\omega_x = \frac{1}{2}\left[\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right]$$

$$\omega_y = \frac{1}{2}\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right]$$

Vorticity. It is defined as the value twice of the rotation and hence it is given as 2ω.

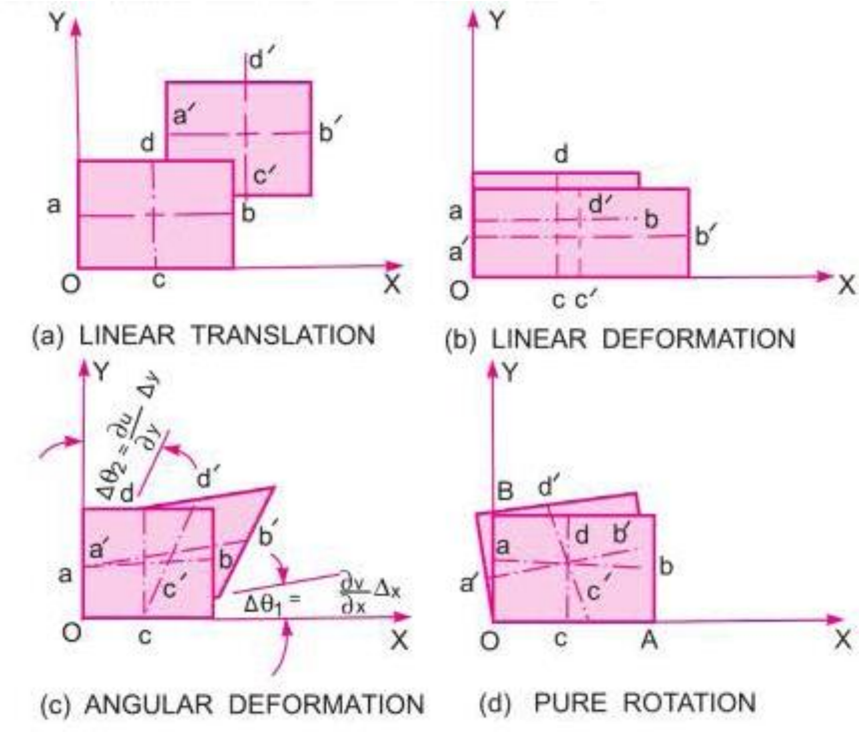


Fig. Displacement of a fluid element.

A fluid flow is give by V = $8x^3$i – $10x^2$yj. Find the shear strain rate and state whether the flow is rotational or irrotational.

Given V = $8x^3$i – $10x^2$yj

∴ $u = 8x^3$, $\frac{\partial u}{\partial x} = 24x^2$, $\frac{\partial u}{\partial y} = 0$

and $v = -10x^2y$, $\frac{\partial v}{\partial x} = 20xy$, $\frac{\partial v}{\partial y} = -10x^2$

Solution:

(i) Shear strain rate is given by equation as

$$= \frac{1}{2}\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right] = \frac{1}{2}(-20xy+0) = -10x.$$

(ii) Rotation in x-y plane is given by equation

$$\omega_z = \frac{1}{2}\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right] = \frac{1}{2}(-20xy-0) = -10xy.$$

As rotation $\omega_z \neq 0$. Hence flow is rotational.

VORTEX FLOW

Vortex flow is defined as the flow of a fluid along a curved path or the flow of a rotating mass of fluid is known a 'Vortex Flow'. The vortex flow is of two types namely:

1. Forced vortex flow and
2. Free vortex flow

Forced Vortex Flow. Forced vortex flow is defined as that type of vortex flow, in which some external torque is required to rotate the fluid mass. The fluid mass in this type of flow, rotates at constant angular velocity, ω. The tangential velocity of any fluid particle is given by

$$v = \omega \times r \qquad \text{.............(5.18)}$$

where r = Radius of fluid particle from the axis of rotation.

Hence angular velocity ω is given by

$$\omega = \frac{v}{r} = \text{Constant} \qquad \text{............(5.19)}$$

Examples of forced vortex are:

1. A vertical cylinder containing liquid which is rotated about its central axis with a constant angular velocity ω , as shown in Fig. 5.12
2. Flow of liquid inside the impeller of a centrifugal pump
3. Flow of water through the runner of a turbine.

EULER'S EQUATION OF MOTION

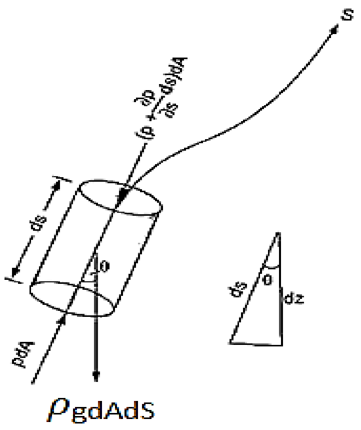
This is equation of motion in which the forces due to gravity and pressure are taken into consideration. This is derived by considering the motion of a fluid element along a stream-line as:

Consider a stream-line in which flow is taking place in s-direction as shown in Fig. Consider a cylindrical element of cross-section dA and length *ds*. The forces acting on the cylindrical element arc:

I. Pressure force pdA in the direction of flow.

2. Pressure force (p+ $\frac{\partial p}{\partial s}$ ds) dA opposite to the direction of flow.

3. Weight of element ρgdAds.



forces on a fluid fluid

Let θ is the angle between the direction of flow and the line of action of the weight of element. The resultant force on the fluid element in the direction of s must be equal to the mass of fluid element x acceleration in the direction s.

$$pdA - (p + \frac{\partial p}{\partial s} ds)dA - \rho gdAds \cos\theta$$

$$= \rho gdAds \times a_s \qquad \text{--------- (1)}$$

where, a_s is the acceleration in the direction of *s*

Now $a_s = \frac{dv}{dt}$, *where v is a function of s and t,*

$$\frac{dv}{ds}\frac{ds}{dt} + \frac{dv}{dt} = \frac{vdv}{ds} + \frac{dv}{dt} \qquad \{\frac{ds}{dt} = v\}$$

If the flow is steady,

$$\frac{dv}{dt} = 0$$

$$a_s = \frac{vdv}{ds},$$

Substituting the value of as in equation (1) and simplifying the equation, we get

$$-\frac{\partial p}{\partial s} dsdA - \rho g\ dAds \cos\theta = \rho dAds \times \frac{dv}{ds}$$

Dividing by ρdsdA,

$$-\frac{\partial p}{\rho \partial s} - g\cos\theta = \frac{vdv}{ds} \quad \text{(Or)}$$

$$\frac{\partial p}{\rho \partial s} + g\cos\theta + \frac{vdv}{ds} = 0$$

But from Fig. 6.1 (b), we have $\cos\theta = \frac{dz}{ds}$

$$\frac{1}{\rho}\frac{dp}{ds} + g\frac{dz}{ds} + \frac{vdv}{ds} = 0 \quad \text{(or)}$$

$$\frac{dp}{\rho} + g\,dz + vdv = 0 \qquad \text{--------- (2)}$$

Equation (2) is known as Euler's equation of motion.

BERNOULLI'S EQUATION FROM EULER'S EQUATION

Bernoulli's equation is obtained by integrating the Euler's equation of motion (2) as

$$\int \frac{dp}{\rho} + \int g dz + \int v dv = \text{constant}$$

If flow is incompressible, ρ is constant and

$$\frac{p}{\rho} + gz + V^2/2 = \text{constant}$$

$$\frac{p}{\rho g} + z + V^2/2g = \text{constant}$$

$$\frac{p}{\rho g} + V^2/2g + z = \text{constant}$$

The above equation is a Bernoulli's equation in which,

$\frac{p}{\rho g}$ = pressure energy per unit weight of fluid or pressure head.

$V^2/2g$ = kinetic energy per unit weight or kinetic head.

z = potential energy per unit weight or potential head.

ASSUMPTIONS BERNOULLI'S EQUATION:

The following are the assumptions made in the derivation of Bernoulli's equation:

(i) The fluid is ideal, i.e., viscosity is zero

(ii) The flow is steady

(iii) The flow is incompressible

(iv) The flow is irrotational.

Water is flowing through a pipe of 5 cm diameter under a pressure of 29.43 N/cm^2 (gauge) and with mean velocity of 2.0 m/s. Find the total head or total energy per unit weight of the water at a cross-section, which is 5 m above the datum line.

Solution: Given:

Diameter of pipe = 5 cm = 0.5 m

Pressure, p = 29.43 N/cm^2 = 29.43 x 10^4 N/m^2

Velocity, v = 2.0 m/s

Datum head, z = 5 m

Total head = [pressure head + kinetic head + datum head]

Pressure head $= \frac{p}{\rho g} = \frac{29.43\ X\ 10^4}{1000\ \ X\ 9.81} =$ 30m

Kinetic head $= \frac{v^2}{2g} = \frac{2\ X\ 2}{2\ \ X\ 9.81}$ = 0.204 m

Total head $= \frac{p}{\rho g} + \frac{v^2}{2g}$ + z = 30 + 0.204 + 5

= 35.204 m. Ans

A pipe, through which water is flowing, is having diameters, 20 cm and 10 cm at the cross-sections 1 and 2 respectively. The velocity of water at section 1 is given 4.0 m/s. Find the velocity head at sections 1 and 2 and also rate of discharge.

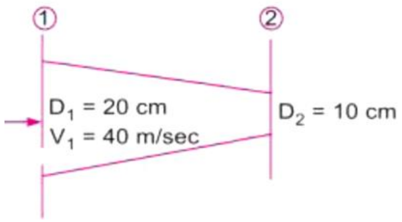
D_1 = 20 cm = 0.2 m

Area, $A_1 = \frac{\pi}{4} {D_1}^2 = \frac{\pi}{4}\ (0.2^2)$ = 0.0314 m^2

V_1 = 4.0 m/s

D_2 = 0.1 m

$A_1 = \frac{\pi}{4} D_2^{\ 2} = (0.1^2) = 0.0785$ m²



Solution

(i)Velocity head at section 1,

$$\frac{V_1^{\ 2}}{2g} = \frac{4.0\ X\ 4.0}{2\ \ X\ 9.81} = \mathbf{0.815\ m}$$

(ii) Velocity head at section 2, $\frac{V_2^{\ 2}}{2g}$

To find V_2, apply continuity equation at 1 and 2,

$A_1V_1 = A_2V_2$ (or)

$$V_2 = \frac{A_1V_1}{V_2} = \frac{0.314}{.00785} \text{ x } 4.0$$

$$\mathbf{V_2 = 16.0\ m/s}$$

Velocity head at section 2, $\frac{V_2^{\ 2}}{2g} = \frac{16.0\ X\ 16.0}{2\ X\ 9.81} = \mathbf{83.047\ m}$

(iii)Rate of discharge , (Q) = A_1V_1 (or) A_2V_2

= 0.0314 x 4.0 = 0.1256 m³/s

Q = 0.1256 m³/s

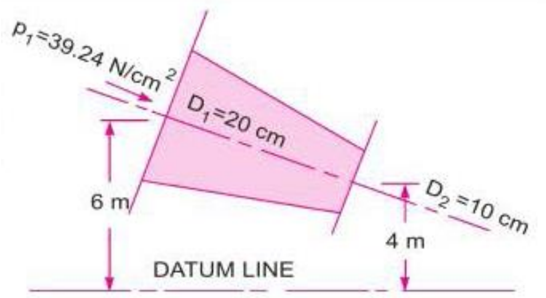
The water is flowing through a pipe having diameters 20 cm and 10 cm at sections 1 and 2 respectively. The rate of flow through pipe is 35 litres/s. The section 1 is 6 m above datum and section 2 is 4 m above datum. If the pressure at section 1 is 39.24 N/cm², find the intensity of pressure at section 2.

Given:

At Section 1

D_1 = 20 cm = 0.2 m

Area, $A_1 = \frac{\pi}{4} D_1{}^2 = \frac{\pi}{4}(0.2^2) = 0.0314$ m^2



P_1 = 39.24 N/cm^2

= 39.24×10^4 N/m^2

Z_1 = 6.0 m

At Section 2

D_2 = 10 cm = 0.1 m

Area, $A_2 = \frac{\pi}{4} D_2{}^2 = \frac{\pi}{4}(0.1^2) = 0.00785$ m^2

Z_2 = 4.0 m

P_2 = ?

Solution:

Rate of flow, Q = 35 lit /s = $\frac{35}{1000}$ = **0.35 m³/s** **Q = 0.35 m³/s**

Now , **Q = A_1V_1 = A_2V_2**

$V_1 = \frac{Q}{A_1} = \frac{.035}{.0314} = 1.114$ m/s **V_1 = 1.114 m/s**

$V_2 = \frac{Q}{A_2} = \frac{35}{.00785} = 4.456$ m/s **V_2 = 4.456 m/s**

Applying Bernoulli's equation at sections 1 and 2, we get,

$$\frac{p_1}{\rho g} + \frac{v_1{}^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2{}^2}{2g} + z_2$$

$$\frac{39.24 \, X \, 10^4}{1000 \; X \, 9.81} + \frac{(1.114)^2}{2 \, X \, 9.81} + 6.0 = \frac{\mathrm{p}_2}{1000 \; \mathrm{X} \, 9.81} + \frac{(4.456)^2}{2 \, X \, 9.81} + 4.0$$

$$40 + 0.063 + 6.0 = \frac{\mathrm{p}_2}{9810} + 1.012 + 4.0$$

$$46.063 = \frac{\mathrm{p}_2}{9810} + 5.012$$

$$\frac{\mathrm{p}_2}{9810} = 46.063 - 5.012 = 41.051$$

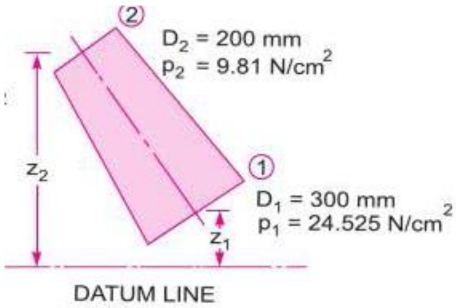
$P_2 = 41.051 \times 9810$ N/ cm^2

$$= \frac{\mathbf{41.051 \times 9810}}{10^4} \ \mathbf{N/cm^2}$$

P_2 = 40.27 N/cm^2

Water is flowing through a pipe having diameter 300 mm and 200 mm at the bottom and upper end respectively. The intensity of pressure at the bottom end is 24.525 N/cm^2 and the pressure at the upper end is 9.8/ N/cm^2. Determine the difference in datum head if the rate of flow through pipe is 40 lit/s.

Given:



At Section 1

D_1 = 300 mm = 0.3 m

P_1 = 24.525 N/cm^2 = 24.525 x 10^4 N/m^2

At Section 2

D_2 = 200 mm = 0.2 m

P_1 = 9.81 N/cm^2 = 9.81 x 10^4 N/m^2

Solution:

Rate of flow (Q) = 40 litres

$$Q = \frac{40}{1000} = 0.04 \ m^3/s$$

$A_1V_1 = A_2V_2$ = Rate of flow (Q) = 0.04

$$V_1 = \frac{Q}{A_1} = \frac{0.04}{A_1} = \mathbf{0.566 \ m/s} \qquad \left\{ A = \frac{\pi}{4} D^2 \right\}$$

$$V_2 = \frac{Q}{A_2} = \frac{0.04}{A_2} = \mathbf{1.274 \ m/s}$$

Applying Bernoulli's equation at sections (1) and (2), we get

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2$$

$$\frac{24.524 \, X \, 10^4}{1000 \;\; X \, 9.81} + \frac{(.566)^2}{2 \, X \, 9.81} + Z_1 = \frac{9.81 \, X \, 10^4}{1000 \;\; X \, 9.81} + \frac{(1.274)^2}{2 \, X \, 9.81} + Z_2$$

$25 + 0.32 + Z_1 = 10 + 1.623 + Z_2$

$25.32 + Z_1 = 11.623 + Z_2$

$Z_2 - Z_1 = 25.32 - 11.623 = 13.697m$

Difference in Datum head, $\mathbf{Z_2 - Z_1 = 13.70 \text{ m}}$

PRACTICAL APPLICATIONS OF BERNOULLI'S EQUATION

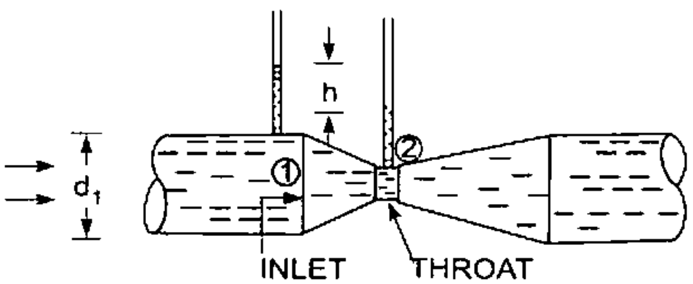
Bernoulli's equation is applied in all problems of incompressible fluid flow where energy considerations are involved. But we shall consider its application to the following measuring devices:

- Venturimeter
- Orifice meter
- Pitot-tube

VENTURIMETER:

A venturimeter is a device used for measuring the rate of a flow of a fluid flowing through a pipe. It consists of three parts:

(i) A short converging part, (ii) Throat, and (iii) Diverging part. It is based on the Principle of Bernoulli's equation



Venturimeter

Expression for rate of flow through venturimeter

Consider a venturimeter fitted in a horizontal pipe through which a fluid is flowing (say water), as shown in Fig.

Let d_1 = diameter at inlet or at section (1)

p_1 = pressure at section (1)

v_1 = velocity of fluid at section (1)

a_1 = area at section (1) = $\frac{\pi}{4} d_1^{\,2}$

and d_2, p_2, v_2 and a_2 are corresponding values at section (2). Applying Bernoulli's equation at sections (1) and (2),

$$\frac{p_1}{\rho g} + \frac{v_1^{\,2}}{2g} + Z_1 = \frac{p_2}{\rho g} + \frac{v_2^{\,2}}{2g} + Z_2$$

As pipe is horizontal, hence $Z_1 = Z_2$

$$Q = \frac{a_1 \; a_2}{\sqrt{a_1^{\,2} - a_2^{\,2}}} \times \sqrt{2gh}$$

The above equation gives the discharge under ideal conditions and is called, theoretical discharge. Actual discharge will be less than theoretical discharge.

$$Q_{act} = C_d \times \frac{a_1 \; a_2}{\sqrt{a_1^{\,2} - a_2^{\,2}}} \times \sqrt{2gh}$$

where C_d = Co-efficient of venturimeter and its value is less than 1.

Value of 'h' given by differential U-tube manometer

Case I: Let the differential manometer contains a liquid which is heavier than the liquid flowing through the pipe. Let,

S_h = Sp. gravity of the heavier liquid

S_a = Sp. gravity of the liquid flowing through pipe

x = Difference of the heavier liquid column in U-tube

then,

$$h = x \left[\frac{s_h}{s_o} - 1 \right]$$

Case II: If the differential manometer contains a liquid which is lighter than the liquid flowing through the pipe, the value of h is given by

$$h = x \left[1 - \frac{s_l}{s_o} \right]$$

where,

S_l = Sp. gr. of lighter liquid in U-Tube

S_o = Sp. gr. of fluid flowing through pipe .

x = Difference of the lighter liquid columns in U-tube.

Case III: Inclined Venturimeter with Differential U-tube manometer. The above two cases are given for a horizontal venturimeter. This case is related to inclined venturimeter having differential U-tube manometer. Let the differential manometer contains heavier liquid then h is given as

$$h = \left(\frac{p_1}{\rho g} + Z_1 \right) - \left(\frac{p_2}{\rho g} + Z_2 \right) = x \left[\frac{s_h}{s_o} - 1 \right]$$

Case IV: Similarly, for inclined venturimeter in which differential manometer contains a liquid which is lighter than the liquid flowing through the pipe, the value of h is given as

$$h = \left(\frac{p_1}{\rho g} + Z_1 \right) - \left(\frac{p_2}{\rho g} + Z_2 \right) = x \left[1 - \frac{s_l}{s_o} \right]$$

A horizontal venturimeter with inlet and throat diameters 30 cm and 15 cm respectively is used to measure the flow of water. The reading of differential manometer connected to the inlet and the throat is 20 cm of mercury. Determine the rate of flow. Take C_d = 0.98.

Solution:

Given :

Diameter at inlet, d_1 = 30 cm = 0.3 m

Area at inlet , $a_1 = \frac{\pi}{4} d_1^2 = \frac{\pi}{4} (30^2)$ = 706.85 cm^2

Diameter at throat, d_2 = 15m = 0.15m

$a_2 = \frac{\pi}{4} d_2^2 = \frac{\pi}{4} (15^2)$ = 176.7 cm^2

C_d = 0.98

Reading of differential manometer = x =20 cm of mercury

Difference of pressure head is , $h = x \left[\frac{s_h}{s_o} - 1 \right]$

Where,

S_h = Sp. gravity of mercury = 13.6

S_o = Sp. Gravity of water = 1

$h = 20 \left[\frac{13.6}{1} - 1 \right]$ = 20 x12.6cm = 252 cm of water

The discharge through venturimeter is given by eqn.

$$Q = C_d \times \frac{a_1 \ a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh}$$

$$= 0.98 \times \frac{706.85 * 176.7}{\sqrt{(706.85)^2 - (176.7)^2}} \times \sqrt{2 * 9.81 * 252}$$

$$= \frac{86067593.36}{\sqrt{499636.9 - 31222.9}} = \frac{86067593.36}{684.4}$$

$$= 125756 \text{ cm}^3/\text{s} = \frac{125756}{1000} \text{ lit/s}$$

Q = 125.756 lit/sec

An oil of sp. gr. 0.8 is flowing through a venturimeter having inlet diameter 20 cm and throat diameter 10 cm. The oil-mercury differential manometer shows a reading of 25 cm. Calculate the discharge of oil through the horizontal venturimeter. Take Cd = 0.98.

