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DEPARTMENT OF ELECTRONICS AND COMMUNICATION ENGINEERING

EC T43 SIGNALS AND SYSTEMS

II YEAR/ IV SEM

Syllabus

EC T43-SIGNALS AND SYSTEMS

COURSE OBJECTIVE

 To introduce the concepts of continuous time and discrete time signals and systems including their classification and properties.

 To comprehend and analyze the frequency domain representation of continuous time signals.

 To learn and investigate the different types of representing continuous time LTI systems and their properties.

 To comprehend and analyze the frequency domain representation of discrete time signals.

 To learn and investigate the different types of representing discrete time LTI systems and their properties

UNIT I

REPRESENTATION AND CLASSIFICATION OF SIGNALS AND SYSTEMS: Continuous time signals - Discrete time signals – Representation of signals – Step, Ramp, Pulse, Impulse, Sinusoidal, Exponential signals, Classification of continuous and discrete time signals -Operations on the signals. Continuous time and discrete time systems: Classification of systems – Properties of systems.

UNIT II

ANALYSIS OF CONTINUOUS TIME SIGNALS: Fourier series: Properties - Trigonometric and Exponential Fourier Series -Parsavel‘s relation for periodic signals - Fourier Transform: Properties - Rayleigh‘s Energy Theorem - Laplace Transformation: Properties, R.O.C -Inverse Laplace transform

UNIT III

ANALYSIS OF DISCRETE TIME SIGNALS: Discrete Time Fourier Transform: Properties; Z-Transformation: Properties – Different methods of finding Inverse Z-Transformation

UNIT IV

CONTINUOUS AND DISCRETE TIME SYSTEMS: LTI continuous time systems- Differential equations – Transfer function and Impulse response – Convolution Integral- Block diagram representation and reduction -State variable techniques – State equations
LTI Discrete time systems – Difference equations – System function and impulse response – Convolution Sum – Block diagram representation – Convolution Sum – State equations for discrete time systems

UNIT V

DISCRETE FOURIER TRANSFORM: DFT – Properties - FFT algorithms –advantages over direct computation of DFT – radix 2 algorithms – DIT and DIF algorithms – Computation of IDFT using FFT.

Text Books:

1. Simon Haykins and Barry Van Veen, ―Signals and Systems‖, Second Edition, John Wiley and Sons, 2002.
2. Allan V. Oppenheim, Allan S.Willsky and S.HamidNawab, ―Signals and Systems‖, Second Edition, PHI Learning, New Delhi, 2007.

Reference Books:

1. Douglas K. Lindner, ―Signals and Systems‖, McGraw-Hill International Edition, 1999.
2. P. Ramesh Babu, ―Signals and Systems‖, Fifth Edition, Scitech Publishers,2014.

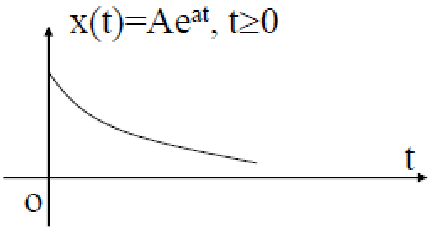
Web References:

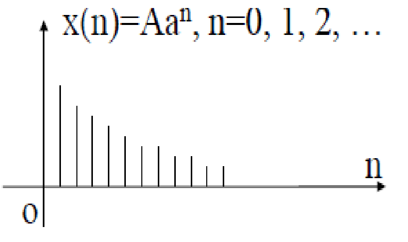
1.http://www.cdeep.iitb.ac.in/nptel/Electrical%20&%20Comm%20Engg/Signals%20and%20System/Course%20Objective.htm
2. http://ocw.mit.edu/resources/res-6-007-signals-and-systems-spring-2011/
3. http://www.ece.jhu.edu/~cooper/courses/214/signalsandsystemsnotes.pdf
4. http://techteach.no/publications/discretetime_signals_systems/discrete.pdf.

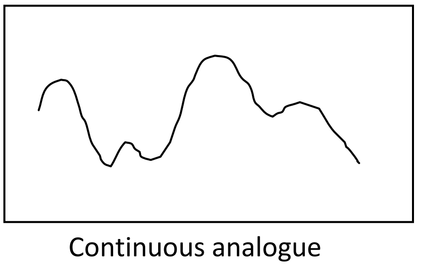
UNIT I

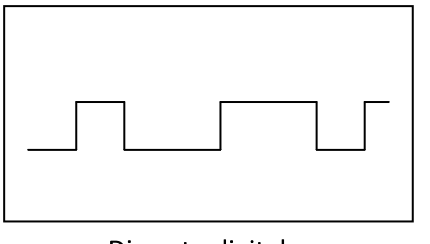
REPRESENTATION AND CLASSIFICATION OF SIGNALS AND SYSTEMS

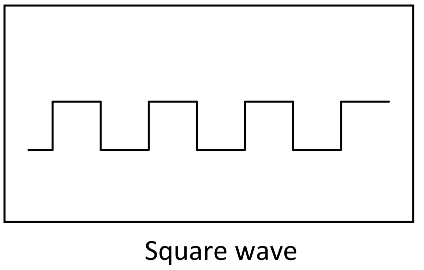
- A **Signal** is the function of one or more independent variables that carries some information to represent a physical phenomenon.
- A '**signal'** may be defined as a physical quantity which varies with time, space or any independent variable Example — voltage, current A 'system may be defined as a combination of devices and networks or subsystem chosen to do a desired action Example Electrical N/W, mechanical system
- A continuous-time signal, also called an analog signal, is defined along a continuum of time. Denoted by x(t)
- A discrete-time signal is defined at discrete times. Denoted by x(n)

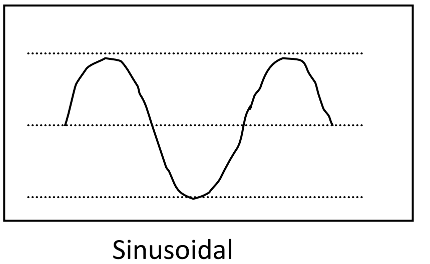


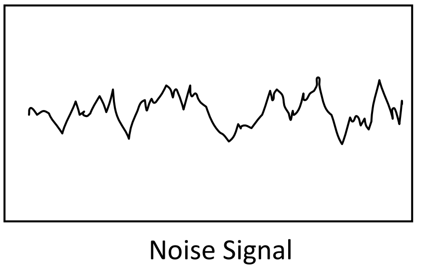


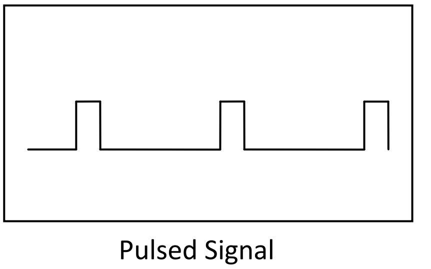

Continuous analogue


Discrete digital


Square wave


Sinusoidal


Noise Signal


Pulsed Signal

Representation of signals

I. Sinusoidal & Exponential Signals

Sinusoids and exponentials are important in signal and system analysis because they arise naturally in the solutions of the differential equations.

Sinusoidal Signals can expressed in either of two ways :

cyclic frequency *form*- $A \sin 2\pi f_o t = A \sin(2\pi / T_o)t$

radian frequency form- $A \sin \omega_o t$

$\omega_o = 2\pi f_o = 2\pi / T_o$

T_o= Time Period of the Sinusoidal Wave

$x(t) = A \sin(2\pi f_o t + \theta) = A \sin(\omega_o t + \theta)$

$x(t) = Ae^{at}$ **Real Exponential**

$= Ae^{j\omega_o t} = A[\cos(\omega_o t) + j \sin(\omega_o t)]$ **Complex Exponential**

θ = Phase of sinusoidal wave A = amplitude of a sinusoidal or exponential signal

f_o= fundamental cyclic frequency of sinusoidal signal ω_o = radian frequency

Discrete Sinusoidal

DT signals can be defined in a manner analogous to their continuous-time counter part

$x[n] = A \sin(2\pi n/N_o + \theta)$ ***SINUSOID***

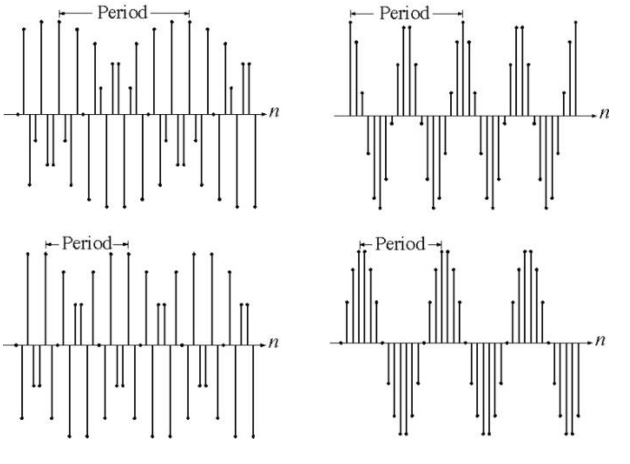
$= A \sin(2\pi F_o n + \theta)$

$x[n] = a^n$ ***EXPONENTIAL***

n = the discrete time *A = amplitude*

θ = phase shifting radians, *N_o = Discrete Period of the wave*

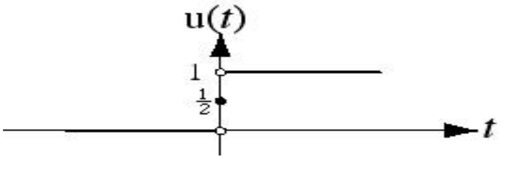
$1/N_0 = F_o = \Omega_o / 2\pi$ = Discrete Frequency



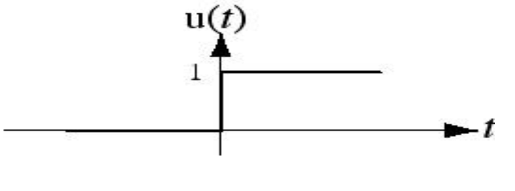
II. Unit Step Function

$$u(t) = \begin{cases} 1 & , \ t > 0 \\ 1/2 & , \ t = 0 \\ 0 & , \ t < 0 \end{cases}$$

Precise Graph

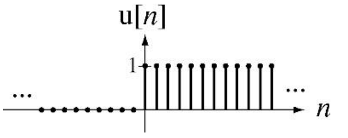


Commonly-Used Graph



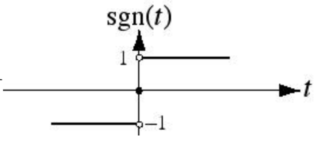
Discrete Unit step

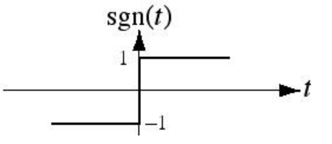
$$\mathrm{u}[n]=\begin{cases}1 \;, & n\geq 0\\ 0 \;, & n<0\end{cases}$$



III. Signum Function

$$\operatorname{sgn}(t)=\begin{Bmatrix}1 & , t>0\\ 0 & , t=0\\ -1 & , t<0\end{Bmatrix} \;=\; 2\mathrm{u}(t)-1$$

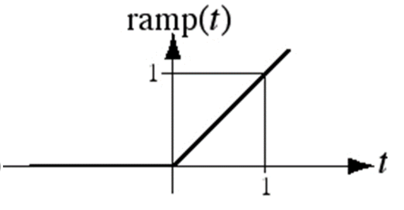




The signum function, is closely related to the unit-step function.

IV. Unit Ramp Function

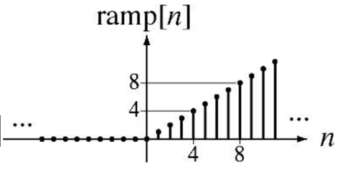
$$\operatorname{ramp}(t)=\begin{Bmatrix}t & , \; t>0\\ 0 & , \; t\leq 0\end{Bmatrix}=\int_{-\infty}^{t}\mathrm{u}(\lambda)\,d\lambda = t\,\mathrm{u}(t)$$



The unit ramp function is the integral of the unit step function.
It is called the unit ramp function because for positive t, its slope is one amplitude unit per time.

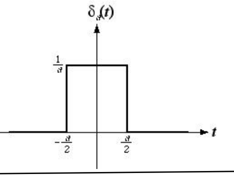
Discrete Ramp

$$\operatorname{ramp}[n]=\begin{Bmatrix}n & , \; n\geq 0\\ 0 & , \; n<0\end{Bmatrix}=\sum_{m=-\infty}^{n}\mathrm{u}[m-1]$$



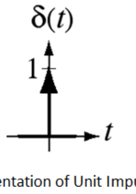
V. Rectangular Pulse or Gate Function

$$\delta_a(t)=\begin{cases}1/a & , \; |t|<a/2\\ 0 & , \; |t|>a/2\end{cases}$$

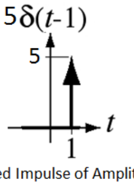


VI. Unit Impulse Function

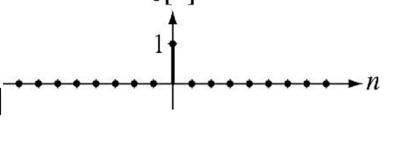
unit impulse function is the derivative of the unit step function or unit step is the integral of the unit impulse function.
The area under an impulse is called its strength or weight. It is represented graphically by a vertical arrow. An impulse with a strength of one is called a unit impulse.



Representation of Unit Impulse



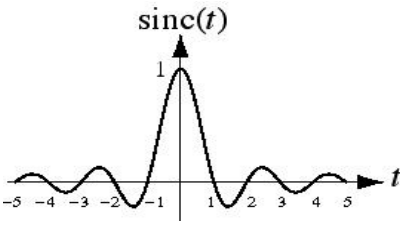
Shifted Impulse of Amplitude5



$$\text{Continuous} \quad \delta(t) = \begin{cases} 1 & for\ t = 0 \\ 0 & for\ t \neq 0 \end{cases} \quad | \ \text{Discrete} \quad \delta[n] = \begin{cases} 1\ , & n = 0 \\ 0\ , & n \neq 0 \end{cases}$$

VII. Sinc Function

$$\operatorname{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$



Operations of Signals

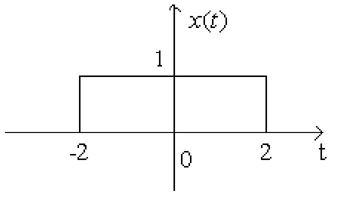
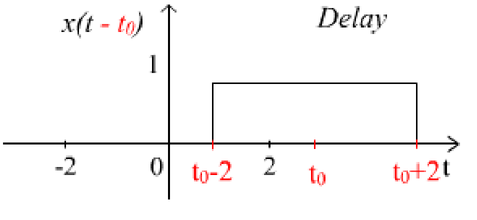
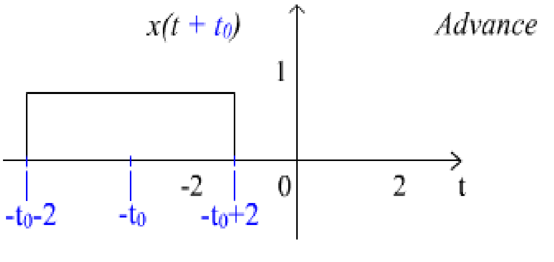
Sometime a given mathematical function may completely describe a signal.

Different operations are required for different purposes of arbitrary signals.

The operations on signals can be

Time Shifting
Time Scaling
Time Inversion or Time Folding

Time Shifting

The original signal $x(t)$ is shifted by an amount t_o.  X(t)→X(t-to) →Signal Delayed→ Shift to the **right** 	X(t)→X(t+to) →Signal Advanced→ Shift to the **left** 

Time Scaling

- For the given function x(t), x(at) is the time scaled version of x(t)
- For a > 1,period of function x(t) reduces and function speeds up. Graph of the function shrinks.
- For a < 1, the period of the x(t) increases and the function slows down. Graph of the function expands.

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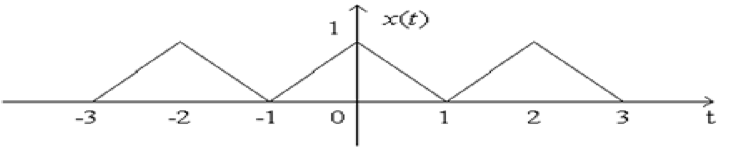
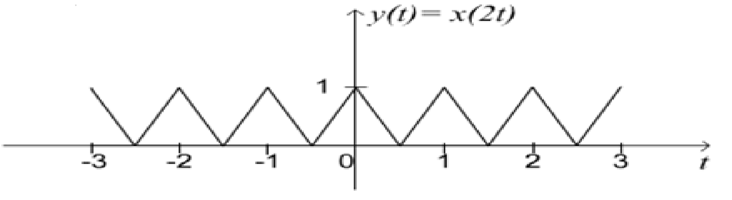
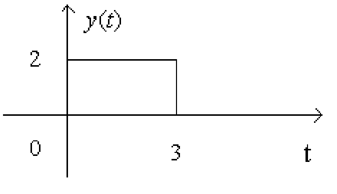
Example: Given *x*(*t*) and we are to find *y*(*t*) = *x*(2*t*).

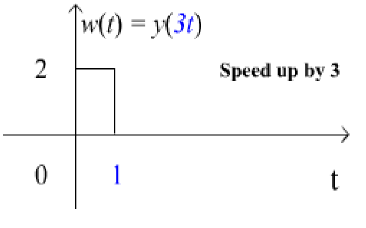


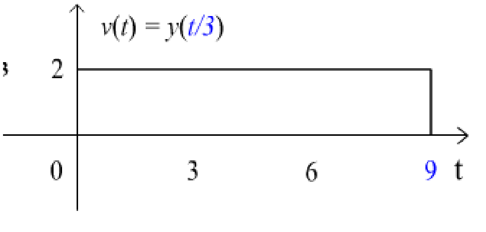


The period of *x*(*t*) is 2 and the period of *y*(*t*) is 1,

Given *y*(*t*), find *w*(*t*) = *y*(3*t*) *and v*(*t*) = *y*(*t*/3).



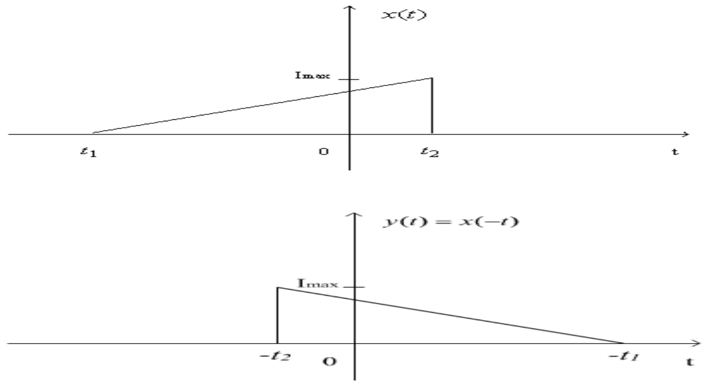




Time Reversal

Time reversal is also called time folding

In Time reversal signal is reversed with respect to time i.e. y(t) = x(-t) is obtained for the given function



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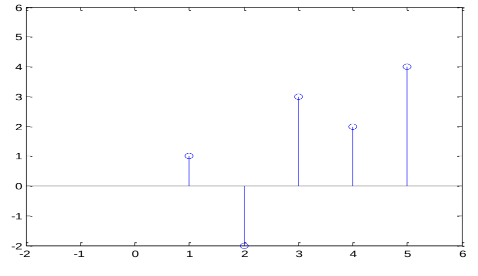
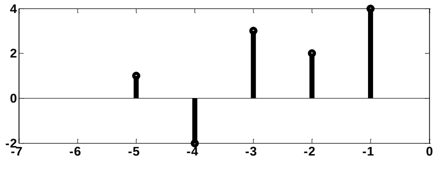
Consider a signal x(n)



X(-n)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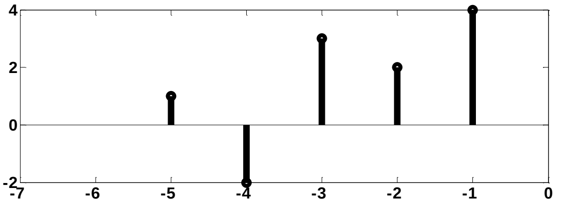
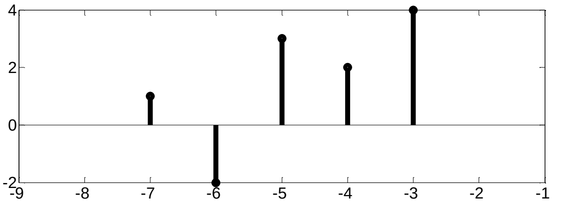
NOTE:

X(-n + 2) Shift right



X(-n-2) shift left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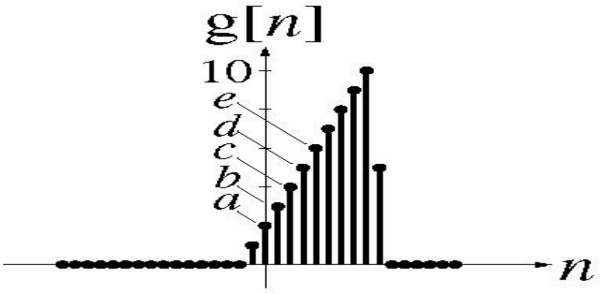
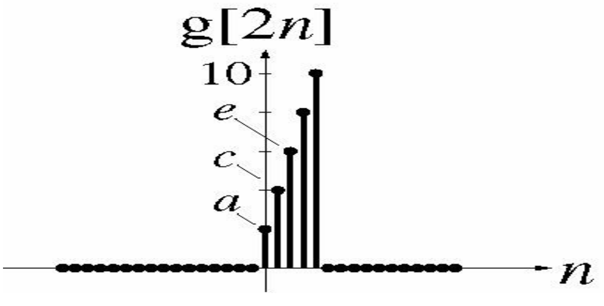
Scaling; Signal Compression

$n \rightarrow Kn$ $\quad\quad$ K an integer > 1





n	$2n$	$g[2n]$
0	0	2
1	2	4
2	4	6
3	6	8
4	8	10

, ***<u>Classification of continuous and discrete time signals</u>***

There are various types of signals Every signal is having its own characteristic The processing of signal mainly depends on the characteristics of that particular signal So classification of signal is necessary Broadly the signal are classified as follows

1 Continuous and discrete time signals

2. Continuous valued and discrete valued signals.

3. Periodic and non periodic signals.

4 Even and odd signals

5. Energy and power signals:

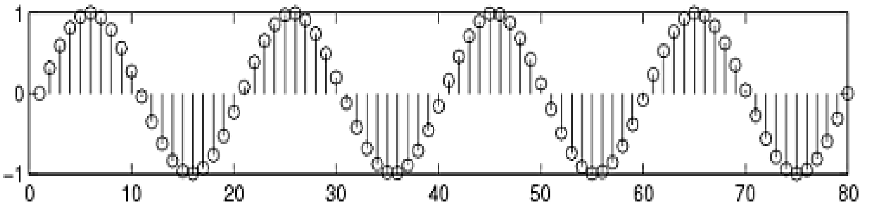
6 Deterministic and random signals

1. **Deterministic & Non Deterministic Signals**

Deterministic signals

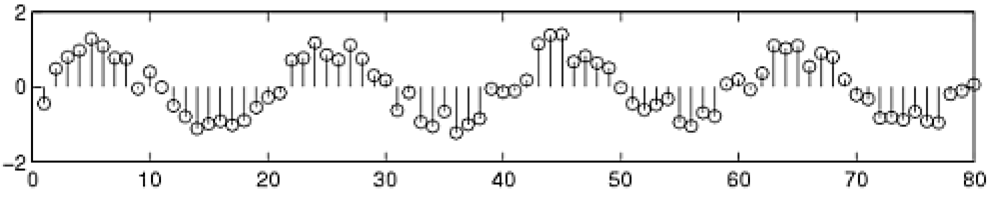
Behavior of these signals is predictable w.r.t time. There is no uncertainty with respect to its value at any time. These signals can be expressed mathematically.

For example $x(t) = \sin(3t)$ is deterministic signal.



Non Deterministic or Random signals

- Behavior of these signals is random i.e. not predictable w.r.t time.
- There is an uncertainty with respect to its value at any time.
- These signals can't be expressed mathematically.
- For example Thermal Noise generated is non deterministic signal.



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2. Periodic and Non-periodic Signals

Given x(t) is a continuous-time signal x (t) is periodic iff $x(t) = x(t+T_o)$ for any T and any integer n.

Example $x(t) = A\cos(wt)$

$x(t+T_o) = A\cos[w(t+T_o)] = A\cos(wt+wT_o)= A\cos(wt+2p) = A\cos(wt)$

Note: $T_o = 1/f_o$; $w=2pf_o$

For non-periodic signals $x(t) \neq x(t+T_o)$

A non-periodic signal is assumed to have a period $T = \infty$

Example of non periodic signal is an exponential signal.

A discrete time signal is periodic if $x(n) = x(n+N)$

For satisfying the above condition the frequency of the discrete time signal should be ratio of two integers

i.e. $f_o = k/N$

Sum of periodic Signals

$X(t) = x1(t) + X2(t)$

$X(t+T) = x1(t+m_1T_1) + X2(t+m_2T_2)$

$m_1T_1=m_2T_2 = T_o$ = Fundamental period

Example: cos(tp/3)+sin(tp/4)

- T1=(2p)/(p/3)=6; T2 =(2p)/(p/4)=8;
- T1/T2=6/8 = ¾ = (rational number) = m2/m1
- $m_1T_1=m_2T_2 \rightarrow$ Find m1 and m2$\rightarrow$
- $6.4 = 3.8 = 24 = T_o$

Sum of periodic Signals – may not always be periodic!

$$x(t) = x_1(t) + x_2(t) = \cos t + \sin\sqrt{2}t$$

T1=(2p)/(1)= 2p; T2 =(2p)/(sqrt(2));

T1/T2= sqrt(2);

- Note: T1/T2 = sqrt(2) is an irrational number
  - X(t) is aperiodic

1.**Determine whether or not each of the following signals is periodic .If the signal is periodic ,specify its fundamental period.**

a) $x(t) = je^{j10t}$

Now for periodicity

$x(t) = je^{j10t} = je^{j10(t + T)}$

so; $e^{j10t} = 1$

we know $e^{j10T} = \cos10T + j\sin10T$

Here;

We need $\cos10T = 1$ and $\sin10T = 0$

For $\cos10T = 1$,from trigonometry $10T = 0, 2\pi, 4\pi, 6\pi, \ldots\ldots$

But we cannot take zero because if we take 0,T becomes zero which is not true

So;

$10T = 2\pi, 4\pi, 6\pi, \ldots\ldots$ (since $10T = 2\pi \times n$)

For fundamental Period;

$10T=2\pi$

$T=2\pi /10$

$T=\pi /5$

b) $x(t) = e^{(-1+j)t}$

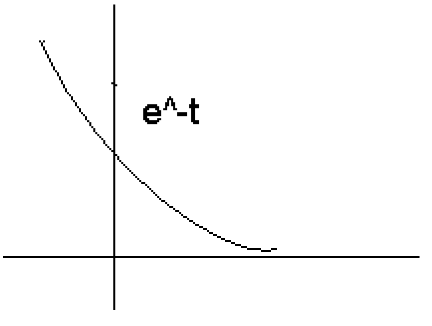
For periodicity;

$e^{(-1+j)t} = e^{(-1+j)(t+T)}$

So ; $e^{(-1+j)T} = 1$

$e^{-T}e^{jT} =1$

Since e^-t is a decaying exponential and e^{jt} is periodic ,the signal is non periodic



c)$x[n] = 3e^{3\pi (n + ½)/5}$

Now;

$x[n] = 3e^{j3\pi \{(n +N) + ½\}/5}$

$e^{j3\pi N/5} = 1=e^{j0}, e^{j2\pi}, e^{j4\pi}, \ldots\ldots, e^{j2\pi k}$

comparing;

$3\pi N/5 = 2\pi k$

$N=10K/3$

$N=10$;putting $K=3$

The fundamental period is 10.

d) $x[n]=e^{j7\pi n}$

For periodicity

$e^{j7\pi n}=e^{j7\pi (n+N)}$

Now;$e^{j7\pi N} = 1=e^{j0}, e^{j2\pi}, \ldots\ldots, e^{j2\pi k}$

So;

$7\pi N=2\pi k$,where k=1,2,3………

$N=2k/7$

$N=2$,putting k=7 since N can only be an integer .