Solution. Given :

Sp. gr. of oil, $S_o = 0.8$

Sp. gr. of mercury, $S_h = 13.6$

Reading of differential manometer, x = 25 cm

Difference of pressure head, $h = x \left[\frac{s_h}{s_o} - 1 \right]$

$$= 25 \left[\frac{\mathbf{13.6}}{\mathbf{0.8}} - 1 \right] \text{ cm of oil} = 25\,[17-1]$$

h = 400 cm of oil

diameter at inlet, $d_1 = 20$ cm

$$a_1 = \frac{\pi}{4} d_1^2 = \frac{\pi}{4} \times 20^2 = 314.16 \text{ cm}^2$$

$d_2 = 10$ cm

$$a_2 = \frac{\pi}{4} d_2^2 = \frac{\pi}{4} \times 10^2 = 78.54 \text{ cm}^2$$

C_d = 0.98

The discharge Q is given by equation

$$Q = C_d \times \frac{a_1 \; a_2}{\sqrt{a_1^2 - a_2^2}} \times \sqrt{2gh}$$

$$= 0.98 \times \frac{\mathbf{314.16 * 78.54}}{\sqrt{(\mathbf{314.16})^2 - (\mathbf{78.54})^2}} \times \sqrt{2} * 9.81 * 400$$

$$= \frac{21421375.68}{\sqrt{98696} - 6168} = \frac{21421375.68}{304} \text{ cm}^3/\text{s}$$

$= 70465 \text{ cm}^3/\text{s}$ **Q = 70.465 litres/sec**

A horizontal venturimeter with inlet diameter 20 cm and throat diameter 10 cm is used to measure the flow of oil of sp. gr. 0.8. The discharge of oil through venturimeter is 60 litres/s. Find the reading of the oil-mercury differential manometer. Take Cd = 0.98.

A horizontal venturimeter with inlet diameter 20 cm and throat diameter 10 cm is used to measure the flow of water. The pressure at inlet is 17.658 N/cm^2 and the vacuum pressure at the throat is 30 cm of mercury. Find the discharge of water through venturimeter. Take Cd = 0.98.

The inlet and throat diameters of a horizontal venturimeter are 30 cm and 10 cm respectively. The liquid flowing through the meter is water. The pressure intensity at inlet is 13.734 N/cm^2 while the vacuum pressure head at the throat is 37 cm of mercury. Find the rate of flow. Assume that 4% of the differential head is lost between the inlet and throat. Find also the value of C_d for the venturimeter.

PROBLEMS ON INCLINED VENTURIMETER

A 20 cm x 10 cm venturimeter is inserted in a vertical pipe carrying oil of sp. gr. 0.8, the flow of oil is in upward direction. The difference of levels between the throat and inlet section is 50 cm. The oil mercury differential manometer gives a reading of 30 cm of mercury. Find the discharge of oil. Neglect losses.

Solution: Given:

Diameter at inlet, d_1 = 20 cm = 0.2 m

Area at inlet , $a_1 = \frac{\pi}{4} {d_1}^2 = \frac{\pi}{4}(20^2)$ = 314.16 cm^2

Diameter at throat, d_2 = 10 cm = 0.10 m

$a_2 = \frac{\pi}{4} {d_2}^2 = \frac{\pi}{4}(10^2)$ = 78.54 cm^2 , C_d = 1.0

Specific gravity of oil, S_o = 0.8

Specific gravity of mercury, S_h = 13.6

Reading of differential manometer, x = 30 cm

$$h = \left(\frac{p_1}{\rho g} + Z_1\right) - \left(\frac{p_2}{\rho g} + Z_2\right) = x\left[\frac{s_g}{s_o} - 1\right]$$

$$= 30\left[\frac{\mathbf{13.6}}{\mathbf{0.8}} - 1\right] = 30\,[17-1] = 30 \times 16$$

h = 480 cm of oil

The discharge,

$$Q = C_d \times \frac{a_1 \; a_2}{\sqrt{a_1{}^2 - a_2{}^2}} \times \sqrt{2gh}$$

$$= 1.0 \times \frac{314.16 * 78.54}{\sqrt{(314.16)^2 - (78.54)^2}} \times \sqrt{2} * 981 * 480 \text{ cm}^3/\text{s}$$

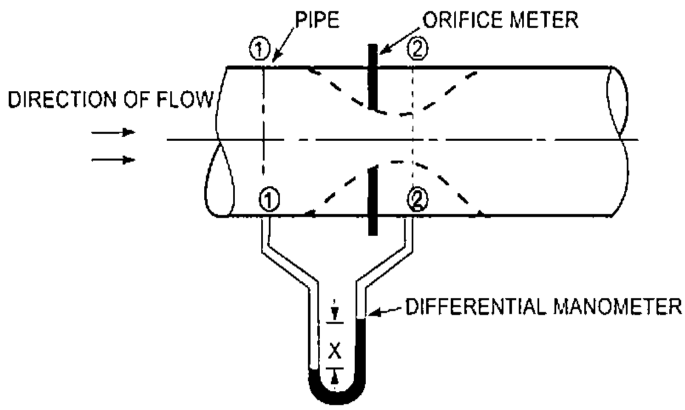
$$= \frac{23932630.7}{304}$$

= 78725.75 cm^3/s **= 78.725 litres/s**

A 30 cm x 15 cm venturimeter is inserted in a vertical pipe carrying water, flowing in the upward direction. A differential mercury manometer connected to the inlet and throat gives a reading of 20 cm. Find the discharge. Take C_d = 0.98

ORIFICE METER (OR) ORIFICE PLATE

- ✓ It is a device used for measuring the rate of flow of a fluid through a pipe. It is a cheaper device as compared to venturimeter.
- ✓ It also works on the same principle as that of venturimeter. It consists of a flat circular plate which has a circular sharp edged hole called orifice, which is concentric with the pipe.
- ✓ The orifice diameter is kept generally 0.5 times the diameter of the pipe, though it may vary from 0.4 to 0.8 times the pipe diameter.



A differential manometer is connected at section (1), which is at a distance of about 1.5 to 2.0 times the pipe diameter upstream from the orifice plate, and at section (2), which is at a distance of about half the diameter of the orifice on the downstream side from the orifice plate.

p_1 = pressure at section (1)

v_1 = velocity at section (1)

a_1 = area of the pipe at section (1) = $\frac{\pi}{4} d_1{}^2$

and d_2, p_2, v_2 and a_2 are corresponding values at section (2). Applying Bernoulli's equation at sections (1) and (2), we get

$$\frac{p_1}{\rho g} + \frac{{v_1}^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{{v_2}^2}{2g} + z_2$$

$$Q = \frac{\boldsymbol{C_d\, a_0 a_1\, \sqrt{2gh}}}{\sqrt{1} - (\frac{a_0}{a_1})} = \frac{\boldsymbol{C_d\, a_0 a_1\, \sqrt{2gh}}}{\sqrt{{a_1}^2 - {a_0}^2}}$$

C_d = coefficient of discharge for orifice meter

An orifice meter with orifice diameter 10 cm is inserted in a pipe of 20 cm diameter. The pressure gauges fitted upstream and downstream of the orifice meter gives readings of 19.62 N/cm^2 and 9.81 N/cm^2 respectively. Co-efficient of discharge for the orifice meter is given as 0.6. Find the discharge of water through pipe.

Given:

Diameter at inlet, d_0 = 10 cm = 0.1 m

Area at inlet, $a_0 = \frac{\pi}{4}\, {d_0}^2 = \frac{\pi}{4}\,(10^2)$ = 78.54 cm^2

Diameter at pipe, d_1 = 20 cm = 0.20 m

$a_1 = \frac{\pi}{4}\, {d_2}^2 = \frac{\pi}{4}\,(20^2)$ = 314.16 cm^2

C_d = 0.6

P_1 = 19.62 N/cm^2 = 19.62 x 10^4 N/m^2

P_2 = 9.81 N/cm^2 = 9.81 x 10^4 N/m^2

Solution:

$$\frac{p_1}{\rho g} = \frac{19.62 * 10^4}{1000 * 9.81}$$ = 20 m of water

$$\frac{p_2}{\rho g} = \frac{9.81 * 10^4}{1000 * 9.81}$$ = 10 m of water

$$h = \frac{p_1}{\rho g} - \frac{p_2}{\rho g} = 20.0 - 10.0$$

= 10 m of water

= 1000 cm of water

The discharge, Q is given by equation,

$$Q = C_d \times \frac{a_0 a_1}{\sqrt{a_1^2 - a_0^2}} \times \sqrt{2gh}$$

$$= 0.6 \times \frac{78.54 * 314.16}{\sqrt{(314.16)^2 - (78.54)^2}} \times \sqrt{2} * 981 * 1000 \text{ cm}^3/\text{s}$$

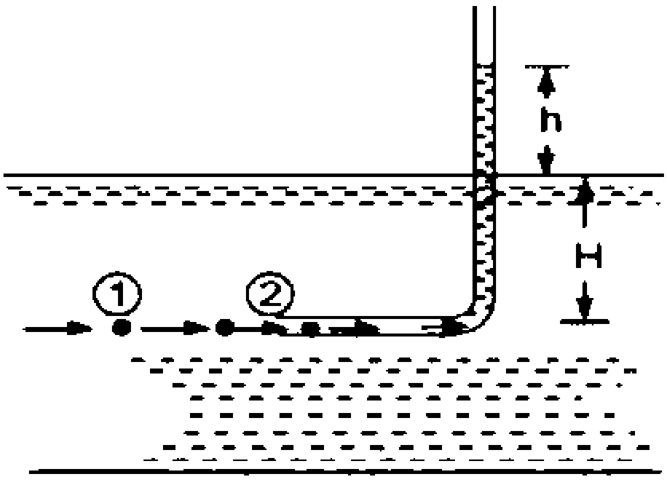
$$= \frac{20736838.09}{304} = 68213.28 \text{ cm}^3/\text{s}$$

Q = 68213.28 cm^3/s. Ans

An orifice meter with orifice diameter 15 cm is inserted in a pipe of 30 cm diameter. The pressure difference measured by a mercury oil differential manometer on the two sides of the orifice meter gives a reading of 50 cm of mercury. Find the rate of flow of oil of sp. gr. 0.9 when the coefficient of discharge of the orifice meter = 0.64.

PITOT-TUBE

It is a device used for measuring the velocity of flow at any point in a pipe or a channel. It is based on the principle that if the velocity of flow at a point becomes zero, the pressure there is increased due to the conversion of the kinetic energy into pressure energy. In its simplest form, the pitot-tube consists of a glass tube, bent at right angles as shown in Fig.



Pitot Tube

The lower end, which is bent through 90° is directed in the up-stream direction as shown in Fig. The liquid rises up in the tube due to the conversion of kinetic energy into pressure energy. The velocity is determined by measuring the rise of liquid in the tube.

Consider two points (1) and (2) at the same level in such a way that point (2) is just as the inlet of the pitot-tube and point (1) is far away from the tube.

Let

p_1 = intensity of pressure at point (1)

V_1= velocity of flow at (I)

p_2 = pressure at point (2)

v_2 = velocity at point (2), which is zero

H = depth of tube in the liquid

h = rise of liquid in the tube above the free surface.

$$h = \frac{V_1^2}{2g} \quad \text{(or)} \quad V_1 = \sqrt{2gh}$$

This is theoretical velocity. Actual velocity is given by,

$(V_1)_{act} = C_v \sqrt{2gh}$

C_v = co-efficient of pitot tube

velocity at any point $V = C_v \sqrt{2gh}$

Velocity of flow in a pipe by pitot-tube:

For finding the velocity at any point in a pipe by pitot- tube, the following arrangements are adopted :

1. Pitot-tube along with a vertical piezometer tube as shown in Fig. (a)

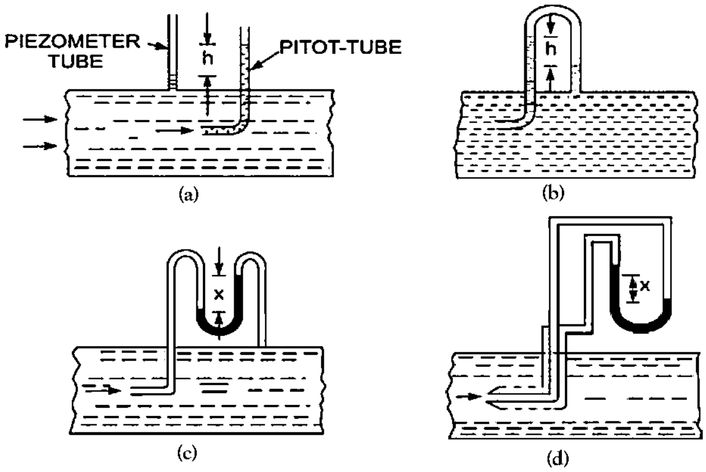
2. Pitot-tube connected with piezometer tube as shown in Fig. (b)

3. Pitot-tube and vertical piezometer tube connected with a differential U-tube manometer as shown in Fig. (c)

4. Pitot-static tube, which consists of two circular concentric tubes one inside the other with some annular space in between as shown in Fig.(d).

The outlet of these two tubes are connected to the differential manometer where the difference of pressure head ‘h’ is measured by knowing the difference of the levels of the manometer liquid say x.

Then, $$h = x\left[\frac{S_g}{S_o} - 1\right]$$



A pitot-static tube placed in the centre of a 300 mm pipe line has one orifice pointing upstream and other perpendicular to it. The mean velocity in the pipe is 0.80 of the central velocity. Find the discharge through the pipe if the pressure difference between the two orifices is 60 mm of water. Take the co-efficient of pi tot tube as Cv = 0.98

Given :

Dia. of pipe, d = 300 mm = 0.30 m

Diff. of pressure head, h = 60 mm of water = .06 m of water

C_v = 0.98

Mean velocity, V = 0.80 x Central velocity

Solution:

Central velocity is given by equation,

v = $C_v\sqrt{2gh}$= 0.98 x tJ2 x 9.81 x .06

v = 1.063 m/s

therefore, V = 0.80 x 1.063 = 0.8504 m/s

discharge, Q = area of the pipe x V

$= \frac{\pi}{4}\,(d^2)$ x V

$= \frac{\pi}{4}\,(0.30^2)$ x 0.8504

Q = 0.06 m^3/s

Find the velocity of the flow of an oil through a pipe, when the difference of mercury level in a differential U-tube manometer connected to the two tappings of the pitot-tube is 100 mm. Take co-efficient of pitot-tube 0.98 and sp. gr. of oil = 0.8.

A sub-marine moves horizontally in sea and has its axis 15 m below the surface of water. A pitot-tube properly placed just in front of the sub-marine and along its axis is connected to the two limbs of a U-tube containing mercury. The difference of mercury level is found to be 170 mm. Find the speed of the sub-marine knowing that the sp. gr. of mercury is 13.6 and that of sea-water is 1.026 with respect of fresh water.

THE MOMENTUM EQUATION

It is based on the law of conservation of momentum or on the momentum principle, which states that the net force acting on a fluid mass is equal to the change in momentum of flow per unit time in that direction. The force acting on a fluid mass 'm' is given by the Newton's second law of motion,

$$F = m \times a$$

where, *a* is the acceleration acting in the same direction as force F.

but, $a = \frac{dv}{dt}$

$$F = m\frac{dv}{dt}$$

$$\mathbf{F = \frac{d(mv)}{dt}}$$ [m is constant and can be taken inside the differential}

The above equation is known as the *momentum principle.*

The above equation can be written as, **F.dt = d(mv)**

which is known as the *impulse-momentum equation* and states that the impulse of a force F acting on a fluid of mass m in a short interval of time dt is equal to the change of momentum d(mv) in the direction of force**.**

UNIT III

REYNOLDS EXPERIMENT

- ✓ The velocity at which the flow changes from the laminar to turbulent for the case of given fluid at a given temperature and given pipe is known as ***critical velocity***.
- ✓ This critical velocity only determines that the flow is either laminar or turbulent. Prof. Osborne Reynolds was first to find that the value of critical velocity is governed by the relationship between the inertia force and viscous force.
- ✓ He derived a ratio of the lie two forces and obtained a dimensionless number called ***Reynolds number.***

Reynolds number, $R_e = \frac{\text{Inertia force}}{\text{Viscous force}}$

Inertia force = Mass x Acceleration

$= \rho \times \text{Volume} \times \text{acceleration}$

$= \rho \times L^3 \times (LT^{-2})$

$= \rho^2 L^2 V^2$

Viscous force = Shear stress x Area

$= \mu \left(\frac{\delta v}{\delta y}\right) \times L^2$

$= \mu V L$

Reynolds number, $R_e = \rho^2 L^2 V^2 / \mu V L$

$$= \frac{\rho V L}{\mu} = \frac{V D}{v} \qquad \left[\frac{\mu}{\rho} = v \right]$$

Where, ρ = Density of the fluid

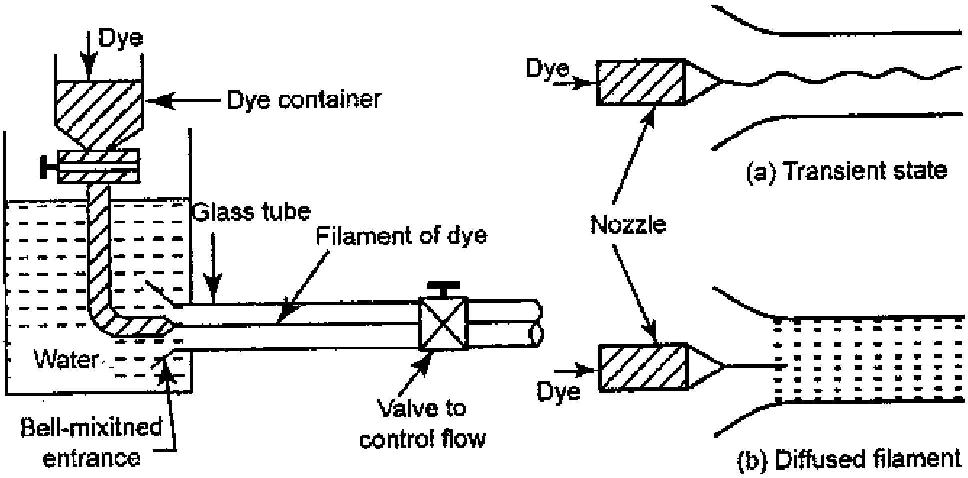
V = Mean velocity of flow

L = Characteristic linear dimension = Diameter of the pipe (D)

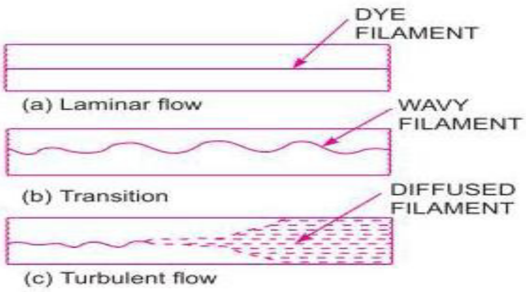
μ = Viscosity of the fluid.

v= Kinematic viscosity of the fluid.

- ✓ The existence of two types of flow (i.e, Laminar and Turbulent flow) was demonstrated by Prof. Reynolds with the help of the following simple experiment. The apparatus used for the experiment is shown in Figure.
- ✓ ***The apparatus*** consists ***of*** (i) A water tank (ii) An arrangement to inject a fine filament of dye into the bell mouthed entrance of a glass tube through which water flows and (iii) A valve to control the flow through the tube.
- ✓ The water was made to flow from the tank through the glass tube into the atmosphere and the velocity of flow was varied by adjusting the regulating and the velocity of flow was varied by adjusting the regulating valve.
- ✓ A fine filament of dye was introduced into the glass tube near the entrance of the tube. It was concluded that when the flow velocity was low, the dye appeared as a straight line parallel to the tube axis characterizing laminar flow.



- As the valve was further opened and greater velocities attained, the dye filament became wavy, this state is called "Transition state". With further increase in the velocity, the fluctuations in the filament of dye became more intense and finally diffusing into the flowing water. Ii shows the turbulent flow.
- From this experiment, Reynolds found that the occurrence of a laminar and turbulent flow was governed by the relative magnitudes of the inertia and the viscous forces.



- At low velocities, viscous forces become predominant and therefore, flow is largely viscous in character. At higher velocities, the inertia forces have predominance over the viscous force.
- After carrying out a series of experiments, Reynolds found that if Reynolds number for a particular flow is less than 2000, the flow is a laminar flow.
- If the Reynolds number is between 2000 and 4000, it is neither laminar flow nor turbulent flow (i.e., Transition state). But, if the Reynolds number exceeds 4000, the flow is a turbulent flow. Experimentally, the value of the lower critical Reynolds number has been found to be approximately 2000. So, simply

$R_e < 2000$ ---> Laminar flow.

$2000 < R_e < 4000$ ---> Transition flow.

$R_e > 4000$ ---> Turbulent flow.

FRICTIONAL LOSS IN PIPE FLOW

- When a liquid is flowing through a pipe, the velocity of the liquid layer adjacent to the pipe wall is zero. The velocity of liquid goes on increasing from the wall and thus velocity gradient and hence shear stresses are produced in the whole liquid due to viscosity.
- This viscous action causes loss of energy which is usually known as frictional loss. On the basis of his experiments, William Froude gave the following laws of fluid fraction for turbulent flow.

The frictional resistance for turbulent flow is:

- ✓ proportional to V^n, where « varies from 1.5 to 2.0
- ✓ proportional to the density of fluid
- ✓ proportional to the area of surface in contact
- ✓ independent of pressure
- ✓ dependent on the nature of the surface in contact.

METHODS OF DETERMINATION OF CO-EFFICIENT OF VISCOSITY

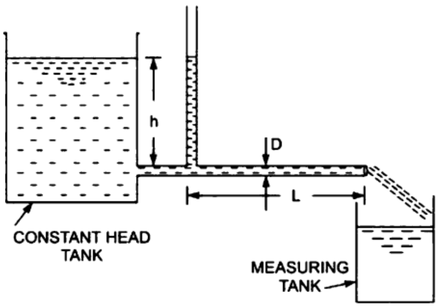
The following are the experimental methods of determining the co-efficient of viscosity of a liquid:

1. Capillary tube method.
2. Falling sphere resistance method,
3. By rotating cylinder method, and
4. Orifice type viscometer.

The apparatus used for determining the viscosity of a liquid is called viscometer.

CAPILLARY TUBE METHOD

- ✓ In capillary tube method, the viscosity of a liquid is calculated by measuring the pressure difference for a given length of the capillary tube. The Hagen Poiseuille law is used for calculating viscosity Fig. Shows the capillary lube viscometer.
- ✓ The liquid whose viscosity is to be determined is filled in a constant head tank. The liquid is maintained at constant temperature and is allowed to pass through the capillary tube from the constant head tank. Then, the liquid is collected in a measuring tank for a given time.
- ✓ Then the rate of liquid collected in the tank per second is determined. The pressure head '/r is measured at a point far away from the tank as shown in Fig.



Then h = Difference of pressure head for length L.

The pressure at outlet is atmospheric.

Let D = Diameter of capillary tube,

L = Length of tube for which difference of pressure head is known,

ρ = Density of fluid.

$$\mu = \frac{\pi \rho g H D^4}{128 Q.L}$$

Measurement of D should be done very accurately.

FALLING SPHERE RESISTANCE METHOD

Theory. This method is based on Stoke's law, according to which the drag force, F on a small sphere moving with a constant velocity, U through a viscous fluid of viscosity, μ for viscous conditions is given by

$$F = 3 \pi \mu U d$$

where d = diameter of sphere

U = velocity of sphere

When the sphere attains a constant velocity U, the drag force is the difference between the weight of sphere and buoyant force acting on it.

Let L = distance travelled by sphere in viscous fluid

t = time taken by sphere to cover distance l

ρ_s = density of sphere

ρ_f = density of fluid

W = weight of sphere

and F_B = buoyant force acting on sphere.

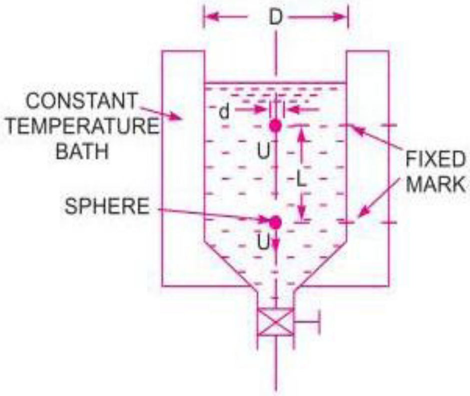


Fig. Falling sphere resistance

$$\mu = \frac{g\ d^2}{18\ U}\left[\frac{\rho_s}{\rho_f}\right]$$

where, ρ_s = density of liquid

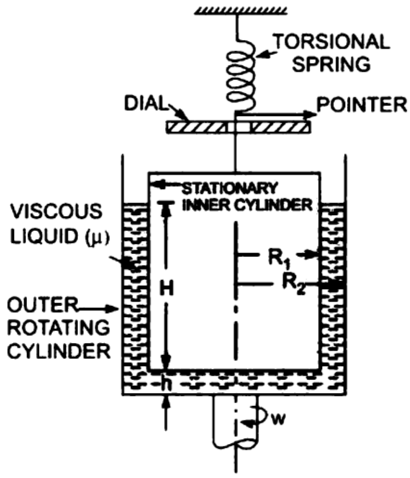
Hence in equation , the values of d, U, ρ_s and ρ_f are known and hence the viscosity of liquid can be determined.

Method: Thus this method consists of a tall vertical transparent cylindrical tank, which is filled with the liquid whose viscosity is to be determined.

- ✓ This tank is surrounded by another transparent tank to keep the temperature of the liquid in the cylindrical tank to be constant.
- ✓ A spherical ball of small diameter'd' is placed on the surface of liquid. Provision is made to release this ball. After a short distance of travel, the ball attains a constant velocity.
- ✓ The time to travel a known vertical distance between two fixed marks on the cylindrical tank is noted to calculate the constant velocity V of the ball. Then with the known values of d, ρ_s, ρ_f, the viscosity μ of the fluid is calculated by using equation.

ROTATING CYLINDER METHOD

- ✓ This method consists of two concentric cylinders of radii R_1 and R_2 as shown in Fig. The narrow space between the two cylinders is filled with the liquid whose viscosity is to be determined.
- ✓ The inner cylinder is held stationary by means of a torsional spring while outer cylinder is rotated at constant angular speed w.
- ✓ The torque T acting on the inner cylinder is measured by the torsional spring. The torque on the inner cylinder must be equal and opposite to the torque applied on the outer cylinder.
- ✓ The torque applied on the outer cylinder is due to viscous resistance provided by liquid in the annular space and at the bottom of the inner cylinder.



$$\mu = \frac{2\,(R_2 - R_1)\,h\,T}{\pi\,R_1^2\,\omega\,[\,4\,H\,h\,R_2^2 + R_1^2\,(R_2 - R_1)]}$$

where,

T = torque measured by the strain of the torsional spring

R_1 and R_2 = radii of inner and outer cylinder

h = clearance at the bottom of cylinders

H = height of liquid in annular space

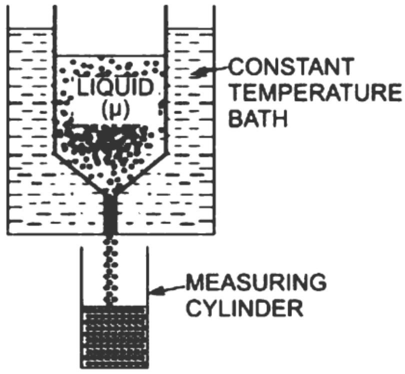
μ = co-efficient of viscosity to be determined

Hence, the value of μ can be calculated from equation.

ORIFICE TYPE VISCOMETER

- ✓ In this method, the time taken by a certain quantity of the liquid whose viscosity is to be determined, to flow through a short capillary tube is noted down.
- ✓ The co-efficient of viscosity is then obtained by comparing with the co-efficient of viscosity of a liquid whose viscosity is known or by the use conversion factors.
- ✓ Viscometers such as Saybolt. Redwood or Engler are usually used. The principle for all the three viscometer is same. In the United Kingdom, Redwood viscometer is used while in U.S.A.. Saybolt viscometer is commonly used.
- ✓ Fig. shows that Say bolt viscometer, which consists of a tank at the bottom of which a short capillary tube is fitted.

- ✓ In this tank the liquid whose viscosity is to be determined is filled. This tank is surrounded by another tank, called ***constant temperature bath.***
- ✓ The liquid is allowed to flow through capillary lube at a standard temperature. The time taken by 60 c.c. of the liquid lo flow through the capillary tube is noted down. The initial height of liquid in the tank is previously adjusted to a standard height. From the time measurement, the kinematic viscosity of liquid is known from the relation,



Saybolt viscometer

$$v = A\,t - \frac{B}{t}$$

where, A = 0.24 , B = 190

t = time noted in seconds

v =vkinematic viscosity in stokes

FLOW THROUGH A PIPE

Here the turbulent flow of fluids through pipes running full will be considered. It the pipes are partially full as in the case of sewer lines, the pressure inside the pipe is same and equal to atmospheric pressure.

Then the flow of fluid in the pipe is not under pressure. This case will be taken in the chapter of flow of water through open channels. Here we will consider flow of fluids through pipes under pressure only.

LOSS OF ENERGY IN PIPES

When a fluid is flowing through a pipe, the fluid experiences some resistance due to which some of the energy of fluid is lost. This loss of energy is classified as:

ENERGY LOSSES

1. Major Energy Losses.

This is due to friction and it is calculated by the following formulae:

(a) Darcy-Weisbach Formula

(b) Chezy's Formula

2. Minor Energy Losses

This is due to

(a) Sudden expansion of pipe

(b) Sudden contraction of pipe

(c) Bend in pipe

(d) Pipe fittings etc.

(e) An obstruction in pipe.

MINOR ENERGY (HEAD) LOSSES

The loss of head or energy due to friction in a pipe is known as major loss while the loss of energy due to change of velocity of die following fluid in magnitude or direction is called minor loss of energy. The minor loss of energy (or head) includes the following cases:

- ✓ Loss of head due to sudden enlargement
- ✓ Loss of head due to sudden contraction
- ✓ Loss of head at the entrance of a pipe
- ✓ Loss of head at the exit of a pipe
- ✓ Loss of head due to an obstruction in a pipe
- ✓ Loss of head due to bend in the pipe
- ✓ Loss of head in various pipe fittings

In case of long pipe the above losses me small as compared with the loss of head due to friction and hence they are called minor losses and even may he neglected without serious error. But in case of a short pipe, these losses are comparable with the loss of head due to friction.

Find the head lost due to friction in a pipe of diameter 300 mm and length 50 m through which water is flowing at a velocity of 3 m/s using (i) Darcy formula, (ii) Chezy's formula for which C = 60.

Take ν for water = 0.01 stoke.

Solution. Given :

Dia. of pipe, $d = 300\text{ mm} = 0.30\text{ m}$

Length of pipe, $L = 50\text{ m}$

Velocity of flow, $V = 3\text{ m/s}$

Chezy's constant, $C = 60$

Kinematic viscosity, $\nu = 0.01\text{ stoke} = 0.01\text{ cm}^2/\text{s}$

$= 0.01 \times 10^{-4}\text{ m}^2/\text{s}.$

(*i*) **Darcy Formula** is given by equation (11.1) as

$$h_f = \frac{4 \cdot f \cdot L \cdot V^2}{d \times 2g}$$

where 'f' = co-efficient of friction is a function of Reynolds number, R_e

But R_e is given by $R_e = \dfrac{V \times d}{\nu} = \dfrac{3.0 \times 0.30}{.01 \times 10^{-4}} = 9 \times 10^5$

$\therefore$ Value of $f = \dfrac{0.079}{R_e^{1/4}} = \dfrac{0.079}{\left(9 \times 10^5\right)^{1/4}} = .00256$

$\therefore$ Head lost, $h_f = \dfrac{4 \times .00256 \times 50 \times 3^2}{0.3 \times 2.0 \times 9.81}$ = **.7828 m. Ans.**

(*ii*) **Chezy's Formula.** Using equation

$$V = C\sqrt{mi}$$

where $C = 60$, $m = \dfrac{d}{4} = \dfrac{0.30}{4} = 0.075\text{ m}$

$\therefore$ $3 = 60\sqrt{.075 \times i}$ or $i = \left(\dfrac{3}{60}\right)^2 \times \dfrac{1}{.075} = 0.03$..

But $i = \dfrac{h_f}{L} = \dfrac{h_f}{50}$

Equating the two values of i, we have $\dfrac{h_f}{50} = .0333$

$\therefore$ $h_f = 50 \times .0333$ = **1.665 m. Ans.**

Find the diameter of a pipe of length 2000 m when the rate of flow of water through the pipe is 200 litres/s and the head lost due to friction is 4 m. Take the value of C = 50 in Chezy's formulae.

Solution. Given

Length of pipe, $L = 2000$ m

Discharge, $Q = 200$ litre/s $= 0.2\ m^3/s$

Head lost due to friction, $h_f = 4$ m

Value of Chezy's constant, $C = 50$

Let the diameter of pipe $= d$

Velocity of flow, $$V = \frac{\text{Discharge}}{\text{Area}} = \frac{Q}{\frac{\pi}{4}d^2} = \frac{0.2}{\frac{\pi}{4}d^2} = \frac{0.2 \times 4}{\pi d^2}$$

Hydraulic mean depth, $$m = \frac{d}{4}$$

Loss of head per unit length, $$i = \frac{h_f}{L} = \frac{4}{2000} = .002$$

Chezy's formula is given by equation as $V = C\sqrt{mi}$

Substituting the values of V, m, i and C, we get

$$\frac{0.2 \times 4}{\pi d^2} = 50\sqrt{\frac{d}{4} \times .002} \text{ or } \sqrt{\frac{d}{4} \times .002}$$

$$= \frac{0.2 \times 4}{\pi d^2 \times 50} = \frac{.00509}{d^2}$$

Squaring both sides, $$\frac{d}{4} \times .002 = \frac{.00509^2}{d^4} = \frac{.0000259}{d^4}$$

$$d^5 = \frac{4 \times .0000259}{.002} = 0.0518$$

$$d = \sqrt[5]{0.0518} = (.0518)^{1/5}$$

$$= 0.553 \text{ m} = \textbf{553 mm. Ans.}$$

An oil of sp. gr. 0.7 is flowing through a pipe of diameter 300 mm at the rate of 500 litres/s. Find the head lost due to friction and power required to maintain the flow for a length of 1000 m. Take ν = .29 stokes.

Solution. Given :

Sp. gr. of oil, $S = 0.7$

Dia. of pipe, $d = 300$ mm $= 0.3$ m

Discharge, $Q = 500$ litres/s $= 0.5\ m^3/s$

Length of pipe, $L = 1000$ m

Velocity, $V = \frac{Q}{\text{Area}} = \frac{0.5}{\frac{\pi}{4}d^2} = \frac{0.5 \times 4}{\pi \times 0.3^2} = 7.073$ m/s

$\therefore$ Reynolds number, $R_e = \frac{V \times d}{\nu} = \frac{7.073 \times 0.3}{0.29 \times 10^{-4}} = 7.316 \times (10)^4$

$\therefore$ Co-efficient of friction, $f = \frac{.079}{R_e^{1/4}} = \frac{0.79}{\left(7.316 \times 10^4\right)^{1/4}} = .0048$

$\therefore$ Head lost due to friction, $h_f = \frac{4 \times f \times L \times V^2}{d \times 2g}$

$$= \frac{4 \times .0048 \times 1000 \times 7.073}{0.3 \times 2 \times 9.81}$$

$$= 163.18 \text{ m}$$

Power required $= \frac{\rho g \,.\, Q \,.\, h_f}{1000}$ kW

where ρ = density of oil = 0.7 × 1000 = 700 kg/m^3

$\therefore$ Power required $= \frac{700 \times 9.81 \times 0.5 \times 163.18}{1000}$

$$= \textbf{560.28 kW. Ans.}$$

Calculate the discharge through a pipe of diameter 200 mm when the difference of pressure head between the two ends of a pipe 500 m apart is 4 m of water. Take the value of 'f' = 0.009 in the formula $h_f = \frac{4 \,.\, f \,.\, L \,.\, V^2}{d \times 2g}$.

Solution. Given :

Dia. of pipe, $d = 200$ mm $= 0.20$ m

Length of pipe, $L = 500$ m

Difference of pressure head, $h_f = 4$ m of water

$f = .009$

Using equation (11.1), we have $h_f = \frac{4 \times f \times L \times V^2}{d \times 2g}$

$$4.0 = \frac{4 \times .009 \times 500 \times V^2}{0.2 \times 2 \times 9.81}$$

$$V^2 = \frac{4.0 \times 0.2 \times 2 \times 9.81}{4.0 \times .009 \times 500} = 0.872$$

Then h_c becomes as $$h_c = \frac{kV_2^2}{2g} = 0.375\ \frac{V_2^2}{2g}$$

If the value of C_c is not given then the head loss due to contraction is taken as

$$= 0.5\ \frac{V_2^2}{2g} \text{ or } h_c = 0.5\ \frac{V_2^2}{2g}.$$

Find the loss of head when a pipe of diameter 200 mm is suddenly enlarged to a diameter of 400 mm. The rate of flow of water through the pipe is 250 litres/s.

Solution. Given :

Dia. of smaller pipe, $D_1 = 200 \text{ mm} = 0.20 \text{ m}$

$\therefore$ Area, $A_1 = \frac{\pi}{4} D_1^2 = \frac{\pi}{4} (.2)^2 = 0.03141 \text{ m}^2$

Dia. of large pipe, $D_2 = 400 \text{ mm} = 0.4 \text{ m}$

$\therefore$ Area, $A_2 = \frac{\pi}{4} \times (0.4)^2 = 0.12564 \text{ m}^2$

Discharge, $Q = 250 \text{ litres/s} = 0.25 \text{ m}^3\text{/s}$

Velocity, $V_1 = \frac{Q}{A_1} = \frac{0.25}{.03141} = 7.96 \text{ m/s}$

Velocity, $V_2 = \frac{Q}{A_2} = \frac{0.25}{.12564} = 1.99 \text{ m/s}$

Loss of head due to enlargement is given by equation (11.5) as

$$h_e = \frac{(V_1 - V_2)^2}{2g} = \frac{(7.96 - 1.99)^2}{2g}$$

$$= \mathbf{1.816 \text{ m of water.}} \textbf{ Ans.}$$

At a sudden enlargement of a water main from 240 mm to 480 mm diameter, the hydraulic gradient rises by 10 mm. Estimate the rate of flow.

Solution. Given :

Dia. of smaller pipe, $D_1 = 240 \text{ mm} = 0.24 \text{ m}$

$\therefore$ Area, $A_1 = \frac{\pi}{4} D_1^2 = \frac{\pi}{4} (.24)^2$

Dia. of large pipe, $D_2 = 480 \text{ mm} = 0.48 \text{ m}$

$\therefore$ Area, $A_2 = \frac{\pi}{4} (0.48)^2$

Rise of hydraulic gradient*, *i.e.*, $\left(z_2 + \frac{p_2}{\rho g}\right) - \left(\frac{p_1}{\rho g} + z_1\right) = 10 \text{ mm} = \frac{10}{1000} = \frac{1}{100} \text{ m}$

Let the rate of flow = Q

Applying Bernoulli's equation to both sections, *i.e.*, smaller pipe section, and large pipe section.

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2 + \text{Head loss due to enlargement} \quad ...(i)$$

But head loss due to enlargement,

$$h_e = \frac{(V_1 - V_2)^2}{2g} \quad ...(ii)$$

From continuity equation, we have $A_1 V_1 = A_2 V_2$

A crude oil of kinematic viscosity 0.4 stoke is flowing through a pipe of diameter 300 mm at the rate of 300 litres per sec. Find the head lost due to friction for a length of 50 m of the pipe.

Solution. Given :

Kinematic viscosity, $\nu = 0.4 \text{ stoke} = 0.4 \text{ cm}^2/\text{s} = .4 \times 10^{-4} \text{ m}^2/\text{s}$

Dia. of pipe, $d = 300 \text{ mm} = 0.30 \text{ m}$

Discharge, $Q = 300 \text{ litres/s} = 0.3 \text{ m}^3/\text{s}$

Length of pipe, $L = 50 \text{ m}$

Velocity of flow, $V = \dfrac{Q}{\text{Area}} = \dfrac{0.3}{\frac{\pi}{4}(0.3)^2} = 4.24 \text{ m/s}$

$\therefore$ Reynolds number, $R_e = \dfrac{V \times d}{\nu} = \dfrac{4.24 \times 0.30}{0.4 \times 10^{-4}} = 3.18 \times 10^4$

As R_e lies between 4000 and 100000, the value of f is given by

$$f = \frac{.079}{(R_e)^{1/4}} = \frac{.079}{\left(3.18 \times 10^4\right)^{1/4}} = .00591$$

$\therefore$ Head lost due to friction, $h_f = \dfrac{4 \,.\, f \,.\, L \,.\, V^2}{d \times 2g}$

$$= \frac{4 \times .00591 \times 50 \times 4.24^2}{0.3 \times 2 \times 9.81}$$

$= \mathbf{3.61 \text{ m. Ans.}}$

MINOR ENERGY (HEAD) LOSSES

The loss of head or energy due to friction in a pipe is known as major loss while the loss of energy due to change of velocity of the following fluid in magnitude or direction is called minor loss of energy. The minor loss of energy (or head) includes the following cases:

1. Loss of head due to sudden enlargement
2. Loss of head due to sudden contraction
3. Loss of head at the entrance of a pipe
4. Loss of head at the exit of a pipe
5. Loss of head due to an obstruction in a pipe
6. Loss of head due to bend in the pipe
7. Loss of head in various pipe fittings.

In case of long pipe the above losses are small as compared with the loss of head due to friction and hence they are called minor losses and even may be neglected without serious error. But in case of a short pipe, these losses are comparable with the loss of head due to friction.

LOSS OF HEAD DUE TO SUDDEN ENLARGEMENT.

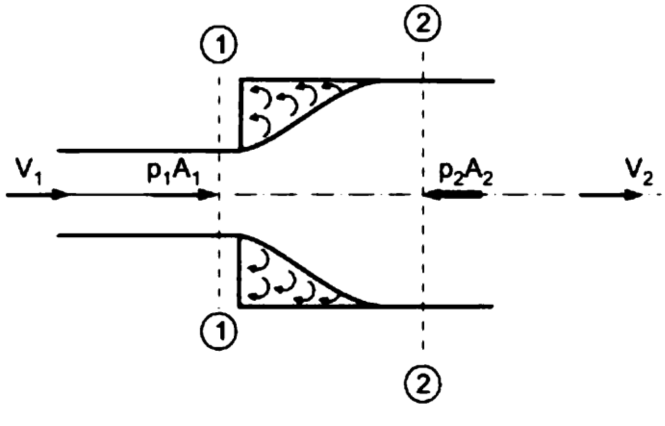
Consider a liquid flowing through a pipe which has sudden enlargement as shown in Fig. Consider two section (1)-(1) and (2)-(2) before and after the enlargement.

Let p_1 = pressure intensity at section 1.1

V_1 = velocity of flow at section 1.1

A_1 = area of pipe at section 1.1

P_2 , V_2 and A_2 = corresponding values at section 2.2.



Due to sudden change of diameter of the pipe from D_1 to D_2 , the liquid flowing from the smaller pipe is not able to follow the abrupt change of the boundary. Thus the flow separates from the boundary and turbulent eddies are formed as sown in Fig. 11.1. The loss of head (or energy) takes place due to the formation of theses eddies.