Hence the signal is periodic with smallest period 2.

e) $x[n] = 3e^{j3/5(n+1/2)}$

for periodicity

$3e^{j3/5(n+1/2)} =3e^{j3/5(n+1/2+N)}$

$=3\{e^{j3/5(n+1/2)}\}\times e^{j\,3/5N}$

$e^{j3/5N}=1=e^{j2\pi k}$,where k is an integer

$N=3/(2\pi \times5\times k)$

Whatever value we put of k and since k is itself an integer ,N doesn't become an integer because of π .So the signal x[n] is not periodic.

2) Determine the fundamental period of the signals.

a) x(t) = 2cos(10t + 1) – sin(4t – 1)

Here fundamental period of cos(10t + 1) is 2π /10 and sin(4t – 1) is 2π /4

Comparison to make both periods equal

cos(10t + 1)	sin(4t – 1)
π /5	< π /2
π ×5/5=π >	π /2
π =	π ×2/2=π

So the fundamental period of the given signal is π .

b) $x[n]=1+e^{4\pi n/7} - e^{j2\pi n/5}$

Here;

$e^{j4\pi n/7}=e^{j4\pi (n+N)/7}$

$e^{4\pi n/7}=e^{j2\pi k}$ where k and n are both integers

N=7k/2

Again;

$e^{j2\pi n/5}=e^{j2\pi (n+N)/5}$

$e^{j2\pi N/5}=1=e^{j2\pi k}$

N=5k

Comparison to make both periods equal

$e^{j4\pi n/7}$	$e^{j2\pi n/5}$
N=7k/2	< N=5k
N=7×10/2	N=5×7
35 =	35

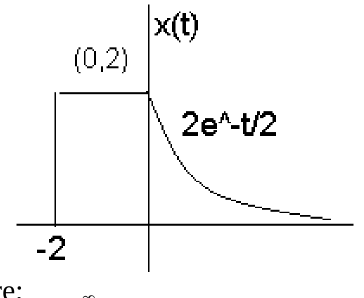
The fundamental period is 35.

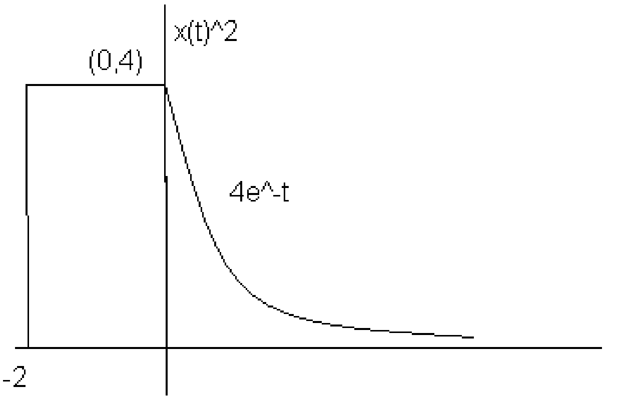
3)State which of the following signal is power signal and which one is energy signal .For power signal ,find the average power ,and for energy signal ,find the total energy of the signal.

a) x(t)=0 <-2

2 -2≤t≤0

$2e^{\wedge -t/2}$





Here;

$E_\infty = \int_{\infty}^{\infty} x(t)dt$

$E_\infty = \int^{0} 4\ dt + \int^{\infty} 4e^{(-t)}\ dt$

-2 0

= 12

Since we got finite energy ,it is energy signal.

b)x(t)=t 0<t<1 with period 1 sec

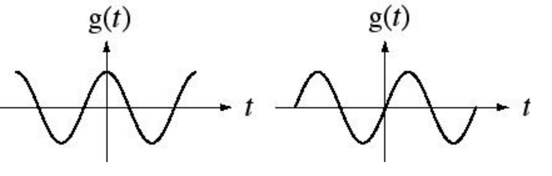
$P_{avg}=1/1\int_0^1 t^2 dt$

= 1/3

Since we got power to be finite it is a power signal.

---

3. Even and Odd Signals



Even function – $g(t)=g(-t)$

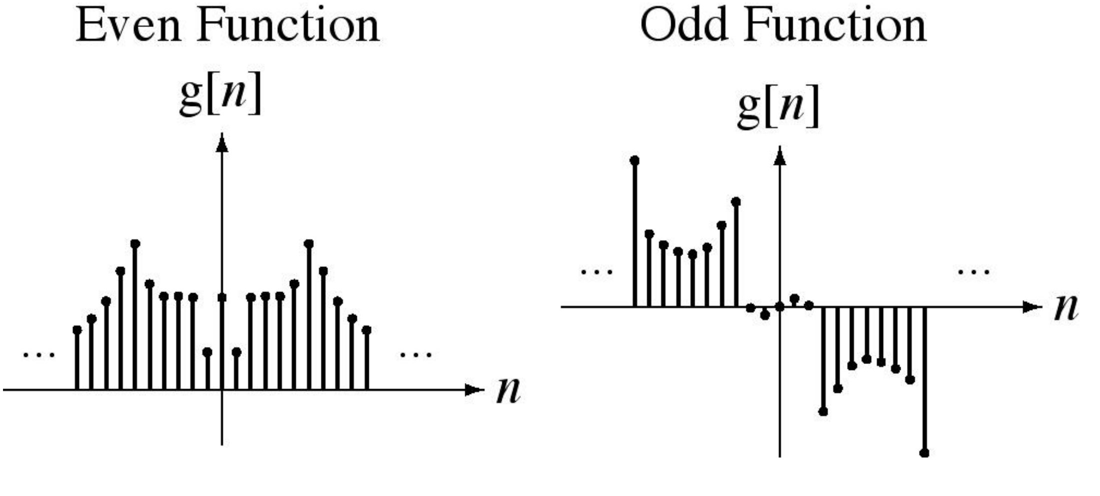
Odd Function -- $g(t)=-g(-t)$

The **even part** of a function is $g_e(t)=\dfrac{g(t)+g(-t)}{2}$

The **odd part** of a function is $g_o(t)=\dfrac{g(t)-g(-t)}{2}$

A function whose even part is zero, is odd and a function whose odd part is zero, is even.

Function type	***Sum***	***Difference***	***Product***	***Quotient***
Both even	*Even*	*Even*	*Even*	*Even*
Both odd	*Odd*	*Odd*	*Even*	*Even*
Even and odd	*Neither*	*Neither*	*Odd*	*Odd*



$$g_e[n] = \frac{g[n]+g[-n]}{2} \quad g_o[n] = \frac{g[n]-g[-n]}{2}$$

Example 1:

$x(t) = \cos(t)$ and $x(t) = t^2 + 4t^4$ are even functions. Verify.

Example 2:

$x(t) = \sin(t)$ and $x(t) = 2t + 3t^3$ are odd functions. Verify.

Example 3:

$x(t) = (t - 2)^2$ is neither odd nor even. Verify.

- Any arbitrary function x(t) can be written as sum of two function $x_e(t)$ and $x_o(t)$ where $x_e(t)$ is an even function and $x_o(t)$ is an odd function.

Let x(t) be an arbitrary function. Let us assume that there exists an even function $x_e(t)$ and an odd function $x_o(t)$ such that

$$x(t) = x_e(t) + x_o(t)$$

then $\quad x(-t) = x_e(-t) + x_o(-t) = x_e(t) - x_o(t)$

By solving these two equations we get

$x_e(t) = 1/2\,[x(t) + x(-t)]$ and $x_o(t) = 1/2\,[x(t) - x(-t)]$

Exercise: Show that $x(t) = (t - 1)^2 + \sin(t)$ is neither even nor odd. Find an even function $x_e(t)$ and an odd function $x_o(t)$ such that

$$x(t) = x_e(t) + x_o(t)$$

---

4. **Energy and Power Signals**

Energy Signal

- A signal with finite energy and zero power is called Energy Signal i.e.for energy signal

$$0<E<\infty \text{ and } P=0$$

- Signal energy of a signal is defined as the *area under the square of the magnitude of the signal*.

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2\, dt$$

- The units of signal energy depends on the unit of the signal.

Power Signal

- Some signals have infinite signal energy. In that caseit is more convenient to deal with **averagesignal power.**
- For power signals

$$0<P<\infty \text{ and } E=\infty$$

- Average power of the signal is given by

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2\, dt$$

- For a periodic signal x(t) the average signal power is

$$P_x = \frac{1}{T}\int_T |x(t)|^2\, dt$$

- T is any period of the signal.
- Periodic signals are generally power signals.

A discrtet time signal with finite energy and zero power is called Energy Signal i.e.for energy signal ***0<E<∞ and P =0***

Energy

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

Power

$$P_x = \lim_{N\to\infty} \frac{1}{2N} \sum_{n=-N}^{N-1} |x[n]|^2$$

Example 1: determine if the following signals are Energy signals, Power signals, or neither,

a) $a(t) = 3\sin(2\pi t),\ -\infty < t < \infty$,

This is a periodic signal, so it must be a power signal. Let us prove it.

$$\begin{aligned}
E_a &= \int_{-\infty}^{\infty} |a(t)|^2\, dt = \int_{-\infty}^{\infty} |3\sin(2\pi t)|^2\, dt \\
&= 9\int_{-\infty}^{\infty} \frac{1}{2}\left[1-\cos(4\pi t)\right]dt \\
&= 9\int_{-\infty}^{\infty} \frac{1}{2}dt - 9\int_{-\infty}^{\infty} \cos(4\pi t)dt \\
&= \infty \quad \text{J}
\end{aligned}$$

Notice that the evaluation of the last line in the above equation is infinite because of the first term. The second term has a value between –2 to 2 so it has no effect in the overall value of the energy.

Since $a(t)$ is periodic with period T = $2\pi/2\pi$ = 1 second, we get

$$\begin{aligned}
P_a &= \frac{1}{1}\int_0^1 |a(t)|^2\, dt = \int_0^1 |3\sin(2\pi t)|^2\, dt \\
&= 9\int_0^1 \frac{1}{2}\left[1-\cos(4\pi t)\right]dt \\
&= 9\int_0^0 \frac{1}{2}dt - 9\int_0^1 \cos(4\pi t)dt \\
&= \frac{9}{2} - \left[\frac{9}{4\pi}\sin(4\pi t)\right]_0^1 \\
&= \frac{9}{2} \quad \text{W}
\end{aligned}$$

So, the energy of that signal is infinite and its average power is finite (9/2). This means that it is a power signal as expected. Notice that the average power of this signal is as expected (square of the amplitude divided by 2)

---

b) $b(t) = 5e^{-2|t|}, \; -\infty < t < \infty,$

Let us first find the total energy of the signal.

$$\begin{aligned}
E_b &= \int_{-\infty}^{\infty} |b(t)|^2 \, dt = \int_{-\infty}^{\infty} \left|5e^{-2|t|}\right|^2 \, dt \\
&= 25\int_{-\infty}^{0} e^{4t} dt + 25\int_{0}^{\infty} e^{-4t} dt \\
&= \frac{25}{4}\left[e^{4t}\right]_{-\infty}^{0} + \frac{25}{4}\left[e^{-4t}\right]_{0}^{\infty} \\
&= \frac{25}{4} + \frac{25}{4} = \frac{50}{4} \quad \text{J}
\end{aligned}$$

The average power of the signal is

$$\begin{aligned}
P_b &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |b(t)|^2 \, dt = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left|5e^{-2|t|}\right|^2 \, dt \\
&= 25 \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{0} e^{4t} dt + 25 \lim_{T\to\infty} \frac{1}{T} \int_{0}^{T/2} e^{-4t} dt \\
&= \frac{25}{4} \lim_{T\to\infty} \frac{1}{T} \left[e^{4t}\right]_{-T/2}^{0} + \frac{25}{4} \lim_{T\to\infty} \frac{1}{T} \left[e^{-4t}\right]_{0}^{T/2} \\
&= \frac{25}{4} \lim_{T\to\infty} \frac{1}{T} \left[1 - e^{-2T}\right] + \frac{25}{4} \lim_{T\to\infty} \frac{1}{T} \left[e^{-2T} - 1\right] \\
&= 0 + 0 = 0
\end{aligned}$$

So, the signal $b(t)$ is definitely an energy signal.
So, the energy of that signal is infinite and its average power is finite (9/2). This means that it is a power signal as expected. Notice that the average power of this signal is as expected (the square of the amplitude divided by 2)

c) $c(t) = \begin{cases} 4e^{+3t}, & |t| \le 5 \\ 0, & |t| > 5 \end{cases},$

d) $d(t) = \begin{cases} \dfrac{1}{\sqrt{t}}, & t > 1 \\ 0, & t \le 1 \end{cases},$

Let us first find the total energy of the signal.

$$\begin{aligned}
E_d &= \int_{-\infty}^{\infty} |d(t)|^2 \, dt = \int_{1}^{\infty} \frac{1}{t} \, dt \\
&= \ln\left[t\right]_{1}^{\infty} \\
&= \infty - 0 = \infty \quad \text{J}
\end{aligned}$$

So, this signal is NOT an energy signal. However, it is also NOT a power signal since its average power as shown below is zero.

The average power of the signal is

$$P_d = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}|d(t)|^2\,dt = \lim_{T\to\infty}\frac{1}{T}\int_{1}^{T/2}\frac{1}{t}dt$$

$$= \lim_{T\to\infty}\left(\frac{1}{T}\ln\left[t\right]_1^{T/2}\right) = \lim_{T\to\infty}\left(\frac{1}{T}\ln\left[\frac{T}{2}\right]-\frac{1}{T}\ln\left[1\right]\right)$$

$$= \lim_{T\to\infty}\left(\frac{1}{T}\ln\left[\frac{T}{2}\right]\right) = \lim_{T\to\infty}\left(\frac{\ln\left[\frac{T}{2}\right]}{T}\right)$$

Using Le'hopital's rule, we see that the power of the signal is zero. That is

$$P_d = \lim_{T\to\infty}\left(\frac{\ln\left[\frac{T}{2}\right]}{T}\right) = \lim_{T\to\infty}\left(\frac{\frac{2}{T}}{1}\right) = 0$$

So, not all signals that approach zero as time approaches positive and negative infinite is an energy signal. They may not be power signals either.

e) $e(t) = -7t^2, \quad -\infty < t < \infty$,

f) $f(t) = 2\cos^2(2\pi t), \quad -\infty < t < \infty$.

g) $g(t) = \begin{cases} 12\cos^2(2\pi t), & -8 < t < 31 \\ 0, & \text{elsewhere} \end{cases}$.

What is System?

- Systems process input signals to produce output signals
- A system is combination of elements that manipulates one or more signals to accomplish a function and produces some output.

input signal → **system** → **output signal**

Examples

- A circuit involving a capacitor can be viewed as a system that transforms the source voltage (signal) to the voltage (signal) across the capacitor
- A communication system is generally composed of three sub-systems, the transmitter, the channel and the receiver. The channel typically attenuates and adds noise to the transmitted signal which must be processed by the receiver
- Biomedical system resulting in biomedical signal processing
- Control systems

Types of Systems

- Causal &Anticausal
- Linear & Non Linear
- Time Variant &Time-invariant
- Stable & Unstable
- Static & Dynamic

1. Causal and anticausal system

Causal system : A system is said to be *causal* if the present value of the output signal depends only on the present and/or past values of the input signal.
Example: $y[n]=x[n]+1/2x[n-1]$

Anticausalsystem : A system is said to be *anticausal* if the present value of the output signal depends only on the future values of the input signal.
Example: $y[n]=x[n+1]+1/2x[n-1]$

Check whether the following systems are causal or non-causal:

i. $y(n) = x(n) + x(n–2)$
ii. $y(n) = x(n) + x(n+2)$
iii. $y(n) = x(3n)$

Solution

i. The given system equation is

$$y(n) = x(n) + x(n–2)$$

The output $y(n)$ depends on the present input $x(n)$ and the previous input $x(n–1)$. Therefore, the system is causal.

ii. The given system equation is

$$y(n) = x(n) + x(n+2)$$

The output $y(n)$ depends on the present input $x(n)$ and the future input $x(n+2)$. The output $y(n)$ does not depend on the previous input. Therefore, the system is non-causal.

iii. The given system equation is
$y(n) = x(3n)$
For $n = 1, y(1) = x(3)$

For $n = 2, y(2) = x(6)$

and so on.
The output $y(n)$ depends on the future input only. Therefore, the system is non-causal.

---

2. linear-Nonlinear system.

A system is called linear, if superposition principle applies to that system. This means that linear system may be defined as one whose response to the sum of the weighted inputs is same as the sum of the weighted responses.

Let us consider two systems defined as follows.

$$y_1(t) = f(x_1(t)) \quad \text{...(1)}$$

Here x1(t) is the input or excitation and y1(t) is its output or response and

$$y_2(t) = f(x_2(t))$$

Here x2 (t) is the input or excitation and y2(t) is its output or response

Then for a linear system

$$f(a_1 x_1(t) + a_2 x_2(t)) = a_1 y_1(t) + a_2 y_2(t)$$

Where a1 and a2 are constants.

Linearity property for both continuous time and discrete time systems may be written as for continuous time system

$$a_1 x_1(t) + a_2 x_2(t) \longrightarrow a_1 y_1(t) + a_2 y_2(t) \quad \text{...(3)}$$

For discrete time system

$$a_1 x_1(n) + a_2 x_2(n) \longrightarrow a_1 y_1(n) + a_2 y_2(n) \quad \text{...(4)}$$

For any non-linear system, the principle of super-position does not hold true and equations (3) and (4) are not satisfied.

Few examples of linear system are filters, communication channels etc.

Determine whether the following systems

i. $y(n) = x(n^3)$ **and**
ii. $y(n) = x^2(n)$ **are linear or non-linear.**

Solution

i. The given equation is

$$y(n) = x(n^3)$$

Let the system produces $y_1(n)$ and $y_2(n)$ for two separate inputs $x_1(n)$ and $x_2(n)$.
Therefore, $y_1(n) = x_1(n^3)$ and $y_2(n) = x_2(n^3)$
The response $y_3(n)$ due to linear combination of inputs is given by

$$
\begin{aligned}
y_3(n) &= T[a_1x_1(n) + a_2x_2(n)] \\
&= T\{a_1x_1(n)\} + T\{a_2x_2(n)\} = a_1T\{x_1(n)\} + a_2T\{x_2(n)\} \\
&= a_1x_1(n^3) + a_2x_2(n^3) \\
&= a_1y_1(n) + a_2y_2(n) \qquad (1)
\end{aligned}
$$

The response $y'_3(n)$ of the system due to linear combination of two outputs will be

$$y'_3(n) = a_1y_1(n) + a_2y_2(n) \qquad (2)$$

From Eq. (1) and (2), we get

$$y_3(n) = y'_3(n)$$

Therefore, the system is linear.

ii. The given equation is $y(n) = x^2(n)$

Let the system produces $y_1(n)$ and $y_2(n)$ for two separate inputs $x_1(n)$ and $x_2(n)$.

$$y_1(n) = x_1^2(n) \text{ and } y_2(n) = x_2^2(n)$$

Therefore,

The response $y_3(n)$ due to linear combination of inputs is given by

$$
\begin{aligned}
y_3(n) &= T[a_1x_1(n) + a_2x_2(n)] \\
&= [a_1x_1(n) + a_2x_2(n)]^2 \\
&= a_1^2x_1^2(n) + 2a_1a_2x_1(n)x_2(n) + a_2^2x_2^2(n) \qquad (3)
\end{aligned}
$$

The response $y'_3(n)$ of the system due to linear combination of two outputs will be

$$y'_3(n) = a_1x_1^2(n) + a_2x_2^2(n) \qquad (4)$$

From equations (3) and (4), we get

$$y_3(n) \neq y'_3(n)$$

Therefore, the system is non-linear.

---

3. Time variant and invariant system

A system is called time invariant if its input output characteristics do not charge

with time. A LTI discrete time system satisfies boths the linearity and the time invariance properties.

To test if any given systems is time invariant, first apply an arbitrary sequence x (n) and find y (n).

y (n) = T [x (n)]

Now delay the input sequence by k samples and find output sequence denote it as. y(n,k) T[x(n-k)]

Delay the output sequence by k samples denote it as

$$y(n, k) = y(n - k)$$

For all possible values of k, the systems is the invariant on the other hand

$$y\ (n,\ k)\ \neq y\ (n-k)$$

Even for one value of k, the system is time variant.

the output.

Even for one value of k, the system is time variant.

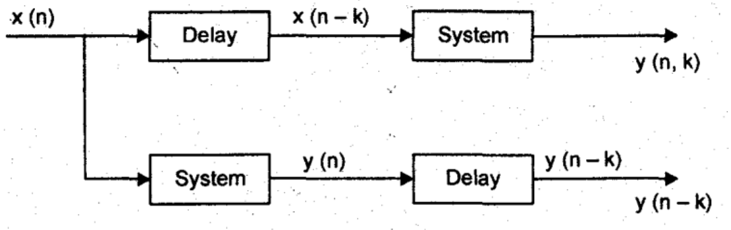


Fig. Time invariant and time variant system.

Determine whether the following signals are shift invariant i.e., time invariant or not.

i. $\boldsymbol{y(n) = x(n) - x(n-2)}$
ii. $\boldsymbol{y(n) = nx(n)}$
iii. $\boldsymbol{y(n) = x(-n)}$

Solution

i. **Here** $\boldsymbol{y(n) = x(n) - x(n-2) = T[x(n)]}$

If the input is delayed by 'k' samples, the output will be

$$y(n, k) = T[x(n-k)] = x(n-k)-x(n-k-2) \quad (1)$$

If we delay $y(n)$ by 'k' samples, we get

$$y(n-k) = x(n-k)-x(n-k-2) \quad (2)$$

From (1) and (2) we get

$$y(n, k) = y(n-k)$$

Therefore, the system is shift variant.

ii. **Here** $\boldsymbol{y(n) = nx(n) = T[x(n)]}$

If the input is delayed by 'k' samples, the output will be

$$y(n, k) = T[x(n-k)] = nx(n-k) -k) \quad (3)$$

because the multiplier n is not a part of input.

If we delay $y(n)$ by 'k' samples, we get

$$y(n-k) = (n-k)-x(n-k) \quad (4)$$

From (3) and (4) we get

$$y(n, k) \neq y(n-k)$$

Therefore, the system is shift invariant.

iii. **Here** $\boldsymbol{y(n) = x(-n) = T[x(n)]}$

If the input is delayed by 'k' samples, the output will be

$$y(n, k) = T[x(n-k)] = x[(-n)-k] = x(-n-k) \quad (5)$$

Here n of $x(n)$ has not been replaced by $n-k$. Here we are delaying $x(n)$ and $x(-n)$ will be delayed by the same amount.

If we delay $y(n)$ by 'k' samples, we get

$$y(n-k) = x[(-n)-k] = x(-n + k) \quad (6)$$

From Eq. (5) and Eq. (6) we get

$$y(n, k) \neq y(n-k)$$

Therefore, the system is shift invariant.

4.Stable and unstable system

A system is said to be *bounded-input bounded-output stable* (BIBO stable) iff every bounded input results in a bounded output.

LTI system is stable if its impulse response is absolutely summablei e

$$\sum_{k=-\infty}^{\infty} (h(k)) < \infty \qquad ...(1)$$

Here h(k)= h(n) is the impulse response of LTI system Thus equation (1) give the

condition of stability in terms of impulse response of the system.

Now the stability factor is denoted by ‘s’.

$$s = \sum_{k=-\infty}^{\infty} |h(k)| < \infty$$

Stable system	Astable System
1. An initialy relexed system is BIBO stable if and only if every bounded input produces bounded output. 2. Stable system shows finite behaviour. 3. When stable system is practically implemented then it cause limited range output.	1. An initially relexed system is said to be unstable if bounded input produces unbounded output. 2. Unstable system shows Eratic and extreme behaviour 3. When unstable system is practically implemented then it cause overflow.

5.Static and Dynamic system

A **static** system is memoryless system
It has no storage devices
its output signal depends on present values of the input signal

For example $i(t) = \frac{1}{R} v(t)$

A d**ynamic** system possesses memory
It has the storage devices

A system is said to possess memory if its output signal depends on past values and future values of the input signal.

$$i(t) = \frac{1}{L} \int_{-\infty}^{t} v(\tau) d\tau$$

For example : $y[n] = x[n] + x[n-1]$

What is a LTI System ?

- LTI Systems are *completely characterized* by its unit sample response
- The LTI System is Linear, Time invariant and stable system which can be static or dynamic
- The output of *any* LTI System is a convolution of the input signal with the unit-impulse response, *i.e.*

$$y[n] = x[n] * h[n]$$

$$= \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

Problems

1. Determine if the following systems are time-invariant, linear, causal, and/or memoryless?

a) $\frac{dy}{dt} + 6y(t) = 4x(t)$

b) $\frac{dy}{dt} + 4ty(t) = 2x(t)$

c) $y[n] + 2y[n-1] = x[n+1]$

d) $y(t) = \sin(x(t))$

e) $\frac{dy}{dt} + y^2(t) = x(t)$

f) $y[n+1] + 4y[n] = 3x[n+1] - x[n]$

g) $y(t) = \frac{dx}{dt} + x(t)$

h) $y[n] = x[2n]$

i) $y[n] = nx[2n]$

j) $\frac{dy}{dt} + \sin(t)y(t) = 4x(t)$

k) $\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 4y(t) = \frac{dx}{dt} + 4x(t)$

Solution(detailed Analysis Refer class Notes)

1. a) $\frac{dy}{dt} + 6y(t) = 4x(t)$

This is an ordinary differential equation with constant coefficients, therefore, it is linear and time-invariant. It contains memory and it is causal.

b) $\frac{dy}{dt} + 4ty(t) = 2x(t)$

This is an ordinary differential equation. The coefficients of 4t and 2 do not depend on y or x, so the system is linear. However, the coefficient 4t is not constant, so it is time-varying. The system is also causal and has memory

c)

$$y[n] + 2y[n-1] = x[n+1]$$

This is a difference equation with constant coefficients; therefore, it is linear and time-invariant. It is noncausal since the output depends on future values of x. Specifically, let x[n] = u[n], then y[-1] = 1.

d) $y(t) = \sin(x(t))$

check linearity:

$y_1(t) = \sin(x_1(t))$

$y_2(t) = \sin(x_2(t))$

Solution to an input of $a_1x_1(t) + a_2x_2(t)$ is $\sin(a_1x_1(t) + a_2x_2(t))$.

This is not equal to $a_1y_1(t) + a_2y_2(t)$.

As a counter example, consider $x_1(t) = \pi$ and $x_2(t) = \pi/2$, $a_1 = a_2 = 1$

the system is causal since the output does not depend on future values of time, and it is memoryless
the system is time-invariant

e) $\frac{dy}{dt} + y^2(t) = x(t)$

The coefficient of y means that this is nonlinear; however, it does not depend explicitly on t, so it is time-invariant. It is causal and has memory.

f) $y[n+1] + 4y[n] = 3x[n+1] - x[n]$

Rewrite the equation as $y[n] + 4y[n-1] = 3x[n] - x[n-1]$ by decreasing the index.
This is a difference equation with constant coefficients, so it is linear and time-invariant. The output does not depend on future values of the input, so it is causal. It has memory.

h) $y[n] = x[2n]$

has memory since the output relies on values of the input at other the the current index n,

causal? Let $x[n] = u[n-2]$, so $x[1] = 0$. Then $y[1] = x[2] = 1$, so not causal.

linear? Let $y_1[n] = x_1[2n]$ and $y_2[n] = x_2[2n]$. The response to an input of $x[n] = ax_1[n]+bx_2[n]$ is

$y[n] = ax_1[2n]+bx_2[2n]$, which is $ay_1[2n]+by_2[2n]$, so this is linear

time-invariant: Let $y_1[n]$ represent the response to an input of $x[n-N]$, so $y_1[n] = x[2(n-N)]$. This is also equal to $y[n-N]$, so the system is time-invariant.

i) $y[n] = nx[2n]$

This is similar to part h), except for the n coefficient. Similar to above, it is noncausal, has memory and is linear. Check time-invariance:

Let $y_1[n]$ represent the response to an input of $x[n-N]$, so $y_1[n] = nx[2(n-N)]$. This is not equal to $y[n-N] = (n-N)x[2(n-N)]$, so the system is time-varying.

j) $\dfrac{dy}{dt} + \sin(t)y(t) = 4x(t)$

This is an ordinary differential equation with coefficients sin(t) and 4. Neither depends on y or x, so it is linear. However, the explicit dependence on t means that it is time-varying. It is causal and has memory.

k) $\dfrac{d^2y}{dt^2} + 10\dfrac{dy}{dt} + 4y(t) = \dfrac{dx}{dt} + 4x(t)$

This is an ordinary differntial equation with constant coefficients, so it is linear and time-invariant. It is also causal and has memory.