Let p^1 = pressure intensity of the liquid eddies on the area $(A_2 - A_1)$

h_e = loss of head due to sudden enlargement

Applying Bernoulli's equation at section 1.1 and 2.2

$$\frac{P_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{P_2}{\rho g} + \frac{V_2^2}{2g} + z_2 + \text{ loss of head due to sudden enlargement}$$

But $z_1 = z_2$ as pipe is horizontal

$\therefore$ $$\frac{P_1}{\rho g} + \frac{V_1^2}{2g} = \frac{P_2}{\rho g} + \frac{V_2^2}{2g} + h_e$$

or $$h_e = \left(\frac{P_1}{\rho g} - \frac{P_2}{\rho g}\right) + \left(\frac{V_1^2}{2g} - \frac{V_2^2}{2g}\right) \quad \text{...........(i)}$$

Consider the control volume of liquid between section 1-1 and 2-2. Then the force acting on the liquid in the control volume in the direction of flow is given by

$$F_x = p_1 A_1 + p'(A_2 - A_1) - P_2 A_2$$

But experimentally it is found that $p' = p_1$

$\therefore$ $$F_x = p_1 A_1 + p_1(A_2 - A_1) - P_2 A_2 = p_1 A_2 + p_2 A_2$$

Momentum of liquid/sec at section 1-1 = mass x velocity

$$= \rho A_1 V_1 \times V_1 = \rho A_1 V_1^2$$

Momentum of liquid/sec at section 2-2 = $\rho A_2 V_2 \times V_2 = \rho A_2 V_2^2$

$\therefore$ Change of momentum/sec = $\rho A_2 V_2^2 - \rho A_1 V_1^2$

But from continuity equation, we have

$$A_1 V_1 = A_2 V_2 \text{ or } A_1 = \frac{A_2 V_2}{V_1}$$

$\therefore$ Change of momentum/sec = $\rho A_2 V_2^2 - \rho \times \frac{A_2 V_2}{V_1} \times V_1^2 = \rho A_2 V_2^2 - \rho A_2 V_1 V_2$

$$= \rho A_2 [V_2^2 - V_1 V_2]$$

Now net force acting on the control volume in the direction of flow must be equal to the rate of change of momentum or change of momentum per second. Hence equating (ii) and (iii)

$$(p_1 - p_2) A_2 = \rho A_2 [V_2^2 - V_1 V_2]$$

or $$\frac{p_1 - p_2}{\rho} = V_2^2 - V_1 V_2$$

Dividing by g on both sides, we have $\frac{p_1 - p_2}{\rho g} = \frac{V_2^2 - V_1 V_2}{g}$ or $\frac{p_1}{\rho g} - \frac{p_2}{\rho g} = \frac{V_2^2 - V_1 V_2}{g}$

Substituting the value of $\left(\frac{p_1}{\rho g} - \frac{p_2}{\rho g}\right)$ in equation (i), we get

$$h_e = \frac{V_2^2 - V_1 V_2}{g} + \frac{V_1^2}{2g} - \frac{V_2^2}{2g} = \frac{2V_2^2 - 2V_1 V_2 + V_1^2 - V_2^2}{2g}$$

$$= \frac{V_2^2 + V_1^2 - 2V_1 V_2}{2g} = \left(\frac{V_1 - V_2}{2g}\right)^2$$

$$h_e \quad = \frac{(V_1 - V_2)^2}{2g} \qquad \text{..............(11.5)}$$

SUDDEN CONTRACTION

Consider two sections 1-1 and 2-2 before and after contraction. As the liquid flows from large pipe to smaller pipe, the area of flow goes on decreasing and becomes minimum at a section C-C as sown in Fig.. This section C-C is called Vena-contracta. After section C-C a sudden enlargement of the area takes place. The loss of head due to sudden contraction is actually due to sudden enlargement from Vena-contracta to smaller pipe.

Let A_c = Area of flow at section C-C

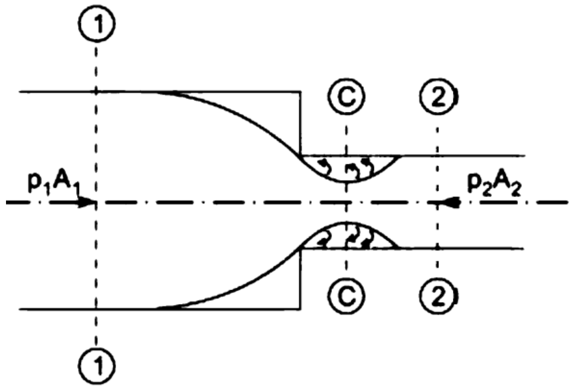
V_c = Velocity of flow at section C-C

A_2 = Area of flow at section 2-2

V_2 = Velocity of flow at section 2-2

h_c = Loss of head due to sudden contraction

now h_c = actual loss of head due to enlargement from section C-C to section 2-2 and is given by equation as



$$= \frac{(V_1 - V_2)^2}{2g} \quad = \quad \frac{V_2^2}{2g}\left[\frac{V_c}{V_2} - 1\right]^2$$

From continuity equation, we have

$$A_cV_c = A_2V_2 \text{ or } \frac{V_c}{V_2} = \frac{A_2}{A_c} = \frac{1}{(A_c/A_2)} = \frac{1}{C_c} \qquad \left[\because\ C_c = \frac{A_c}{A_2}\right]$$

Substituting the value of $\frac{V_c}{V_2}$ in (i) we get

$$h_c = \frac{V_2^2}{2g}\left[\frac{1}{C_c} - 1\right]^2$$

$$= \frac{kV_2^2}{2g}, \text{ where } k = \left[\frac{1}{C_c} - 1\right]^2$$

If the value of C_c is assumed to be equal to 0.62, then

$$k = \left[\frac{1}{0.62} - 1\right]^2 = 0.375$$

Then h_c becomes as $h_c = \frac{kV_2^2}{2g} = 0.375\frac{V_2^2}{2g}$

If the value of C_c is not given then the head loss due to contraction is taken as

$$= 0.5\frac{V_2^2}{2g} \text{ or } h_c = 0.5\frac{V_2^2}{2g}$$

Find the loss of head when a pipe of diameter 200 mm is suddenly enlarged to a diameter of 400 mm. The rate of flow of water through the pipe is 250 litres/s.

Solution. Given:

Dia. Of smaller pipe, D_1 = 200 mm = 0.20 m

∴ Area, $A_1 = \frac{\pi}{4}D_1^2 = \frac{\pi}{4}(.2)^2 = 0.03141 \text{ m}^2$

Dia. Of large pipe, $D_2 = 400\ mm = 0.4$m

∴ Area, $A_2 = \frac{\pi}{4} \text{ x } (0.4)^2 = 0.12564 \text{ m}^2$

Discharge, Q = 250 litres/s = 0.25 m^3/s

Velocity, $V_1 = \frac{Q}{A_1} = \frac{0.25}{.03141} = 7.96$ m/s

Velocity, $V_2 = \frac{Q}{A_2} = \frac{0.25}{.12564} = 1.99$ m/s

Loss of head due to enlargement is given by equation (11.5) as

$$h_e = \frac{(V_1 - V_2)^2}{2g} = \frac{(7.96 - 1.99)^2}{2g} = 1.816 \text{ m of water.}$$

At a sudden enlargement of a water main from 240 mm to 480 mm diameter, the hydraulic gradient rises by 10 mm. Estimate the rate of flow.

Solution. Given:

Dia. Of smaller pipe, D_1 = 240 mm = 0.24 m

∴ Area, $A_1 = \frac{\pi}{4}D_1^2 = \frac{\pi}{4}(.24)^2$

Dia. Of large pipe, $D_2 = 480\ mm = 0.48$ m

∴ Area, $A_2 = \frac{\pi}{4} \text{ x } (0.48)^2$

Rise of hydraulic gradient*, i.e., $\left[z_2 + \frac{p_2}{\rho g}\right] - \left[\frac{p_1}{\rho g} + z_1\right] = 10 \text{ mm} = \frac{10}{1000} = \frac{1}{100}$ m

Let the rate of flow = Q

Applying Bernoulli's equation to both section, i.e., smaller pipe section, and large pipe section

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2 + \text{Head loss due to enlargement}$$

But head loss due to enlargement,

$$h_e = \frac{(V_1 - V_2)^2}{2g}$$

From continuity equation, we have $A_1V_1 = A_2V_2$

$$\therefore \qquad V_1 = \frac{A_2V_2}{A_1} = \frac{\frac{\pi}{4}D_2^2 \text{ x } V_2}{\frac{\pi}{4}D_1^2} = \left(\frac{D_2}{D_1}\right)^2 \text{ x } V_2 = \left(\frac{.48}{.24}\right)^2 \text{ x } V_2 = 2^2 \text{ x } V_2 = 4V_2$$

Substituting this value in (ii), we get

$$h_e = \frac{(4V_2 - V_2)^2}{2g} = \frac{(3V_2)^2}{2g} = \frac{9V_2^2}{2g}$$

Now substituting the value of h_e and V_1 in equation (i),

$$\frac{p_1}{\rho g} + \frac{(4V_2)^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2 + \frac{9V_2^2}{2g}$$

$$\frac{16V_2^2}{2g} - \frac{V_2^2}{2g} - \frac{9V_2^2}{2g} = \left[\frac{p_2}{\rho g} + z_2\right] - \left[\frac{p_1}{\rho g} + z_1\right]$$

But hydraulic gradient rise = $\left[\frac{p_2}{\rho g} + z_2\right] - \left[\frac{p_1}{\rho g} + z_1\right] = \frac{1}{100}$

$$\therefore \qquad \frac{16V_2^2}{2g} - \frac{V_2^2}{2g} - \frac{9V_2^2}{2g} = \frac{1}{100} \text{ or } \frac{6V_2^2}{2g} = \frac{1}{100}$$

$$\therefore \qquad V_2 = \sqrt{\frac{2 \text{ x } 9.81}{6 \text{ x } 100}} = 0.1808 = 0.181 \text{ m/s.}$$

∴ Discharge, $Q = A_2 \text{ x } V_2$

$= \frac{\pi}{4}D_2^2 \text{ x } V_2 = \frac{\pi}{4}(.48)^2 \text{ x } .181$

$= 0.03275 \text{ m}^3\text{/s.}$

= 32.75 litres/s.

FLOW THROUGH PIPES IN SERIES OR FLOW THROUGH COMPOUND PIPES

Pipes in series or compound pipes are defined as the pipes of different lengths and different diameters connected end to end (in series) to from a pipe line as shown in Fig. 11.16.

Let L_1, L_2, L_3 = length of pipes 1, 2 and 3 respectively

d_1, d_2, d_3 = diameter of pipes 1, 2 and 3 respectively

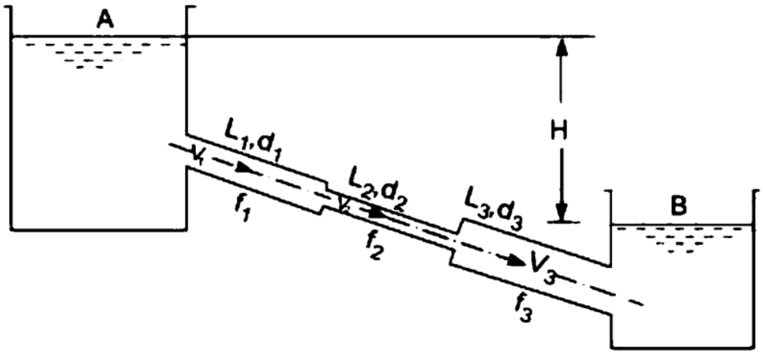
V_1, V_2, V_3 = velocity of flow through pipes 1,2,3

f_1, f_2, f_3 = co-efficient of frictions for pipes 1,2,3

H = difference of water level in the two tanks

The discharge passing through each pipe is same.

$$\therefore \qquad Q = A_1V_1 = A_2V_2 = A_2V_3$$



The difference in liquid surface levels is equal to the sum of the total head loss in the pipes.

$$\therefore \quad H = \frac{0.5V_1^2}{2g} + \frac{4f_1L_1V_1^2}{d_1 \text{ x } 2g} + \frac{0.5V_2^2}{2g} + \frac{4f_2L_2V_2^2}{d_2 \text{ x } 2g} + \frac{(V_2-V_3)^2}{2g} + \frac{4f_3L_3V_3^2}{d_3 \text{ x } 2g} + \frac{V_3^2}{2g} \quad ....(11.12)$$

If minor losses are neglected, then above equation becomes as

$$H = \frac{4f_1L_1V_1^2}{d_1 \text{ x } 2g} + \frac{4f_2L_2V_2^2}{d_2 \text{ x } 2g} + \frac{4f_3L_3V_3^2}{d_3 \text{ x } 2g} \quad ....(11.13)$$

If the co-efficient of friction is same for all pipes

i.e., $f_1 = f_2 = f_3 = f$ then equation (11.13) becomes as

$$H = \frac{4fL_1V_1^2}{d_1 \text{ x } 2g} + \frac{4fL_2V_2^2}{d_2 \text{ x } 2g} + \frac{4fL_3V_3^2}{d_3 \text{ x } 2g}$$

$$= \frac{4f}{2g}\left[\frac{L_1V_1^2}{d_1} + \frac{L_2V_2^2}{d_2} + \frac{L_3V_3^2}{d_3}\right] \quad ....(11.14)$$

The difference in water surface levels in two tanks, which are connected by three pipes in series of lengths 300 m , 170 m and 210 m and of diameters 300 mm , 200 mm and 400 mm respectively, is 12 m. Determine the rate of flow of water if co-efficient of friction are .005 , .0052 and .0048 respectively, considering: (i) minor losses also (ii) neglecting minor losses.

Solution. Given:

Difference of water level, $H = 12$ m

Length of pipe 1, $L_1 = 300$ m and dia., $d_1 = 300$ mm = 0.3 m

Length of pipe 2, $L_2 = 170$ m and dia., $d_2 = 200$ mm = 0.2 m

Length of pipe 3, $L_3 = 210$ m and dia., $d_3 = 400$ mm = 0.4 m

Also, $f_1 = .005$, $f_2 = .0052$ and $f_3 = .0048$

(i) Considering Minor Losses. Let V_1 , V_2 and V_3 are the velocities in the 1st , 2nd and 3rd pipe respectively,

From continuity, we have $A_1V_1 = A_2V_2 = A_3V_3$

$$\therefore \quad V_2 = \frac{A_1V_1}{A_2} \; \frac{\frac{\pi}{4}d_1^2}{\frac{\pi}{4}d_2^2} V_1 = \frac{d_1^2}{d_2^2} V_1 = \left(\frac{0.3}{.2}\right)^2 \text{ x } V_1 = 2.25\, V_1$$

and $$V_2 = \frac{A_1V_1}{A_2} = \frac{d_1^2}{d_3^2} V_1 = \left(\frac{0.3}{0.4}\right)^2 \text{ x } V_1 = 0.5625\, V_1$$

Now using equation (11.12), we have

$$H = \frac{0.5V_1^2}{2g} + \frac{4f_1L_1V_1^2}{d_1 \text{ x } 2g} + \frac{0.5V_2^2}{2g} + \frac{4f_2L_2V_2^2}{d_2 \text{ x } 2g} + \frac{(V_2-V_3)^2}{2g} + \frac{4f_3L_3V_3^2}{d_3 \text{ x } 2g} + \frac{V_3^2}{2g}$$

Substituting V_2 and V_3 $12.0 = \frac{0.5V_1^2}{2g} + \frac{4\text{x}.005\text{x}300\text{x}V_1^2}{0.3\text{x } 2g} + \frac{0.5+ (2.25V_1^2)^2}{2g} + 4\text{x}0.0052\text{x}170 \text{ x } \frac{(2.25V_1)^2}{2g}$

$$+ \frac{(2.25V_1-.562V_1)^2}{2g} + \frac{4\text{x}0048\text{x}210\text{x}(.5625V_1)^2}{0.4\text{x } 2g} + \frac{(.5625V_1)^2}{2g}$$

or $12.0 = \frac{V_1^2}{2g}$ [0.5+20.0+2.53+89.505+2.847+3.189+0.316]

$= \frac{V_1^2}{2g}$ [118.887]

∴ $V_1 = \sqrt{\frac{12+2+9.81}{118.887}} = 1.407$ m/s

∴ Rate of flow, Q = Area x Velocity = A_1 x V_1

$= \frac{\pi}{4}(d_1^2)$ x $V_1 = \frac{\pi}{4}(.3)^2$ x 1.407 = 0.09945 m^3/s.

= 99.45 litres/s.

(ii) Neglecting Minor Losses. Using equation (11.13), we have

$$H = \frac{4fL_1V_1^2}{d_1 \text{ x } 2g} + \frac{4fL_2V_2^2}{d_2 \text{ x } 2g} + \frac{4fL_3V_3^2}{d_3 \text{ x } 2g}$$

or $12.0 = \frac{V_1^2}{2g}\left[\frac{4\text{x}.005\text{x}300}{0.3} + \frac{4 \text{ x } .0052 \text{ x } 170 \text{ x}(2.25)^2}{0.2} + \frac{4 \text{ x } .0048 \text{ x } 210 \text{ x}(.5625)^2}{0.4}\right]$

$= \frac{V_1^2}{2g}$ [20.0 + 89.505+3.189] = $\frac{V_1^2}{2g}$ x 112.694

∴ $V_1 = \sqrt{\frac{2\text{x}9.81\text{x}12.0}{112.694}} = 1.445$ m/s

∴ Discharge, Q = V_1 x A_1= 1.445 x $\frac{\pi}{4}(.3)^2$ = 0.1021 m^3/s = 102.1 litres/s.

Three pipes of 400mm and 300 mm diameters have lengths of 400 m, 200 m and 300 m respectively. They are connected in series to make a compound pipe. The ends of this compound pipe are connected with two tanks whose difference of water levels is 16 m. If co-efficient of friction for these pipes is same and equal to 0.005, determine the discharge through the compound pipe neglecting first the minor losses and then including them.

Solution. Given:

Difference of water levels, H = 16 m

Length and dia. of pipe 1, L_1 = 400 m and , d_1 = 400 mm = 0.4 m

Length and dia. of pipe 2, L_2 = 200 m and d_2 = 200 mm = 0.2 m

Length and dia. of pipe 3, L_3 = 300 m and d_3 = 300 mm = 0.3 m

Also, $f_1 = f_2 = f_3 = 0.005$

(i) Discharge through the compound pipe first neglecting minor losses.

Let V_1 , V_2 and V_3 are the velocities in the 1st , 2nd and 3rd pipe respectively,

From continuity, we have $A_1V_1 = A_2V_2 = A_3V_3$

$\therefore \quad V_2 = \frac{A_1 V_1}{A_2} \quad \frac{\frac{\pi}{4} d_1^2}{\frac{\pi}{4} d_2^2} \times V_1 = \frac{d_1^2}{d_2^2} V_1 = \left(\frac{0.4}{0.2}\right)^2 \times V_1 = 4\, V_1$

and $\quad V_2 = \frac{A_1 V_1}{A_2} \quad = \frac{\frac{\pi}{4} d_1^2}{\frac{\pi}{4} d_3^2} \times V_1 = \frac{d_1^2}{d_2^2} V_1 \left(\frac{0.4}{0.2}\right)^2 \times V_1 = 1.77 V_1$

Now using equation (11.13) we have

$$H = \frac{4fL_1V_1^2}{d_1 \times 2g} + \frac{4fL_2V_2^2}{d_2 \times 2g} + \frac{4fL_3V_3^2}{d_3 \times 2g}$$

or
$$16 = \frac{4 \times 0.005 \times 400 \times V_1^2}{0.4 \times 2 \times 9.81} + \frac{4 \times 0.005 \times 200 \times (4V_1)^2}{0.2 \times 2 \times 9.81} + \frac{4 \times 0.005 \times 300}{0.3 \times 2 \times 9.81} \times (1.77V_1)^2$$

$$= \frac{V_1^2}{2 \times 9.81}\left[\frac{4 \times 0.005 \times 400}{0.4} + \frac{4 \times 0.005 \times 200 \times 16}{0.2} + \frac{4 \times 0.005 \times 300 \times 3.175}{0.3}\right]$$

$$16 = \frac{V_1^2}{2 \times 9.81}(20+320+63.14) = \frac{V_1^2}{2 \times 9.81} \times 403.14$$

$$\therefore \quad V_1 = \sqrt{\frac{16 \text{x} 2 \text{x} 9.81}{403.14}} = 0.882 \text{ m/s}$$

$\therefore$ Discharge, $Q = V_1 \times A_1 = \frac{\pi}{4}(0.4)^2 \times 0.882 = 0.1108\ m^3/s$.

(ii) Discharge through the compound pipe considering minor losses also.

Minor losses are:

(a) At inlet, $\quad h_i = \frac{0.5\, V_1^2}{2g}$

(b) Between 1st pipe and 2nd pipe, due to contraction,

$$H_c = \frac{0.5\, V_2^2}{2g} = \frac{0.5\, V_1^2}{2g} \qquad (\because V_2 = 4V_1)$$

$$= \frac{0.5 \times 16 \times V_1^2}{2g} = 8 \times \frac{V_1^2}{2g}$$

(c) Between 2nd pipe and 3rd pipe, due to sudden enlargement,

$$h_e = \frac{(V_2 - V_3)^2}{2g} = \frac{(4V_1 - 1.77V_1)^2}{2g} \qquad (\because V_3 = 1.77V_1)$$

$$= (2.23)^2 \times \frac{V_1^2}{2g} = 4.973\, \frac{V_1^2}{2g}$$

(d) At the outlet of 3rd pipe, $h_o = \frac{V_3^2}{2g} = \frac{(1.77V_1)^2}{2g}$

$$= 1.77^2 \times \frac{V_1^2}{2g} = 3.1329\, \frac{V_1^2}{2g}$$

The major losses are = $\frac{4fL_1V_1^2}{d_1 \times 2g} + \frac{4fL_2V_2^2}{d_2 \times 2g} + \frac{4fL_3V_3^2}{d_3 \times 2g}$

$$= \frac{4 \times 0.005 \times 400 \times V_1^2}{0.4 \times 2 \times 9.81} + \frac{4 \times 0.005 \times 200 \times (4V_1)^2}{0.2 \times 2 \times 9.81} + \frac{4 \times 0.005 \times 300\,(1.77V_1)^2}{0.3 \times 2 \times 9.81}$$

$$= 403.14 \times \frac{V_1^2}{2 \times 9.81}$$

$\therefore$ sum of minor losses and major losses

$$= \left[\frac{0.5\, V_1^2}{2g} + 8 \times \frac{V_1^2}{2g} + 4.973\frac{V_1^2}{2g} + 3.1329\frac{V_1^2}{2g}\right] + 403.14\frac{V_1^2}{2g}$$

$$= 419.746\, \frac{V_1^2}{2g}$$

But total loss must be equal to *H* (or 16m)

$\therefore$ $419.746 \times \frac{V_1^2}{2g} = 16$ $\therefore$ $V_1 = \sqrt{\frac{16 \times 2 \times 9.81}{419.746}} = 0.864$ m/s

$\therefore$ Discharge, $Q = A_1 \times V_1 = \frac{\pi}{4}(0.4)^2 \times 0.864$ **= 0.1085 m³/s.**

EQUIVALENT PIPE

This is defined as the pipe of uniform diameter having loss of head and discharge equal to the loss of head and discharge of a compound pipe consisting of several pipes of different lengths and diameters. The uniform diameter of the equivalent pipe is called equivalent size of the pipe. The length of equivalent pipe is equal to sum of lengths of the compound pipe consisting of different pipes.

Let L_1 = length of pipe 1 and d_1 = diameter of pipe 1

L_2 = length of pipe 2 and d_2 = diameter of pipe 2

L_3 = length of pipe 3 and d_3 = diameter of pipe 3

H = total head loss

L = length of equivalent pipe

d = diameter of the equivalent pipe

Then $L = L_1 + L_2 + L_3$

Total head loss in the compound pipe, neglecting minor losses

$$H = \frac{4fL_1V_1^2}{d_1 \times 2g} + \frac{4fL_2V_2^2}{d_2 \times 2g} + \frac{4fL_3V_3^2}{d_3 \times 2g}$$

Assuming $f_1 = f_2 = f_3 = f$

Discharge, $Q = A_1V_1 = A_2V_2 = A_3V_3 = \frac{\pi}{4}d_1^2V_1 = \frac{\pi}{4}d_2^2V_2 = \frac{\pi}{4}d_3^2V_3$

$\therefore$ $V_1 = \frac{4Q}{\pi d_1^2}$, $V_2 = \frac{4Q}{\pi d_2^2}$, $V_3 = \frac{4Q}{\pi d_3^2}$

Substituting these values in equation (11.14A), we have

$$H = \frac{4fL_1 \times \left(\frac{4Q}{\pi d_1^2}\right)^2}{d_1 \times 2g} + \frac{4fL_2 \times \left(\frac{4Q}{\pi d_2^2}\right)^2}{d_2 \times 2g} + \frac{4fL_3 \times \left(\frac{4Q}{\pi d_3^2}\right)^2}{d_3 \times 2g}$$

$$= \frac{4 \times 16fQ^2}{\pi^2 \times 2g}\left[\frac{L_1}{d_1^5} + \frac{L_2}{d_2^5} + \frac{L_3}{d_3^5}\right]$$

Head loss in the equivalent pipe, $H = \frac{4f \cdot L \cdot V^2}{d \times 2g}$ [Taking same value of *f* as in compound pipe]

where $V = \frac{Q}{A} = \frac{Q}{\frac{\pi}{4}d^2} = \frac{Q}{\pi d^2}$

$$\therefore \quad H = \frac{4f.L.\left[\frac{4Q}{\pi d^2}\right]^2}{d \times 2g} = \frac{4 \times 16Q^2f}{\pi^2 \times 2g}\left[\frac{L}{d^5}\right]$$

Head loss in compound pipe and in equivalent pipe is same equating equations (11.15) and (11.16) we have

$$\frac{4 \times 16Q^2f}{\pi^2 \times 2g}\left[\frac{L_1}{d_1^5} + \frac{L_2}{d_2^5} + \frac{L_3}{d_3^5}\right] = \frac{4 \times 16Q^2f}{\pi^2 \times 2g}\left[\frac{L}{d^5}\right]$$

or $\frac{L_1}{d_1^5} + \frac{L_2}{d_2^5} + \frac{L_3}{d_3^5} = \frac{L}{d^5}$ or $\frac{L}{d^5} = \frac{L_1}{d_1^5} + \frac{L_2}{d_2^5} + \frac{L_3}{d_3^5}$

Equation is known as Dupuit's equation. In this equation $L = L_1+L_2+L_3$ and d_1, d_2 and d_3 are known. Hence the equivalent size of the pipe, i.e. value of d can be obtained.

Three pipes of lengths 800 m , 500 m and 400 m and of diameters 50 mm, 40mm and 300 mm respectively are connected in series. These pipes are to be replaced by a single pipe of length 1700 m. Find the diameter of the single pipe.

Solution. Given:

Length of pipe 1, L_1 = 800 m and dia., of d_1 = 500mm = 0.5 m

Length of pipe 2, L_2 = 500 m and dia., of d_2 = 400mm = 0.4 m

Length of pipe 3, L_3 = 400 m and dia., of d_3 = 300mm = 0.3 m

Length of single pipe, L = 1700

Let the diameter of equivalent single pipe = d

Applying equation(11.17), $\frac{L}{d^5} = \frac{L_1}{d_1^5} + \frac{L_2}{d_2^5} + \frac{L_3}{d_3^5}$

or $\frac{1700}{d^5} = \frac{800}{.5^2} + \frac{500}{.4^2} + \frac{400}{0.3^2}$ = 25600 + 48828.125+164609 = 239037

∴ $d^5 = \frac{1700}{239037}$ = .007118

∴ $d = (.007188)^{0.2}$

= 0.3718 **= 371.8 mm.**

UNIT IV

DIMENSIONAL AND MODEL ANALYSIS

INTRODUCTION

Dimensional analysis is a method of dimensions. It is a mathematical technique used in research work for design and for conducting model tests. It deals with the dimensions of the physical quantities involved in the phenomenon.

All physical quantities are measured by comparison, which is made with respect to an arbitrarily fixed value. Length L, mass M and time T are three fixed dimensions which are of importance in Fluid Mechanics.

If in any problem of fluid mechanics, heat is involved then temperature is also taken as fixed dimension. These fixed dimensions are called *fundamental dimensions or fundamental quantity.*

SECONDARY OR DERIVED QUANTITIES

Secondary or derived quantities are those quantities which possess more than one fundamental dimension. For example, velocity is denoted by distance per unit time (L/T), density by mass per unit volume (M/L^3) and acceleration by distance per second square (L/T^2).

Then velocity, density and acceleration become as secondary or derived quantities. The expressions (L/T), (M/L^3) and (L/T^2) are called the dimensions of velocity, density and acceleration respectively. The dimensions of mostly used physical quantities in Fluid Mechanics are given in Table.

S. No.	Physical Quantity	Symbol	units	Dimensions
	(a) Fundamental quantities			
1	Mass	M	kg	M
2	Length	L	m	L
3	Time	T	s	T
	(b)Geometric quantities			
4	Area	A	m^2	L^2
5	Volume	V	m^3	L^3
6	Moment of inertia	I	m^4	L^4
	(c)Kinetic quantities			
7	Velocity	v	m/s	LT^{-1}
8	Angular velocity	ω	rad/sec	T^{-1}
9	Acceleration	a	m/s^2	LT^{-2}

10	Angular acceleration	α	rad/sec^2	T^{-2}
11	Gravity	g	m/s^2	LT^{-2}
12	Discharge	Q	m^3/s	L^3T^{-1}
13	Kinematic viscosity	υ	m^2/s	L^2T^{-1}
	(d)Dynamic quantities			
14	Force	F	N (kg.m/s^2)	MLT^{-2}
15	Weight	W	N (kg.m/s^2)	MLT^{-2}
16	Specific weight	w	N/m^3	$ML^{-2} T^{-2}$
17	Density	ρ	kg/m^3	ML^{-3}
18	Dynamic viscosity	μ	N-s/m^2	$ML^{-1}T^{-1}$
19	Pressure Intensity	p	N/m^2	$ML^{-1}T^{-2}$
20	Modulus Of Elasticity	K or E		$ML^{-1}T^{-2}$
21	Work	W	N-m or J	ML^2T^{-2}
22	Energy	E	N-m or J	ML^2T^{-2}
23	Power	P	watts	$Ml^2 T^{-2}$
24	Torque	T	N-m or J	$ML^2 T^{-2}$
25	Momentum	M	kg-m/s	MLT^{-1}
26	Surface tension	σ	N/m	ML^{-2}
27	Shear stress	τ		$ML^{-1}T^{-2}$

DIMENSIONAL HOMOGENEITY

The law of Fourier principle of dimensional homogeneity states "an equation which expresses a physical phenomenon of fluid flow should be algebraically correct and dimensionally homogeneous".

Dimensionally homogeneous means, the dimensions of the terms of left hand side should be same as the dimensions of the terms on right hand side.

Let us consider the equation, $V=\sqrt{2gh}$

Dimension of L.H.S, $V=\frac{L}{T} = LT^{-1}$

Dimension of R.H.S, $\sqrt{2gh} = \sqrt{\frac{L}{T2}} \text{ x } L = \sqrt{\frac{L2}{T2}} = \frac{L}{T} = LT^{-1}$

Dimension of L.H.S = Dimension of R.H.S = LT^{-1}

Therefore equation,

$V=\sqrt{2gh}$ is dimensionally homogeneous. So it can be used in any system of units.

Uses of Dimensional Homogeneity

- To check the dimensional homogeneity of the given equation.
- To determine the dimension of a physical variable.
- To convert units from one system to another through dimensional homogeneity.
- It is a step towards dimensional analysis

Points to Be Remembered While Deriving Expressions Using Dimensional Analysis

1. First, the variables controlling the phenomenon should be identified and expressed in terms of primary dimensions.
2. Any mathematical equation should be dimensionally homogeneous.
3. In typical cases, a suitable mathematical model is constructed to simplify the problem with suitable assumptions.

METHODS OF DIMENSIONAL ANALYSIS

There are two methods of dimensional analysis used.

(i) Rayleigh's method

(ii) Buckingham π Theorem

RAYLEIGH'S METHOD

In this method, the expression is determined for a variable depending upon maximum three or four variables only. If the number of independent variables becomes more than four, it is very difficult to find the expression for the dependent variable. So, a functional relationship between variables is expressed in exponential form of equations.

Steps involved in Rayleigh's method

1. First, the functional relationship is written with the given data. ,

   Consider X as a variable which depends on $X_1, X_2, X_3, \ldots X_n$

   So, the functional equation is written

   $$X=f(X_1, X_2, X_3, \ldots X_n)$$

2. Then the equation is expressed in terms of a constant with exponents like powers of a, b, c ...

   Therefore, the equation is again written as

$X = \phi(X_1^a, X_2^b, X_3^c, ... X_n^z)$

Here, (ϕ)= Constant

$a, b, c, ... z$ = Arbitrary powers

3. The values of ***a, b,* c, ... *z*** are determined with the help of dimensional homogeneity. It means, the powers of the fundamental dimensions on both sides are compared to obtain the values of exponents.
4. Finally, these exponents/power values are substituted in the functional equation and simplified to obtain the suitable form.

BUCKINGHAM Π THEOREM

Rayleigh method is not helpful when the number of independent variables is more than three or four. This difficulty is eliminated in Buckingham π Theorem

It states that if there are 'n' variables in a dimensionally homogeneous equation and if these variables contain 'm' fundamental dimensions (M, L, T), then they are grouped into (n - m), dimensionless independent π-terms.

Let $X_1, X_2, X_3, ..., X_n$ are the variables involved in a physical phenomenon. Let X_1 be the variables involved in a physical phenomenon. Let X_1 be the dependent variables and $X_2, X_3, ... X_n$ are the independent variables on which X_1 depends. Then X_1 is a function of $X_2, X_3, ... X_n$ and mathematically, it is expressed as

$$X_1 = f(X_2, X_3, ... X_n) \quad .... \quad (1)$$

Equation (1) can also be written as

$$F_1 (X_1, X_2, X_3, ... X_n) = 0 \quad .... \quad (2)$$

This equation is a dimensionally homogeneous equation. It contains *n* variables. If there are *'m'* fundamental dimensions then according to Buckingham-π-theorem, equation (2) can be written interms in which number of π -terms is equal to (n - *m)*. Hence, equation (2) becomes

$$F(\pi_1, \pi_2, ... \pi_{n-m}) = 0 \quad .... \quad (3)$$

Each of π terms is dimensionless and independent of the system. Division or multiplication by a constant does not change the character of the π term. Each of π term contains (m +1) variables, where *m* is the number of Fundamental dimensions and is also called *repeating variables.* Let *'m'* in the above case X_2, X_3 and X_4 are repeating variables, if the fundamental dimensions (M, L, *T)* = 3 then each π term is written as

$$\pi_1 = X_2^{a1}, X_3^{b1}, X_4^{c1} . X^1$$

$$\pi_1 = X_2^{a2}, X_3^{b2}, X_4^{c2} . X^5$$

....

....

$$\pi_{n-m} = X_2^{a\,n-m}, X_3^{b\,n-m},\ X_4^{c\,n-m}\ .\ X^n \quad .... \qquad (4)$$

Each equation is solved by the principle of dimensional homogeneity and values of a_1, b_1, c_1 etc. are obtained. These values are substituted in equation (4) and values of $\boldsymbol{\pi_1, \pi_2, \pi_3}$...... π_{n-m} are obtained. These values of $\boldsymbol{\pi}$'s are substituted in equation (3). The final equation for the phenomenon is obtained by expressing any one of the π-terms as a function of others as

$$\mathbf{\Pi_1 = \varphi\ [\pi_2, \pi_3, ...\ \pi_{n-m}]}$$

$$\mathbf{\Pi_2 = \varphi\ [\pi_1, \pi_3, ...\ \pi_{n-m}]}$$

SELECTION OF REPEATING VARIABLES

There is no separate rule for selecting repeating variables. Hut the number of repeating variables is equal to the fundamental dimensions of the problem. Generally, ρ, υ, l or ρ, υ, D) are choosen as repeating variables.

It means, one refers to fluid property (ρ), one refers to flow property (υ) and the other one refers to geometric property (l or D). In addition to this, the following points should be kept in mind while selecting the repeating variables:

1. No variables should be dimensionless.
2. The selected two repeating variables should not have the same dimensions.
3. The selected repeating variables should be independent as far as possible.

STEPS TO BE FOLLOWED IN BUCKINGHAM Π METHOD

1. First, the variables involved in a given analysis are listed to study about given phenomenon thoroughly.
2. Then, these variables are expressed in terms of primary dimensions.
3. Next, the repeating variables are chosen according to the hint given in selection of repeating variables. Once, the repeating variables should be checked either those are independent or dependent variables because all should be independent variables.
4. Then the dimensionless parameters are obtained by adding one at a time repeating variables.
5. The number of π-terms involved in dimensional analysis is calculated using, n - m = Number of π terms.

Where, n = Total number of variables involved in given analysis.

m = Number of fundamental variables.

6. Finally, each equation in exponential form is solved which means the coefficients of exponents are found by comparing both sides exponents. Then these dimensionless parameters are recombined and arranged suitably.

In most of the fluid mechanics problems, the choice of repeating variables may be (i) d, v, ρ (ii) l, v, ρ (iii) l, v, μ or (iv) d, v, μ

The efficiency η of a fan depends on density ρ, dynamic viscosity μ of the fluid, angular velocity ω, diameter D of the rotor and the discharge Q. Express η in terms of dimensionless parameters.

Solution:

η is a function of ρ, μ,ω,D and Q

$\eta = f(\rho, \mu, \omega, D, Q)$ or $f_1(\eta, \rho, \mu, \omega, D, Q) = 0$ **…..... (i)**

Hence total number of variables, n = 6.

The value of m, i.e., number of fundamental dimensions for the problem is obtained by writing dimensions of each variable.

Dimensions of each variable are,

η = Dimensionless,

[$\rho = ML^{-3}$, $\mu = ML^{-1}T^{-1}$, $\omega = T^{-1}$, $D = L$ and $Q = L^{-3}T^{-1}$]

Number of π-terms = n - m = 6 - 3 = 3

Equation (i) is written as $(\pi_1, \pi_2, \pi_3) = 0$ **……. (ii)**

Each π -term contains *(m* + 1) variables, where *m* is equal to three and is also repeating variable. Choosing D, ω and ρ as repeating variables, we have

$\pi_1 = D^{a1} . \omega^{b1} . \rho^{c1} . \eta$

$\pi_2 = D^{a2} . \omega^{b2} . \rho^{c2} . \mu$

$\pi_3 = D^{a3} . \omega^{b3} . \rho^{c3} . Q$

First π – Term

$$\pi_1 = D^{a1} . \omega^{b1} . \rho^{c1} . \eta$$

Substituting dimensions on both sides of π_1,

$$M^0 L^0 T^0 = L^{a1} . (T^{-1})^{b1} . (ML^{-3})^{c1} . M^0 L^0 T^0$$

Equating the powers of M, L, T on both sides

Power of *M*, $0 = c_1 + 0$, $\mathbf{c_1 = 0}$

Power of *L*, $0 = a_1 + 0$, $\mathbf{a_1 = 0}$

Power of *T*, $0 = - b_1 + 0$, $\boldsymbol{b_1 = 0}$

Substituting the values of a_1, b_1 and c_1 in π_1, we get

$$\pi_1 = D^0 . \omega^0 . \rho^0 . \eta = \eta$$

[If a variable is dimensionless, it itself is a π - term. Here the variable η is a dimensionless and hence η is a π- term. As it exists in first π- term and hence $\pi_1 = \eta$ .Then there is no need of equating the powers. Directly the value can be obtained.]

Second π- Term

$$\boldsymbol{\pi_2 = D^{a2} . \omega^{b2} . \rho^{c2} . \mu}$$

Substituting dimensions on both sides of π_2,

$$M^0 L^0 T^0 = L^{a2} . (T^{-1})^{b2} . (ML^{-3})^{c2} . M L^{-1} T^{-1}$$

Equating the powers of M, L, T on both sides

Power of *M*, $0 = c_2 + 1$, $c_1 = -1$ $\boldsymbol{c_1 = -1}$

Power of *L*, $0 = a_2 - 3c_2 - 1$, $a_2 = 3c_2 + 1 = -3 + 1 = -2$ $\boldsymbol{a_2 = -2}$

Power of *T*, $0 = - b_2 + -1$, $b_1 = -1$ $\boldsymbol{b_1 = -1}$

Substituting the values of a_2, b_2 and c_2 in π_2, we get

$$\pi_2 = D^{-2} . \omega^{-1} . \rho^{-1} . \mu = \frac{\mu}{D^2\ \omega\ \rho}$$

Third π- Term

$$\pi_3 = D^{a3} . \omega^{b3} . \rho^{c3} . Q$$

Substituting dimensions on both sides of π_3,

$$M^0 L^0 T^0 = L^{a3} . (T^{-1})^{b3} . (ML^{-3})^{c3} . L^{-3} T^{-1}$$

Equating the powers of M, L, T on both sides

Power of *M*, $0 = c_3$, $c_3 = 0$ $\boldsymbol{c_3 = 0}$

Power of *L*, $0 = a_3 - 3c_3 + 3$, $a_3 = 3c_3 - 3 = -3$ $\boldsymbol{a_3 = -3}$

Power of *T*, $0 = - b_3 - 1$, $b_3 = -1$ $\boldsymbol{b_3 = -1}$

Substituting the values of a_2, b_2 and c_2 in π_2, we get

$$\pi_3 = D^{-3} . \omega^{-1} . \rho^0 . Q = \frac{Q}{D^2\ \omega}$$

Substituting the values of π_1, π_2 and π_3 in equation (ii), we get

$$f_1\left(\eta , \frac{\mu}{D^2\ \omega\ \rho} , \frac{Q}{D^2\ \omega}\right) = 0$$

$$\boldsymbol{\eta = \Phi\left(\frac{\mu}{D^2\ \omega\ \rho} , \frac{Q}{D^2\ \omega}\right)}$$ **Ans**

The pressure difference Δp in a pipe of diameter D and length l due to the turbulent flow depends on the velocity V, viscosity μ, density ρ and roughness k. Using Buckingham's π –theorem, obtain the expression for Δp.

Solution:

Δp is a function of D, l, V, μ, ρ, k

$\Delta p = f(D, l, V, \mu, \rho, k)$ (or) $f_1(\Delta p, D, l, V, \mu, \rho, k) = 0$ **…..... (i)**

Hence total number of variables, n = 7.

Number of fundamental dimensions, m = 3.

Writing dimensions of each variable,

Dimension of, [$\Delta p = ML^{-1}T^{-2}$, $D = l$, $l = L$, $V = LT^{-1}$,

$\mu = ML^{-1}T^{-1}$ $\rho = ML^{-3}$, $k = L$]

Number of π-terms = n - m = 7 - 3 = 4.

Now equation (i), can be grouped in 4 π terms as,

Equation (i) is written as $(\pi_1, \pi_2, \pi_3, \pi_4) = 0$ **……. (ii)**

Each π -term contains *(m* + 1) variables or 3+1 variables, out of four variables, three are repeating variable. Choosing D, V and ρ as repeating variables, we have four π terms as,

$\pi_1 = D^{a1}. V^{b1}. \rho^{c1}. \Delta p$

$\pi_2 = D^{a2}. V^{b2}. \rho^{c2}. l$

$\pi_3 = D^{a3}. V^{b3}. \rho^{c3}. \mu$

$\pi_4 = D^{a4}. V^{b4}. \rho^{c4}. k$

First π – Term

$$\pi_1 = D^{a1}. V^{b1}. \rho^{c1}. \Delta p$$

Substituting dimensions on both sides of π_1,

$M^0 L^0 T^0 = L^{a1}. (LT^{-1})^{b1}. (ML^{-3})^{c1}. ML^{-1}T^{-2}$

Equating the powers of M, L, T on both sides

Power of *M*, $0 = c_1 + 1$, $\mathbf{c_1 = -1}$

Power of *L*, $0 = a_1 + b_1 - 3c_1 - 1$ $\mathbf{a_1 =} - b_1 + 3c_1 - 1 = 2-3+1 = 0$

Power of *T*, $0 = -b_1 - 2$, $\mathbf{b_1 = -2}$

Substituting the values of a_1, b_1 and c_1 in π_1, we get

$$\pi_1 = D^0 . V^{-2} . \rho^{-1} . \Delta p$$

$$\boldsymbol{\pi_1 = \frac{\Delta p}{\rho V^2}}$$

Second π- Term

$$\boldsymbol{\pi_2 = D^{a2} . V^{b2} . \rho^{c2} . l}$$

Substituting dimensions on both sides of π_2,

$$M^0 L^0 T^0 = L^{a2} . (LT^{-1})^{b2} . (ML^{-3})^{c2} . L$$

Equating the powers of M, L, T on both sides

Power of *M*,	$0 = c_2$,	$c_1 = 0$	$\boldsymbol{c_1 = 0}$
Power of *L*,	$0 = a_2 - b_2 - 3c_2 + 1$,	$a_2 = b_2 + 3c_2 - 1 = -1$	$\boldsymbol{a_2 = -1}$
Power of *T*,	$0 = - b_2$	$b_2 = 0$	$\boldsymbol{b_2 = 0}$

Substituting the values of a_2, b_2 and c_2 in π_2, we get

$$\pi_2 = D^{-1} . V^0 . \rho^0 . l \quad = \frac{l}{D}$$

Third π- Term

$$\boldsymbol{\pi_3 = D^{a3} . V^{b3} . \rho^{c3} . \mu}$$

Substituting dimensions on both sides of π_3,

$$M^0 L^0 T^0 = L^{a3} . (LT^{-1})^{b3} . (ML^{-3})^{c3} . ML^{-1} T^{-1}$$

Equating the powers of M, L, T on both sides

Power of *M*,	$0 = c_3 + 1$,	$c_3 = -1$	$\boldsymbol{c_3 = -1}$
Power of *L*,	$0 = a_3 + b_3 - 3c_3 - 1$,	$a_3 = -b_3 + 3c_3 + 1 = 1-3+1 = -1$	$\boldsymbol{a_3 = -1}$
Power of *T*,	$0 = - b_3 - 1$,	$b_3 = -1$	$\boldsymbol{b_3 = -1}$

Substituting the values of a_3, b_3 and c_3 in π_3, we get

$$\pi_3 = D^{-1} . V^{-1} . \rho^{-1} . \mu \quad = \frac{Q}{DV\rho}$$

Fourth π- Term

$$\boldsymbol{\pi_4 = D^{a4} . V^{b4} . \rho^{c4} . k}$$

Substituting dimensions on both sides of π_3,

$$M^0 L^0 T^0 = L^{a4} . (LT^{-1})^{b4} . (ML^{-3})^{c4} . L$$

Equating the powers of M, L, T on both sides

Power of *M*,	$0 = c_4$,	$c_4 = 0$	$\boldsymbol{c_4 = 0}$

Power of L, $0 = a_4 - b_4 - 3c_4 + 1$, $a_3 = b_4 + 3c_4 - 1 = -1$ $\mathbf{a_4 = -1}$

Power of T, $0 = -b_4$, $b_4 = 0$ $\mathbf{b_4 = 0}$

Substituting the values of a_3, b_3 and c_3 in π_4, we get

$$\pi_4 = D^{-1}.V^0.P^0.k = \frac{\mathbf{k}}{\mathbf{D}}$$

Substituting the values of π_1, π_2, π_3 and π_4 in equation (ii), we get

$$f_1\left(\frac{\Delta p}{\rho V^2}, \frac{l}{D} . \frac{\mathbf{Q}}{\mathbf{DV\rho}} . \frac{\mathbf{k}}{\mathbf{D}}\right) = 0$$

$$\frac{\Delta p}{\rho V^2} = \boldsymbol{\Phi}\left(\frac{l}{D}, \frac{\mathbf{Q}}{\mathbf{DV\rho}}, \frac{\mathbf{k}}{\mathbf{D}}\right) \quad \textbf{Ans}$$

The pressure difference Δp in a pipe of diameter D and length l due to the viscous flow depends on the velocity V, viscosity μ, density ρ. Using Buckingham's π –theorem, obtain the expression for Δp.