UNIT II

ANALYSIS OF CONTINUOUS TIME SIGNALS

Fourier Series

The basis of the Fourier Series

Any periodic signal with time period *T* can be written as a sum of sines and cosines

$$x(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\right]$$

The fundamental frequency for this time period *T* is

$$\omega_0 = \frac{2\pi}{T} \quad \frac{radians}{\sec ond}$$

The dc term is $$a_0 = \frac{2}{T}\int_0^T x(t)\, dt\,,$$

and the other terms are

$$a_n = \frac{2}{T}\int_0^T x(t)\cos(n\omega_0 t)\, dt\,,$$

$$b_n = \frac{2}{T}\int_0^T x(t)\sin(n\omega_0 t)\, dt\,.$$

Note that the limits of integration can be taken form *–T/2* to *T/2* instead of *0* to *T*. The calculation of a_n or b_n is done using the orthogonality properties of sines and cosines, i.e.,

$$\frac{2}{T}\int_0^T \cos(n\omega_0 t)\cos(m\omega_0 t)dt = \begin{cases} 1 & if\ n = m \\ 0 & if\ n \neq m \end{cases}$$

$$\frac{2}{T}\int_0^T \sin(n\omega_0 t)\sin(m\omega_0 t)dt = \begin{cases} 1 & if\ n = m \\ 0 & if\ n \neq m \end{cases}$$

and

$$\frac{2}{T}\int_0^T \sin(n\omega_0 t)\cos(m\omega_0 t)dt = 0 \quad for\ all\ n\ \&m..$$

The fact that identical functions integrate to one indicates that they are *orthonormal.* For instance, if we have a signal *x(t)* with time period *T*, then we can write it like Eq. (4.2.1). So when we calculate the a_n,

$$a_m = \frac{2}{T}\int_0^T \left[\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\right] \right]\cos(m\omega_0 t)\, dt\,.$$

The integral of $a_0 \cos(m\omega_0 t)$ over one period *T* will be zero. Similarly the integral of $\cos(m\omega_0 t)$ with any sine term will be zero. And the integral of $\cos(m\omega_0 t)$ with any other cosine except *m=n* will be zero. There will only be one term left:

$$\begin{aligned} a_m &= \frac{2}{T}\int_0^T a_m \cos(m\omega_0 t)\cos(m\omega_0 t)\, dt \\ &= \frac{2}{T}a_m \int_0^T \left[\frac{1}{2} - \frac{1}{2}\cos(2m\omega_0 t) \right] dt \\ &= \frac{2}{T}a_m \int_0^T \frac{1}{2}\, dt = \frac{2}{T}a_m \frac{1}{2}T = a_m \end{aligned}$$

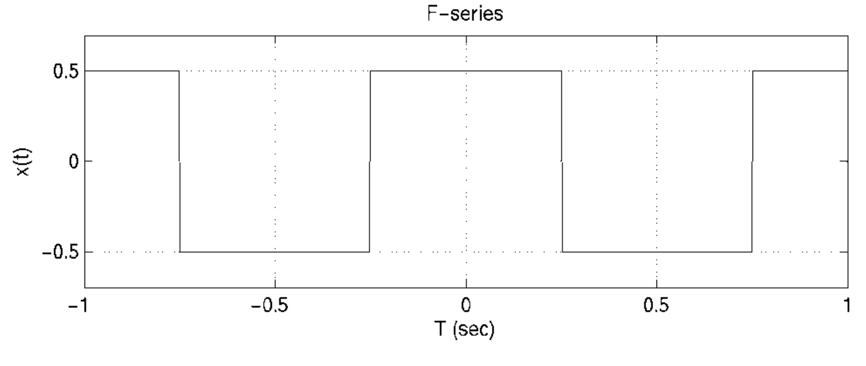
We would obtain a similar result for any of the *b* terms.

Note that the cosine functions (and the function 1) are even, while the sine functions are odd.

If f (x) is even (f (–x) = + f (x) for all x), then $b_n = 0$ *for all n, leaving a Fourier cosine series (and perhaps a constant term) only for f (x).*

If f (x) is odd (f (–x) = – f (x) for all x), then $a_n = 0$ *for all n, leaving a Fourier sine series only for f (x).*

Example Calculate the Fourier series for the rectangular series shown in Fig.



A periodic time-domain signal.

Solution

There are a couple things we can do to simplify the calculation. First of all, we will add a dc term of *1/2*, and then just leave the calculation of a_0 off. And using symmetry, we will calculate over the interval 0 to T/2, and double it.

$$\begin{aligned}
a_n &= \frac{2}{T} 2 \left\{ \int_0^{T/4} \cos\left(n\omega_o t\right) dt \right\} \\
&= \frac{4}{T} \left\{ \frac{1}{n\omega_o} \left[\sin\left(n\omega_o t\right) \right] \Big|_0^{T/4} \right\} \\
&= \frac{4}{2\pi n} \left\{ \sin\left(\frac{n 2\pi T}{T 4} \right) \right\} = \frac{2}{\pi n} \sin\left(\frac{n\pi}{2} \right)
\end{aligned}$$

$$a_n = \begin{cases} 2/n\pi & n = 1,5,9 \\ -2/n\pi & n = 3,7,11 \\ 0 & n = 0,2,4 \end{cases}$$

Notice that just the first three non-zero terms of the Fourier series result in a pretty good approximation (Fig. 1). As more and more terms are added, the series comes closer to the rectangular function (Fig. 2.)

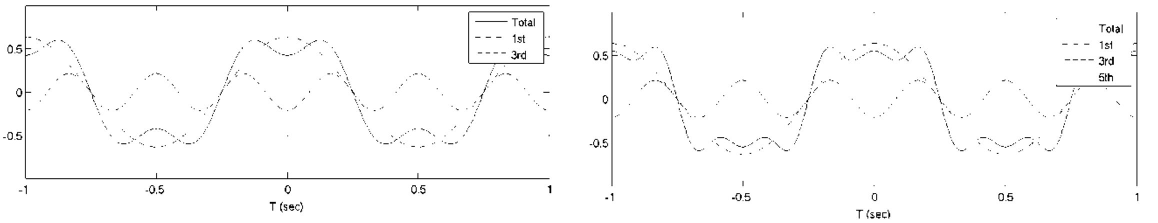


Figure 1. Fourier series reconstruction using two terms (left) and three terms (right).

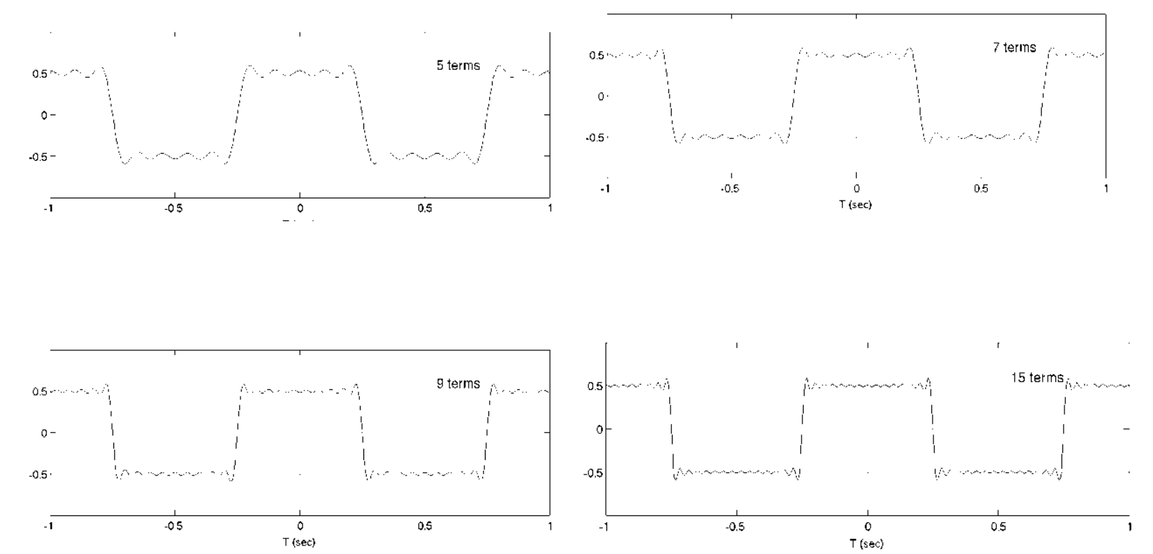
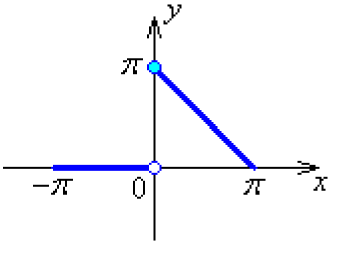


Fig 2 Reconstruction of the series of fig 1. using an increasing number of terms.

<u>Example 2</u>

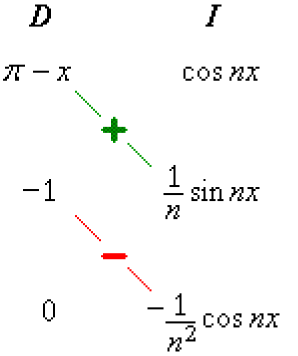
Expand $f(x) = \begin{cases} 0 & (-\pi < x < 0) \\ \pi - x & (0 \le x < +\pi) \end{cases}$ in a Fourier series.

$L = \pi.$



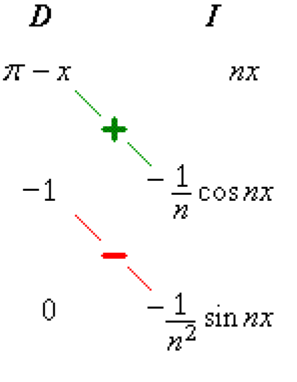
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{\pi}\int_{0}^{\pi} (\pi - x)\,dx$$

$$= 0 + \frac{1}{\pi}\left[\frac{(\pi - x)^2}{-2}\right]_0^{\pi} = \frac{\pi}{2}$$



$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0 + \frac{1}{\pi}\int_{0}^{\pi} (\pi - x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{n(\pi - x)\sin nx - \cos nx}{n^2}\right]_0^{\pi} = \frac{1 - (-1)^n}{n^2 \pi}$$



$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0 + \frac{1}{\pi}\int_{0}^{\pi} (\pi - x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{n(\pi - x)\cos nx + \sin nx}{-n^2}\right]_0^{\pi} = \frac{1}{n}$$

Therefore the Fourier series for $f(x)$ is

$$\boxed{f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n^2\pi}\cos nx + \frac{1}{n}\sin nx\right) \qquad (-\pi < x < +\pi)}$$

Complex Series

In general, even if we only have one frequency, say $\omega_n = n\omega_0$, we still need two numbers, a_n and b_n, to describe the *nth* series term. Since we know that the sine and cosine terms are orthonormal, we might wonder if we could change the two real numbers to one complex number. Start with

$$x_n(t) = a_n \cos(n\omega_n t) + b_n \sin(n\omega_n t),$$

and use Euler's equations:

$$x_n(t) = a_n\left[\frac{e^{jn\omega_n t} + e^{-jn\omega_n t} +}{2}\right] + b_n\left[\frac{e^{jn\omega_n t} - e^{-jn\omega_n t} +}{2j}\right]$$

We will begin by grouping the positive and negative frequency components

$$\begin{aligned} x_n(t) &= \left(\frac{a_n}{2} - j\frac{b_n}{2}\right)e^{jn\omega_n t} + \left(\frac{a_n}{2} + j\frac{b_n}{2}\right)e^{-jn\omega_n t} \\ &= X_n e^{jn\omega_n t} + X^*{}_n\left(e^{-jn\omega_n t}\right) \end{aligned}$$

The original series

$$x(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\right]$$

can be written

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}.$$

Notice that the new series goes between plus and minus infinity because the Euler equations used plus and minus terms. We determine the X_n in the same way

$$X_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_o t}dt.$$

This depends on a similar orthonormality condition

$$\frac{1}{T}\int_0^T e^{jn\omega_o t}e^{jm\omega_o t}dt = \frac{1}{T}\int_0^T e^{j(n-m)\omega_o t}dt = 1 \quad if \ n = m$$

otherwise

$$\begin{aligned}
&\frac{1}{T}\int_0^T e^{jn\omega_o t}e^{jm\omega_o t}dt = \frac{1}{T}\int_0^T e^{j(n-m)\omega_o t}dt \\
&= \frac{1}{T}\frac{1}{j(n-m)\omega_o}e^{j(n-m)\omega_o t}\Big|_{t=0}^{t=T} \\
&= \frac{1}{T}\frac{1}{j(n-m)\omega_o}\left(e^{j(n-m)2\pi}-1\right) = 0.
\end{aligned}$$

So when we try to calculate the *n* coefficient

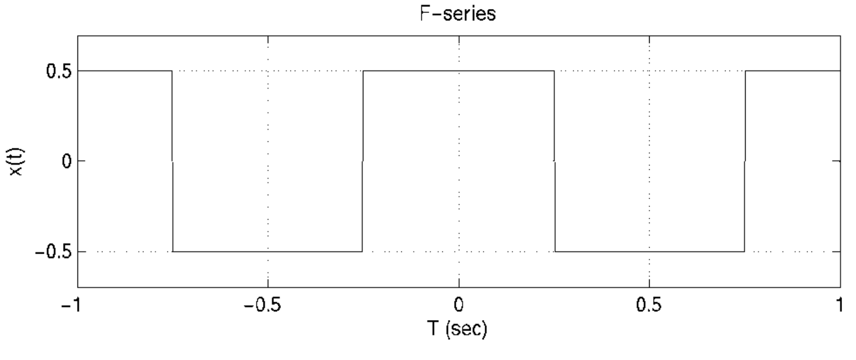
$$X_n = \frac{1}{T}\int_0^T \sum_{m=-\infty}^{\infty} X_m e^{jm\omega_0 t}e^{-jn\omega_0 t}dt \ ,$$

the only term that survives is *m* = *n*

$$X_n = \frac{1}{T}\int_0^T X_n e^{jn\omega_o t}e^{-jn\omega_o t}dt = \frac{1}{T}\int_0^T X_n dt = X_n \, .$$

Note that the complex form has plus and minus values. But $X_{-n} = X^*_n$, so if we know the positive one, we know the negative one. Once again we can plot the coefficients X_n out, but since they are complex, we will have to plot magnitude and phase. This is called the *line spectra.*

Example Redo the previous example of using the complex series.



<u>Solution</u>

As before, it won't hurt to add a dc term, and then just leave it out of the series. So now we calculate

$$
\begin{aligned}
X_n &= \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jn\omega_o t}dt \\
&= \frac{1}{T}\int_{-T/4}^{T/4} e^{-jn\omega_o t}dt = \frac{1}{T}\frac{1}{jn\omega_o}e^{-jn\omega_o t}\Big|_{-T/4}^{T/4} \\
&= \frac{1}{T}\frac{1}{jn\omega_o}\left[e^{-jn\omega_o T/4} - e^{jn\omega_o T/4}\right]
\end{aligned}
$$

Remember that

$$\omega_o = \frac{2\pi}{T}$$

So

$$
\begin{aligned}
X_n &= \frac{1}{T}\frac{T}{jn2\pi}\left[e^{-jn\pi/2} - e^{jn\pi/2}\right] \\
&= \frac{1}{jn2\pi}\left[2j\sin\left(\frac{n\pi}{2}\right)\right] = \frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right)
\end{aligned}
$$

$$X_1 = X_{-1} = \frac{1}{\pi}$$

$$X_2 = \frac{1}{n\pi}\sin\left(\frac{2\pi}{2}\right) = 0\,,$$

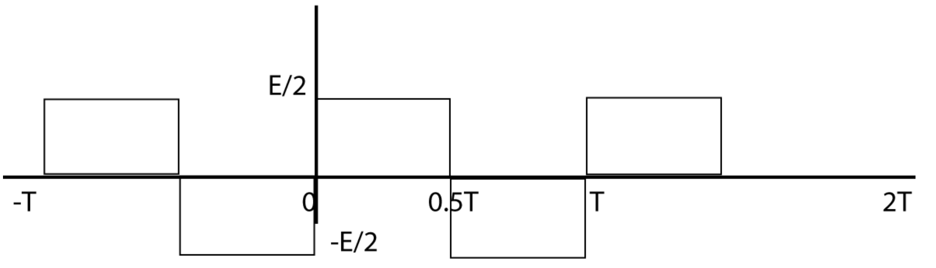
as will all even terms. Furthermore,

$$X_3 = X_{-3} = \frac{1}{3\pi}\sin\left(\frac{3\pi}{2}\right) = -\frac{1}{3\pi}$$

Since the positive and negative terms are the same.

$$
\begin{aligned}
x(t) &= \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t} = \sum_{n=1,3,5}^{\infty} X_n\left(e^{jn\omega_0 t} + e^{-jn\omega_0 t}\right) \\
&= \sum_{n=1,3,5}^{\infty} 2X_n\cos\left(n\omega_0 t\right) = \frac{2}{\pi}\left[\cos\left(\omega_0 t\right) - \frac{1}{3}\cos\left(3\omega_0 t\right) + \frac{1}{5}\cos\left(5\omega_0 t\right) - \ldots\right]
\end{aligned}
$$

Example Find the Fourier series of the function below



Solution

Step 1 $$x(t) = -\frac{E}{2} p_{T/2}(t + T/4) + \frac{E}{2} p_{T/2}(t - T/4)$$

Step 2

$$X(\omega) = \left(-e^{j\omega T/4} + e^{-j\omega T/4}\right)\frac{E}{2}\frac{T}{2}\frac{\sin\left(\frac{\omega T}{4}\right)}{\frac{\omega T}{4}}$$

Step 3

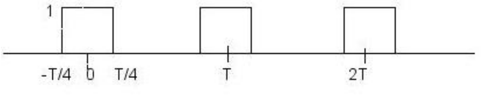
$$X_n = \frac{1}{T} X(\omega = n\omega_0 = 2\pi n) = \left(-e^{j\pi n/2} + e^{-j\pi n/2}\right)\frac{E}{2}\frac{\sin\left(\frac{n\pi}{2}\right)}{n\pi}$$
$$= -jE\sin\left(\frac{\pi n}{2}\right)\frac{\sin\left(\frac{n\pi}{2}\right)}{n\pi} = -\frac{jE}{n\pi}\sin^2\left(\frac{n\pi}{2}\right)$$

Step 4

Since $X_{-n} = -X_n$ we will convert the exponential to the sine series

$$x(t) = \sum_{n=1}^{\infty} X_n\left(e^{jn\omega_0 t} - e^{-jn\omega_0 t}\right) = \sum_{n=1}^{\infty} \frac{2E}{\pi}\sin^2\left(\frac{n\pi}{2}\right)\sin(n\omega_0 t)$$
$$= \frac{2E}{\pi}\left[\sin(\omega_0 t) + \frac{1}{3}\sin(3\omega_0 t) + \frac{1}{5}\sin(5\omega_0 t) + \ldots\right]$$

Example Determine the Fourier series of the function in Fig. 1. (T = 1).



Old Way:

We know the fundamental frequency is

$$\omega_0 = \frac{2\pi}{T} = 2\pi$$

Now to calculate the X_n

$$\begin{aligned} X_n &= \frac{1}{T}\int_{-T/4}^{T/4} 1e^{-jn\omega_0 t}dt \\ &= \frac{1}{T}\frac{1}{-jn\omega_0}\left[e^{-j\frac{n\omega_0 T}{4}} - e^{j\frac{n\omega_0 T}{4}}\right] \\ &= \frac{1}{T}\frac{2}{n\omega_0}\sin\left(\frac{n\omega_0 T}{4}\right). \end{aligned}$$

New Way:

Now, to return to the problem in the figure: We can recognize that

$$x_0(t) = p^h_{T/2}(t),$$

and its Fourier Transform is

$$X_0(\omega) = \tau\frac{\sin\left(\frac{\omega\tau}{2}\right)}{\frac{\omega\tau}{2}} = \frac{T}{2}\frac{\sin\left(\frac{\omega T}{4}\right)}{\left(\frac{\omega T}{4}\right)} = \frac{2}{\omega}.$$

Therefore,

$$X_n = \frac{1}{T}\frac{T}{2}\frac{\sin\left(\frac{n\omega_0 T}{4}\right)}{\left(\frac{n\omega_0 T}{4}\right)} = \frac{1}{T}\frac{2}{n\omega_0}\sin\left(\frac{n\omega_0 T}{4}\right),$$

and since $\omega_0 = \frac{2\pi}{T}$

$$X_n = \frac{\sin(n\pi/2)}{(n\pi)}.$$

$$X_0 = 1/2, \quad X_1 = X_{-1} = \frac{1}{\pi}, \quad X_3 = X_{-3} = \frac{-1}{3\pi}, \quad X_5 = X_{-5} = \frac{1}{5\pi}$$

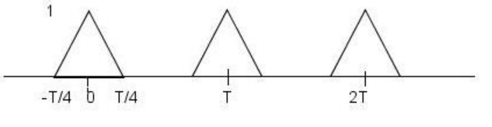
So my series is

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn2\pi t} = X_0 + 2X_1 \cos(2\pi t) + 2X_3 \cos(6\pi t) + \ldots$$
$$= \frac{1}{2} + \frac{2}{\pi}\cos(2\pi t) - \frac{2}{3\pi} X_3 \cos(6\pi t) + \frac{2}{5\pi} X_3 \cos(10\pi t)$$

The bottom line is that we can bring to bear everything we have learned about FT to help us calculate the Fourier series.

Example

Write the Fourier series of the function in (T = 1)



<u>Solution</u>

From the table, one of the triangles has the FT

$$\mathcal{F}\{\Delta_\tau\} = \mathcal{F}\{\Delta_{T/2}\}$$
$$= X(\omega) = \frac{\tau}{2}\sin c^2\left(\frac{\omega\tau}{4}\right) = \frac{1}{4}\sin c^2\left(\frac{\omega}{8}\right)$$

$$X_n = \frac{1}{T} X\left(jn\omega_0\right) = \frac{1}{4}\sin c^2\left(\frac{n\omega_0}{8}\right)$$

Notice that $X_n = X_{-n}$, and the dc term $X(\omega=0)=1/4$ so the series is

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t} = \frac{1}{4} + \sum_{n=1}^{\infty} 2X_n \cos(n\omega_0 t)$$

Example Find the Fourier series of the function given by

$$x(t) = \sum_{n=-\infty}^{\infty} e^{-2|t-nT|} \quad T = 1$$

<u>Solution</u>

Look at just the term centered at t = 0

$$x_0(t) = e^{-2|t|}$$

We know that its Fourier transform is

$$X(\omega) = \frac{2 \cdot 2}{2^2 + \omega^2}.$$

So the X_n terms are

$$X_n(\omega) = \frac{1}{T}\frac{4}{4+(n\omega_0)^2} = \frac{4}{4+(2\pi n)^2} = \frac{1}{1+(\pi n)^2}.$$

Obviously, these are even, except for the dc term, which is

$$X_0(\omega) = \frac{1}{1+(\pi 0)^2} = 1$$

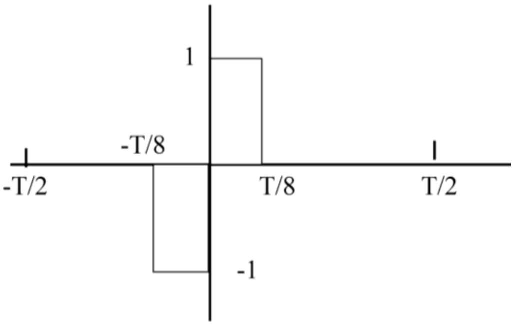
So we can write

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t} = 1 + \sum_{n=1}^{\infty} X_n \left(e^{jn\omega_0 t} + e^{-jn\omega_0 t}\right)$$

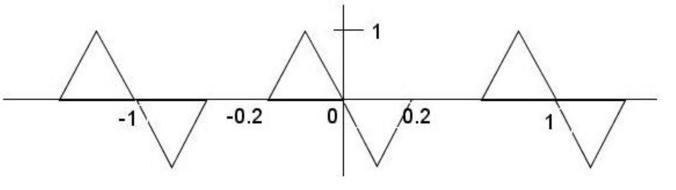
$$= 1 + \sum_{n=1}^{\infty} 2X_n \cos(n\omega_0 t).$$

Practice Problems

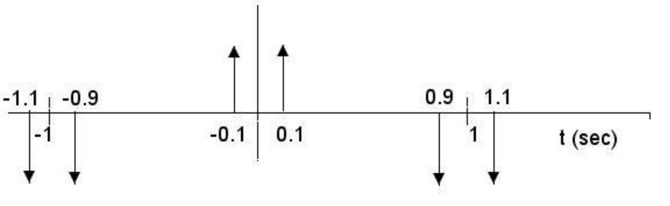
1. The diagram below represents one period of a time series that extends infinitely in each direction. Write the Fourier series of this signal. T= 1 second. Your answer should be a real series (i.e., not complex functions).



2. The pattern below extends infinitely in each direction. The interval is 1 second. Write a Fourier Series. {Your final answer should be a sine series.}



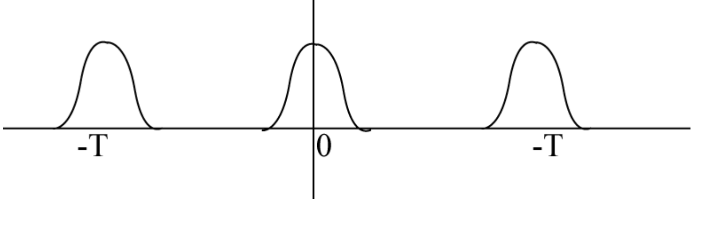
3. Write a Fourier series to describe the function below. You may assume that it extends infinitely in both directions. The amplitude of the delta functions is one. The time scale is seconds. Your final answer should be a sine and/or cosines series.



4. The series below is made up of function of the form

$$f(t) = e^{-50t^2}.$$

Write the Fourier series for T = 0.1 sec.

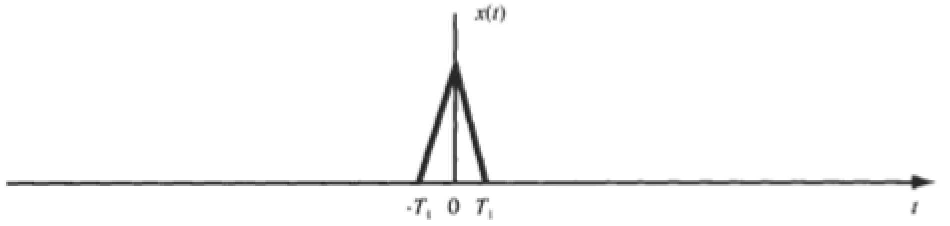


Fourier Transform

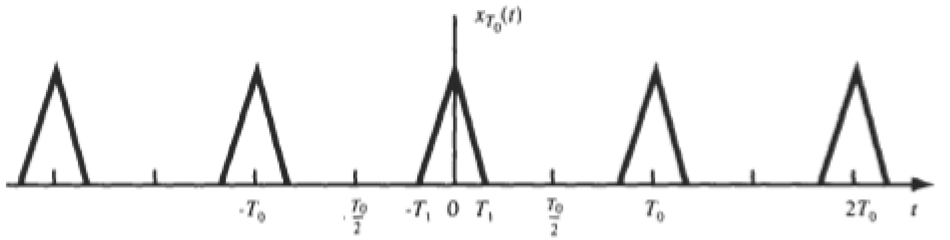
The Fourier Transform

Let $x(t)$ be a nonperiodic signal of finite duration, i.e.,

$$x(t) = 0 \qquad |t| > T_1$$



(a)



Let us form a periodic signal by extending $x(t)$ to $x_{T_0}(t)$ as,

$$\lim_{T_0 \to \infty} x_{T_0}(t) = x(t) \ , \qquad \text{[i.e., the period is infinity]}$$

Then,

$$x_{T_0}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}$$
$$c_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x_{T_0}(t) e^{-jk\omega_0 t} dt \qquad (01)$$

Or,

$$c_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-jk\omega_0 t} dt = \frac{1}{T_0} \int_{-\infty}^{\infty} x(t) e^{-jk\omega_0 t} dt$$

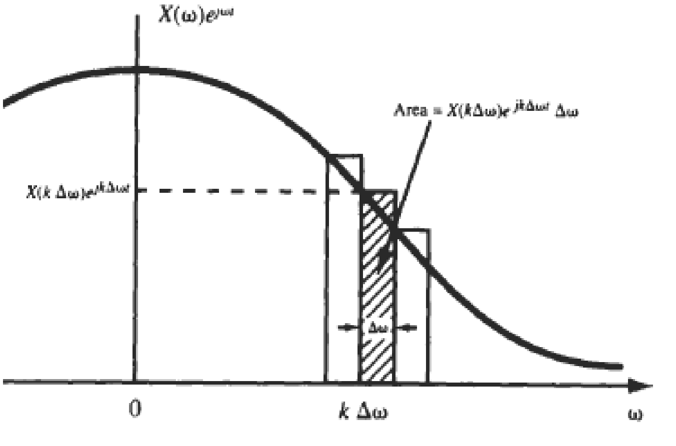
Let us now define $X(\omega)$ as, $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

Thus,

$$c_k = \frac{1}{T_0} X(k\omega_0) .$$

Substituting this in equation (01) we get,

$$x_{T_0}(t) = \sum_{k=-\infty}^{\infty} \frac{X(k\omega_0)}{T_0} e^{jk\omega_0 t} = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \frac{X(k\omega_0)}{T_0} e^{jk\omega_0 t} \omega_0$$



As $T_0 \to \infty$, $\omega_0 \to 0$. Let us assume $\omega_0 = \Delta\omega$.

Thus, $\lim\limits_{T_0 \to \infty} x_{T_0}(t) = \lim\limits_{\Delta\omega \to 0} \dfrac{1}{2\pi} \sum\limits_{k=-\infty}^{\infty} X(k\Delta\omega) e^{jk\omega_0 t} \Delta\omega = x(t)$

Or, $x(t) = \dfrac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$ (02)

$x(t)$ in equation (02) is called the Fourier Integral. Thus a finite duration signal is represented by Fourier integral instead of Fourier series.

The function $X(\omega)$ is called the Fourier transform of $x(t)$.

Symbolically these two pairs are represented as,

$$X(\omega) = F\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

And $$x(t) = F^{-1}\{X(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

Alternatively, $x(t) \xleftrightarrow{F.T.} X(\omega)$.

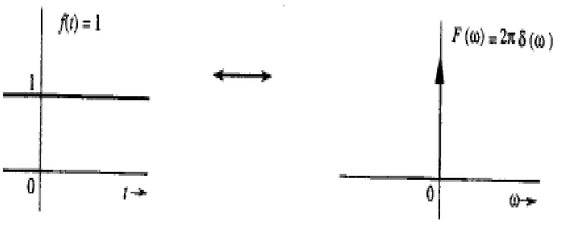
Example

1. Find the Fourier transform of $e^{-at}u(t) \quad a > 0$.

$$X(\omega) = \int_{-\infty}^{\infty} e^{-at} u(t) e^{-j\omega t} dt = \int_{0}^{\infty} e^{-(a+j\omega)t} dt = \frac{1}{a + j\omega}.$$

2. Find the Fourier transform of $\delta(t)$.

$$F\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$



3. Find the inverse Fourier transform of $\delta(\omega)$.

$$F^{-1}\{\delta(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega) e^{-j\omega t} d\omega = \frac{1}{2\pi}.$$

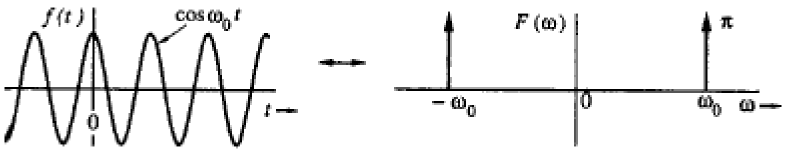
Thus, $\dfrac{1}{2\pi} \xleftrightarrow{F.T.} \delta(\omega)$ or, $1 \xleftrightarrow{F.T.} 2\pi \cdot \delta(\omega)$.

4. Find the inverse Fourier transform of $\delta(\omega - \omega_0)$.

$$F^{-1}\{\delta(\omega - \omega_0)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega t} d\omega = \frac{1}{2\pi} e^{j\omega_0 t}$$

Thus, $\boxed{e^{j\omega_0 t} \xleftrightarrow{F.T.} 2\pi \cdot \delta(\omega - \omega_0)}$

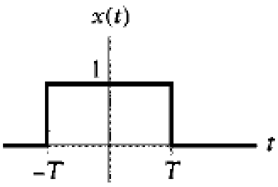
We know, $\cos\omega_0 t = \frac{1}{2}\left(e^{j\omega_0 t} + e^{j\omega_0 t}\right)$; Thus, $\boxed{\cos\omega_0 t \xleftrightarrow{F.T.} \pi\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]}$

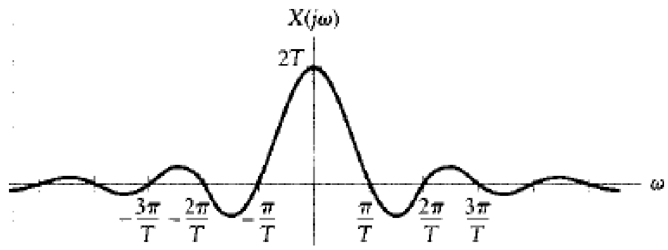


5. Find the Fourier transform of the rectangular pulse $x(t)$ shown in Figure.

$$X(\omega) = \int_{-T}^{T} e^{-j\omega t}dt = 2\frac{\sin\omega T}{\omega} = 2T\frac{\sin\omega T}{\omega T}$$

$$= 2T\sin c\left(\frac{\omega T}{\pi}\right)$$

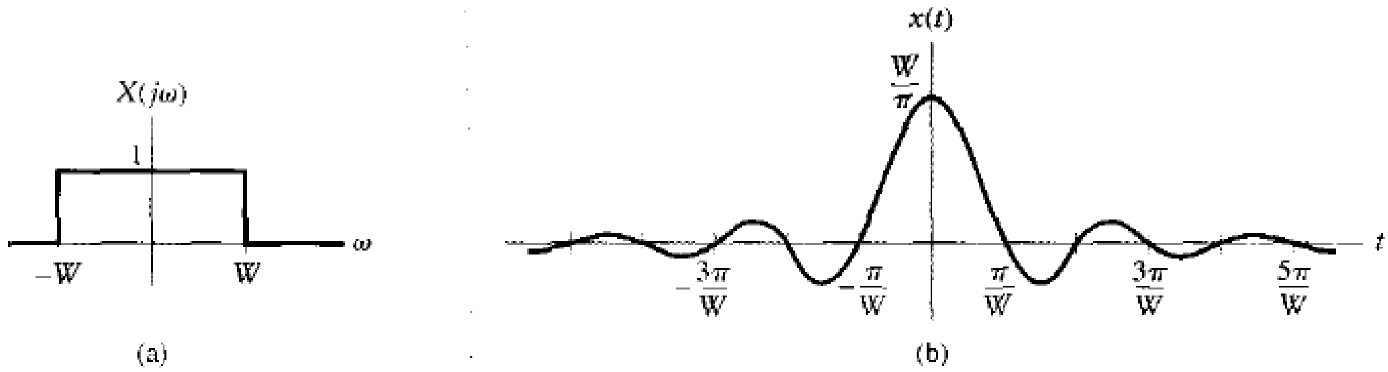




The magnitude spectrum is, $\left|X(\omega)\right| = 2\left|\frac{\sin\omega T}{\omega}\right|$, and the phase spectrum is,

$$\arg\{X(\omega)\} = \begin{cases} 0, & \frac{\sin(\omega T)}{\omega} > 0 \\ \pi, & \frac{\sin(\omega T)}{\omega} < 0 \end{cases}.$$

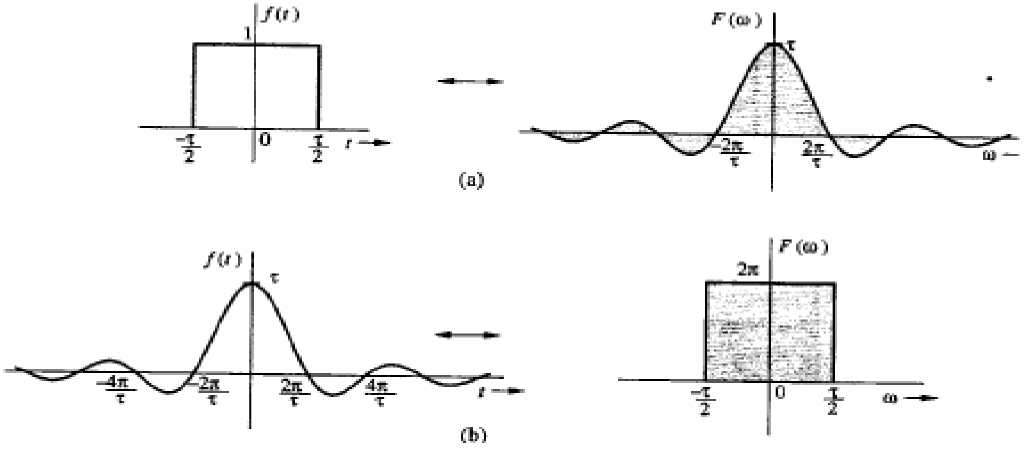
6. Find the inverse Fourier transform of the rectangular spectrum shown below.



$x(t) = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}d\omega = \frac{1}{\pi t}\sin(Wt) = \frac{W}{\pi}\text{sinc}\left(\frac{Wt}{\pi}\right)$. The plot is shown in Figure above.

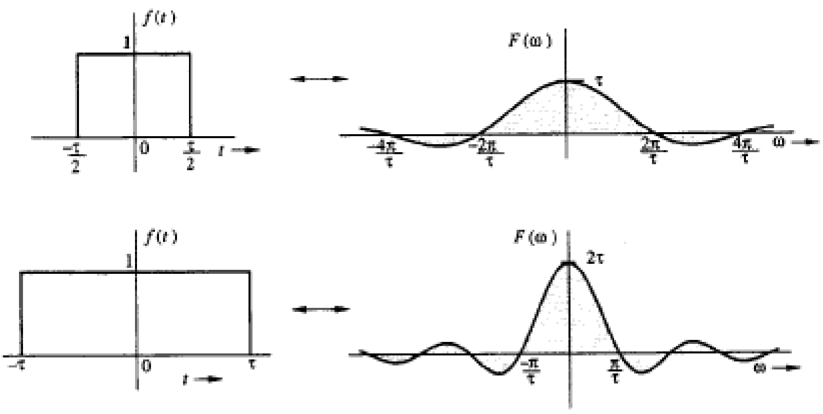
Some Properties of Fourier Transform

1. Symmetry property: If $f(t) \Leftrightarrow F(\omega)$ then $F(t) \Leftrightarrow 2\pi f(-\omega)$. (duality property)



Example: Apply symmetry property to show that $\delta(t+t_0)+\delta(t-t_0) \Leftrightarrow 2\cos t_0\omega$.

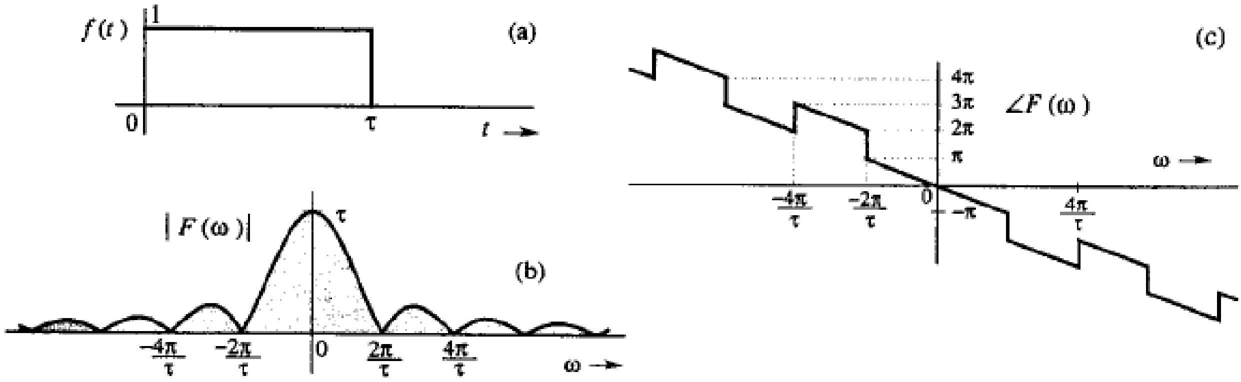
2. **Scaling Property:** If $f(t) \Leftrightarrow F(\omega)$ then $f(at) \Leftrightarrow \frac{1}{|a|}F(\omega/a)$.



3. Time-shifting Property: If $f(t) \Leftrightarrow F(\omega)$ then $f(t-t_0) \Leftrightarrow e^{-j\omega t_0}F(\omega)$.

4. **Frequency-shifting Property:** If $f(t) \Leftrightarrow F(\omega)$ then $f(t)e^{j\omega_0 t} \Leftrightarrow F(\omega-\omega_0)$.

Example: Find the Fourier transform of the gate pulse shown in Figure below.

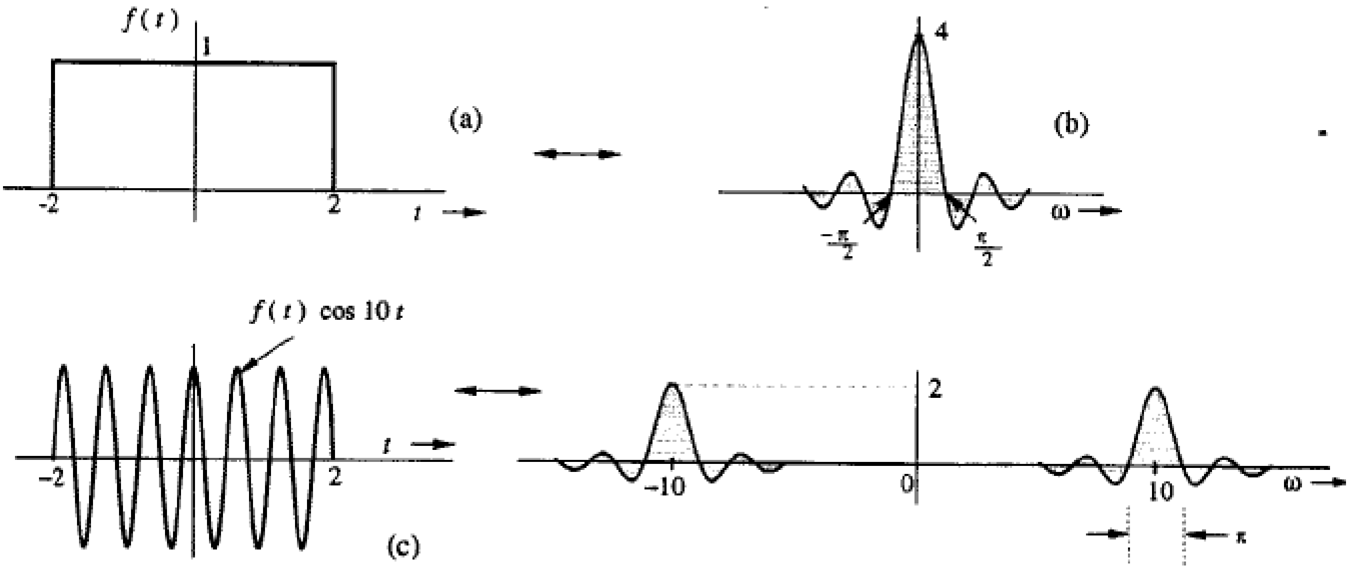


We get the Fourier transform by applying time-delay property to the F.T. of rectangular pulse (symmetrical).

Thus, $F(\omega) = \tau \operatorname{sinc}\left(\frac{\omega\tau}{2\pi}\right) e^{-j\omega\tau/2}$.

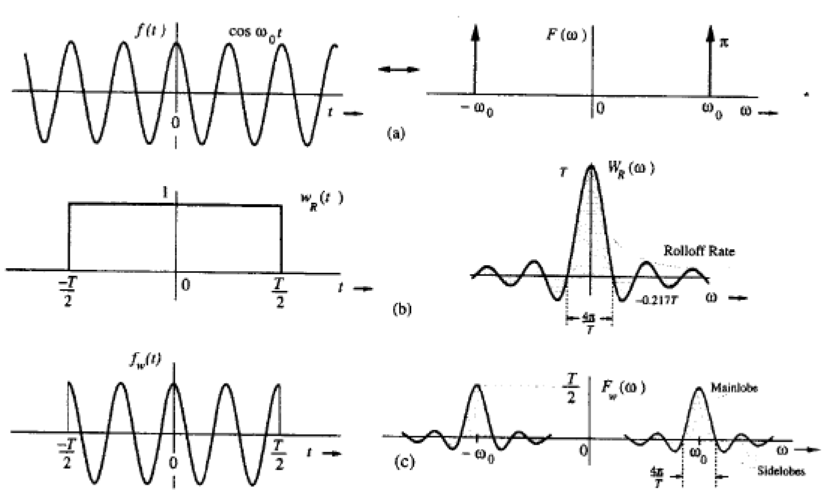
Example: Sketch the Fourier transform of $f(t)\cos 10t$ using frequency shifting property. **[property 4]**

$f(t)\cos 10t = f(t)\left[\frac{1}{2}e^{j10t} + \frac{1}{2}e^{-j10t}\right]$. Therefore, $f(t)\cos 10t \Leftrightarrow \frac{1}{2}\left[F(\omega-10) + F(\omega+10)\right]$. The sketch is shown in Figure below. Here, $f(t) \Leftrightarrow 4\sin c\left(\frac{2\omega}{\pi}\right)$.



5. Time and Frequency convolution:

$f_1(t) * f_2(t) \Leftrightarrow F_1(\omega)F_2(\omega)$ and $f_1(t)f_2(t) \Leftrightarrow \frac{1}{2\pi}F_1(\omega) * F_2(\omega)$.



6. Time differentiation and time integration:

$$\frac{df(t)}{dt} \Leftrightarrow j\omega F(\omega); \quad \int_{-\infty}^{t} f(\tau)d\tau \Leftrightarrow \frac{F(\omega)}{j\omega} + \pi F(0)\delta(\omega).$$

(a) $f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega \Rightarrow \frac{df(t)}{dt} = j\omega \cdot \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega = j\omega \cdot f(t)$

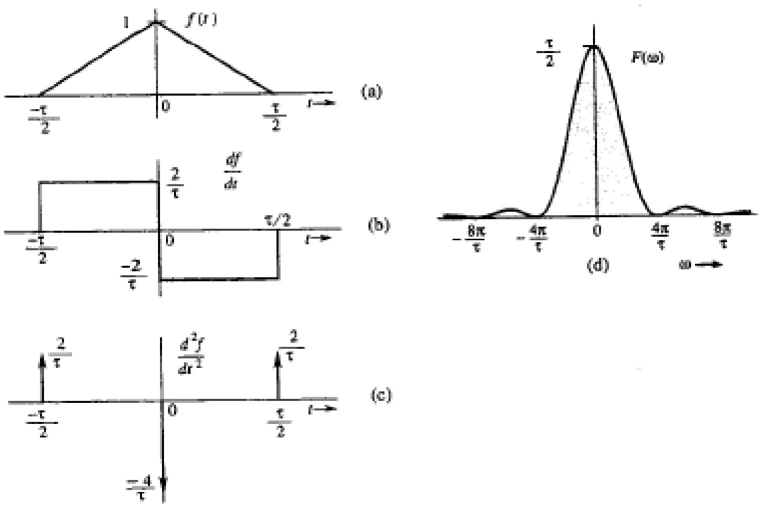
Therefore, $F\left\{\frac{df(t)}{dt}\right\} = j\omega F\left\{f(t)\right\}$, or, $\frac{df(t)}{dt} \Leftrightarrow j\omega F(\omega)$.

(b) $f(t) * u(t) = \int_{-\infty}^{\infty} f(\tau)u(t-\tau)d\tau = \int_{-\infty}^{t} f(\tau)d\tau$

Using convolution property, $F\left\{\int_{-\infty}^{t} f(\tau)d\tau\right\} = F(\omega)\left[\frac{1}{j\omega} + \pi\delta(\omega)\right]$

Therefore, $\int_{-\infty}^{t} f(\tau)d\tau \Leftrightarrow \frac{F(\omega)}{j\omega} + \pi F(0)\delta(\omega)$.

Example: Using the time-differentiation property, find the F.T. of the triangle illustrated in figure below.



$$\frac{d^2 f(t)}{dt^2} = \frac{2}{\tau}\left[\delta(t+\tau/2) + \delta(t-\tau/2) - 2\delta(t)\right]$$

$\delta(t) \Leftrightarrow 1 \quad \delta(t-\tau/2) \Leftrightarrow e^{-j\omega\tau/2}$

Performing F.T. of the first equation,

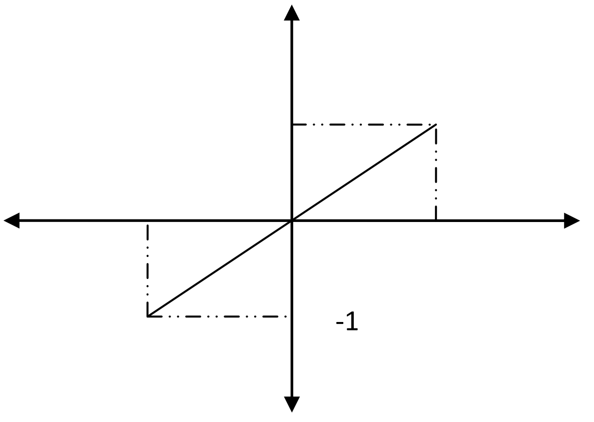
$$(j\omega)^2 F(\omega) = \frac{2}{\tau}\left[e^{j\omega\tau/2} + e^{-j\omega\tau/2} - 2\right]$$

$$\Rightarrow -\omega^2 F(\omega) = \frac{4}{\tau}\left[\cos\left(\frac{\omega\tau}{2}\right) - 1\right] = -\frac{8}{\tau}\sin^2\left(\frac{\omega\tau}{4}\right) \Rightarrow F(\omega) = \frac{8}{\tau\omega^2}\sin^2\left(\frac{\omega\tau}{4}\right) = \frac{8}{\tau\omega^2}\frac{\sin^2\left(\frac{\omega\tau}{4}\cdot\frac{\pi}{\pi}\right)}{\left(\frac{\omega\tau}{4\pi}\right)^2}\cdot\left(\frac{\omega\tau}{4\pi}\right)^2$$

$$\Rightarrow F(\omega) = \frac{\tau}{2}\cdot\left[\frac{\sin\left(\frac{\omega\tau}{4}\right)}{\left(\frac{\omega\tau}{4}\right)}\right]^2 = \frac{\tau}{2}\cdot \mathrm{sinc}^2\left(\frac{\omega\tau}{4\pi}\right)$$

Example:

Calculate the Fourier transform X(jw) for the signal x(t) .



x(t) = t **for-1<t<1**

Solution:

$y(t)=dx(t)/dt$

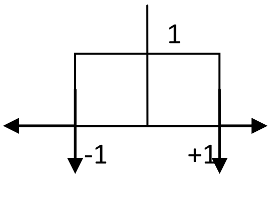
We know,

Analysis Equation is given by:

$$X(jw) = \int_{-\infty}^{\infty} x(t)\ e^{-jwt}\ dt$$

$$= \int_{-1}^{+1} t\ e^{-jwt}\ dt$$

To calculate derivative we need to calculate discontinuity, if discontinuity occurs we have impulse at that point,



Hence y(t) is the sum of a rectangular pulse and two impulses at -1 and +1

$$Y(jw) = -e^{-jw} - e^{jw} + \int_{-1}^{+1} e^{-jwt}\ dt$$

$$= -2\cos wt + \{2\ (e^{-jw} - e^{jw)}/-2jw\}$$

=-2cosw + 2sinw/w

Note that Y (0) = -2cos (0) + 2*1 {sin x/x=1}

=-2+2

=0

Using Integration Property, we obtain

$$X(jw)=Y(jw)/jw + \pi Y(0)\,\delta(w)$$

With Y (0) =0

We have

$$X(jw) = 2\sin w/jw^2 - 2\cos w/jw$$

This expression for X (jw) is purely imaginary and odd, which is consistent with the fact that x (t) is real and odd.

Rayleigh's Energy Theorem

Energy in time domain is equal to energy in frequency domain.

Energy $E = \int_{-\infty}^{\infty} |g(t)|^2\, dt = \int_{-\infty}^{\infty} |G(f)|^2\, df$

Consider the time domain energy

$$E = \int_{-\infty}^{\infty} |g(t)|^2\, dt = \int_{-\infty}^{\infty} g(t)g^*(t)dt$$

$$= \int_{-\infty}^{\infty} g(t)g^*(t)e^{-j2\pi ft}dt \;\Big|_{f=0}$$

$$= G(f)\otimes G^*(-f)\,|_{f=0}$$

$$= \int_{-\infty}^{\infty} G(\lambda)G^*(f-(-\lambda))d\lambda \;\Big|_{f=0}$$

$$= \int_{-\infty}^{\infty} G(\lambda)G^*(\lambda)d\lambda$$

$$= \int_{-\infty}^{\infty} |G(f)|^2\, df$$

Rayleigh's energy theorem or Parseval's theorem for Fourier transform

Define energy spectral density of $g(t)$ as $E_g(f) = |G(f)|^2$

Ex : consider a Sinc pulse

$$g(t) = ASinc(2Wt) \Leftrightarrow \frac{A}{2W}\Pi(\frac{f}{2W})$$

method (i):

$$E = \int_{-\infty}^{\infty} | g(t) |^2 \ dt = A^2 \int_{-\infty}^{\infty} g(t) g^*(t) dt$$

$$= A^2 \int_{-\infty}^{\infty} Sinc^2(2Wt) dt = A^2 \times \frac{1}{2W} \int_{-\infty}^{\infty} Sinc^2(u) du = \frac{A^2}{2W}$$

method(ii): Applying Rayleigh's energy theorem

$$E = (\frac{A}{2W})^2 \int_{-\infty}^{\infty} \Pi^2(\frac{f}{2W}) df = (\frac{A}{2W})^2 \int_{-W}^{W} df = \frac{A^2}{2W}$$

Parseval's Theorem:

$$\int_{-\infty}^{\infty} g_1(t) g_2{}^*(t) dt = \int_{-\infty}^{\infty} G_1(f) G_2{}^*(f) df$$

If $g_1(t) = g_2(t)$, then the theorem reduces to Rayleigh's energy theorem.

Laplace Transformation

Why Laplace Transforms?