Using Buckingham's π –theorem, show that the velocity through a circular orifice is given by V = $\sqrt{2gH}\ \Phi\left[\frac{D}{H}, \frac{\mu}{\rho V H}\right]$, where H is the head causing flow, D is the diameter of the orifice, μ is the coefficient of viscosity, ρ is mass density, g is the acceleration due to gravity.

Solution:

V is a function of (H, D, μ, ρ, g)

$V = f(H, D, \mu, \rho, g)$ (or) $f_1(V, H, D, \mu, \rho, g) = 0$ **…….. (i)**

Hence total number of variables, n = 6.

Number of fundamental dimensions, m = 3.

Writing dimensions of each variable,

Dimension of, [$V = LT^{-1}$, $H = L$, $D = L$, $\rho = ML^{-3}$,

$\mu = ML^{-1}T^{-1}$, $g = LT^{-2}$]

Total Number of π-terms = n - m = 6 - 3 = 3.

Now equation (i), can be grouped in 3 π terms as,

Equation (i) is written as $(\pi_1, \pi_2, \pi_3) = 0$ **……. (ii)**

Each π -term contains *(m* + 1) variables or 3+1 variables, (m = 3) and also is equal to repeating variables. Choosing H, g and ρ as repeating variables, we have three π terms as,

$\pi_1 = H^{a1} . g^{b1} . \rho^{c1} . V$

$\pi_2 = H^{a2} . g^{b2} . \rho^{c2} . D$

$\pi_3 = H^{a3} . g^{b3} . \rho^{c3} . \mu$

First π – Term

$$\boldsymbol{\pi_1 = H^{a1} . g^{b1} . \rho^{c1} . V}$$

Substituting dimensions on both sides of π_1,

$M^0 L^0 T^0 = L^{a1} . (LT^{-2})^{b1} . (ML^{-3})^{c1} . (LT^{-1})$

Equating the powers of M, L, T on both sides

Power of *M,*	$0 = c_1$,	$c_1 = 0$	$\mathbf{c1 = 0}$
Power of L,	$0 = a_1 + b_1 - 3c_1 + 1$	$a_1 = - b_1 + 3c_1 - 1 = \frac{1}{2} - 1$	$\mathbf{a1 = -\frac{1}{2}}$
Power of T,	$0 = - 2b_1 - 1$,	$b_1 = -\frac{1}{2}$	$\mathbf{b_1 = -\frac{1}{2}}$

Substituting the values of a_1, b_1 and c_1 in π_1, we get

$$\pi_1 = H^{-(1/2)} . g^{-(1/2)} . \rho^0 . V$$

$$\boldsymbol{\pi_1 = \frac{V}{\sqrt{gH}}}$$

Second π- Term

$$\pi_2 = H^{a2} . g^{b2} . \rho^{c2} . D$$

Substituting dimensions on both sides of π_2,

$M^0 L^0 T^0 = L^{a2} . (LT^{-2})^{b2} . (ML^{-3})^{c2} . L$

Equating the powers of M, L, T on both sides

Power of *M,*	$0 = c_2$,	$c_1 = 0$	$\mathbf{c_1 = 0}$
Power of L,	$0 = a_2 + b_2 - 3c_2 + 1$,	$a_2 = b_2 + 3c_2 - 1 = -1$	$\mathbf{a_2 = -1}$
Power of T,	$0 = - 2b_2$	$b_2 = 0$	$\mathbf{b_2 = 0}$

Substituting the values of a_2, b_2 and c_2 in π_2, we get

$$\pi_2 = H^{-1} . g^0 . \rho^0 . D$$

$$\boldsymbol{\pi_2 = \frac{D}{H}}$$

Third π- Term

$$\pi_3 = H^{a3}. g^{b3}. \rho^{c3}. \mu$$

Substituting dimensions on both sides of π_3,

$$M^0 L^0 T^0 = L^{a3} . (LT^{-2})^{b3} . (ML^{-3})^{c3} . \; ML^{-1} T^{-1}$$

Equating the powers of M, L, T on both sides

Power of *M*, $0 = c_3 + 1$, $c_3 = -1$ **$c_3 = -1$**

Power of *L*, $0 = a_3 + b_3 - 3c_3 - 1$, $a_3 = -b_3 + 3c_3 + 1 = \frac{1}{2} - 3 + 1$ **$a_3 = -\frac{3}{2}$**

Power of *T*, $0 = -2b_3 - 1$, $b_3 = -\frac{1}{2}$ **$b_3 = -\frac{1}{2}$**

Substituting the values of a_3, b_3 and c_3 in π_3, we get

$$\pi_3 = H^{-(3/2)}. g^{-(1/2)}. \rho^{-1}. \mu$$

$$= \frac{\mu}{H^{(3/2)}\rho\sqrt{g}} = \frac{\mu}{H\,\rho\,\sqrt{gH}}$$ (multiply and divide by V)

$$= \frac{\mu V}{H\,\rho V\,\sqrt{gH}} = \frac{\mu}{H\,\rho V}.\,\pi_1$$ $$\frac{V}{\sqrt{gH}} = \pi_1$$

Substituting the values of π_1, π_2 and π_3 in equation (ii), we get

$$f_1\left(\frac{V}{\sqrt{gH}}, \frac{D}{H}, \pi_1 \frac{\mu}{H\,\rho V}\right) = 0$$

$$\frac{V}{\sqrt{gH}} = \Phi\left[\frac{D}{H}, \pi_1 \frac{\mu}{H\,\rho V}\right]$$

$V = \sqrt{2gH}\,\Phi\left[\frac{D}{H}, \frac{\mu}{\rho V H}\right]$ Ans

[multiplying by a constant does not change the character of π – terms]

Using Buckingham's π –theorem, show that the discharge Q consumed by an oil ring is given by,

$$Q = Nd^3\,\Phi\left[\frac{\mu}{\rho N d^2}, \frac{\sigma}{\rho N^2 d^3}, \frac{\omega}{\rho N^2 d}\right]$$

where d is the internal diameter of the ring, N id rotational speed, ρ is density, viscosity μ, σ is surface tension and ω is the specific weight of oil.

Solution:

Q is a function of (d, N, ρ, μ, σ, ω)

Q = f (d, N, ρ, μ, σ, ω) (or) f_1(Q, d, N, ρ, μ, σ, ω) = 0 **…..... (i)**

Hence total number of variables, n = 7.

Number of fundamental dimensions, m = 3.

Writing dimensions of each variable,

Dimension of, [$Q = L^3 T^{-1}$, $d = L$, $N = T^{-1}$, $\rho = ML^{-3}$,

$\mu = ML^{-1} T^{-1}$, $\sigma = MT^{-2}$, $\omega = ML^{-2} T^{-2}$]

Total Number of π-terms = n - m = 7 - 3 = 4.

Now equation (i), can be grouped in 4 π terms as,

Equation (i) is written as $(\pi_1, \pi_2, \pi_3, \pi_4) = 0$ **……. (ii)**

Each π -term contains *(m* + 1) variables or 3+1 variables, out of four variables, three are repeating variable. Choosing d, N and ρ as repeating variables, we have four π terms as,

$$\pi_1 = d^{a1} . N^{b1} . \rho^{c1} . Q$$

$$\pi_2 = d^{a2} . N^{b2} . \rho^{c2} . \mu$$

$$\pi_3 = d^{a3} . N^{b3} . \rho^{c3} . \sigma$$

$$\pi_4 = d^{a4} . N^{b4} . \rho^{c4} . \omega$$

First π – Term

$$\boldsymbol{\pi_1 = d^{a1} . N^{b1} . \rho^{c1} . Q}$$

Substituting dimensions on both sides of π_1,

$$M^0 L^0 T^0 = L^{a1} . (T^{-1})^{b1} . (ML^{-3})^{c1} . L^{-3} T^{-1}$$

Equating the powers of M, L, T on both sides

Power of *M,*	$0 = c_1$,	$c_1 = 0$	$c_1 = 0$
Power of *L,*	$0 = a_1 - 3c_1 + 3$	$a_1 = 3c_1 - 3 = 0 - 3 = -3$	$a_1 = -3$
Power of *T,*	$0 = -b_1 - 1$,	$b_1 = -1$	$b_1 = -1$

Substituting the values of a_1, b_1 and c_1 in π_1, we get

$$\pi_1 = d^{-3} . N^{-1} . \rho^0 . Q$$

$$\boldsymbol{\pi_1 = \frac{Q}{d^3 N}}$$

Second π- Term

$$\pi_2 = d^{a2} . N^{b2} . \rho^{c2} . \mu$$

Substituting dimensions on both sides of π_2,

$$M^0 L^0 T^0 = L^{a2} . (T^{-1})^{b2} . (ML^{-3})^{c2} . ML^{-1} T^{-1}$$

Equating the powers of M, L, T on both sides

Power of *M*, $0 = c_2 + 1$, $c_1 = -1$ $\mathbf{c_1 = -1}$

Power of *L*, $0 = a_2 - 3c_2 - 1$, $a_2 = 3c_2 + 1 = -3+1$ $\mathbf{a_2 = -2}$

Power of *T*, $0 = - b_2 - 1$ $b_2 = -1$ $\boldsymbol{b_2 = -1}$

Substituting the values of a_2, b_2 and c_2 in π_2, we get

$$\pi_2 = d^{-2} . N^{-1} . \rho^{-1} . \mu = \frac{\mu}{\rho N d^2}$$

Third π- Term

$$\boldsymbol{\pi_3 = d^{a3} . N^{b3} . \rho^{c3} . \sigma}$$

Substituting dimensions on both sides of π_3,

$$M^0 L^0 T^0 = L^{a3} . (T^{-1})^{b3} . (ML^{-3})^{c3} . ML\,T^{-2}$$

Equating the powers of M, L, T on both sides

Power of *M*, $0 = c_3 + 1$, $c_3 = -1$ $\mathbf{c_3 = -1}$

Power of *L*, $0 = a_3 - 3c_3$, $a_3 = 3c_3 = -3$ $\mathbf{a_3 = -3}$

Power of *T*, $0 = - b_3 - 2$, $b_3 = -2$ $\boldsymbol{b_3 = -2}$

Substituting the values of a_3, b_3 and c_3 in π_3, we get

$$\pi_3 = d^{-2} . N^{-1} . \rho^{-1} . \boldsymbol{\sigma} = \frac{\sigma}{d^3 N^2 \rho}$$

Fourth π- Term

$$\boldsymbol{\pi_4 = d^{a4} . N^{b4} . \rho^{c4} . \omega}$$

Substituting dimensions on both sides of π_3,

$$M^0 L^0 T^0 = L^{a4} . (T^{-1})^{b4} . (ML^{-3})^{c4} . ML^{-2} T^{-2}$$

Equating the powers of M, L, T on both sides

Power of *M*, $0 = c_4 + 1$, $c_4 = -1$ $\mathbf{c_4 = -1}$

Power of *L*, $0 = a_4 - 3c_4 - 2$, $a_4 = 3c_4 + 2 = -3+2$ $\mathbf{a_4 = -1}$

Power of *T*, $0 = - b_4 - 2$, $b_4 = -2$ $\boldsymbol{b_4 = -2}$

Substituting the values of a_3, b_3 and c_3 in π_4, we get

$$\pi_4 = d^{-1} . N^{-2} . \rho^{-1} . \omega = \frac{\omega}{d\, N^2 \rho}$$

Substituting the values of π_1, π_2, π_3 and π_4 in equation (ii), we get

$$f_1\left(\frac{Q}{d^3\, N}, \frac{\mu}{\rho N d^2} . \frac{\sigma}{d^3 N^2 \rho} . \frac{\omega}{d\, N^2 \rho}\right) = 0$$

$$\frac{Q}{d^3 N} = f_1 \left(\frac{\mu}{\rho N d^2} \cdot \frac{\sigma}{d^3 N^2 \rho} \cdot \frac{\omega}{d N^2 \rho}\right)$$

$$\mathbf{Q} = \boldsymbol{d^3 N} \ \Phi \left(\frac{\mu}{\rho N d^2} \cdot \frac{\sigma}{d^3 N^2 \rho} \cdot \frac{\omega}{d N^2 \rho}\right). \qquad \textbf{Ans}$$

MODEL ANALYSIS

- ✓ For predicting the performance of the hydraulic structures (such as dams, spillways etc.) or hydraulic machines (such as turbines, pumps etc.), before actually constructing or manufacturing.
- ✓ Models of the structures or machines are made and tests are performed on them to obtain the desired formation.
- ✓ The model is the small scale replica of the actual structure or machine. The actual structure or machine is called Prototype. It is not necessary that the models should be smaller than the prototypes (though in most of cases it is), they may be larger than the prototype.
- ✓ The study of models of actual machines is called ***Model analysis***. Model analysis is actually an experimental method of finding solutions of complex flow problems. Exact analytical solutions are possible only for a limited number of flow problems.

The followings are the advantages of the dimensional and model analysis:

1. The performance of the hydraulic structure or hydraulic machine can be easily predicted, in advance, from its model.
2. With the help of dimensional analysis, a relationship between the variables influencing a flow probem in terms of dimensionless parameters is obtained. This relationship helps in conducting tests on the model.
3. The merits of alternative designs can be predicted with the help of model testing. The most economical and safe design may be, finally, adopted.
4. The tests performed on the models can be utilized for obtaining, in advance, useful information about the performance of the prototypes only if a complete similarity exists between the model and the prototype.

CLASSIFICATION OF MODELS

Generally, hydraulic models are classified into two types.

- ✓ Undistorted models.
- ✓ Distorted models

1. UNDISTORTED MODELS

The model which is geometrically similar to its prototype is known as undistorted models. In such models, the conditions of similitude are fully satisfied. So, the results obtained from the model are used to predict the performance of the prototype easily. Based on this, design, construction and interpretation of (he model are simpler.

2. DISTORTED MODELS

A model which is not geometrically similar to its prototype but it may be similar in appearance with its prototype. So, different scale ratios are used for linear dimensions such as length, breadth and height.

Usually, the following distortions may occur in distorted models:

- ✓ Geometrical distortion.
- ✓ Material distortion.
- ✓ Distortion of hydraulic quantities

Geometrical distortion.

The distortion occurs either in dimensions or in configuration. It can be corrected by using different scale values for vertical and horizontal dimensions.

Material distortion:

It arises due to the use of different materials for the model and prototype. To avoid this, the same materials have to be used as much as possible.

Distortion of hydraulic quantities:

Due to uncontrollable hydraulic quantities, the distortion may occur. Example: Velocity, discharge etc.

Reasons of adopting distorted models

- ✓ To maintain accuracy.
- ✓ To maintain turbulent flow.
- ✓ To accommodate available facilities
- ✓ To obtain suitable bed materials.
- ✓ To obtain required roughness condition.

Advantages of distorted models

- ✓ Accurate measurements can be possible.
- ✓ Surface tension can be minimized as much as possible.
- ✓ The operation is simplified due to small model size.
- ✓ Reynolds number of flow is increased sufficiently.

Disadvantages of distorted models

- ✓ Exit pressure and velocity distributions are not true.

- ✓ A model wave may differ from that of prototype.
- ✓ Both extrapolation and interpolation of results are difficult.

SIMILITUDE-TYPES OF SIMILARITIES

Similitude is defined as the similarity between the model and its prototype in every respect, which means that the model and prototype have similar properties or model and prototype are completely similar.

Three types of similarities must exist between the model and prototype. They are

1 Geometric Similarity, 2. Kinematic Similarity, and 3. Dynamic Similarity.

Geometric Similarity: The geometric similarity is said to exist between the model and the prototype. The ratio of all corresponding linear dimension in the model and prototype are equal. Let,

L_m = Length of model,

b_m = Breadth of model,

D_m - Diameter of model,

A_m - Area of model,

V_m = Volume of model,

L_P, b_p, D_p, A_p, V_p = Corresponding values of the prototype,

For geometric similarity between model and prototype, we must have the relation,

$$\frac{L_p}{L_m} = \frac{b_p}{b_m} = \frac{D_p}{D_m} = L_r$$

L_r is called the scale ratio.

Kinematic Similarity:

- ✓ Kinematic similarity means the similarity of motion between model and prototype. Thus kinematic similarity is said to exist between the model and the prototype if the ratios velocity and acceleration at the corresponding points in the model and at the corresponding points in the prototype are the same.
- ✓ Since velocity and acceleration are vector quantities, hence not only the ratio of magnitude of velocity and acceleration at the corresponding points in model and prototype should be same, but the directions of velocity and accelerations at the corresponding points in the model and prototype also should be parallel.
- ✓ All the direction of the velocities in the model and prototype should be same.

Dynamic Similarity:

- ✓ Dynamic similarity means the similarity of forces between the model and prototype. Thus dynamic similarity is said to exist between the model and the prototype if the ratios of the corresponding forces acting at the corresponding points are equal.
- ✓ Also the directions of the corresponding forces at the corresponding points should be same.

TYPES OF FORCES ACTING IN MOVING FLUID

For the fluid flow problems, the forces acting on a fluid mass may be any one, or a combination of the several of the following forces:

- ✓ Inertia force, F_i
- ✓ Viscous force, F_v
- ✓ Gravity force, F_g
- ✓ Pressure force, F_p
- ✓ Surface tension force, F_y
- ✓ Elastic force, F_e

Inertia force (fv):

It is equal to the product of mass and acceleration of the flowing fluid and acts in the direction opposite to the direction of acceleration. It is always existing in the fluid flow problems.

Viscous Force (Fv):

It is equal to the product of shear stress (x) due to viscosity and surface area of the flow. It is present in fluid flow problems where viscosity is having an important role to play.

Gravity Force (F_g):

It is equal to the product of mass and acceleration due to gravity of the flowing fluid. It is present in case of open surface flow.

Pressure Force (F_p):

It is equal to the product of pressure intensity and cross-sectional area of -e flowing fluid. It is present in case of pipe-flow.

Surface Tension Force (F_s):

It is equal to the product of surface tension and length of surface of Lie flowing fluid.

Elastic Force (F_e):

It is equal to the product of elastic stress and area of the flowing fluid.

For a flowing fluid, the above-mentioned forces may not always be present. And also the forces, which are present in a fluid flow problem, are not of equal magnitude. There are always one or two forces which dominate the other forces. These dominating forces govern the flow of fluid.

DIMENSIONLESS NUMBERS

Dimensionless numbers are those numbers .which are obtained by dividing the inertia force by viscous force or gravity force or pressure force or surface tension force or elastic force. As this is a ratio of one force to the other force, it will be a dimensionless number.

These dimensionless numbers also called non-dimensional parameters. The followings are the important dimensionless numbers:

- ✓ Reynold's number
- ✓ Froude's number
- ✓ Euler's number
- ✓ Weber's number
- ✓ Mach's number

Reynold's Number (Re): It is defined as the ratio of inertia force of a flowing fluid and r viscous force of the fluid. The expression for Reynold's number is obtained as,

$$R_e = \sqrt{\frac{inertia\ force}{viscous\ force}}$$

$$Re = \frac{V * L}{v} \qquad [v = \frac{\mu}{\rho}]$$

In case of pipe flow, the linear dimension L is taken as diameter, d. Hence Reynold's number for pipe flow,

$$R_e = \frac{V * d}{v} \quad (\text{ or }) = \frac{\rho V d}{\mu}$$

Froude's Number (Fe): The Froude's number is defined as the square root of the ratio of inertia force of a flowing fluid to the gravity force. Mathematically, it is expressed as,

$$Fe = \sqrt{\frac{inertia\ force}{gravity\ force}} = \sqrt{\frac{V}{Lg}}$$

Euler's Number (E_u): It is defined as the square root of the ratio of the inertia force of a flowing fluid to the pressure force. Mathematically, it is expressed as,

$$E_u = \sqrt{\frac{inertia\ force}{pressure\ force}} = \frac{V}{\sqrt{\frac{p}{\rho}}}$$

Weber's Number (W_e): It is defined as the square root of the ratio of the inertia force of a flowing fluid to the surface tension force. Mathematically, it is expressed as,

$$W_e = \sqrt{\frac{inertia\ force}{surface\ tension\ force}}$$

$$= \frac{V}{\sqrt{\frac{\sigma}{\rho\ L}}}$$

Mach's Number (M): Mach's number is defined as the square root of the ratio of the inertia force of a flowing fluid to the elastic force. Mathematically, it is defined as,

$$M = \sqrt{\frac{inertia\ force}{elastic\ force}}$$

$$= \frac{M}{C}$$

MODEL LAWS OR SIMILARITY LAWS

✓ For the dynamic similarity between the model and the prototype, the ratio of the corresponding forces acting at the corresponding points in the model and prototype should be equal. The ratio of the forces are dimensionless numbers.