1) Converts differential equations to algebraic equations- facilitates combination of multiple components in a system to get the total dynamic behavior (through addition & multiplication)

2) Can gain insight from the solution in the transform domain ("s")- inversion of transform not necessarily required

3) Allows development of an analytical model which permits use of a discontinuous (piecewise continuous) forcing function and the use of an integral term in the forcing function (important for control)

Definition of a Laplace Transform

$$F(s) = L[f(t)] = \int_{0}^{\infty} f(t) \ e^{-st} dt$$

$$L^{-1}[F(s)] = f(t)$$

Examples of Evaluating Laplace Transforms using the definition

(1) $x(t)$=1 and step function $x(t)=u(t)$

$$L[x(t)=u(t)]=\int_0^\infty x(t)e^{-st}dt=\int_0^\infty e^{-st}dt=-\frac{1}{s}\int_{t=0}^{t=\infty} e^{-st}d(-st)$$

$$=-\frac{e^{-st}}{s}\Bigg|_{t=0}^{t=\infty}=-\frac{e^{-\sigma t}}{s}e^{-j\omega t}\Bigg|_{t=0}^{t=\infty}=-\frac{e^{-\sigma\infty}}{s}e^{-j\omega\infty}+\frac{e^{-\sigma 0}}{s}e^{-j\omega 0}$$

$$(|e^{-j\omega\infty}|=1 \qquad e^{-\sigma\infty}=0, \quad (if \quad \sigma>0) \qquad e^{-\sigma\infty}\to\infty, \quad (if \quad \sigma<0)$$

$$=\frac{1}{s}(\cos 0-j\sin 0)=\frac{1}{s}$$

$$\Rightarrow L(1)=L[u(t)]=\frac{1}{s} \qquad\qquad (\mathrm{Re(s)})>0$$

(2) $x(t)=e^{-\alpha t}u(t)$

$$L[e^{-\alpha t}u(t)]=\int_0^\infty e^{-\alpha t}e^{-st}dt=\int_0^\infty e^{-(\alpha+s)t}dt$$

Define a new complex variable $\tilde{s}=s+\alpha$

$$\Rightarrow \int_0^\infty e^{-\tilde{s}t}dt$$

we know $\int_0^\infty e^{-st}dt=\frac{1}{s}$ $\qquad\qquad \mathrm{Re}(s)>0$

$$\Rightarrow \int_0^\infty e^{-\tilde{s}t}dt=\frac{1}{\tilde{s}} \qquad\qquad \mathrm{Re}(\tilde{s})>0$$

$$\Rightarrow \int_0^\infty e^{-(\alpha+s)t}dt=\frac{1}{s+\alpha} \qquad\qquad \mathrm{Re}(s+\alpha)>0$$

$$\Rightarrow L[e^{-\alpha t}u(t)]=\frac{1}{s+\alpha} \qquad\qquad \mathrm{Re}(s+\alpha)>0 \quad or \qquad \mathrm{Re}(s)>-\mathrm{Re}(\alpha)$$

$$\Rightarrow L[e^{-\alpha t}]=\frac{1}{s+\alpha}$$

(3) $x(t) = \delta(t)$

$$L[\delta(t)] = \int_0^{\infty} \delta(t) e^{-st} dt$$
$$= e^{-st}\Big|_{t=0} = e^{-\sigma t} e^{-j\omega t}\Big|_{t=0}$$
$$= e^{-\sigma t}(\cos\omega t - j\omega \sin\omega t)\Big|_{t=0}$$
$$= 1$$

No constraint on s.

(4)Find $L(\cos\omega_0 t)$

Key to solution : express $(\cos\omega_0 t)$ as linear combination of $\delta(t)$, $u(t)$,

and/or $e^{-\alpha t}$:

$$L[\delta(t)] = 1$$
$$L[u(t)] = \frac{1}{s}$$
$$L[e^{-\alpha t}] = \frac{1}{s+\alpha}$$

let $\alpha = j\omega_0$

$$L[e^{-j\omega_0 t}] = \frac{1}{s + j\omega_0}$$

let $\alpha = -j\omega_0$

$$L[e^{j\omega_0 t}] = L[e^{-(-j\omega_0)t}] = \frac{1}{s - j\omega_0}$$

Can we use $e^{-j\omega_0 t}$ and $e^{j\omega_0 t}$ to express $\cos(\omega_0 t)$?

$$e^{-j\omega_0 t} = \cos(-\omega_0 t) + j\sin(-\omega_0 t)$$
$$= \cos(\omega_0 t) - j\sin(\omega_0 t)$$
$$e^{j\omega_0 t} = \cos(\omega_0 t) + j\sin(\omega_0 t)$$

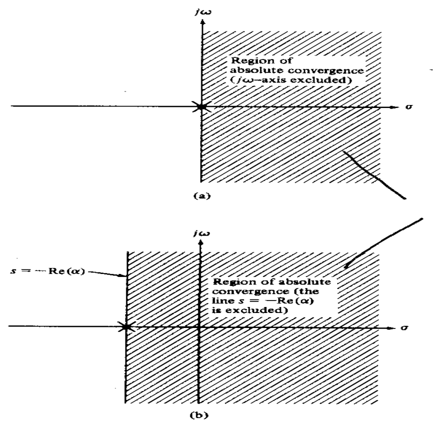
$$e^{-j\omega_0 t} + e^{j\omega_0 t} = 2\cos(\omega_0 t)$$
$$\Rightarrow \cos(\omega_0 t) = \frac{e^{-j\omega_0 t} + e^{j\omega_0 t}}{2}$$
$$\Rightarrow L[\cos(\omega_0 t)] = \frac{1}{2}[L(e^{-j\omega_0 t}) + L(e^{j\omega_0 t})]$$
$$= \frac{1}{2}\left[\frac{1}{s + j\omega_0} + \frac{1}{s - j\omega_0}\right]$$
$$= \frac{1}{2}\left[\frac{(s - j\omega_0) + (s + j\omega_0)}{(s + j\omega_0)(s - j\omega_0)}\right]$$
$$= \frac{s}{s^2 + {\omega_0}^2}$$

H.W. Find $L[\sin \omega_0 t]$

Convergence of the Laplace Transform

(1) To assure $\int_0^\infty x(t)e^{-st}dt = \int_0^\infty x(t)e^{-\sigma t}e^{-j\omega t}dt$ converge, $\sigma = \mathrm{Re}(s)$ must be psotive enough such that $x(t)e^{-\sigma t}$ goes to zero when t goes to positive infinite

(2) Region of absolute convergence and pole



For any s in the region, the Laplace transform is finite

FIGURE 5-1. Regions of absolute convergence for the Laplace transforms of (a) $u(t)$ and (b) $e^{-\alpha t}u(t)$.

(3) How to obtain Fourier transform form Laplace transform:

$$L[x(t)] = X(s) \overset{s=j\omega}{\Rightarrow} X(j\omega) = F(x(t))$$

Important: why introduce Laplace transform; definition of Laplace transform as a modification of Fourier transform; find the Laplace transforms of the three basic functions based on the (mathematical) definition of Laplace transform.

(4) Properties of Laplace Transform

I. Properties of Laplace Transform

Property	Original Function	Transformed Function
Linearity	$af(t)+bg(t)$	$aF(s)+bG(s)$
Shifting	$f(t-a)\,u(t-a)$	$e^{-as}F(s)$
	$e^{at}f(t)$	$F(s-a)$
Scaling	$f(at)$	$\frac{1}{a}F(\frac{s}{a})$
Differentiation	$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)-s^{n-3}f''(0)-\ldots-f^{(n-1)}(0)$

	$(-t)^n f(t)$	$\dfrac{d^n F(s)}{ds^n}$
Integration	$\int_0^t f(\tau)d\tau$	$\dfrac{1}{s}F(s)$
	$\dfrac{1}{t}f(t)$	$\int_s^\infty F(s)ds$
Convolution	$\int_0^t f(\tau)g(t-\tau)d\tau$	$F(s)G(s)$
Periodic Function	$f(t)=f(t+T)$	$\dfrac{1}{1-e^{-sT}}\int_0^T f(t)e^{-st}dt$
Initial Value Theorem	$\lim_{t\to 0} f(t)=\lim_{s\to\infty} sF(s)$	
	$\lim_{t\to 0}\dfrac{f(t)}{g(t)}=\lim_{s\to\infty}\dfrac{F(s)}{G(s)}$	
Final Value Theorem	$\lim_{t\to\infty} f(t)=\lim_{s\to 0} sF(s)$	
	$\lim_{t\to\infty}\dfrac{f(t)}{g(t)}=\lim_{s\to 0}\dfrac{F(s)}{G(s)}$	

1. Linearity

$$\mathcal{L}\,[af(t)+bg(t)]=\int_0^\infty [af(t)+bg(t)]e^{-st}dt=a\int_0^\infty f(t)e^{-st}dt+b\int_0^\infty g(t)e^{-st}dt=aF(s)+bG(s)$$

Ex. 1

Find the Laplace transform of $\cos^2 t$.

$$\text{Solution}:\mathcal{L}\,[\cos^2 t]=\mathcal{L}\,[\frac{1+\cos 2t}{2}]=\frac{1}{2}\left(\frac{1}{s}+\frac{s}{s^2+2^2}\right)=\frac{s^2+2}{s(s^2+4)}$$

---

2. Shifting

$$(a)\ \mathcal{L}\,[f(t-a)u(t-a)]=\int_0^\infty f(t-a)u(t-a)e^{-st}dt=\int_a^\infty f(t-a)e^{-st}dt$$

Let $\tau=t-a$, then

$$\mathcal{L}\,[f(t-a)u(t-a)]=\int_0^\infty f(\tau)e^{-s(\tau+a)}d\tau=e^{-sa}\int_0^\infty f(\tau)e^{-s\tau}d\tau=e^{-sa}F(s)$$

$$(b)\ F(s-a)=\int_0^\infty f(t)e^{-(s-a)t}dt=\int_0^\infty [e^{at}f(t)]e^{-st}dt=\mathcal{L}\,[e^{at}f(t)]$$

Ex. 2

What is the Laplace transform of the function: $f(t) = \begin{cases} 0, & t < 4 \\ 2t^3, & t \geq 4 \end{cases}$.

Solution: $f(t) = 2t^3 u(t-4)$

$$\mathcal{L}[f(t)] = \mathcal{L}\{2[(t-4)^3 + 12(t-4)^2 + 48(t-4) + 64]\, u(t-4)\}$$

$$= 2e^{-4s}\left(\frac{3!}{s^4} + 12\times\frac{2!}{s^3} + 48\times\frac{1}{s^2} + \frac{64}{s}\right) = 4e^{-4s}\left(\frac{3}{s^4} + \frac{12}{s^3} + \frac{24}{s^2} + \frac{32}{s}\right)$$

---

3. Scaling

$$\mathcal{L}\,[f(at)] = \int_0^\infty f(at)e^{-st}\,dt$$

Let $\tau = at$, then

$$\mathcal{L}\,[f(at)] = \int_0^\infty f(\tau)e^{-s\frac{\tau}{a}}\,d\frac{\tau}{a} = \frac{1}{a}\int_0^\infty f(\tau)e^{-\frac{s}{a}\tau}\,d\tau = \frac{1}{a}F(\frac{s}{a})$$

Ex. 3

Find the Laplace transform of cos2t.

Solution : $\because \mathcal{L}\,[\cos t] = \dfrac{s}{s^2+1}$

$$\therefore \mathcal{L}\,[\cos 2t] = \frac{1}{2}\,\frac{\frac{s}{2}}{(\frac{s}{2})^2+1} = \frac{s}{s^2+4}$$

---

4. Derivative

(a) Derivative of original function

$$\mathcal{L}\,[f'(t)] = \int_0^\infty f'(t)e^{-st}\,dt = f(t)e^{-st}\Big|_0^\infty - (-s)\int_0^\infty f(t)e^{-st}\,dt \qquad ($$

(1) If $f(t)$ is continuous, equation (2.1) reduces to

$$\mathcal{L}\,[f'(t)] = -f(0) + sF(s) = sF(s) - f(0)$$

(2) If $f(t)$ is not continuous at $t = a$, equation (2.1) reduces to

$$\mathcal{L}\,[f'(t)] = f(t)e^{-st}\Big|_0^{a^-} + f(t)e^{-st}\Big|_{a^+}^{\infty} + sF(s) = [f(a^-)\,e^{-sa} - f(0)] + [0 - f(a^+)\,e^{-sa}] + sF(s)$$

$$= sF(s) - f(0) - e^{-sa}\,[f(a^+) - f(a^-)]$$

(3) Similarly, if $f(t)$ is not continuous at $t = a_1, a_2, \ldots, \ldots, a_n$, equation (2.1) reduces to

$$\mathcal{L}\,[f'(t)] = sF(s) - f(0) - \sum_{i=1}^{n} e^{-sa_i}\,[f(a_i^+) - f(a_i^-)]$$

[Deduction] If $f(t), f'(t)$, $f''(t)$, …, $f^{(n-1)}(t)$ are continuous, and $f^{(n)}(t)$ is piecewise continuous, and all of them are exponential order functions, then

$$\mathcal{L}\,[f^{(n)}(t)] = s^n F(s) - \sum_{i=1}^{n} s^{n-i} f^{(i-1)}(0)$$

(b) Derivative of transformed function

$$\frac{dF(s)}{ds} = \frac{d}{ds}\int_0^\infty f(t)e^{-st}dt = \int_0^\infty \frac{\partial}{\partial s}[f(t)e^{-st}]dt = \int_0^\infty (-t)f(t)e^{-st}dt = \mathcal{L}\ [(-t)f(t)]$$

[Deduction] $\dfrac{d^nF(s)}{ds^n} = \mathcal{L}\ [(-t)^n f(t)]$

Ex. 4

Find the Laplace transform of te^t.

Solution : $\mathcal{L}\ (e^t) = \dfrac{1}{s-1} \Rightarrow \mathcal{L}\ (te^t) = -\dfrac{d}{ds}\left(\dfrac{1}{s-1}\right) = \dfrac{1}{(s-1)^2}$

Ex. 5

$f(t) = \begin{cases} t^2, & 0 \le t \le 1 \\ 0, & t > 1 \end{cases}$, find $\mathcal{L}\ [f'(t)]$.

Solution : $f(t) = t^2[u(t) - u(t-1)]$

$$\begin{aligned}
\mathcal{L}\ [f(t)] &= \mathcal{L}\ [t^2u(t)] - \mathcal{L}\ [t^2u(t-1)] = \frac{2!}{s^3} - \mathcal{L}\ \{[(t-1)+1]^2u(t-1)\} \\
&= \frac{2}{s^3} - \mathcal{L}\ \{[(t-1)^2 + 2(t-1) + 1]u(t-1)\} \\
&= \frac{2}{s^3} - e^{-s}\left(\frac{2}{s^3} + 2\frac{1}{s^2} + \frac{1}{s}\right) \\
\mathcal{L}\ [f'(t)] &= sF(s) - f(0) - e^{-s}[f(1^+) - f(1^-)] \\
&= \left[\frac{2}{s^2} - e^{-s}\left(\frac{2}{s^2} + \frac{2}{s} + 1\right)\right] - 0 - e^{-s}(0-1) = \frac{2}{s^2} - e^{-s}\left(\frac{2}{s^2} + \frac{2}{s}\right)
\end{aligned}$$

5. Integration

(a) Integral of original function

$$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \int_0^\infty \int_0^t f(\tau)d\tau e^{-st}dt$$

$$= \frac{1}{-s}\left[e^{-st}\int_0^t f(\tau)d\tau \Big|_0^\infty - \int_0^\infty f(t)e^{-st}dt \right] = \frac{1}{s}F(s)$$

$$\Rightarrow \mathcal{L}\left[\int_0^t\int_0^t \cdots \int_0^t f(t)dtdt\cdots dt\right] = \frac{1}{s^n}F(s)$$

(b) Integration of Laplace transform

$$\int_s^\infty F(s)ds = \int_s^\infty \int_0^\infty f(t)e^{-st}dtds = \int_0^\infty f(t)\int_s^\infty e^{-st}dsdt$$

$$= \int_0^\infty f(t)\frac{e^{-st}}{-t}\Big|_s^\infty dt = \int_0^\infty \frac{f(t)}{t}e^{-st}dt = \mathcal{L}\left[\frac{f(t)}{t}\right]$$

$$\Rightarrow \int_s^\infty \int_s^\infty \cdots \int_s^\infty F(s)dsds\cdots ds = \mathcal{L}\left[\frac{1}{t^n}f(t)\right]$$

Ex. 6

Find (a) $\mathcal{L}\left[\frac{1-e^{-t}}{t}\right]$ (b) $\mathcal{L}\left[\frac{1-e^{-t}}{t^2}\right]$.

Solution : (a) $\mathcal{L}[1-e^{-t}] = \frac{1}{s} - \frac{1}{s+1}$

$$\mathcal{L}\left[\frac{1-e^{-t}}{t}\right] = \int_s^\infty \left(\frac{1}{s} - \frac{1}{s+1}\right)ds = \ln s - \ln(s+1)\Big|_s^\infty = \ln\frac{s}{s+1}\Big|_s^\infty$$

$$= 0 - \ln\frac{s}{s+1} = \ln\frac{s+1}{s}$$

$$(b)\ \mathcal{L}\left[\frac{1-e^{-t}}{t^2}\right] = \int_s^\infty \ln\frac{s+1}{s}ds = s\ln\frac{s+1}{s}\Big|_s^\infty - \int_s^\infty s\left(\frac{1}{s+1} - \frac{1}{s}\right)ds$$

$$= s\ln\frac{s+1}{s}\Big|_s^\infty + \int_s^\infty \frac{1}{s+1}ds = \left[s\ln\frac{s+1}{s} + \ln(s+1)\right]_s^\infty$$

$$= \left[(s+1)\ln(s+1) - s\ln s\right]_s^\infty = s\ln s - (s+1)\ln(s+1)$$

Ex. 7

Find (a) $\int_0^\infty \frac{\sin kt\, e^{-st}}{t}dt$ (b) $\int_{-\infty}^\infty \frac{\sin x}{x}dx$.

$$\text{Solution : }(a)\int_0^{\infty}\frac{\sin kte^{-st}}{t}dt = \mathcal{L}\left[\frac{\sin kt}{t}\right]$$

$$\because \mathcal{L}[\sin kt] = \frac{k}{s^2+k^2}$$

$$\mathcal{L}\left[\frac{\sin kt}{t}\right] = \int_s^{\infty}\frac{k}{s^2+k^2}ds = \frac{1}{k}\int_s^{\infty}\frac{1}{(\frac{s}{k})^2+1}ds$$

$$= \tan^{-1}\frac{s}{k}\bigg|_s^{\infty} = \frac{\pi}{2} - \tan^{-1}\frac{s}{k}$$

$$(b)\int_{-\infty}^{\infty}\frac{\sin x}{x}dx = 2\int_0^{\infty}\frac{\sin x}{x}dx$$

$$= 2\lim_{\substack{k\to 1\\ s\to 0}}\int_0^{\infty}\frac{\sin kte^{-st}}{t}dt$$

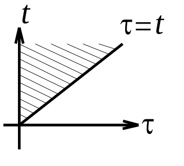
$$= 2\lim_{\substack{k\to 1\\ s\to 0}}\left(\frac{\pi}{2} - \tan^{-1}\frac{s}{k}\right) = \pi$$

---

6. Convolution theorem

$$\mathcal{L}\left[\int_0^t f(\tau)g(t-\tau)d\tau\right] = \int_0^{\infty}\int_0^t f(\tau)g(t-\tau)d\tau e^{-st}dt$$

$$= \int_0^{\infty}\int_{\tau}^{\infty} f(\tau)g(t-\tau)e^{-st}dtd\tau = \int_0^{\infty} f(\tau)\int_{\tau}^{\infty} g(t-\tau)e^{-st}dtd\tau$$



Let $u = t-\tau$, $du = dt$, then

$$\mathcal{L}\left[\int_0^t f(\tau)g(t-\tau)d\tau\right] = \int_0^{\infty} f(\tau)\int_0^{\infty} g(u)e^{-s(u+\tau)}dud\tau$$

$$= \int_0^{\infty} f(\tau)e^{-s\tau}d\tau\int_0^{\infty} g(u)e^{-su}du = F(s)G(s)$$

Ex. 8

Find the Laplace transform of $\int_0^t e^{t-\tau}\sin 2\tau\, d\tau$.

$$\text{Solution : }\because \mathcal{L}[e^t] = \frac{1}{s-1},\ \mathcal{L}[\sin 2t] = \frac{2}{s^2+4}$$

$$\therefore \mathcal{L}\left[\int_0^t e^{t-\tau}\sin 2t\, d\tau\right] = \mathcal{L}[e^t * \sin 2t] = \mathcal{L}[e^t]\cdot\mathcal{L}[\sin 2t]$$

$$= \frac{1}{s-1}\cdot\frac{2}{s^2+4} = \frac{2}{(s-1)(s^2+4)}$$

7. Periodic Function: $f(t+T) = f(t)$

$$\mathcal{L}[f(t)] = \int_0^{\infty} f(t)e^{-st}dt = \int_0^T f(t)e^{-st}dt + \int_T^{2T} f(t)e^{-st}dt + \cdots\cdots$$

$$\text{and } \int_T^{2T} f(t)e^{-st}dt = \int_0^T f(u+T)e^{-s(u+T)}du = e^{-sT}\int_0^T f(u)e^{-su}du$$

Similarly,

$$\int_{2T}^{3T} f(t)e^{-st}dt = e^{-2sT}\int_0^T f(u)e^{-su}du$$

$$\therefore \mathcal{L}\ [f(t)] = (1+e^{-sT}+e^{-2sT}+\cdots\cdots)\int_0^T f(t)e^{-st}dt$$

$$= \frac{1}{1-e^{-sT}}\int_0^T f(t)e^{-st}dt$$

Ex. 9

Find the Laplace transform of $f(t)=\frac{k}{p}t,\ 0<t<p,\ f(t+p)=f(t)$.

$$\text{Solution}: \mathcal{L}\ [f(t)] = \frac{1}{1-e^{-ps}}\int_0^p \frac{k}{p}te^{-st}dt$$

$$= \frac{1}{1-e^{-ps}}\frac{k}{p}[\frac{1}{-s}(te^{-st}\Big|_0^p - \int_0^p e^{-st}dt)]$$

$$= \frac{-k}{ps(1-e^{-ps})}(te^{-st}+\frac{1}{s}e^{-st})\Big|_0^p$$

$$= \frac{-k}{ps(1-e^{-ps})}(pe^{-sp}+\frac{e^{-sp}}{s}-\frac{1}{s})$$

---

8. ***Initial Value Theorem***:

$$\because \mathcal{L}\ [f'(t)] = sF(s)-f(0) \Rightarrow \lim_{s\to\infty}\int_0^\infty f'(t)e^{-st}dt = \lim_{s\to\infty} sF(s)-f(0) \Rightarrow 0 = \lim_{s\to\infty} sF(s)-f(0)$$

we get initial value theorem $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$ ***9.***

Deduce general initial value theorem: $\lim_{t\to 0}\frac{f(t)}{g(t)} = \lim_{s\to\infty}\frac{F(s)}{G(s)}$

Final Value Theorem:

$$\mathcal{L}\ [f'(t)] = sF(s)-f(0) \Rightarrow \lim_{s\to 0}\int_0^\infty f'(t)e^{-st}dt = \lim_{s\to 0} sF(s)-f(0) \Rightarrow$$

$\lim_{t\to\infty} f(t)-f(0) = \lim_{s\to 0} sF(s)-f(0) \Rightarrow$ final value theorem: $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

General final value theorem: $\lim_{t\to\infty}\frac{f(t)}{g(t)} = \lim_{s\to 0}\frac{F(s)}{G(s)}$

Ex. 10

Find $\mathcal{L}\ [\int_0^t \frac{\sin x}{x}dx]$.

$$\text{Solution : Let } f(t)=\int_0^t \frac{\sin x}{x}dx \Rightarrow f'(t)=\frac{\sin t}{t},\ f(0)=0$$

$$\mathcal{L}\ [tf'(t)]=\mathcal{L}\ [\sin t]=\frac{1}{s^2+1}$$

$$-\frac{d}{ds}\mathcal{L}\ [f'(t)]=\frac{1}{s^2+1}$$

$$-\frac{d}{ds}[sF(s)-f(0)]=\frac{1}{s^2+1}\Rightarrow\frac{d}{ds}[sF(s)]=-\frac{1}{s^2+1}$$

$$sF(s)=-\tan^{-1}s+C$$

From the initial value theorem, we get

$$\lim_{t\to 0}f(t)=\lim_{s\to\infty}sF(s)$$

$$0=-\frac{\pi}{2}+C\quad\therefore C=\frac{\pi}{2}$$

$$sF(s)=\frac{\pi}{2}-\tan^{-1}s=\tan^{-1}\frac{1}{s}$$

$$F(s)=\frac{1}{s}\tan^{-1}\frac{1}{s}$$

EG. Find $\mathcal{L}\left[\int_t^\infty \frac{e^{-x}}{x}dx\right]$.

$$\text{Solution : Let } f(t)=\int_x^\infty \frac{e^{-x}}{x}dx \Rightarrow f'(t)=-\frac{e^{-t}}{t},\ \lim_{t\to\infty}f(t)=0$$

$$\mathcal{L}\ [tf'(t)]=\mathcal{L}\ [-e^{-t}]=-\frac{1}{s+1}$$

$$-\frac{d}{ds}[sF(s)-f(0)]=-\frac{1}{s+1}$$

$$\frac{d}{ds}[sF(s)]=\frac{1}{s+1}$$

$$sF(s)=\ln(s+1)+C$$

$$\text{From the final value theorem : }\lim_{t\to\infty}f(t)=\lim_{s\to 0}sF(s)$$

$$0=0+C\Rightarrow C=0,\text{ and } F(s)=\frac{\ln(s+1)}{s}$$

[Note] $\int_0^t \frac{\sin x}{x}dx$, and $\int_t^\infty \frac{e^{-x}}{x}dx$ are called sine, and exponential integral function, respectively.

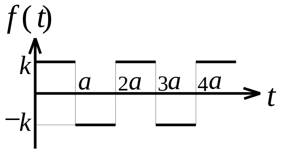
[Exercises] Find the Laplace transform of the problems:1–4, and 7

1. $e^{-\alpha t}(A\cos\beta t + B\sin\beta t)$ 2. $t^2\cos t$ 3. $u(t-\pi)\cos t$

4. $\int_t^\infty \frac{\cos x}{x}dx$ (cosine integral function) 5. Find the value of the integral $\int_0^\infty te^{-2t}\cos t dt$

6. Find the value of the integral $\int_0^\infty \frac{e^{-t}-e^{-3t}}{t}dt$

7.



[Ans.] 1. $\frac{A(s+\alpha)+B\beta}{(s+\alpha)^2+\beta^2}$ 2. $\frac{2s(s^2-3)}{(s^2+1)^3}$ 3. $\frac{-se^{-\pi s}}{s^2+1}$

4. $\frac{\ln(s^2+1)}{2s}$ 5. $\frac{3}{25}$ 6. $\ln 3$ 7. $\frac{k}{s}\tanh\frac{as}{2}$

Inverse Laplace Transform

I. Inversion from Basic Properties

1. Linearity

Ex. 1.

$(a)\mathcal{L}^{-1}[\frac{2s+1}{s^2+4}]$ $(b)\mathcal{L}^{-1}[\frac{4(s+1)}{s^2-16}]$.

Solution: $(a)\mathcal{L}^{-1}[\frac{2s+1}{s^2+4}] = \mathcal{L}^{-1}[2\frac{s}{s^2+2^2}+\frac{1}{2}\frac{2}{s^2+2^2}] = 2\cos 2t + \frac{1}{2}\sin 2t$

$(b)\mathcal{L}^{-1}[\frac{4(s+1)}{s^2-16}] = \mathcal{L}^{-1}[4\frac{s}{s^2-4^2}+\frac{4}{s^2-4^2}] = 4\cosh 4t + \sinh 4t$

2. Shifting

Ex. 2.

$(a)\mathcal{L}^{-1}[\frac{e^{-\pi s}}{s^2+2s+2}]$ $(b)\mathcal{L}^{-1}[\frac{2s+3}{s^2+3s+2}]$.