- ✓ It means for dynamic similarity between the model and prototype, the dimensionless numbers should be same for model and the prototype.
- ✓ But it is quite difficult to satisfy the condition that all the dimensionless numbers (i.e., R_e, F_e, W_e, E_u and M) are the same for the model and prototype. Hence models are designed on the basis of ratio of the force, which is dominating in the phenomenon.

The laws on which the models are designed for dynamic similarity are called model laws or laws of similarity. The followings are the model laws: Reynold's model law

- ✓ Froude model law
- ✓ Euler model law
- ✓ Weber model law
- ✓ Mach model law

Reynolds's Model Law:

Reynolds's model law is the law in which models are based on Reynolds's number. Models based on Reynolds's number includes:

(i) Pipe flow

(ii) Resistance experienced by sub-marines, airplanes, fully immersed bodies etc.

As defined earlier that Reynolds number is the ratio of inertia force and viscous force, and hence fluid flow problems where viscous forces alone are predominant, the models are designed for dynamic similarity on Reynolds law, which states that the Reynolds number for the model must be equal to the Reynolds number for the prototype.

Let

V_m = Velocity of fluid in model

ρm= Density of fluid in model

L_m = Length or linear dimension of the model

μ_m = Viscosity or fluid in model

And ($V_{m,}$ ρm, L_m and μ_m) are the corresponding values of velocity, density, linear dimension and viscosity of fluid in prototype. Then according to Reynolds's model law,

$$Re_{(p)} = Re_{(m)}$$

Froude Model Law:

Froude model law is the law in which the models are based on Froude number which means for dynamic similarity between the model and prototype, the Froude number for both of them should be equal.

Froude model law is applicable when the gravity force is only predominant force which controls the flow in addition to the force of inertia. Froude model law is applied in the following fluid flow problems:

- ✓ Free surface flows such as flow over spillways, weirs, sluices, channels etc.
- ✓ Flow of jet from an orifice or nozzle,

- ✓ Where waves are likely to be formed on surface,
- ✓ Where fluids of different densities flow over one another.

$$(F_e)_{model} = (F_e)_{prototype}$$

Euler's Model Law:

Euler's model law is the law in which the models are designed on Euler's number which means for dynamic similarity between the model and prototype, the Euler number for model arid prototype should be equal. Euler's model law is applicable when the pressure forces are alone predominant in addition to the inertia force. According to this law:

$$(E_u)_{model} = (E_u)_{prototype}$$

If

V_m = Velocity of fluid in model

P_m = Pressure of fluid in model

ρ_m = Density of fluid in model

Then V_m, P_m, ρ_m = Corresponding values in prototype

$$Vm/\sqrt{P_m} = Vp/\sqrt{P_p}$$

Euler's model law is applied for fluid flow problems where flow is taking place in a closed pipe in which case turbulence is fully developed so that viscous forces are negligible and gravity force and surface tension force is absent. This law is also used where the phenomenon of cavitation takes place.

Weber Model Law:

Weber model law is the law in which models are based on Weber's number, which is the ratio of the square root of inertia force to surface tension force.

Hence where surface tension effects predominate in addition to inertia force, the dynamic similarity between these model and prototype is obtained by equating the Weber number of the model and its prototype.

Hence according to this law:

$$(W_e)model = (W_e)prototype$$

Where, W_e is Weber number and = $V/\sqrt{\sigma/\rho_L}$

If

V_m = Velocity of fluid in model

σ_m= Surface tensile force in model

ρ_m = Density of fluid in model

L_m = Length of surface in model

And ($V_m, \sigma_m, \rho_m, L_m$) = Corresponding values of fluid in prototype.

Then according to Weber law, we have

$$\frac{Vm}{\sqrt{\sigma\ m/\ \rho m, Lm}} = \frac{V}{\sqrt{\sigma\ p/\ \rho p, Lp}}$$

Weber model law is applied in following cases:

- ✓ Capillary rise in narrow passages
- ✓ Capillary movement of water in soil
- ✓ Capillary waves in channels
- ✓ Flow over weirs for small head

Mach Model Law:

Mach model law is the law in which models are designed on Mach number, which is the ratio of the square root of inertia force to elastic force of a fluid.

Hence where the forces due to elastic compression predominate in addition to inertia force, the dynamic similarity between the model and its prototype is obtained by equating the Mach number of the model and its prototype.

Hence according to this law:

$$(M)_{model} = (M)_{prototype}$$

where , M = Mach number = $V/\sqrt{K/\rho}$

If

V_m = Velocity of fluid in model

K_m = Elastic stress for model

ρ_m = Density of fluid in model

And V_p, K_P and ρ_P= Corresponding values for prototype.

Then according to Mach law,

$$\frac{Vm}{\sqrt{Km/\ \rho m}} = \frac{V}{\sqrt{Kp/\ \rho p}}$$

Mach model law is applied in the following cases:

- ✓ Flow of aeroplane and projectile through air at supersonic speed, ie., at a velocity more than the velocity of sound.
- ✓ Aerodynamic testing
- ✓ Under water testing of torpedoes
- ✓ Water hammer problems

UNIT V
BOUNDARY LAYER

INTRODUCTION

The variation of velocity from zero to free-stream velocity in the direction normal to the boundary takes place in a narrow region in the vicinity of solid boundary.

This narrow region of the fluid is called boundary layer. The theory dealing with boundary layer flows is called ***boundary layer theory.***

1. A very thin layer of the fluid called the boundary layer in the immediate neighbourhood of the solid boundary, where the variation of velocity from zero at the solid boundary to free-stream velocity in the direction normal to the boundary takes place.
2. The remaining fluid which is outside the boundary layer. The velocity outside the boundary layer is constant and equal to free-stream velocity.

Laminar boundary layer:

- ✓ The leading edge of the surface of the plate where the thickness is small, the flow in the boundary layer is laminar though the main flow is turbulent.
- ✓ This layer of the fluid is said to be laminar boundary layer.

Turbulent boundary layer:

- ✓ The laminar boundary layer becomes unstable and motion of fluid within, it is disturbed and irregular which leads to a transition from laminar to turbulent boundary layer.
- ✓ This short length over which the boundary layer flow changes from laminar to turbulent is called transition zone.
- ✓ Further downstream the transition zone the boundary layer is turbulent and continues to grow in thickness. This layer of boundary is called turbulent boundary layer.

Laminar sub-layer:

- ✓ The region in the turbulent boundary layer zone, adjacent to the solid surface of the plate. In this zone the velocity variation is influenced only by viscous effects.
- ✓ Though the velocity distribution would be a parabolic curve in the laminar sub-layer zone but in view of the very small thickness. That velocity variation is linear and so the velocity gradient can be considered constant.
- ✓ Therefore, the shear stress in the laminar sub-layer would be constant and equal to the boundary shear stress $\tau 0$.

Boundary layer thickness:

- ✓ It is defined as the distance from the boundary of the solid body measured in the y-direction to the point, where the velocity of the fluid is approximately equal to 0.99 times the free stream velocity of the fluid.
- ✓ It is denoted by the symbol δ.

Displacement thickness:

- ✓ It is defined as the distance measured perpendicular to the boundary of the solid body by which the boundary should be displaced to compensate for the reduction in flow rate on account of boundary layer formation. It is denoted by δ*.

$$\delta^* = \int [1 – (u/U)]\, dy$$

It is also defined as:

- ✓ The distance perpendicular to the boundary by which the free stream is displaced due to the formation of boundary layer.

Momentum thickness:

- ✓ It is defined as the distance, measured perpendicular to the boundary of the solid body, by which the boundary should be displaced to compensate for the reduction in momentum of the flowing fluid on account of boundary layer formation. It is denoted by θ.

$$\theta = \int [(u/U) – (u/U)^2]\, dy$$

Energy thickness:

- ✓ It is defined as the distance measured perpendicular to the boundary of the solid body by which the boundary should be displaced to compensate for the reduction in kinetic energy of the flowing fluid on account of boundary layer formation. It is denoted by δ**.

$$\delta^{**} = \int [(u/U) – (u/U)^3]\, dy$$

Boundary condition for the velocity profiles:

1. At y = 0,u = 0 and $dudy$ has some finite value.

2. At y = δ,u=U.

3. At y = δ, / dy = 0

Turbulent boundary layer on a flat plate:

- ✓ The thickness of the boundary layer, drag force on one side of the plate and co-efficient of drag due to turbulent boundary layer on a smooth plate at zero pressure gradient are determined as in case of laminar boundary layer provider the velocity profile is known.
- ✓ Blasius on the basis of the experiment given the following velocity profile for a turbulent boundary layer.

$uU = (y\delta)n$

Where n=1/7 for Re < 10^7 but more than 5× 10^5

$uU = (y\delta)17$

The above equation is not applicable very near the boundary, where the thin laminar sub-layer of thickness δ` exist. Here velocity distribution is influenced by viscous effects.

Analysis of turbulent boundary layer:

(a) If Reynold number is more than 5×105 and less than 107the thickness of boundary layer and drag co-efficient are given as:

δ = 0.37(Rex)1/5 and CD = 0.072(ReL)1/5

Where x=distance from the leading edge

Rex = reynold number for length x

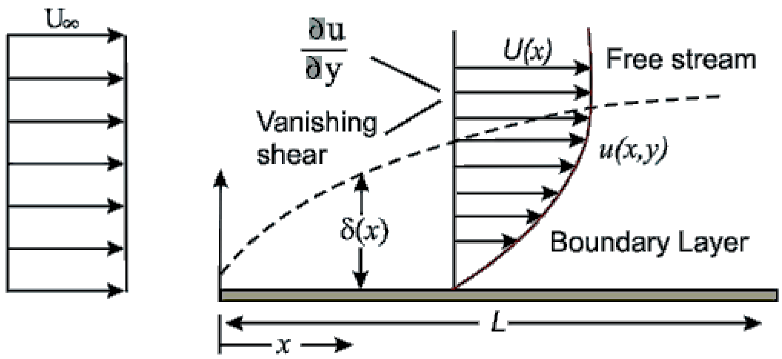
Rel = reynold number at the end of the plate

(b) If reynold number is more than 10^7 but less than 10^9 , gave the empirical equation as

CD = 0.455(log10 ReL)

SEPARATION OF BOUNDARY LAYER:

- ✓ The loss of kinetic energy is recovered from the intermediate fluid layer in contact with the layer adjacent to solid surface through momentum exchange process. Thus the velocity of the layer goes on decreasing.



- ✓ Along the length of the solid body, at a certain point a stage may come when the boundary layer may not be able to keep sticking to the solid body if it can't provide kinetic energy to overcome the resistance offer by the solid body, the boundary layer will be separated from the surface.
- ✓ This phenomenon is called boundary layer separation. The point on the body at which the boundary layer is on the verge of separation from the surface is called as the point of separation.

EFFECT OF PRESSURE GRADIENT ON THE BOUNDARY LAYER SEPARATION:

- ✓ Effect of pressure gradient ($dpdx$) on the boundary layer separation can be explained by considering the flow over a curved surface. The area of flow decreases and hence velocity increases. This means that flow gets accelerated in this region. Due to increase in the velocity, the pressure decreases in the direction of the flow and hence pressure gradient ($dpdx$) is negative.

Location of separation point:

The separation point is determined from the condition, ( )=0=0

For a given velocity profile, it can be determine whether the boundary layer has separated or verge of separation or will not separate from the following condition.

1.If (∂u ∂y)y=0 is negative…the flow has separated.

2.If (∂u ∂y)y=0 = 0 the flow is on the verge of separation.

3.If (∂u ∂y)y=0 is positive ….the flow will not separate or flow will remain attached with the surface.

Methods of preventing the separation of boundary layer:

- ✓ When the boundary layer separates from the surface, a certain portion adjacent to the surface has a back flow and eddies are continuously formed in this region and hence continuous loss of energy takes place. Thus separation of boundary layer is undesirable and attempts should be made to avoid separation by various methods.

The following are the methods for preventing the separation of boundary layer:

1. Suction of the slow moving fluid by a suction slot.
2. Supplying additional energy from a blower.
3. Providing a bypass in the slotted wing.
4. Rotating boundary in the direction of flow.
5. Providing small divergence in a diffuser.
6. Providing guide-blades in a bend.
7. Providing a trip-wire ring in the laminar region for the flow over a sphere.

Find the displacement thickness, the momentum thickness and energy thickness for the velocity distribution in the boundary layer given by $\frac{u}{U} = \frac{y}{\delta}$, where u is the velocity at a distance y from the plate and u = U at y = δ , where δ = boundary layer thickness. Also calculate the value of $\delta*/\theta$.

Given:

Velocity distribution $\frac{u}{U} = \frac{y}{\delta}$

(i) Displacement thickness δ* is given by equation,

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dy = \int_0^\delta \left(1 - \frac{y}{\delta}\right) dy \qquad \{\because \frac{u}{U} = \frac{y}{\delta}\}$$

$$= \left[y - \frac{y^2}{2\delta}\right]_0^\delta \qquad \{\delta\ is\ constant\ across\ a\ section\}$$

$$= \delta - \frac{\delta^2}{2\delta} = \delta - \frac{\delta}{2}$$

$$\boldsymbol{\delta^* = \frac{\delta}{2}}$$

(ii) Momentum thickness, θ is given by equation,

$$\theta = \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$$

Substituting the value of $\frac{u}{U} = \frac{y}{\delta}$,

$$\theta = \int_0^\delta \frac{y}{\delta}\left(1 - \frac{y}{\delta}\right) dy = \int_0^\delta \left(\frac{y}{\delta} - \frac{y^2}{\delta^2}\right) dy$$

$$= \left[\frac{y^2}{2\delta} - \frac{y^3}{3\delta^2}\right]_0^\delta = \frac{\delta^2}{2\delta} - \frac{\delta^3}{3\delta^2}$$

$$= \frac{\delta}{2} - \frac{\delta}{3} = \frac{3\delta - 2\delta}{6}$$

$$\boldsymbol{\theta = \frac{\delta}{6}}$$

(iii) Energy thickness δ ** is given by equation, as

$$\delta^{**} = \int_0^\delta \frac{u}{U}\left(1 - \frac{u^2}{U^2}\right) dy = \int_0^\delta \frac{y}{\delta}\left(1 - \frac{y^2}{\delta^2}\right) dy \qquad \{\because \frac{u}{U} = \frac{y}{\delta}\}$$

$$= \int_0^\delta \left[\frac{y}{\delta} - \frac{y^3}{\delta^3}\right] dy = \left[\frac{y^2}{2\delta} - \frac{y^4}{4\delta^3}\right]_0^\delta = \frac{\delta^2}{2\delta} - \frac{\delta^4}{4\delta^3}$$

$$= \frac{\delta}{2} - \frac{\delta}{4} = \frac{2\delta - \delta}{4} = \frac{\delta}{4}$$

(iv)

$$\frac{\delta^*}{\theta} = \frac{\left(\frac{\delta}{2}\right)}{\left(\frac{\delta}{6}\right)} = \frac{\delta}{2} \times \frac{6}{\delta}$$

$$\boldsymbol{\frac{\delta^*}{\theta} = 3}$$

Find the displacement thickness, the momentum thickness and energy thickness for the velocity distribution in the boundary layer given by $\frac{u}{U} = 2\left[\frac{y}{\delta}\right] - \left[\frac{y}{\delta}\right]^2$

Solution: Given:

Velocity distribution $\frac{u}{U} = 2\left[\frac{y}{\delta}\right] - \left[\frac{y}{\delta}\right]^2$

(i) Displacement thickness δ * is given by equation,

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dy$$

Substituting the value of $\frac{u}{U} = 2\left[\frac{y}{\delta}\right] - \left[\frac{y}{\delta}\right]^2$, we have

$$\delta^* = \int_0^\delta \left\{1 - \left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right]\right\} dy$$

$$= \int_0^\delta \left\{1 - 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right\} dy = \left[\frac{2y^2}{2\delta} + \frac{y^3}{3\delta^2}\right]_0^\delta$$

$$= \delta - \frac{\delta^2}{\delta} + \frac{\delta^3}{3\delta^2} = \delta - \delta + \frac{\delta}{3}$$

$$\boldsymbol{\delta^* = \frac{\delta}{3}}$$

(ii) Momentum thickness θ, is given by equation,

$$\theta = \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy = \int_0^\delta \left(\frac{2y}{\delta} - \frac{y^2}{2\delta}\right)\left[1 - \left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)\right] dy$$

$$= \int_0^\delta \left[\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right]\left[1 - \frac{2y}{\delta} + \frac{y^2}{\delta^2}\right] dy$$

$$= \int_0^\delta \left[\frac{2y}{\delta} - \frac{4y^2}{\delta^2} + \frac{2y^3}{\delta^3} - \frac{y^2}{\delta^2} - \frac{2y^3}{\delta^3} - \frac{y^4}{\delta^4}\right] dy$$

$$= \int_0^\delta \left[\frac{2y}{\delta} - \frac{5y^2}{\delta^2} + \frac{4y^3}{\delta^3} - \frac{y^4}{\delta^4}\right] dy = \left[\frac{2y^2}{2\delta} - \frac{5y^3}{3\delta^2} + \frac{4y^4}{4\delta^3} - \frac{y^5}{5\delta^4}\right]_0^\delta$$

$$= \left[\frac{\delta^2}{\delta} - \frac{5\delta^3}{3\delta^2} + \frac{\delta^4}{\delta^3} - \frac{\delta^5}{5\delta^4}\right] = \delta - \frac{5\delta}{\delta} + \delta - \frac{5\delta}{\delta}$$

$$= \frac{15\delta - 25\delta + 15\delta - 3\delta}{15} = \frac{30\delta - 28\delta}{15}$$

$$\boldsymbol{\theta = \frac{2\delta}{15}}$$

(ii) (iii) Energy thickness δ is given by equation,**

$$\delta^{**} = \int_0^\delta \frac{u}{U}\left(1 - \frac{u^2}{U^2}\right) dy = \int_0^\delta \left(\frac{2y}{\delta} - \frac{y^2}{2\delta}\right)\left(\left[1 - \left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)\right]^2\right) dy$$

$$= \int_0^\delta \left(\frac{2y}{\delta} - \frac{y^2}{2\delta}\right)\left(1 - \left[\frac{4y^2}{\delta^2} + \frac{y^4}{\delta^4} - \frac{4y^3}{\delta^3}\right]\right) dy$$

$$= \int_0^\delta \left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)\left(1 - \frac{4y^2}{\delta^2} - \frac{y^4}{\delta^4} - \frac{4y^3}{\delta^3}\right) dy$$

$$= \int_0^\delta \left(\frac{2y}{\delta} - \frac{8y^3}{\delta^3} - \frac{2y^5}{\delta^5} + \frac{8y^4}{\delta^4} - \frac{y^2}{\delta^2} + \frac{4y^4}{\delta^4} + \frac{y^6}{\delta^6} - \frac{4y^5}{\delta^5}\right) dy$$

$$= \int_0^\delta \left(\frac{2y}{\delta} - \frac{y^2}{\delta^2} - \frac{8y^3}{\delta^3} - \frac{12y^4}{\delta^4} - \frac{6y^5}{\delta^5} + \frac{y^6}{\delta^6}\right) dy$$

$$= \left(\frac{2y}{2\delta} - \frac{y^3}{3\delta^2} - \frac{8y^4}{4\delta^3} - \frac{12y^5}{5\delta^4} - \frac{6y^6}{6\delta^5} + \frac{y^7}{7\delta^6}\right)_0^\delta$$

$$= \frac{\delta^2}{\delta} - \frac{\delta^3}{3\delta^3} - \frac{2\delta^4}{\delta^3} + \frac{12\delta^5}{5\delta^5} - \frac{\delta^6}{\delta^5} + \frac{\delta^7}{7\delta^6}$$

$$= \delta - \frac{\delta}{3} - 2\delta + \frac{12}{5}\delta - \delta + \frac{\delta}{7}$$

$$= -2\delta - \frac{\delta}{3} + \frac{12}{5}\delta + \frac{\delta}{7}$$

$$= \frac{-210\delta - 35\delta + 252\delta + 15\delta}{105}$$

$$= \frac{-245\delta + 267\delta}{105}$$

$$\boldsymbol{\delta^{**} = \frac{22\delta}{105}}$$

DRAG FORCE ON A FLAT PLATE DUE TO BOUNDARY LAYER

Consider the flow of a fluid having free-stream velocity equal to U, over a thin plate a shown in Fig. The drag force on the plate can be determined if the velocity profile near the plate is known. Consider a small length Δ*x* of the plate at a distance of *x* from the leading edge as shown in Fig.(a). The enlarged view of the small length of the plate is shown in Fig.(b)

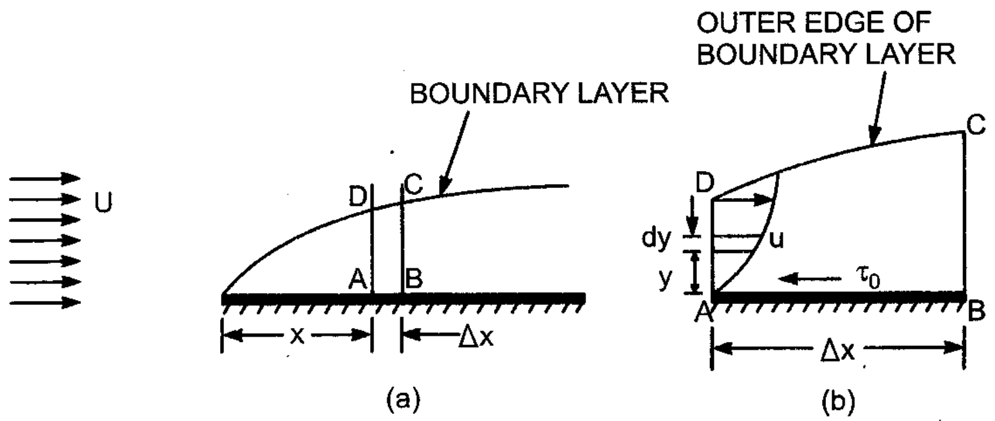


Fig. Drag force on a plate due to boundary layer

The shear stress τ_0 is given by τ_0 = $\left(\frac{du}{dy}\right)_{y=0}$, where $\left(\frac{du}{dy}\right)_{y=0}$ is the velocity distribution near the plate at y=0.