Solution : $(a)\ \mathcal{L}^{-1}[\frac{e^{-\pi s}}{s^2+2s+2}]=\mathcal{L}^{-1}[\frac{e^{-\pi s}}{(s+1)^2+1}]$

$$\because \mathcal{L}^{-1}[\frac{1}{(s+1)^2+1}]=e^{-t}\sin t$$

and $\mathcal{L}[f(t-a)u(t-a)]=e^{-as}F(s)$

$$\therefore \mathcal{L}^{-1}[\frac{e^{-\pi s}}{(s+1)^2+1}]=e^{-(t-\pi)}\sin(t-\pi)u(t-\pi)=-e^{-(t-\pi)}\sin t u(t-\pi)$$

$$(b)\ \mathcal{L}^{-1}[\frac{2s+3}{s^2+3s+2}]=\mathcal{L}^{-1}[\frac{2(s+\frac{3}{2})}{(s+\frac{3}{2})^2-(\frac{1}{2})^2}]=2e^{-\frac{3}{2}t}\cosh\frac{t}{2}$$

3. Scaling

Ex. 3.

$\mathcal{L}^{-1}[\frac{4s}{16s^2-4}]$.

Solution : $\mathcal{L}^{-1}[\frac{4s}{16s^2-4}]=\mathcal{L}^{-1}[\frac{4s}{(4s)^2-2^2}]=\frac{1}{4}\cosh 2\cdot\frac{1}{4}t=\frac{1}{4}\cosh\frac{t}{2}$

4. Derivative

Ex. 4.

$(a)\mathcal{L}^{-1}[\frac{1}{(s^2+\omega^2)^2}]$ $\qquad$ $(b)\mathcal{L}^{-1}[\ln\frac{s+a}{s+b}]$.

solution : $(a)\mathcal{L}[\sin\omega t]=\frac{\omega}{s^2+\omega^2}\Rightarrow\mathcal{L}[t\sin\omega t]=-\frac{d}{ds}(\frac{\omega}{s^2+\omega^2})=\frac{2\omega s}{(s^2+\omega^2)^2}$

Let $F(t)=t\sin\omega t\Rightarrow\mathcal{L}[F'(t)]=s\cdot\frac{2\omega s}{(s^2+\omega^2)^2}-F(0)$

$$\mathcal{L}[F'(t)]=2\omega\frac{s^2}{(s^2+\omega^2)^2}=2\omega[\frac{(s^2+\omega^2)-\omega^2}{(s^2+\omega^2)^2}]=2\omega[\frac{1}{s^2+\omega^2}-\frac{\omega^2}{(s^2+\omega^2)^2}]$$

$$=2\mathcal{L}[\sin\omega t]-\frac{2\omega^3}{(s^2+\omega^2)^2}$$

$$\frac{1}{(s^2+\omega^2)^2}=\frac{1}{2\omega^3}\cdot\mathcal{L}[2\sin\omega t-F'(t)]$$

$$\mathcal{L}^{-1}[\frac{1}{(s^2+\omega^2)^2}]=\frac{1}{2\omega^3}\cdot[2\sin\omega t-F'(t)]=\frac{1}{2\omega^3}(\sin\omega t-\omega t\cos\omega t)$$

(b) Let $\mathcal{L}[f(t)]=\ln\frac{s+a}{s+b}=\ln(s+a)-\ln(s+b)$

$$\mathcal{L}[tf(t)]=-\frac{d}{ds}[\ln(s+a)-\ln(s+b)]=\frac{1}{s+b}-\frac{1}{s+a}=\mathcal{L}[e^{-bt}-e^{-at}]$$

$$\therefore f(t)=\frac{e^{-bt}-e^{-at}}{t}$$

5. Integration

Ex. 5.

$$(a)\mathcal{L}^{-1}[\frac{1}{s^2}(\frac{s-1}{s+1})] \qquad (b)\mathcal{L}^{-1}[\ln\frac{s+a}{s+b}].$$

$$\text{Solution : }(a)\mathcal{L}^{-1}[\frac{1}{s^2}(\frac{s-1}{s+1})] = \mathcal{L}^{-1}[\frac{1}{s(s+1)} - \frac{1}{s^2(s+1)}] = \int_0^t e^{-t}dt - \int_0^t\int_0^t e^{-t}dtdt$$

$$= -(e^{-t}-1) + \int_0^t (e^{-t}-1)dt = -(e^{-t}-1) - (e^{-t}-1) - t = 2 - 2e^{-t} - t$$

$$(b)\mathcal{L}[e^{-bt} - e^{-at}] = \frac{1}{s+b} - \frac{1}{s+a}$$

$$\mathcal{L}[\frac{e^{-bt}-e^{-at}}{t}] = \int_s^\infty (\frac{1}{s+b} - \frac{1}{s+a})ds = \ln\frac{s+b}{s+a}\bigg|_s^\infty = \ln\frac{s+a}{s+b}$$

$$\therefore \mathcal{L}^{-1}[\ln\frac{s+a}{s+b}] = \frac{e^{-bt}-e^{-at}}{t}$$

6. Convolution

Ex. 6.

$$(a)\mathcal{L}^{-1}[\frac{1}{(s^2+\omega^2)^2}] \qquad (b)\mathcal{L}^{-1}[\frac{s}{(s^2+\omega^2)^2}].$$

$$\text{Solution : }(a)\mathcal{L}[\sin\omega t] = \frac{\omega}{s^2+\omega^2} \Rightarrow \mathcal{L}[\frac{1}{\omega}\sin\omega t] = \frac{1}{s^2+\omega^2}$$

$$\mathcal{L}^{-1}[\frac{1}{(s^2+\omega^2)^2}] = \frac{1}{\omega^2}\int_0^t \sin\omega\tau\sin\omega(t-\tau)d\tau$$

$$= \frac{1}{\omega^2}\int_0^t \frac{1}{2}[\cos(\omega\tau - \omega t + \omega\tau) - \cos(\omega\tau + \omega t - \omega\tau)]d\tau$$

$$= \frac{1}{2\omega^2}\int_0^t [\cos(2\omega\tau - \omega t) - \cos\omega t]d\tau = \frac{1}{2\omega^2}\left[\frac{1}{2\omega}\sin(2\omega\tau - \omega t) - \tau\cos\omega t\right]_0^t$$

$$= \frac{1}{2\omega^2}\{[\frac{1}{2\omega}(\sin\omega t - \sin(-\omega t)] - t\cos\omega t\} = \frac{1}{2\omega^3}(\sin\omega t - \omega t\cos\omega t)$$

$$(b)\mathcal{L}[\frac{1}{\omega}\sin\omega t] = \frac{1}{s^2+\omega^2} \qquad \mathcal{L}[\cos\omega t] = \frac{s}{s^2+\omega^2}$$

$$\mathcal{L}^{-1}[\frac{s}{(s^2+\omega^2)^2}] = \frac{1}{\omega}\int_0^t \sin\omega\tau\cos\omega(t-\tau)d\tau$$

$$= \frac{1}{\omega}\int_0^t \frac{1}{2}[\sin(\omega\tau + \omega t - \omega\tau) + \sin(\omega\tau - \omega t + \omega\tau)]d\tau$$

$$= \frac{1}{2\omega}\int_0^t [\sin\omega t + \sin(2\omega\tau - \omega t)]d\tau = \frac{1}{2\omega}\left[\tau\sin\omega t + \frac{-1}{2\omega}\cos(2\omega\tau - \omega t)\right]_0^t$$

$$= \frac{1}{2\omega}\{t\sin\omega t - \frac{1}{2\omega}[\cos\omega t - \cos(-\omega t)]\} = \frac{t}{2\omega}\sin\omega t$$

[Exercises] 1. $\mathcal{L}^{-1}[s^{-3/2}]$ 2. $\mathcal{L}^{-1}[\dfrac{2n\pi T}{T^2s^2+(2n\pi)^2}]$ 3. $\mathcal{L}^{-1}[\dfrac{s-4}{s^2-8s-9}]$

4. $\mathcal{L}^{-1}[\dfrac{se^{-2\pi s/3}}{s^2+1}]$ 5. $\mathcal{L}^{-1}[\dfrac{1}{s}\tan^{-1}\dfrac{1}{s}]$ 6. $\mathcal{L}^{-1}[\ln\dfrac{s-1}{s+1}]$ 7. $\mathcal{L}^{-1}[\dfrac{s+1}{s^2+s+1}]$

8. $\mathcal{L}^{-1}[\ln(1+\dfrac{1}{s^2})]$ 9. $\mathcal{L}^{-1}[\dfrac{1}{s^2(s+1)^2}]$ 10. $\mathcal{L}^{-1}[\dfrac{\sqrt{s}+1}{s^3}]$

[Ans.] 1. $2\sqrt{t}/\Gamma(\dfrac{1}{2})$ or $2\sqrt{\dfrac{t}{\pi}}$ 2. $\sin\dfrac{2n\pi t}{\mathrm{T}}$ 3. $e^{4t}\cosh 5t$ 4. $\cos(t-\dfrac{2\pi}{3})u(t-\dfrac{2\pi}{3})$

5. $\displaystyle\int_0^t \frac{\sin u}{u}\,du$ 6. $\dfrac{2\sinh t}{t}$ 7. $e^{-\frac{t}{2}}(\cos\dfrac{\sqrt{3}}{2}t+\dfrac{1}{\sqrt{3}}\sin\dfrac{\sqrt{3}}{2}t)$ 8. $\dfrac{2(1-\cos t)}{t}$

9. $e^{-t}(t+2)+t-2$ 10. $\dfrac{4}{3\sqrt{\pi}}t^{3/2}+\dfrac{1}{2}t^2$

II. Partial Fraction

Ex. 7.

$$\mathcal{L}^{-1}[\frac{s+1}{s^3+s^2-6s}].$$

Solution : $\dfrac{s+1}{s^3+s^2-6s}=\dfrac{s+1}{s(s-2)(s+3)}=\dfrac{A_1}{s}+\dfrac{A_2}{s-2}+\dfrac{A_3}{s+3}$

$$A_1=\lim_{s\to 0}\frac{s+1}{(s-2)(s+3)}=-\frac{1}{6}$$

$$A_2=\lim_{s\to 2}\frac{s+1}{s(s+3)}=\frac{3}{10}$$

$$A_3=\lim_{s\to -3}\frac{s+1}{s(s-2)}=\frac{-2}{15}$$

$$\mathcal{L}^{-1}[\frac{s+1}{s^3+s^2-6s}]=\frac{-\frac{1}{6}}{s}+\frac{\frac{3}{10}}{s-2}+\frac{\frac{-2}{15}}{s+3}=-\frac{1}{6}+\frac{3}{10}e^{2t}-\frac{2}{15}e^{-3t}$$

$$\mathcal{L}^{-1}[\frac{s^4-7s^3+13s^2+4s-12}{s^2(s-1)(s-2)(s-3)}].$$

$$\text{Solution}: \frac{s^4-7s^3+13s^2+4s-12}{s^2(s-1)(s-2)(s-3)}=\frac{C_2}{s^2}+\frac{C_1}{s}+\frac{A_1}{s-1}+\frac{A_2}{s-2}+\frac{A_3}{s-3}$$

$$C_2=\lim_{s\to 0}\frac{s^4-7s^3+13s^2+4s-12}{(s-1)(s-2)(s-3)}=\frac{-12}{-6}=2$$

$$C_1=\lim_{s\to 0}\frac{d}{ds}[\frac{s^4-7s^3+13s^2+4s-12}{(s-1)(s-2)(s-3)}]$$

$$=\frac{4(-1)(-2)(-3)-(-12)[(-2)(-3)+(-1)(-3)+(-1)(-2)]}{[(-1)(-2)(-3)]^2}=\frac{-24+12\times 11}{6^2}=3$$

$$A_1=\lim_{s\to 1}\frac{s^4-7s^3+13s^2+4s-12}{s^2(s-2)(s-3)}=\frac{-1}{2}$$

$$A_2=\lim_{s\to 2}\frac{s^4-7s^3+13s^2+4s-12}{s^2(s-1)(s-3)}=\frac{8}{-4}=-2$$

$$A_3=\lim_{s\to 3}\frac{s^4-7s^3+13s^2+4s-12}{s^2(s-1)(s-2)}=\frac{9}{18}=\frac{1}{2}$$

$$\mathcal{L}^{-1}[\frac{s^4-7s^3+13s^2+4s-12}{s^2(s-1)(s-2)(s-3)}]=2t+3-\frac{1}{2}e^t-2e^{2t}+\frac{1}{2}e^{3t}$$

Ex. 8.

$$\mathcal{L}^{-1}[\frac{s^2}{s^4+4}].$$

$$\text{Solution}: \frac{s^2}{s^4+4}=\frac{s^2}{(s^2)^2+2\cdot s^2\cdot 2+2^2-2\cdot s^2\cdot 2}=\frac{s^2}{(s^2+2)^2-(2s)^2}$$

$$=\frac{s^2}{(s^2+2s+2)(s^2-2s+2)}=\frac{A_1 s+B_1}{(s+1)^2+1}+\frac{A_2 s+B_2}{(s-1)^2+1}$$

$$\lim_{s\to -1+i}\frac{s^2}{(s-1)^2+1}=A_1(-1+i)+B_1\Rightarrow\frac{-2i}{4-4i}=(-A_1+B_1)+iA_1$$

$$\frac{8-8i}{32}=(-A_1+B_1)+iA_1\Rightarrow A_1=-\frac{1}{4},B_1=0$$

$$\lim_{s\to 1+i}\frac{s^2}{(s+1)^2+1}=A_2(1+i)+B_2\Rightarrow\frac{2i}{4+4i}=(A_2+B_2)+iA_2$$

$$\frac{8+8i}{32}=(A_2+B_2)+iA_2\Rightarrow A_2=\frac{1}{4},B_2=0$$

$$\mathcal{L}^{-1}[\frac{s^2}{s^4+4}]=\mathcal{L}^{-1}[\frac{-\frac{1}{4}(s+1)+\frac{1}{4}}{(s+1)^2+1}+\frac{\frac{1}{4}(s-1)+\frac{1}{4}}{(s-1)^2+1}]$$

$$=\frac{e^{-t}}{4}(-\cos t+\sin t)+\frac{e^t}{4}(\cos t+\sin t)$$

Ex. 9.

$$\mathcal{L}^{-1}[\frac{s^3-3s^2+6s-4}{(s^2-2s+2)^2}].$$

$$\text{Solution : }\frac{s^3-3s^2+6s-4}{(s^2-2s+2)^2}=\frac{As+B}{[(s-1)^2+1]^2}+\frac{cs+D}{(s-1)^2+1}$$

$$\lim_{s\to 1+i}(s^3-3s^2+6s+4)=A(1+i)+B$$

$$2i=(A+B)+iA\Rightarrow A=2, B=-2$$

$$\lim_{s\to 1+i}\frac{d}{ds}(s^3-3s^2+6s+4)=A+[c(1+i)+D]\lim_{s\to 1+i}\frac{d}{ds}[(s-1)^2+1]$$

$$0=A+(c+ic+D)2i=(A-2c)+2i(c+D)$$

$$c=1, D=-1$$

$$\mathcal{L}^{-1}[\frac{s^3-3s^2+6s-4}{(s^2-2s+2)^2}]=\mathcal{L}^{-1}\{\frac{2(s-1)}{[(s-1)^2+1]^2}\}+\mathcal{L}^{-1}[\frac{s-1}{(s-1)^2+1}]$$

$$=e^t(2\cdot\frac{t}{2}\sin t+\cos t)=e^t(t\sin t+\cos t)$$

[Exercises] 1. $\mathcal{L}^{-1}[\dfrac{s+1}{bs^2+7s+2]}$ 2. $\mathcal{L}^{-1}[\dfrac{s-1}{(s+3)(s^2+2s+2)}]$

3. $\mathcal{L}^{-1}[\dfrac{s}{(s^2-2s+2)(s^2+2s+2)}]$ 4. $\mathcal{L}^{-1}[\dfrac{11s^3-47s^2+56s+4}{(s-2)^3(s+2)}]$

5. $\mathcal{L}^{-1}[\dfrac{s^2}{s^4+4}]$ 6. $\mathcal{L}^{-1}[\dfrac{1}{(s^2-1)^3}]$

[Ans] 1. $\dfrac{1}{2}e^{-\frac{t}{2}}-\dfrac{1}{3}e^{-\frac{2}{3}t}$ 2. $\dfrac{1}{5}e^{-t}(4\cos t-3\sin t)-\dfrac{4}{5}e^{-3t}$ 3. $\dfrac{1}{2}\sin t\sinh t$

4. $e^{2t}(2t^2-t+5)+6e^{-2t}$ 5. $\dfrac{1}{2}(\cosh t\sin t+\sinh t\cos t)$ 6. $\dfrac{1}{8}[(3+t^2)\sinh t-3t\cosh t]$

UNIT III

ANALYSIS OF DISCRETE TIME SIGNALS

Discrete Time Fourier Transform

Discrete Time Fourier Transform.

The Discrete Time Fourier Transform (DTFT) $X(e^{jw})$ of a discrete line signal x(n) is expressed as

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x(n)e^{-jwn}$$

or DTFT $x(n) = X(e^{jw})$

Symbolically, this may be expressed as

$$x(n) \xleftarrow{\text{DTFT}} X(e^{jw})$$

DTFT is periodic units period 2π . So any interval of length 2π is sufficient for the complete specification of the spectrum. Generally, we draw the spectrum in the fundamental internal $(-\pi, \pi)$

---

linearity property of DTFT

If

$$x_1(n) \xleftrightarrow{\text{DTFT}} x_1(w)$$

$$x_2(n) \xleftrightarrow{\text{DTFT}} x_2(w)$$

$$a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow{\text{DTFT}} a_1 x_1(w) + a_2 x_2(w)$$

According to definition of DTFT

$$x(w) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-jwn}$$

Here input sequence, $x(n) = a_1 x_1(n) + a_2 x_2(n)$

$$\therefore \quad x(w) = a_1 \sum_{n=-\infty}^{\infty} [a_1 x_1(n) + a_2 x_2(n)]\, e^{-jwn}$$

$$x(w) = a_1 \sum_{n=-\infty}^{\infty} x_1(n) e^{-jwn} + a_2 \sum_{n=-\infty}^{\infty} x_2(n) e^{-jwn}$$

Comparing each summation term with definition of DTFT then we can write

Problems

1. **x(n) = δ(n)**

   If $x[n] = \delta[n]$, then $X(\Omega) = 1$ and

$$Y(\Omega) = H(\Omega)X(\Omega) = \frac{1}{1 - \frac{1}{2}e^{-j\Omega}},$$

   so

$$y[n] = (\tfrac{1}{2})^n u[n]$$

---

2. **x(n) = $e^{j\Omega n}$**

   $X(\Omega) = e^{-j\Omega n_0}$, so

$$Y(\Omega) = \frac{e^{-j\Omega n_0}}{1 - \frac{1}{2}e^{-j\Omega}}$$

   and, using the delay property of the Fourier transform,

$$y[n] = (\tfrac{1}{2})^{n-n_0} u[n - n_0]$$

3. x(n) = (3/4)^n u(n)

If $x[n] = (\frac{3}{4})^n u[n]$, then

$$X(\Omega) = \frac{1}{1 - \frac{3}{4}e^{-j\Omega}},$$

$$Y(\Omega) = \left(\frac{1}{1 - \frac{1}{2}e^{-j\Omega}}\right)\left(\frac{1}{1 - \frac{3}{4}e^{-j\Omega}}\right) = \frac{-2}{1 - \frac{1}{2}e^{-j\Omega}} + \frac{3}{1 - \frac{3}{4}e^{-j\Omega}},$$

so

$$y[n] = -2(\tfrac{1}{2})^n u[n] + 3(\tfrac{3}{4})^n u[n]$$

(a) $$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} (\tfrac{1}{4})^n u[n]e^{-j\Omega n}$$

$$= \sum_{n=0}^{\infty} (\tfrac{1}{4}e^{-j\Omega})^n$$

$$= \frac{1}{1 - \frac{1}{4}e^{-j\Omega}}$$

Here we have used the fact that

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \qquad \text{for } |a| < 1$$

(b) $x[n] = (a^n \sin \Omega_0 n)u[n]$

We can use the modulation property to evaluate this signal. Since

$$\sin \Omega_0 n \overset{\mathcal{F}}{\longleftrightarrow} \frac{2\pi}{2j}[\delta(\Omega - \Omega_0) - \delta(\Omega + \Omega_0)],$$

periodically repeated, then

$$X(\Omega) = \frac{1}{2j}\left[\frac{1}{1 - ae^{-j(\Omega-\Omega_0)}} - \frac{1}{1 - ae^{-j(\Omega+\Omega_0)}}\right]$$

periodically repeated.

(c) $X(\Omega) = \sum_{n=0}^{3} e^{-j\Omega n}$

$= \frac{1 - e^{-j4\Omega}}{1 - e^{-j\Omega}}$,

using the identity

$$\sum_{n=0}^{N-1} a^n = \frac{1 - a^N}{1 - a}$$

Alternatively, we can use the fact that $x[n] = u[n] - u[n-4]$, so

$$X(\Omega) = \frac{1}{1 - e^{-j\Omega}} - \frac{e^{-j4\Omega}}{1 - e^{-j\Omega}} = \frac{1 - e^{-j4\Omega}}{1 - e^{-j\Omega}}$$

(d) $x[n] = (\frac{1}{4})^n u[n+2]$

$= (\frac{1}{4})^{n+2}(\frac{1}{4})^{-2}u[n+2]$

$= 16(\frac{1}{4})^{n+2}u[n+2]$

We know that

$$16\left(\frac{1}{4}\right)^n u[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{16}{1 - \frac{1}{4}e^{-j\Omega}},$$

so

$$16\left(\frac{1}{4}\right)^{n+2} u[n+2] \overset{\mathcal{F}}{\longleftrightarrow} \frac{16e^{j2\Omega}}{1 - \frac{1}{4}e^{-j\Omega}}$$

Fourier Transform Pairs

Sequence	DTFT
$\delta[n-n_o]$	$e^{-j\omega n_o}$
1	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
$a^n u[n]$ $\|a\|<1$	$\frac{1}{1 - ae^{-j\omega}}$
$u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
$\frac{\sin(\omega_c n)}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1 & \|\omega\| < \omega_c \\ 0 & \omega_c < \|\omega\| \le \pi \end{cases}$
$x[n] = \begin{cases} 1 & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
$e^{j\omega_o n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_o + 2\pi k)$
$\cos(\omega_o n + \phi)$	$\sum_{k=-\infty}^{\infty} \left[\pi e^{j\phi}\delta(\omega - \omega_o + 2\pi k) + \pi e^{-j\phi}\delta(\omega + \omega_o + 2\pi k)\right]$

Properties of DTFT

Periodicity: $X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$

Linearity: $ax_1[n] + bx_2[n] \longleftrightarrow aX_1(e^{j\omega}) + bX_2(e^{j\omega})$

Time Shifting: $x[n-n_0] \longleftrightarrow e^{-j\omega n_0} X(e^{j\omega})$

Conjugate Symmetry: $x[n]$ real $\Rightarrow X(e^{j\omega}) = X^*(e^{-j\omega})$

$|X(e^{j\omega})|$ and $\Re e\left\{X(e^{j\omega})\right\}$ are even functions
$\angle X(e^{j\omega})$ and $\Im m\left\{X(e^{j\omega})\right\}$ are odd functions

Convolution Property

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$y[n] = h[n] * x[n]$$

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

$$H(e^{j\omega}) = \text{DTFT of } h[n]$$

Frequency response $=$ DTFT of the unit sample response

Multiplication Property

$$y[n] = x_1[n] \cdot x_2[n]$$

$$\begin{aligned} Y(e^{j\omega}) &= \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta \\ &= \frac{1}{2\pi} X_1(e^{j\omega}) \otimes X_2(e^{j\omega}) \\ &\hookrightarrow \text{ Periodic Convolution} \end{aligned}$$

Parseval's Relation

$$\underbrace{\sum_{n=-\infty}^{\infty} |x[n]|^2}_{\text{Total energy in time domain}} = \underbrace{\frac{1}{2\pi} \int_{2\pi} |X(e^{j\omega})|^2 d\omega}_{\text{Total energy in frequency domain}}$$

Z- Transformation

The Z-transform of a discrete time signal x(n) is defined as the power series

where z is a complex variable. The Z-transform of a signal x(n) is denoted by

whereas the relationship between x(n) and X(z) is indicated by

The z-transform is a infinite power series, it exists only for those values of z for

which this series converges. The region of convergence (ROC) of X(z) is the set of all s values ofz for which X(z) attains a finite value. Thus any time we cite a z-transform. We should also indicate its ROC.

What is Region of convergence?

Ans. The z-transform is an infinite power series, it exists only for those values of z

for which the series converges. The region of convergence (ROC) of X (z) is set of all values of z for which X (z) attains a finite value. The ROC of a finite duration signal is the entire z-plane, except possibly the point . These points are excluded because z^{-n} (when n > 0) becomes unbounded for z =∞ andz^{n} (when n > 0) becomes unbounded for z = 0.

What is the relationship between Z transform and the Discrete Fourier transform?

Ans. Let us consider a sequence x(n) having z-transforrn with ROC that includes the

unit circle. If X(z) is sampled at the N equally spaced points on the unit circle. If X(z) is

sampled at N equally spaced pomts on the unit circle.

We obtain

Expression is (2) identical to the Fourier transform X(w) evaluated at the N. equally spaced. Frequencies

If the sequence x(n) has a finite duration of length N or less, the sequence can be

recovered from its N-point DFT. Hence its Z-transform is uniquely determined by its N-point DFI'. Consequently, X(z) can be expressed as a function of the DFT {X(k)} as

follows

When evaluated on the unit circle (3) yields the Fourier transform of the finite duration sequence in terms of its DFT in the form:

This expression for Fourier transform is a polynomial interpolation formula for X(w)

expressed in terms of the, values {x(k)) of the polynomial at a set of equally spaced

discrete frequencies

What are the application's of z-transform?

Ans. 1. z-transform is an important tool in the analysis of signals and linear time invarient systems.

2. It is used for the analysis of discrete time systems in frequency domain which in generally more efficient than time domain analysis.

3. It is used for filtering process.

4. Causality of discrete time LTL system.

5. Stability of discrete time LTI system.

6. Determination of poles and zeros of rational z-transform.

Properties of the *z* transform

- **Linearity:**

  $Z\{af_n + bg_n\} = aF(z) + bG(z)$. and ROC is $R_f \cap R_g$

  which follows from definition of *z*-transform.

- **Time Shifting**

  If we have $f[n] \Leftrightarrow F(z)$ **then** $f[n-n_0] \Leftrightarrow z^{-n_0}F(z)$

  The ROC of $Y(z)$ is the same as $F(z)$ except that there are possible pole additions or deletions at $z = 0$ or $z = \infty$.

  Proof:

  Let $y[n] = f[n-n_0]$ then

  $$Y(z) = \sum_{n=-\infty}^{\infty} f[n-n_0]z^{-n}$$

  Assume $k = n - n_0$ then $n = k + n_0$, substituting in the above equation we have:

$$Y(z) = \sum_{k=-\infty}^{\infty} f[k] z^{-k-n_0} = z^{-n_0} F[z]$$

- **Multiplication by an Exponential Sequence**

Let $y[n] = z_0^n f[n]$ then $Y(z) = X\left(\frac{z}{z_0}\right)$

Proof:

$$Y(z) = \sum_{n=-\infty}^{\infty} z_0^n x[n] z^{-n} = \sum_{n=-\infty}^{\infty} x[n]\left(\frac{z}{z_0}\right)^{-n} = X\left(\frac{z}{z_0}\right)$$

The consequence is pole and zero locations are scaled by z_0. If the ROC of $X(z)$ is $rR<|z|<rL$, then the ROC of $Y(z)$ is$rR< |z/z_0| <rL$, *i.e.*, $|z_0|rR < |z| < |z_0|rL$

- **Differentiation of *X*(*z*)**

If we have $f[n] \Leftrightarrow F(z)$ then

$$nf[n] \xleftrightarrow{z} -z\frac{dF(z)}{z}$$ and ROC = R_f

Proof:

$$F(z) = \sum_{n=-\infty}^{\infty} f[n] z^{-n}$$

$$-z\frac{dF(z)}{dz} = -z\sum_{n=-\infty}^{\infty} -n\, f[n] z^{-n-1} = \sum_{n=-\infty}^{\infty} -n\, f[n] z^{-n}$$

$$-z\frac{dF(z)}{dz} \xleftrightarrow{z} nf[n]$$

- **Conjugation of a Complex Sequence**

If we have $f[n] \Leftrightarrow F(z)$

then $f^*[n] \xleftrightarrow{z} F^*(z^*)$ and ROC = R_f

- **Time Reversal**

If we have $f[n] \Leftrightarrow F(z)$ then

$$f^*[-n] \overset{Z}{\longleftrightarrow} F^*\left(1/z^*\right)$$

A comprehensive summery for the *z*-transform properties is shown in Table

Table Summery of z-transform properties

Property	Sequence	z-Transform	Region of Convergence
Linearity	$ax(n) + by(n)$	$aX(z) + bY(z)$	Contains $R_x \cap R_y$
Shift	$x(n - n_0)$	$z^{-n_0}X(z)$	R_x
Time reversal	$x(-n)$	$X(z^{-1})$	$1/R_x$
Exponentiation	$\alpha^n x(n)$	$X(\alpha^{-1}z)$	$\lvert\alpha\rvert R_x$
Convolution	$x(n) * y(n)$	$X(z)Y(z)$	Contains $R_x \cap R_y$
Conjugation	$x^*(n)$	$X^*(z^*)$	R_x
Derivative	$nx(n)$	$-z\dfrac{dX(z)}{dz}$	R_x

Note: Given the z-transforms $X(z)$ and $Y(z)$ of $x(n)$ and $y(n)$, with regions of convergence R_x and R_y, respectively, this table lists the z-transforms of sequences that are formed from $x(n)$ and $y(n)$.

Problems on Z Transforms

1. Example 1

.

The X (z) is finite for all values of because

The ROC is entire z-.plane.

2. **Example 2** **unit step sequence *u(n)*:**

$$x(nT)=\begin{cases}1 & n\geq 0\\ 0 & n<0\end{cases}$$

Solution :

$$\begin{aligned}X(z)&=x(0)+x(T)z^{-1}+\cdots\\&=1+z^{-1}+z^{-2}+\cdots\\&=\frac{1}{1-z^{-1}}\end{aligned}$$

---

3. **Example 3 :**

$$x(nT)=e^{-\alpha t}\big|_{t=nT}=e^{-\alpha nT}=(e^{-\alpha T})^{n}\qquad(\alpha>0)$$
$$=k^{n}\qquad(k=e^{\alpha T})$$

Solution:

$$\begin{aligned}X(z)&=x(0)+x(T)z^{-1}+x(2T)z^{-2}+\cdots\\&=1+kz^{-1}+k^{2}z^{-2}+\cdots\\&=1+(k^{-1}z)^{-1}+(k^{-1}z)^{-2}+\cdots\\&\overset{\delta=k^{-1}z}{=}1+\delta^{-1}+\delta^{-2}+\cdots\\&=\frac{1}{1-\delta^{-1}}=\frac{1}{1-kz^{-1}}\end{aligned}\qquad\begin{pmatrix}\left|k^{-1}z\right|^{-1}<1\Rightarrow\\ k=e^{-\alpha T}<|z|\end{pmatrix}$$

$$\Rightarrow X(z)=\frac{1}{1-e^{-\alpha T}z^{-1}}$$

---

4. **Example** Find the *z* transform of $3n + 2 \times 3^{n}$.

SolutionFrom the linearity property

$$Z\{3n + 2 \times 3^n\} = 3Z\{n\} + 2Z\{3^n\}$$

$$Z\{n\} = \frac{z}{(z-1)^2} \text{ and } Z\{3^n\} = \frac{z}{(z-3)}$$

(r^n with $r = 3$). Therefore

$$Z\{3n + 2 \times 3^n\} = \frac{3z}{(z-1)^2} + \frac{2z}{(z-3)}$$

---

5. **Example** :Find the z-transform of each of the following sequences:
   (a) $\boldsymbol{x(n) = 2^n u(n) + 3(½)^n u(n)}$
   (b) $\boldsymbol{x(n) = \cos(n\omega_0)u(n)}$.

Solution:

(a) Because $x(n)$ is a sum of two sequences of the form $\alpha^n u(n)$, using the linearity property of the z-transform, and referring to Table 1, the z-transform pair

$$X(z) = \frac{1}{1-2z^{-1}} + \frac{3}{1-\frac{1}{2}z^{-1}} = \frac{4-\frac{13}{2}z^{-1}}{(1-2z)\left(1-\frac{1}{2}z^{-1}\right)}$$

(b) For this sequence we write

$$x(n) = \cos(n\omega_0)\, u(n) = ½(e^{jn\omega 0} + e^{-jn\omega 0})\, u(n)$$

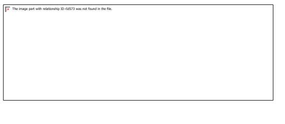
Therefore, the *z*-transform is

$$X(z) = \frac{1}{2}\frac{1}{1-e^{jn\omega_0}z^{-1}} + \frac{1}{2}\frac{1}{1-e^{-jn\omega_0}z^{-1}}$$

with a region of convergence |z| >1. Combining the two terms together, we have

$$X(z) = \frac{1-(\cos\omega_0)\,z^{-1}}{1-2(\cos\omega_0)\,z^{-1}+z^{-2}}$$

---

6. **Example Find z transform of** 

Ans. We have standard z-transform pair.

7. **Example Determine to z-transform of the following signal**

Ans.

Z Transform of some important functions

	Sequence	z – transform
1	$\delta[n]$	1
2	$u[n]$	$\frac{z}{z-1}$
3	b^n	$\frac{z}{z-b}$
4	$b^{n-1}u[n-1]$	$\frac{1}{z-b}$
5	e^{an}	$\frac{z}{z-e^a}$
6	n	$\frac{z}{(z-1)^2}$
7	n^2	$\frac{z(z+1)}{(z-1)^3}$
8	$b^n n$	$\frac{bz}{(z-b)^2}$
9	$e^{an} n$	$\frac{ze^a}{(z-e^a)^2}$
10	$\sin(an)$	$\frac{\sin(a)z}{z^2-2\cos(a)z+1}$
11	$b^n\sin(an)$	$\frac{\sin(a)bz}{z^2-2\cos(a)bz+b^2}$
12	$\cos(an)$	$\frac{z(z-\cos(a))}{z^2-2\cos(a)z+1}$
13	$b^n\cos(an)$	$\frac{z(z-b\cos(a))}{z^2-2\cos(a)bz+b^2}$

The Inverse *z*-Transform

The *z*-transform is a useful tool in linear systems analysis. However, just as important as techniques for finding the z-transform of a sequence are methods that may be used to invert the z-transform and recover the sequence***x*(*n*)**from ***X*(*z*)**. Three possible approaches are described below.

What are the various methods to find out inverse z transform?

Ans. (a) Cauchy Rihemen's theorem

(b) Long division method.

(c) Partial function.

I. Partial Fraction Expansion

Example 1 : Suppose that a sequence *x*(*n*)has a *z*-transform

$$X(z)=\frac{4-\frac{7}{4}z^{-1}+\frac{1}{4}z^{-2}}{1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}}=\frac{4-\frac{7}{4}z^{-1}+\frac{1}{4}z^{-2}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}$$

Solution:

With a region of convergence $|z|>½$. Because $p = q = 2$, and the two poles are simple, the partial fraction expansion has the form

$$X(z)=C+\frac{A_1}{1-\frac{1}{2}z^{-1}}+\frac{A_2}{1-\frac{1}{4}z^{-1}}$$

The constant *C* is found by long division:

$$\begin{array}{r|l} & \quad\quad\quad\quad 2 \\ \hline \frac{1}{8}z^{-2}-\frac{3}{4}z^{-1}+1 & \frac{1}{4}z^{-2}-\frac{7}{4}z^{-1}+4 \\ & \underline{\frac{1}{4}z^{-2}-\frac{3}{2}z^{-1}+2} \\ & \quad\quad -\frac{1}{4}z^{-1}+2 \end{array}$$

Therefore, *C* = 2 and we may write *X*(*z*) as follows:

$$X(z) = 2 + \frac{2 - \frac{1}{4}z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{4}z^{-1}\right)}$$

Next, for the coefficients $A_1$and A_2we have

$$A_1 = \left[\left(1 - \tfrac{1}{2}z^{-1}\right)X(z)\right]_{z^{-1}=2} = \left.\frac{4 - \frac{7}{4}z^{-1} + \frac{1}{4}z^{-2}}{1 - \frac{1}{4}z^{-1}}\right|_{z^{-1}=2} = 3$$

and

$$A_2 = \left[\left(1 - \tfrac{1}{4}z^{-1}\right)X(z)\right]_{z^{-1}=4} = \left.\frac{4 - \frac{7}{4}z^{-1} + \frac{1}{4}z^{-2}}{1 - \frac{1}{2}z^{-1}}\right|_{z^{-1}=4} = -1$$

Thus, the complete partial fraction expansion becomes

$$X(z) = 2 + \frac{3}{1 - \frac{1}{2}z^{-1}} - \frac{1}{1 - \frac{1}{4}z^{-1}}$$

Finally, because the region of convergence is the exterior of the circle $|z| > 1$, $x(n)$ is the right-sided sequence

$$x(n) = 2\delta(n) + 3\left(\tfrac{1}{2}\right)^n u(n) - \left(\tfrac{1}{4}\right)^n u(n)$$

II. Power Series

The z-transform is a power series expansion,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} = \cdots + x(-2)\,z^2 + x(-1)\,z + x(0) + x(1)\,z^{-1} + x(2)\,z^{-2} + \cdots$$

where the sequence values $x(n)$are the coefficients of z^{-n} in the expansion. Therefore, if we can find the power series expansion for $X(z)$, the sequence values $x(n)$may be found by simply picking off the coefficients of z^{-n}.

Example 2 : Consider the z-transform

$$X(z) = \log(1 + az^{-1}) \qquad |z| > |a|$$

Solution:

The power series expansion of this function is

$$\log(1 + az^{-1}) = \sum_{n=1}^{\infty} \frac{1}{n}(-1)^{n+1} a^n z^{-n}$$

Therefore, the sequence x(n) having this z-transform is

$$x(n) = \begin{cases} \frac{1}{n}(-1)^{n+1} a^n & n > 0 \\ 0 & n \leq 0 \end{cases}$$

III. Contour Integration

Another approach that may be used to find the inverse z-transform of $X(z)$ is to use contour integration. This procedure relies on Cauchy's integral theorem, which states that if C is a closed contour that encircles the origin in a counterclockwise direction,

$$\frac{1}{2\pi j}\oint_C z^{-k}dz = \begin{cases} 1 & k=1 \\ 0 & k \neq 1 \end{cases}$$

With

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

Cauchy's integral theorem may be used to show that the coefficients $x(n)$ may be found from $X(z)$ as follows:

$$x(n) = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz$$

where$C$is a closed contour within the region of convergence of $X(z)$ that encircles the origin in a counterclockwise direction. Contour integrals of this form may often by evaluated with the help of Cauchy's residue theorem,

$$x(n) = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}\,dz = \sum\left[\text{residues of } X(z)z^{n-1}\text{at the poles inside } C\right]$$

If $X(z)$ is a rational function of z with a first-order pole at $z = \alpha_k$,

$$\text{Res}\left[X(z)z^{n-1} \text{ at } z=\alpha_k\right] = \left[(1-\alpha_k z^{-1})X(z)z^{n-1}\right]_{z=\alpha_k}$$

Contour integration is particularly useful if only a few values of *x*(*n*) are needed.

Example 3:

Find the inverse of each of the following z-transforms:

$$(a)\ X(z) = 4 + 3(z^2 + z^{-2}) \qquad 0 < |z| < \infty$$

$$(b)\ X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{3}{1-\frac{1}{3}z^{-1}} \qquad |z| > \tfrac{1}{2}$$

$$(c)\ X(z) = \frac{1}{1+3z^{-1}+2z^{-2}} \qquad |z| > 2$$

$$(d)\ X(z) = \frac{1}{(1-z^{-1})(1-z^{-2})} \qquad |z| > 1$$

Solution:

a) Because $X(z)$ is a finite-order polynomial, $x(n)$ is a finite-length sequence. Therefore, $x(n)$ is the coefficient that multiplies z^{-1} in $X(z)$. Thus, $x(0) = 4$ and $x(2) = x(-2) = 3$.

b) This z-transform is a sum of two first-order rational functions of z. Because the region of convergence of $X(z)$ is the exterior of a circle, $x(n)$ is a right-sided sequence. Using the z-transform pair for a right-sided exponential, we may invert $X(z)$ easily as follows:

$$x(n) = \left(\tfrac{1}{2}\right)^n u(n) + 3\left(\tfrac{1}{3}\right)^n u(n)$$

c) Here we have a rational function of z with a denominator that is a quadratic in z. Before we can find the inverse z-transform, we need to factor the denominator and perform a partial fraction expansion:

$$\begin{aligned} X(z) &= \frac{1}{1+3z^{-1}+2z^{-2}} = \frac{1}{(1+2z^{-1})(1+z^{-1})} \\ &= \frac{2}{1+2z^{-1}} - \frac{1}{1+z^{-1}} \end{aligned}$$

Because $x(n)$ is right-sided, the inverse z-transform is

$$x(n) = 2(-2)^n u(n) - (-1)^n u(n)$$

d) One way to invert this z-transform is to perform a partial fraction expansion. With

$$\begin{aligned} X(z) &= \frac{1}{(1-z^{-1})(1-z^{-2})} = \frac{1}{(1-z^{-1})^2(1+z^{-1})} \\ &= \frac{A}{1+z^{-1}} + \frac{B_1}{1-z^{-1}} + \frac{B_2}{(1-z^{-1})^2} \end{aligned}$$

the constants A, B_1, and $B_2$are as follows:

$$\begin{aligned} A &= [(1+z^{-1})X(z)]_{z=-1} = \tfrac{1}{4} \\ B_1 &= \left[\frac{d}{dz}(1-z^{-1})^2 X(z)\right]_{z=1} = \left[\frac{z^{-2}}{(1+z^{-1})^2}\right]_{z=1} = \tfrac{1}{4} \\ B_2 &= [(1-z^{-1})^2 X(z)]_{z=1} = \tfrac{1}{2} \end{aligned}$$

Inverse transforming each term, we have

$$x(n) = \tfrac{1}{4}[(-1)^n + 1 + 2(n+1)]u(n)$$

Example 4 Find the inverse z-transform of the second-order system

$$X(z) = \frac{1 + \frac{1}{4}z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)^2} \qquad |z| > \tfrac{1}{2}$$

Here we have a second-order pole at $z = ½$. The partial fraction expansion for $X(z)$ is

$$X(z) = \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{\left(1 - \frac{1}{2}z^{-1}\right)^2}$$

The constant A_1 is

$$A_1 = \tfrac{1}{2}\left[\frac{d}{dz}\left(1 - \tfrac{1}{2}z^{-1}\right)^2 X(z)\right]_{z=1/2} = \tfrac{1}{2}\left[-\tfrac{1}{4}z^{-2}\right]_{z=1/2} = -\tfrac{1}{2}$$

and the constant A_2 is

$$A_2 = \left[\left(1 - \tfrac{1}{2}z^{-1}\right)^2 X(z)\right]_{z=1/2} = \tfrac{3}{2}$$

Therefore,

$$X(z) = -\frac{\frac{1}{2}}{1 - \frac{1}{2}z^{-1}} + \frac{\frac{3}{2}}{\left(1 - \frac{1}{2}z^{-1}\right)^2}$$

and

$$x(n) = -\left(\tfrac{1}{2}\right)^{n+1}u(n) + 3(n+1)\left(\tfrac{1}{2}\right)^{n+1}u(n)$$

Example 5 Find the inverse z-transform of $X(z) = \sin z$.

Solution

To find the inverse z-transform of $X(z) = \sin z$, we expand $X(z)$ in a Taylor series about $z = 0$ as follows:

$$X(z) = X(z)\Big|_{z=0} + z\frac{dX(z)}{dz}\Big|_{z=0} + \frac{z^2}{2!}\frac{d^2X(z)}{dz^2}\Big|_{z=0} + \cdots + \frac{z^n}{n!}\frac{d^nX(z)}{dz^n}\Big|_{z=0} + \cdots$$
$$= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!}$$

Because

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

we may associate the coefficients in the Taylor series expansion with the sequence values $x(n)$. Thus, we have

$$x(n) = (-1)^n\frac{1}{(2|n|+1)!} \qquad n = -1, -3, -5, \ldots$$

Example 6: Evaluate the following integral:

$$\frac{1}{2\pi j}\oint_C \frac{1 + 2z^{-1} - z^{-2}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - \frac{2}{3}z^{-1}\right)} z^3 dz$$

where the contour of integration C is the unit circle.

Solution:

Recall that for a sequence $x(n)$ that has a z-transform $X(z)$, the sequence may be recovered using contour integration as follows:

$$x(n) = \frac{1}{2\pi j}\oint_c X(z)z^{n-1}dz$$

Therefore, the integral that is to be evaluated corresponds to the value of the sequence *x*(*n*) at *n* = 4 that has a *z*-transform

$$X(z) = \frac{1 + 2z^{-1} - z^{-2}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - \frac{2}{3}z^{-1}\right)}$$

Thus, we may find *x*(*n*) using a partial fraction expansion of *X*(*z*) and then evaluate the sequence at *n* = 4. With this approach, however, we are finding the values of *x*(*n*) for all n. Alternatively, we could perform long division and divide the numerator of *X*(*z*) by the denominator. The coefficient multiplying z^{-4} would then be the value of *x*(*n*) at *n* = 4, and the value of the integral. However, because we are only interested in the value of the sequence at *n* = 4, the easiest approach is to evaluate the integral directly using the Cauchy integral theorem. The value of the integral is equal to the sum of the residues of the poles of $X(z)z^3$ inside the unit circle. Because

$$X(z)z^3 = z^3 \frac{z^2 + 2z - 1}{\left(z - \frac{1}{2}\right)\left(z - \frac{2}{3}\right)}$$

has poles at z =1/2 and z =2/3,

$$\text{Res}[X(z)z^3]_{z=\frac{1}{2}} = \left[z^3 \frac{z^2 + 2z - 1}{z - \frac{2}{3}}\right]_{z=\frac{1}{2}} = -\frac{3}{16}$$

and

$$\text{Res}[X(z)z^3]_{z=\frac{2}{3}} = \left[z^3 \frac{z^2 + 2z - 1}{z - \frac{1}{2}}\right]_{z=\frac{2}{3}} = \frac{112}{81}$$

Therefore, we have

$$\frac{1}{2\pi j} \oint_c X(z)z^3 dz = \frac{112}{81} - \frac{3}{16} = 1.1952$$

UNIT IV

CONTINUOUS AND DISCRETE TIME SYSTEMS

Differential equations	Difference equations
Analysis of LTI circuits gives a relationship between input x(t) and output y(t) in the form of a differential equation: $b_0 y(t) + b_1 \frac{dy(t)}{dt} + b_2 \frac{d^2 y(t)}{dt^2} + \cdots$ $= a_0 x(t) + a_1 \frac{dx(t)}{dt} + a_2 \frac{d^2 x(t)}{dt^2} + \cdots$ whose system (or transfer) function is of the form: $H_a(s) = \frac{a_0 + a_1 s + a_2 s^2 + \ldots + a_N s^N}{b_0 + b_1 s + b_2 s^2 + \ldots + b_M s^M}$ This is a ratio of polynomials in s . The order of the system function is max(N,M). Replacing s by jω gives the frequency-response $H_a(j\omega)$, where ω denotes frequency in radians/second. For values of s with non-negative real parts, $H_a(s)$ is the Laplace Transform of the analogue filter's impulse response $h_a(t)$. H(s) may be expressed in terms of its poles and zeros as: $H_a(s) = k \frac{(s - z_1)(s - z_2)\ldots(s - z_N)}{(s - p_1)(s - p_2)\ldots(s - p_M)}$ The solution is composed of a homogeneous response (natural response) and a particular solution (forced response) of the system. $y(t) = y_h(t) + y_p(t),$	The processing of discrete-time signals is performed by discrete-time systems. Similar to the continuous-time case, we may represent a discrete-time system either by a set of **difference equations** or by a block diagram of its implementation. For example, consider the following difference equation. y(n) = y(n - 1) + x(n) + x(n - 1) + x(n - 2) This equation represents a discrete-time system. It operates on the input signal x(n) to produce the output signal y(n). We use the notation *y(n)* = *T*[*x(n)*] to denote a discrete-time system *T* with input signal *x*(*n*) and output signal *y*(*n*). Notice that the input and output to the system are the complete signals for **all** time *n*. This is important since the output at a particular time can be a function of past, present and future values of *x*(*n*). It is usually quite straightforward to write a computer program to implement a discrete-time system from its difference equation. In fact, programmable computers are one of the easiest and most cost effective ways of implementing discrete-time systems. The general form is $\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k],$ A general solution to Equation can be expressed as the sum of a homogeneous solution (natural response) to and a particular solution (forced response), $y[n] = y_h[n] + y_p[n].$

Transfer function and Impulse response

PROPERTIES OF TRANSFER FUNCTION (TF)

The properties of transfer function are given below:

- The ratio of Laplace transform of output to Laplace transform of input assuming all initial conditions to be zero.
- The transfer function of a system is the Laplace transform of its impulse response under assumption of zero initial conditions.
- Replacing '*s*' variable with linear operation D= d/dt in transfer function of a system, the differential equation of the system can be obtained.
- The transfer function of a system does not depend on the inputs to the system.
- The system poles and zeros can be determined from its transfer function.
- Stability can be found from characteristic equation.
- Transfer function cannot be defined for non-linear systems. It can be defined for linear systems only.

Example :Find the impulse response of the following second order system:

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \delta(t).$$

<u>Solution</u>

The characteristic equation is

$$s^2 + 4s + 3 = (s+3)(s+1) = 0$$

so the homogenous solution will be of the form

$$y(t) = \left(Ae^{-3t} + Be^{-t}\right)u(t).$$

The first derivative is

$$\frac{y(t)}{dt} = \left(-3Ae^{-3t} - Be^{-t}\right)u(t) + (A+B)\delta(t)$$

and the second derivative is

$$\frac{y^2(t)}{dt^2} = \left(9Ae^{-3t} + Be^{-t}\right)u(t) + \left(-3A - B\right)\delta(t) + (A+B)\delta^{(1)}(t)$$

Putting these back into Eq. (2.1.6) gives

$$\begin{aligned}&\left(9Ae^{-3t} + Be^{-t}\right)u(t) + \left(-3A - B\right)\delta(t) + (A+B)\delta^{(1)}(t)\\ &+4\left[\left(-3Ae^{-3t} - Be^{-t}\right)u(t) + (A+B)\delta(t)\right]\\ &+3\left[\left(Ae^{-3t} + Be^{-t}\right)u(t)\right] = \delta(t).\end{aligned}$$

Putting Eq. (2.1.7) in Eq. (2.1.6), we will wind up with three types of functions. If Eq. (2.1.6) is to hold true, then the coefficients for the different types of functions must satisfy Eq. (2.1.6), so we get three equations

The $\delta^{(1)}(t)$ terms give $$(A+B)\delta^{(1)}(t)=0$$
or $$A=-B.$$
The $\delta(t)$ terms give $$(-3A-B)\delta(t)+4(A+B)\delta(t)=\delta(t)$$
or $$(-3A+A)=1 \quad \Rightarrow A=-\frac{1}{2}, \quad B=\frac{1}{2}.$$
The $u(t)$ terms $$\left(9Ae^{-3t}+Be^{-t}\right)+4(-3Ae^{-3t}-Be^{-t})+3\left(Ae^{-3t}+Be^{-t}\right)=0,$$
But this is redundant, because our choice of the homogeneous equation insured it.

So we can conclude

$$h(t)=\frac{1}{2}\left(e^{-t}-e^{-3t}\right)u(t).$$

What would be the response for the input $x(t)=u(t)$?

$$y(t)=h(t)*u(t)=\int_0^t h(\tau)\,d\tau$$
$$=\int_0^t \frac{1}{2}\left(e^{-t}-e^{-3t}\right)d\tau=\frac{1}{2}\left\{\left(1-e^{-t}\right)-(1-e^{-3t})\right\}u(t)$$
$$=\frac{1}{2}\left(e^{-3t}-e^{-t}\right)u(t).$$

This problem was not so difficult because the characteristic equation separated into two simple real roots. In general, it will be much easier to use Laplace transforms.

System function and impulse response

Example : What is the frequency response of a discrete LTI system? Derive the frequency response of a system whose impulse response is given by h(n) = a” u(n —1) for (a) <1.

Ans. The frequency response of a linear time invariant discrete time system can be obtained by applying a spectrum of the input sinusoids to the system. The frequency response gives the gain and phase response of the system to the input sinusoids at all frequencies. Let us consider, the inpulse response of an LTI discrete time system is h(n) and the input x(n) to the system is complex exponential e1u. The output of the system y(n) can be

Given

Example : Determine the system function

Ans. Taking z-transform of both sides.

Example : Determine the pole-zero plot for the system described by difference equation

Ans. Taking z-transform of both sides.

The ROC & pole zero plot shown in Fig. below

From the following figure, we can observe the following1.ROC of the system function include unit circle.2. ROC of the system function cannot have any poles.

Example : Consider the causal second-order system described by

$$\frac{d^2 y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt} + 3x(t)$$

and with initial conditions $\frac{dy(0^-)}{dt} = 2$, $y(0^-) = 1$ Suppose that this system is subjected to the input signal: $x(t) = e^{-5t}u(t).$ what is the output?