The drag force or shear force on a small distance Δ_x is given by

ΔF_D = shear stress x area

= τ_0 x Δ_x x b …………(1) [Taking width of plate = b]

Where ΔF_D=drag force on distance Δ_x

The drag force ΔF_D must also be equal to the rate of change of momentum over the distance Δ_x.
Consider the flow over the small distance Δ_x . Let ABCD is the control volume of the fluid over the distance Δ_x as shown in Fig. (b). The edge DC represents the outer edge of the boundary layer.

Let u = velocity at any point within the boundary layer

b = width of plate

Then mass rate of flow entering through the side AD

$$= \int_0^\delta \rho \text{ x velocity x area of strip of thickness dy}$$

$$= \int_0^\delta \rho \text{ x u x b x dy} \qquad [\because Area\ of\ strip = b\text{x}dy]$$

$$= \int_0^\delta \rho \text{ub.dy}$$

Mass rate of flow leaving the side BC

= mass through AD + $\frac{\partial}{\partial x}$ (mass through AD) x Δ_x

$$= \int_0^\delta \rho\text{ubdy} \frac{\partial}{\partial x}\left[\int_0^\delta (\rho\text{ubdy})\right] \text{ x } \Delta_x$$

From continuity equation for a steady incompressible fluid flow, we have

Mass rate of flow entering AD + mass rate of flow entering DC

= mass rate of flow leaving BC

∵ Mass rate of flow entering DC = mass rate of flow through BC-mass rate of flow through AD

$$= \int_0^\delta \rho \text{ubdy} + \frac{\partial}{\partial x}\left[\int_0^\delta (\rho \text{ubdy})\right] \text{x } \Delta_x \text{ - } \int_0^\delta \rho \text{ubdy}$$

$$= \frac{\partial}{\partial x}\left[\int_0^\delta (\rho \text{ubdy})\right] \text{x } \Delta_x$$

The fluid is entering through side DC with a uniform velocity U.

Now let us calculate momentum flux through control volume.

Momentum flux entering through AD

$$= \int_0^\delta \text{momentum flux through strip of thickness by}$$

$$= \int_0^\delta \text{mass through strip x velocity } = \int_0^\delta (\rho \text{ubdy}) \text{ xu} = \int_0^\delta (\rho \text{u}^2 \text{bdy})$$

Momentum flux leaving the side BC = $\int_0^\delta \rho \text{u}^2\text{bdy} + \frac{\partial}{\partial x}\left[\int_0^\delta \rho \text{u}^2\text{bdy}\right] \text{x } \Delta_x$

Momentum flux entering the side DC = mass rate through DC x velocity

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \rho \text{ubdy}\right] \text{x } \Delta_x \text{ x U} \qquad (\because \text{Velocity = U})$$

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \rho \text{ubdy}\right] \text{x } \Delta_x$$

As U is constant and so it can be taken inside the differential and integral

∴ ***Rate of change of momentum of the control volume***

= Momentum flux through BC – Momentum flux through AD – momentum flux through DC

$$= \int_0^\delta \rho \text{u}^2\text{bdy} + \frac{\partial}{\partial x}\left[\int_0^\delta \rho \text{u}^2\text{bdy}\right] \text{x } \Delta_x \text{ - } \int_0^\delta \rho \text{u}^2\text{bdy - } \frac{\partial}{\partial x}\left[\int_0^\delta \rho \text{uUbdy}\right] \text{x } \Delta_x$$

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \rho \text{u}^2\text{bdy}\right] \text{x } \Delta_x \text{ - } \frac{\partial}{\partial x}\left[\int_0^\delta \rho \text{uUbdy}\right] \text{x } \Delta_x$$

$$= \frac{\partial}{\partial x}\left[\int_0^\delta \rho \text{u}^2\text{bdy}\right] \text{ - } \left[\int_0^\delta \rho \text{uUbdy}\right] \text{x } \Delta_x$$

$$= \frac{\partial}{\partial x}\left[\int_0^\delta (\rho \text{u}^2\text{b} - \rho \text{uUb})\text{dy}\right] \text{x } \Delta_x$$

$$= \frac{\partial}{\partial x}\left[\rho \text{b} \int_0^\delta (\text{u}^2 - \text{uU})\text{dy}\right] \text{x } \Delta_x$$

{For incompressible fluid ρ is constant}

$$= \rho \text{b } \frac{\partial}{\partial x}\left[\int_0^\delta (\text{u}^2 - \text{uU})\text{dy}\right] \text{x } \Delta_x \qquad \text{……….(2)}$$

Now the rate of change of momentum on the control volume ABCD must be equal to the total force on the control volume in the same direction according to the momentum principle.

But for a flat plate $\frac{\partial p}{\partial x}$=0.

Which means there is no external pressure force on the control volume. Also the force on the side DC is negligible as the velocity is constant and velocity gradient is zero approximately.

The only external force acting on the control volume is the shear force acting on the side AB in the direction from B to A as shown in Fig.(b). The value of this force is given by equation (1) as,

$$\Delta F_D = \tau_0 \text{ x } \Delta x \text{ x b}$$

∴ The external force in the direction of rate of change of momentum

$$= \text{ - } \tau_0 \text{ x } \Delta x \text{ x b} \qquad \text{………….(3)}$$

According to momentum principle, the two values given by equations (3) and (2) should be the same.

$\therefore \quad -\tau_0 \text{ x } \Delta x \text{ x b} = \rho b \frac{\partial}{\partial x}\left[\int_0^\delta (u^2 - uU)dy\right] \text{ x } \Delta_x$

Cancelling Δx x b , to both sides, we have

$$-\tau_0 = \rho \frac{\partial}{\partial x}\left[\int_0^\delta (u^2 - uU)dy\right]$$

or

$$\tau_0 = -\rho \frac{\partial}{\partial x}\left[\int_0^\delta (u^2 - uU)dy\right] = \rho \frac{\partial}{\partial x}\left[\int_0^\delta (uU - u^2)dy\right]$$

$$= \rho \frac{\partial}{\partial x}\left[\int_0^\delta U^2\left(\frac{u}{U} - \frac{u^2}{U^2}\right)dy\right]$$

$$= \rho U^2 \frac{\partial}{\partial x}\left[\int_0^\delta \frac{u}{U}\left[1 - \frac{u}{U}\right]dy\right]$$

$$\frac{\tau_0}{\rho U^2} = \frac{\partial}{\partial x}\left[\int_0^\delta \frac{u}{U}\left[1 - \frac{u}{U}\right]dy\right] \qquad \text{......... (4)}$$

In equation (4), the expression $\int_0^\delta \frac{u}{U}\left[1 - \frac{u}{U}\right]$ dy is equal to momentum thickness θ. Hence equation (4) is also written as

$$\frac{\tau_0}{\rho U^2} = \frac{\partial \theta}{\partial x} \qquad \text{..........(5)}$$

Equation (5) is known as ***Von Karman momentum integral equation*** for boundary layer flows.

This is applied to:

1. Laminar boundary layers
2. Transition boundary layers and
3. Turbulent boundary layer flows.

For a given velocity profile in laminar zone, transition zone or turbulent zone of a boundary layer, the shear stress τ_0 is obtained from equation (4) or (5). Then drag force on a small distance Δ_x of the plate is obtained from equation (1) as

$$\Delta F_D = -\tau_0 \text{ x } \Delta x \text{ x b}$$

Then total drag on the plate of length L on one side is

$$F_D = \int \Delta F_D = \int_0^L \tau_0 \text{ x b x } dx \qquad \{\text{change } \Delta x = dx\} \qquad \text{........(6)}$$

Local Co-efficient of Drag [$C_D *$].

It is defined as the ratio of the shear stress τ_0 to the quantity $\frac{1}{2} \rho U^2$. It is denoted by C_D^*

Hence $\quad C_D^* = \frac{\tau_0}{\frac{1}{2}\rho U^2}$ ....(7)

Where A = Area of the surface (or plate)

U = Free-stream velocity

ρ = Mass density of fluid

Boundary Conditions for the Velocity Profiles. The followings are the boundary conditions which must be satisfied by any velocity profile, whether it is in laminar boundary layer zone, or in turbulent boundary layer zone:

1. At y=0, u=0 and $\frac{du}{dy}$ has some finite value
2. At y=δ, u=U
3. At y=δ, $\frac{du}{dy}$=0

For the velocity profile given in, find the thickness of boundary layer at the end of the plate and the drag force on one side of a plate 1m long and 0.8m wide when placed in water flowing with a velocity of 150 mm per second. Calculate the value of co-efficient of drag also. Take μ for water = 0.01 poise.

Given:

Length of plate, L = 1m

Width of plate, b = 0.8m

Velocity of fluid (water) U = 150mm/s = 0.15 m/s

μ for water μ = 0.01 poise = $\frac{0.01}{10}\frac{Ns}{m^2}$ = 0.001 $\frac{Ns}{m^2}$

Solution:

Reynold number at the end of the plate i.e., at a distance of 1m from leading edge is given by

$$R_{e_L} = \frac{\rho UL}{\mu} = 1000 \times \frac{0.15 \times 1.0}{.001} \qquad (\because \rho = 1000)$$

$$= \frac{1000 \times 0.15 \times 1.0}{0.001} = 150000$$

(i) As laminar boundary layer exists upto Reynold number = 2 x 10^5. Hence this is the case of laminar boundary layer. Thickness of boundary layer at x = 1.0m is given by equation as,

$$\delta = 5.48 \frac{x}{\sqrt{R_{e_L}}} = \frac{5.48 \times 1.0}{\sqrt{150000}} = 0.01415\text{m} = 14.15\text{mm}.$$

(ii) Drag force on one side of the plate is given by equation,

$$F_D = 0.73\, b\mu U\sqrt{\frac{\rho UL}{\mu}}$$

$$= 0.73 \times 0.8 \times 0.001 \times 0.15 \times \sqrt{150000} \qquad \{\because \frac{\rho UL}{\mu} = R_{e_L}\}$$

$$\mathbf{F_D = 0.0338N.}$$

(iii) Co-efficient of drag. C_D is given by equation as,

$$C_D = \frac{1.46}{\sqrt{R_{e_L}}} = \frac{1.46}{\sqrt{150000}} = 0.00376$$

$$\mathbf{C_D = 0.00376}$$

S.No	Velocity Distribution	δ	C_D
1	$\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$	$5.48\ x/\sqrt{R_{e_x}}$	$1.46/\sqrt{R_{e_L}}$
2	$\frac{u}{U} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$	$4.64\ x/\sqrt{R_{e_x}}$	$1.292/\sqrt{R_{e_L}}$
3	$\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - 2\left(\frac{y}{\delta}\right)^3 + \left(\frac{y}{\delta}\right)^4$	$5.84\ x/\sqrt{R_{e_x}}$	$1.36/\sqrt{R_{e_L}}$
4	$\frac{u}{U} = sin\left(\frac{\pi}{2}\ \frac{y}{\delta}\right)$	$4.79\ x/\sqrt{R_{e_x}}$	$1.31/\sqrt{R_{e_L}}$
5	Blasius's Solution	$4.91\ x/\sqrt{R_{e_x}}$	$1.328/\sqrt{R_{e_L}}$

For the velocity profile in laminar boundary layer as, $\frac{u}{U} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$ find the thickness of the boundary layer and the shear stress 1.5m from the leading edge of a plate. The plate is 2m long and 1.4 m wide and is placed in water which is moving with a velocity of 200 mm per second. Find the total drag force on the plate if μ for water = 0.01 poise.

Given:

Velocity profile is $$\frac{u}{U} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$$

Distance of *x* from leading edge, x = 1.5m

Length of plate, L = 2m

Width of plate, b = 1.4m

Velocity of plate U = 200 mm/s = 0.2 m/s

Viscosity of water, μ = 0.01 poise = $\frac{0.01}{10}$ = 0.001 Ns/m^2

Solution:

For the given velocity profile, thickness of boundary layer is given by equation as

$$\delta = \frac{4.46\ x}{\sqrt{R_{e_x}}}$$

[Here, $R_{e_x} = \frac{\rho U x}{\mu}$ = 1000 x $\frac{0.2x1.5}{0.001}$ = 300000]

$$\delta = \frac{4.46\ x\ 1.5}{\sqrt{300000}} = 0.0127\ m$$

δ = 12.27 mm.

Shear stress (τ_0) is given by $\tau_0 = 0.323\ \frac{Ux}{x}\sqrt{R_{e_x}}$

$$= 0.323 \times 0.001 \times \frac{0.2}{1.5}\sqrt{300000}$$

τ_0 = 0.0235 N/m^2.

Drag Force (F_D) on the side of the plate is given by as

$$F_D = 0.646\, \mu U \sqrt{\frac{\rho U L}{\mu}} \times b$$

$$= 0.646 \times 0.001 \times 0.2 \times \sqrt{1000 \times \frac{0.2 x 2.0}{0.001}} \times 1.4$$

$$= .646 \times 0.001 \times 0.2 \times \sqrt{400000} \times 1.4 = 0.1138N$$

$$F_D = 0.1138N$$

∴ **Total drag force** = Drag force on both sides of the plate

= 2x0.1138 = 0.2276 N

Total drag force = 0.2276 N

Air is flowing over a smooth plate with a velocity of 10 m/s. The length of the plate is 1.2 m and width 0.8 m. If laminator boundary layer exists up to a value of R_e = 2 x 10^5, find the maximum distance from the leading edge upto which laminar boundary layer exists. Find the maximum thickness of laminar boundary layer if the velocity profile is given by $\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$. Take kinematic viscosity for air = 0.15 strokes.

Given:

Velocity of air, U = 10 m/s

Length of plate, L = 1.2 m

Width of plate, b = 0.8 m

Reynold number upto which laminar boundary exists = 2 x 10^5

υ for air = 0.15 stokes = 0.5x10^{-4} m^2/s

ρ = 1.24 kg/m^3

Solution:

If R_{e_x} =2x10^5 , then x denotes the distance from leading edge upto which laminar boundary layer exists

∴ $$2x10^5 = \frac{10\ x\ x}{0.15 x 10^{-4}}$$

∴ $$x = \frac{2x10^5 x\ 0.15x10^{-4}}{10} = 0.30\ m$$

x = 300mm.

Maximum thickness of the laminar boundary for the velocity profile, $\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$ is given by equation as

$$\delta = \frac{5.48 x\ x}{\sqrt{R_{e_x}}} = \frac{5.48 x 0.30}{\sqrt{2x10^5}}$$

= 0.00367 m

$\boldsymbol{\delta}$ **= 3.67mm**

Air is flowing over a flat plate 500 mm long and 600 mm wide with a velocity of 4 m/s. The kinematic viscosity of air is given as 0.15x10^{-4} m^2/s^2. Find (i) the boundary layer thickness at the end of the plate, (ii) Shear stress at 200mm from the leading edge and (iii) drag force on one side of the plate. Take the velocity profile over the plate as $\frac{u}{U}$ = sin$\left(\frac{\pi}{2}\ ,\frac{y}{\delta}\right)$ and density of air 1.24 kg/m^3.

Given:

Length of plate, L = 500 mm = 0.5m

Width of plate, b = 600 mm = 0.6 m

Velocity of air, U = 4 m/s

Kinematic viscosity, υ = 0.15 x 10^{-4} m^2/s

Mass density, ρ = 1.24 kg/m^3

For the velocity profile, $\frac{u}{U}$ = sin$\left(\frac{\pi}{2}\ ,\frac{y}{\delta}\right)$, we have

Solution:

(i) Boundary layer thickness at the end of the plate means value of δ at x = 0.5m.

First find Reynold number.

$$R_{e_x} = \frac{\rho U x}{\mu} = \frac{Ux}{\upsilon} = \frac{4x0.5}{0.15x10^{-4}} = 1.33x10^5.$$

Hence boundary layer is laminar over the entire length of the plate as Reynold number at the end of the plate is 1.33x10^5.

∴ δ at x = 0.5 m for the given velocity profile is given by equation as

$$\delta = \frac{4.795\ x}{\sqrt{R_{e_x}}} = \frac{4.795x0.5}{\sqrt{1.33x10^5}} = 0.00656 \text{ m} = 6.56 \text{ mm}.$$

(ii) Shear stress at any distance from leading edge is given by

$$\boldsymbol{\tau_0 = 0.327\,\frac{\mu U}{x}\sqrt{R_{e_x}}}$$

At x =200 mm = 0.2 m, $R_{e_x} = \frac{U\ x\ x}{\upsilon} = \frac{4x0.2}{0.15x10^{-4}}$ = 1.33x10^5 = 53333

$$\tau_0 = \frac{0.327 \text{ x } \mu \text{ x } 4 \text{ x } \sqrt{53333}}{0.2}$$

But $\mu = \upsilon \text{ x } \rho$ $\{\because\ \upsilon = \frac{u}{\rho},\ \therefore \mu = \upsilon x \rho\}$

= 0.15 x 10^{-4} x 1.24 = 0.186x10^{-4}

$$\tau_0 = \frac{0.327 \text{ x } 0.186 \text{ x } 10^{-4} \text{ x } 4 \text{ x } \sqrt{53333}}{0.2} = 0.02805 \text{ N/m}^2$$

(iii) Drag force on one side of the plate is given by equation

$$F_D = 0.655 \text{ x } \mu U \text{ x b x } \sqrt{\frac{\rho U L}{\upsilon}}$$

$$= 0.655 \times 0.186 \times 10^{-4} \times 4.0 \times 0.6 \times \sqrt{\frac{UL}{\upsilon}} \qquad \{\because \upsilon = \frac{u}{\rho}\}$$

$$= 0.29234 \times 10^{-4} \times \sqrt{\frac{4 \times 0.5}{.15 \times 10^{-4}}}$$

F_D **= 0.01086 N**

A thin plate is moving in still atmospheric air at a velocity of 5 m/s. The length of the plate is 0.6m and width 0.5m. Calculate (i) the thickness of the boundary layer at the end of the plate, and (ii) drag force on one side of the plate. Take density of air as 1.24 kg/m^3 and kinematic viscosity 0.15 stokes.

Given:

Velocity of air, $U = 5$ m/s

Length of plate, $L = 0.6$ m

Width of plate, $b = 0.5$ m

Density of air, $\rho = 1.24$ kg/m^3

Kinematic viscosity, $\upsilon = 0.15$ stokes $= 0.15 \times 10^{-4}$ m^2/s

Solution:

Reynold number, $R_e = \frac{UL}{\upsilon} = \frac{5x0.6}{0.15x10^{-4}}$ **= 200000.**

As R_e is less than 5×10^5 , hence boundary layer is laminar over the entire length of the plate.

(i) Thickness of boundary layer at the end of the plate by Blasius's solution is

$$\delta = \frac{4.91\,x}{\sqrt{R_{e_x}}} = \frac{4.91\,L}{\sqrt{R_{e_x}}} = \frac{4.91\,x\,0.6}{\sqrt{200000}}$$

= .00658 m

$\boldsymbol{\delta}$ **= 6.58 mm**

(iii) Drag force on one side of the plate is given by equation as,

$$\mathbf{F_D = \frac{1}{2} \rho A U^2 \times C_D}$$

therefoere , $C_D = \frac{F_D}{\frac{1}{2}\,\rho\,A\,U^2}$

Where C_D from Blasius's solution, $C_D = \frac{1.328}{\sqrt{R_{e_L}}} = \frac{1.328}{\sqrt{200000}}$

= 0.002969

$\boldsymbol{C_D}$ **= .00297**

$F_D = \frac{1}{2} \rho A U^2 \times C_D$

$= \frac{1}{2} \times 1.24 \times 0.6 \times 0.5 \times 5^2 \times .002970$

$\mathbf{F_D}$ **= 0.013773 N.**

Note. *If no velocity profile is given in the numerical problem but boundary layer is laminar, then Blasius's solution is used.*

A plate of 600 mm length and 400 mm wide is immersed in a fluid of sp.gr. 0.9 and kinematic viscosity (υ=) 10^{-4} m^2/s. The fluid is moving with a velocity of 6 m/s. Determine (i) boundary layer thickness (ii) shear stress at the end of the plate, and (iii) drag force on one side of the plate.

As no velocity profile is given in the above problem, hence Blasius's solution will be used.

Given:

length of plate, L = 600 mm = 0.60 m

Width of plate, b = 400 mm = 0.40 m

Sp.gr.of fluid, S = 0.9

∴ Density, ρ = 0.9 x 1000 = 900 kg/m^3

Velocity of fluid U = 6 m/s

Kinematic viscosity υ = 10^{-4} m^2/s

Solution:

Reynold number, $R_{e_L} = \frac{U\ x\ L}{\upsilon} = \frac{6\ x\ 0.6}{10^{-4}}$ = 3.6 x 10^4.

As R_{e_L} is less than 5 x 10^5, hence boundary layer is laminar over the entire length of the plate.

(i) Thickness of boundary layer at the end of the plate from Blasius's solution is

$$\delta = \frac{4.91\ x}{\sqrt{R_{e_x}}}$$

where x = 0.6m and R_{e_x}= 3.6x10^4

$$= \frac{4.91\ x\ 0.6}{\sqrt{3.6\ x\ 10^{-4}}} = 0.0155 \text{ m}$$

$\boldsymbol{\delta}$ **= 15.5 mm**

(ii) Shear stress at the end of the plate is

$$\tau_0 = 0.33\frac{\rho U^2}{\sqrt{R_{e_L}}} = \frac{0.332\ x\ 900\ x\ 6^2}{\sqrt{3.6\ x\ 10^4}} = \mathbf{56.6\ N/m^2.}$$

(iii) Drag force ($\boldsymbol{F_D}$) on one side of the plate is given by

$$F_D = \frac{1}{2}\rho A U^2 \text{ x } C_D$$

Where from Blasius's solution is $C_D = \frac{1.328}{\sqrt{R_{e_L}}} = \frac{1.328}{\sqrt{3.6\ x\ 10^4}}$ = 0.00699

$$F_D = \frac{1}{2}\rho A U^2 \text{ x } C_D$$

$$= \frac{1}{2} \text{ x 900 x 0.6 x 0.4 x } 6^2 \text{ x .00699 } [\because A = Lxb = 0.6x.4]$$

F_D = 26.78 N