Solution :

$$\left(s^2 + 3s + 2\right)\mathcal{Y}(s) = \left(s+3\right)\mathcal{X}(s) + sy(0^-) + 3y(0^-) + \frac{dy(0^-)}{dt}$$

$$\mathcal{Y}(s) = \frac{\left(s+3\right)\mathcal{X}(s)}{s^2+3s+2} + \frac{sy(0^-) + 3y(0^-) + \frac{dy(0^-)}{dt}}{s^2+3s+2}$$

We have,

$$\mathcal{X}(s) = \frac{1}{s+5}, \quad \operatorname{Re}\{s\} > -5,$$

and thus,

$$\begin{aligned}\mathcal{Y}(s) &= \frac{s+3}{\left(s^2+3s+2\right)\left(s+5\right)} + \frac{s+5}{s^2+3s+2}, \quad \operatorname{Re}\{s\} > -1 \\ &= \frac{s^2+11s+28}{\left(s^2+3s+2\right)\left(s+5\right)}, \quad \operatorname{Re}\{s\} > -1 \\ &= \frac{\frac{9}{2}}{s+1} - \frac{\frac{10}{3}}{s+2} - \frac{\frac{1}{6}}{s+5}.\end{aligned}$$

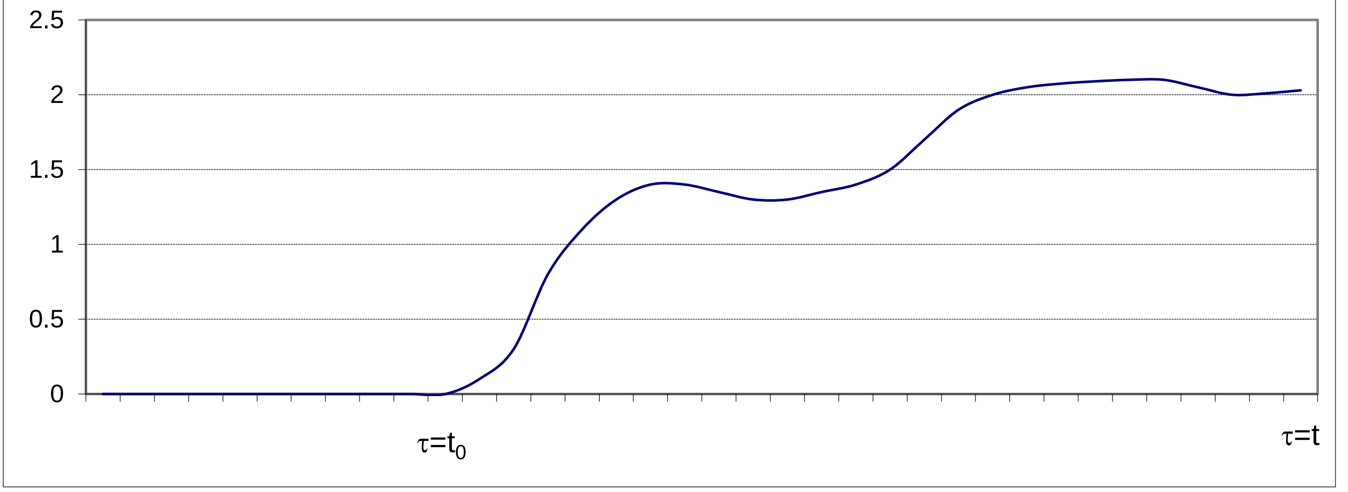
Taking Inverse we have

$$y(t) = \left[\frac{9}{2}e^{-t} - \frac{10}{3}e^{-2t} - \frac{1}{6}e^{-5t}\right]u(t).$$

The Convolution Integral

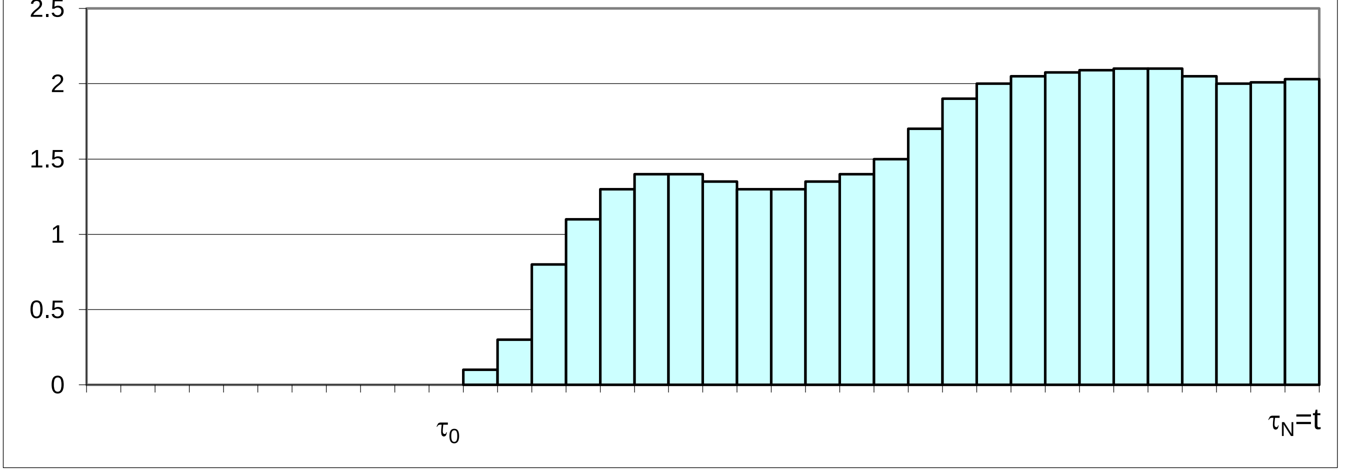
Assume that the input, $x(\tau)$, to an LTI system started at time t_0 (the input was zero for all time prior to t_0) and has continued to the present time, t, as shown below.

Figure 1: $x(\tau)$, the Input



We can approximate this input as a series of rectangular pulses having the same area under the curve as shown in Figure 2.

Figure 2: $x(\tau_n)$, the Approximation to the Input



These graphs are given in terms of the variable τ, the variable t is reserved for the time of observation of the output signal. The interval from $\tau = t_0$ to $\tau = t$ is divided into subintervals of width $\Delta\tau$ each centered about an value of $\tau_n = t_0 + n*\Delta\tau$.

Now perform the following experiment. Apply a rectangular pulse of unit strength and width $\Delta\tau$ to the input of our LTI system. Lets call the resulting output $f(t, \tau_n)$.

- $f(t, \tau_n)$ is the output at time t due to a rectangular pulse of unit amplitude and width $\Delta\tau$ that occurred at time $\tau = \tau_n$.

The output of the system at time t due to the n^{th} pulse of the approximate input is then the value of the input at time τ_n ,which is $x(\tau_n)$, times $f(t, \tau_n)$. Using superposition the total output from the system at time t is then approximated by the sum:

$$y(t) \approx \sum_{n=0}^{N} x(\tau_n) * f(t, \tau_n)$$

Multiplying and dividing each term in the sum by $\Delta\tau$ yields:

$$y(t) \approx \sum_{n=0}^{N} x(\tau_n) * \left[\frac{1}{\Delta\tau} f(t, \tau_n) \right] \Delta\tau$$

Note that the term $\left[\frac{1}{\Delta\tau} f(t, \tau_n) \right]$ is the output of the time t due to a pulse of amplitude $\frac{1}{\Delta\tau}$ that occurred at time $\tau = \tau_n$. The area of this input pulse ise equal to unity. Our approximation gets better as $\Delta\tau$ approaches zero so take the limit of y(t) as $\Delta\tau \to 0$ changing the sum to an integral.

$$y(t) = \int_{\tau=t_0}^{t} x(\tau) * \begin{bmatrix} \frac{1}{\Delta\tau} f(t, \tau_n) \\ \lim \Delta\tau \to 0 \end{bmatrix} d\tau$$

The term in the brackets becomes the output at time, t, of the system to $\delta(t-\tau)$, a Dirac Delta function or impulse that occurred at time τ . It is usually denoted as $h(t, \tau)$, or since our system is time invariant simply $h(t-\tau)$. This function, **h(t)**, is called the **UnitImpulse Response** of the system (which happens to be the **Inverse Fourier Transform** of the **Transfer Function, H(jω)**). The output then is given by:

$$y(t) = \int_{\tau=t_0}^{t} x(\tau) * h(t-\tau) d\tau$$

or, where it is up to you to determine the limits on the integral from the nature of the two functions:

$$y(t) = \int_{\tau=-\infty}^{\infty} x(\tau) * h(t-\tau) d\tau$$

This is known as the **Convolution Integral** and is denoted as:

$$y(t) = x(t) \otimes h(t)$$

Note: The meaning of **Convolution** is that an LTI system can be modeled as having a memory that stores all past input. Acording to this model, the LTI system determines its output by performing a weighted sum of all past inputs using the **Impulse Response** as the weighting factor.

Continuous systems seldom actually function this way, but this model accurately determines the output. Many **Discrete-Time** LTI systems **AREbuilt** according to the **Convolution** model. They are called **Finite Impulse Response** systems since their memory has a limited capacity.

Properties of Convolution

Commutative Law

$$x(t) \otimes y(t) = y(t) \otimes x(t)$$

Proof:

$$x(t)\otimes y(t)=\int_{\tau=-\infty}^{\infty}x(\tau)*y(t-\tau)d\tau$$

Let u = t - τ , therefore τ = t – u and

$$x(t)\otimes y(t)=\int_{u=\infty}^{-\infty}x(t-u)*y(u)(-du)$$

Reversing the limits is the same as multiplying by –1

$$x(t)\otimes y(t)=\int_{u=-\infty}^{\infty}x(t-u)*y(u)du=\int_{u=-\infty}^{\infty}y(u)*x(t-u)du=y(t)\otimes x(t)$$

Distributive Law

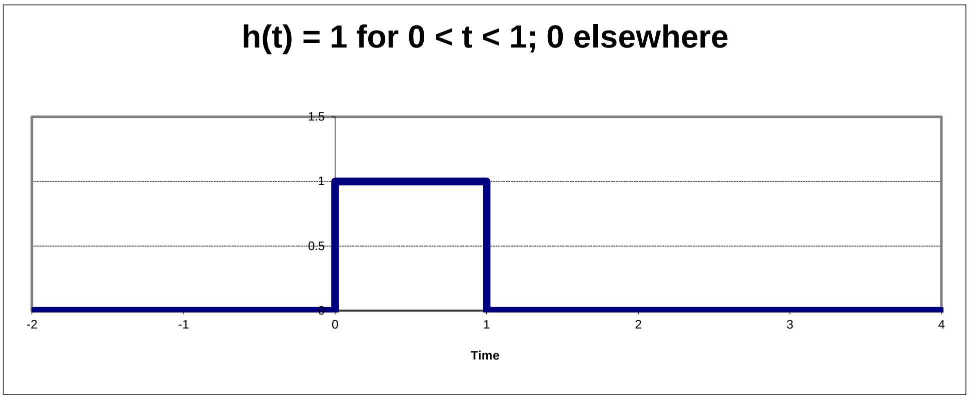
$$x(t)\otimes[y(t)+z(t)]=x(t)\otimes y(t)+x(t)\otimes z(t)$$

Associative Law

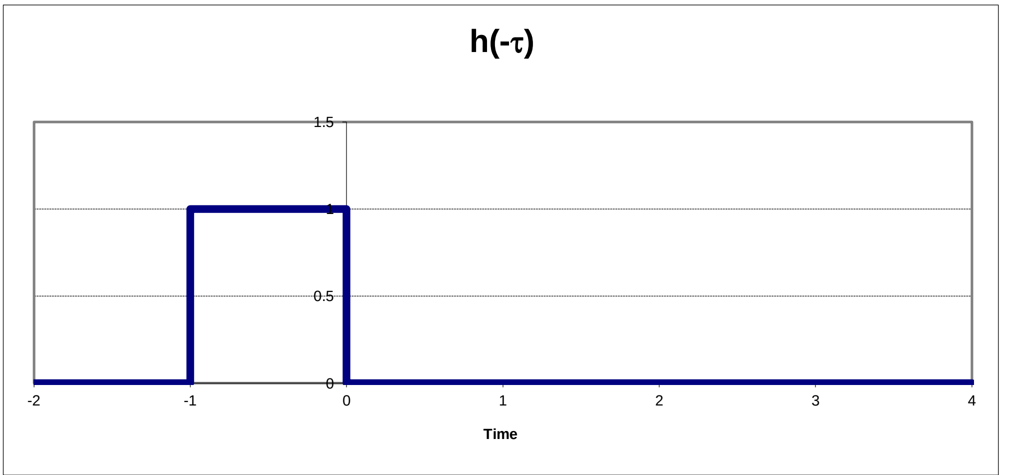
$$x(t)\otimes[y(t)\otimes z(t)]=[x(t)\otimes y(t)]\otimes z(t)$$

Example Convolutions

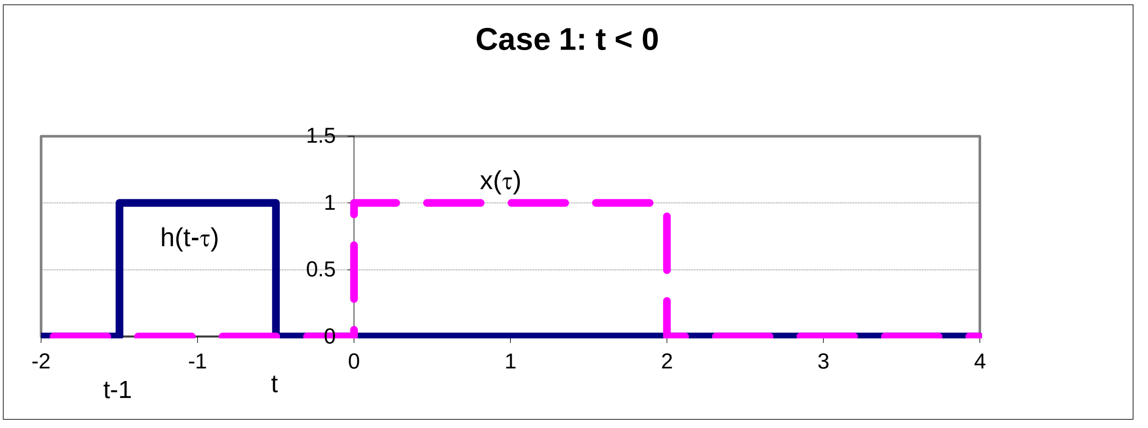
Convolution Example 1: Simple Rectangular Functions

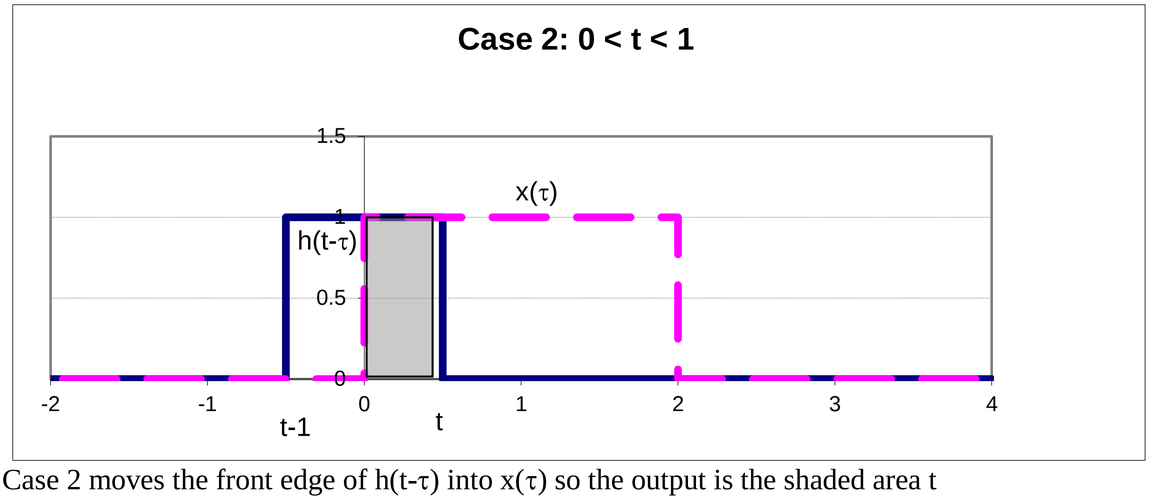


First flip h(t) by letting t = -τ



Now shift h(-τ) to the time for Case 1 by replacing (–τ) with t - τ

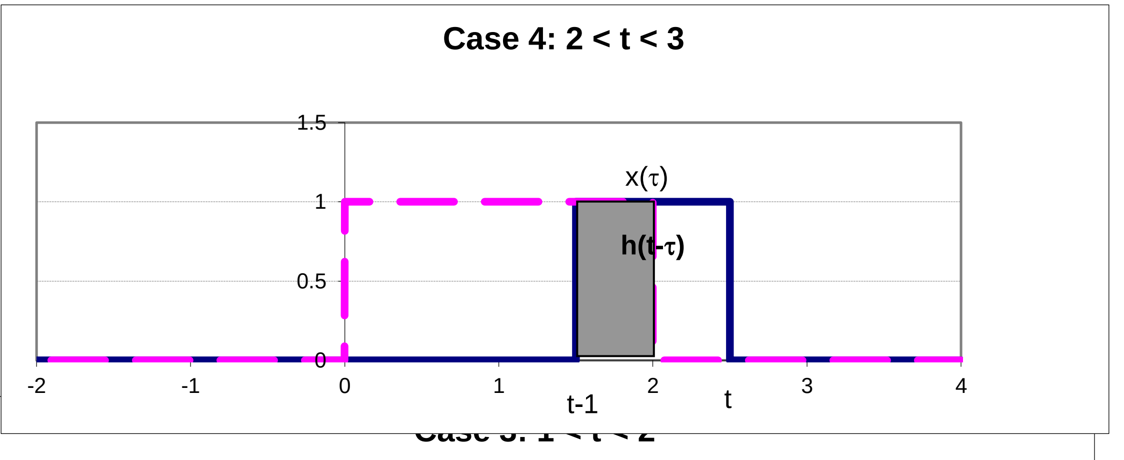


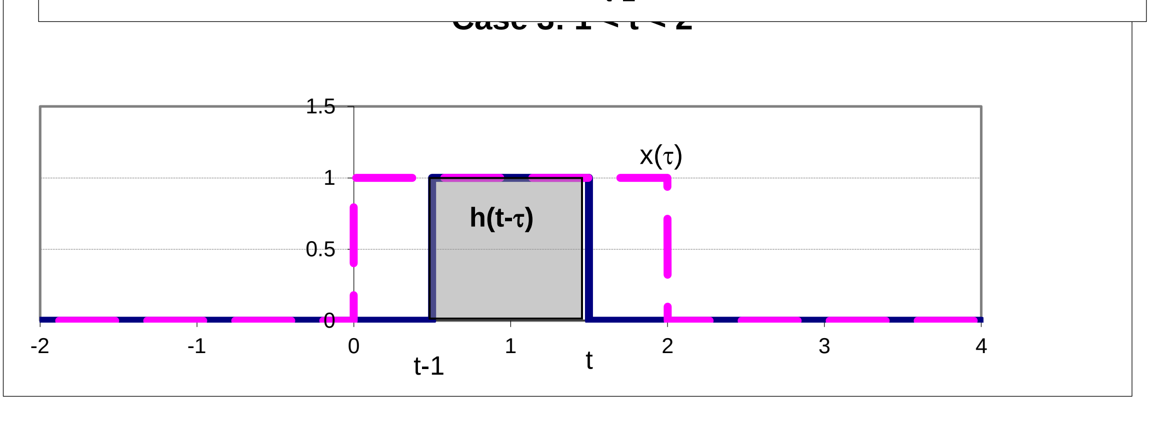


Case 2 moves the front edge of h(t-τ) into x(τ) so the output is the shaded area t

For all of Case 3 h(t-τ) is fully within x(τ) so the output is 1

Case 4 h(t-τ) is exiting x(τ) so the output is [2 – (t-1)]*1 or (3-t)

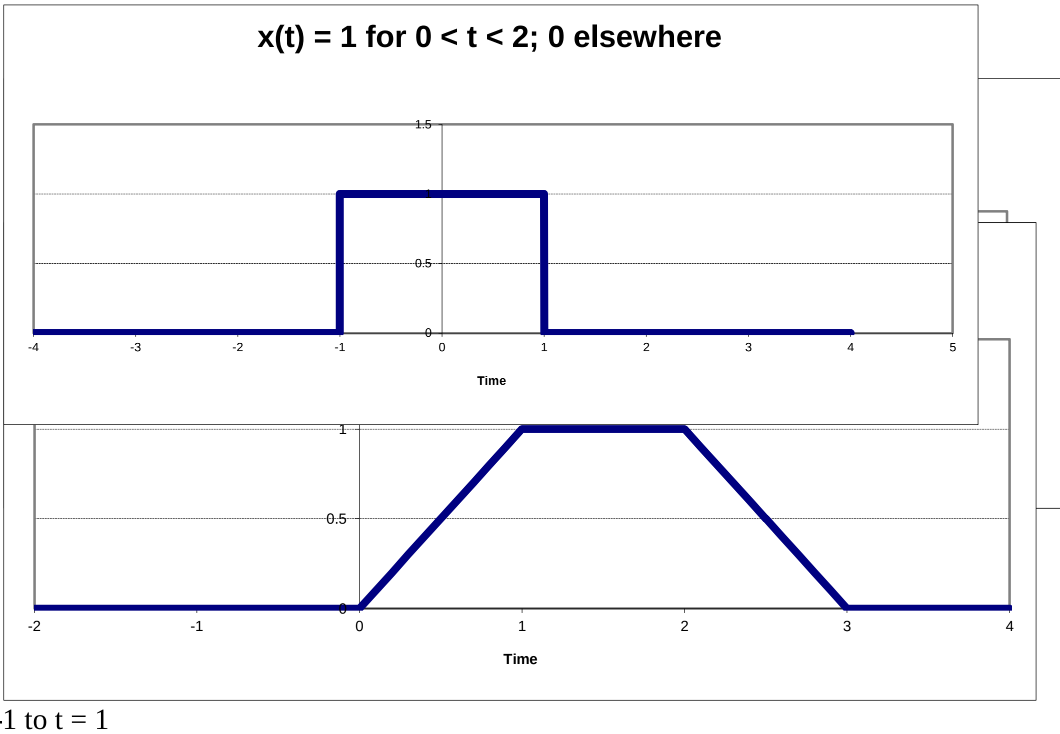




For all of the last Case t > 3 and there is no overlap so the output is 0

So now we can plot the output, $y(t) = x(t) \otimes h(t) = \int_{-\infty}^{\infty} x(\tau) * h(t-\tau)d\tau$ and we are done.

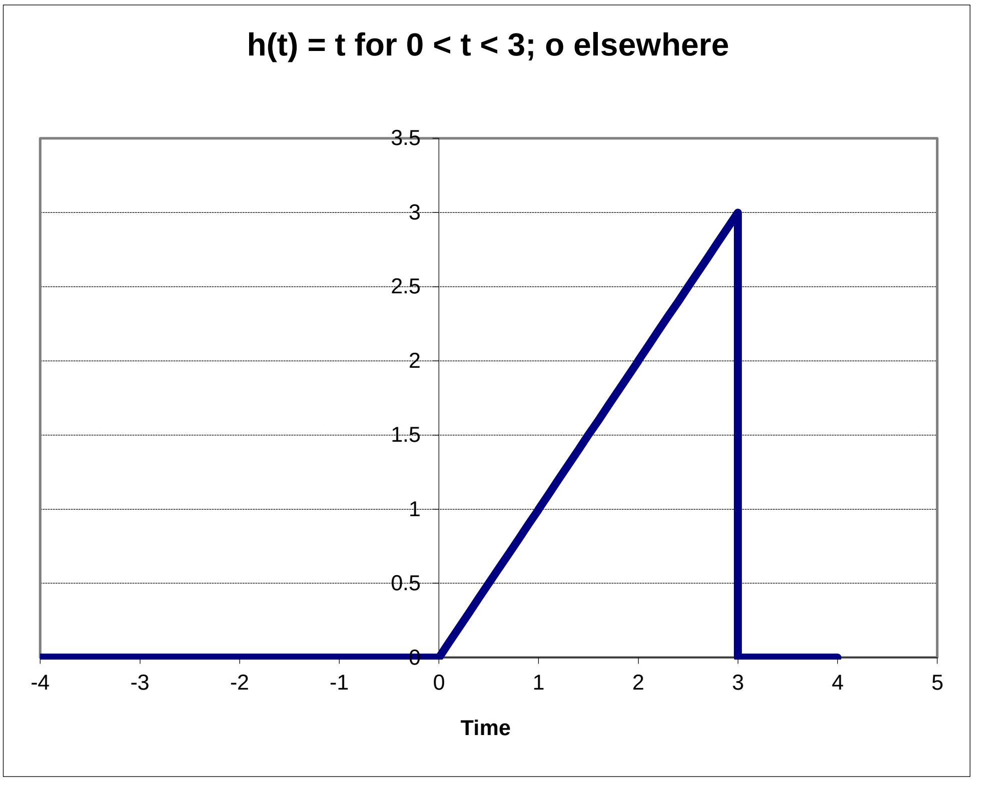
Convolution Example 2: A Triangular Function

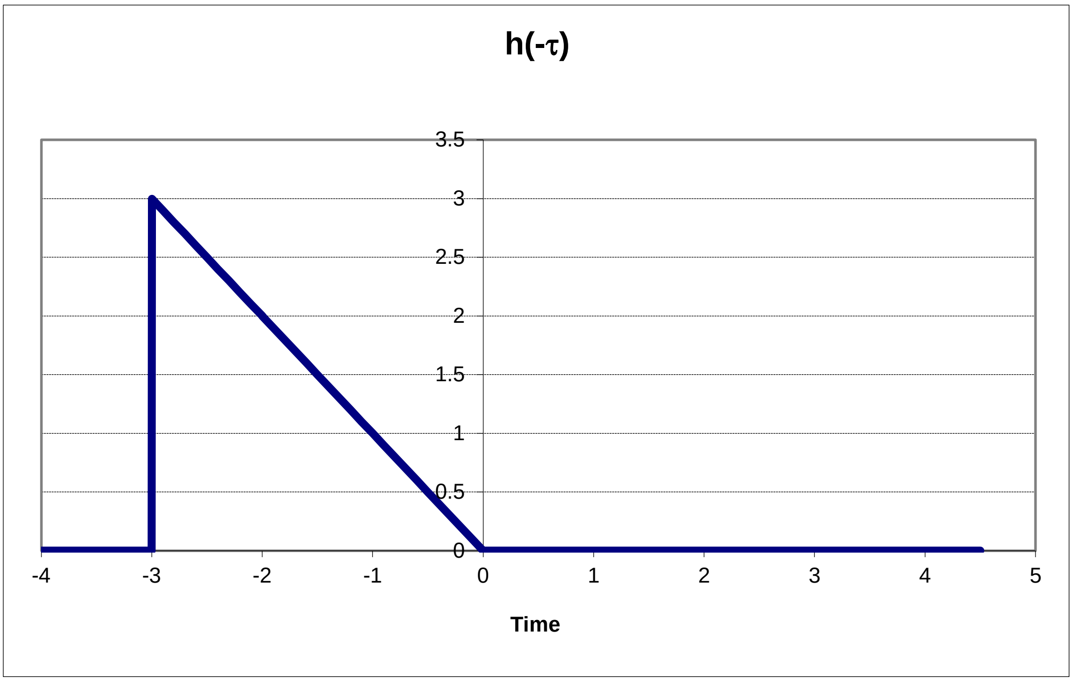


The input signal x(t) is a unit rectangular pulse from t = -1 to t = 1

The system Impulse Response is triangular

Find the output, $y(t) = x(t) \otimes h(t)$

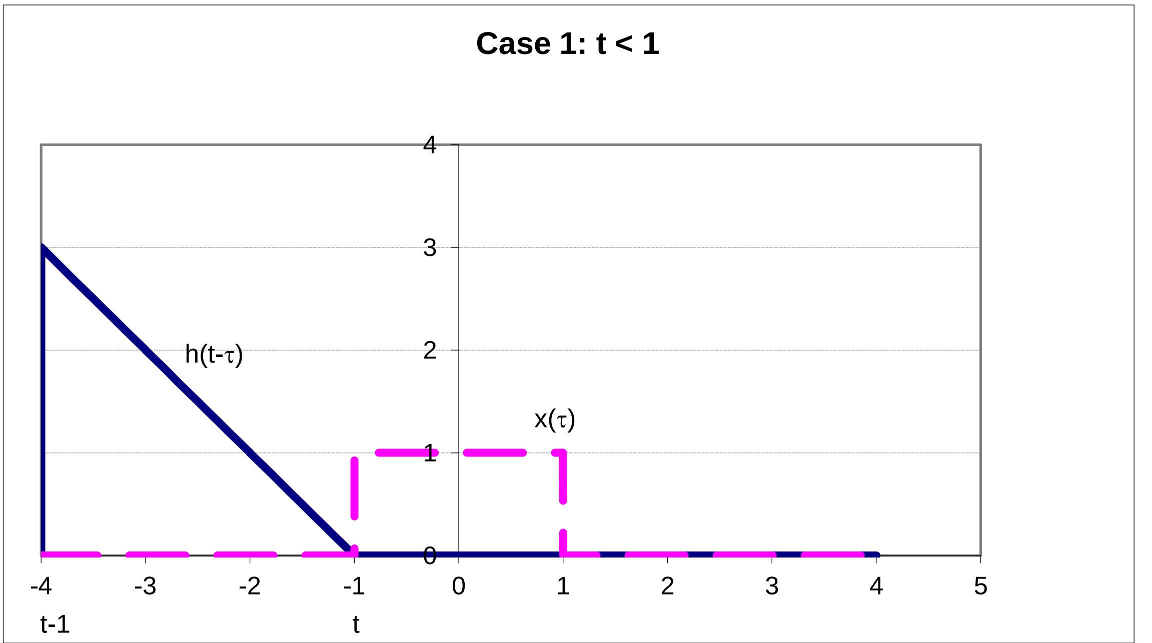




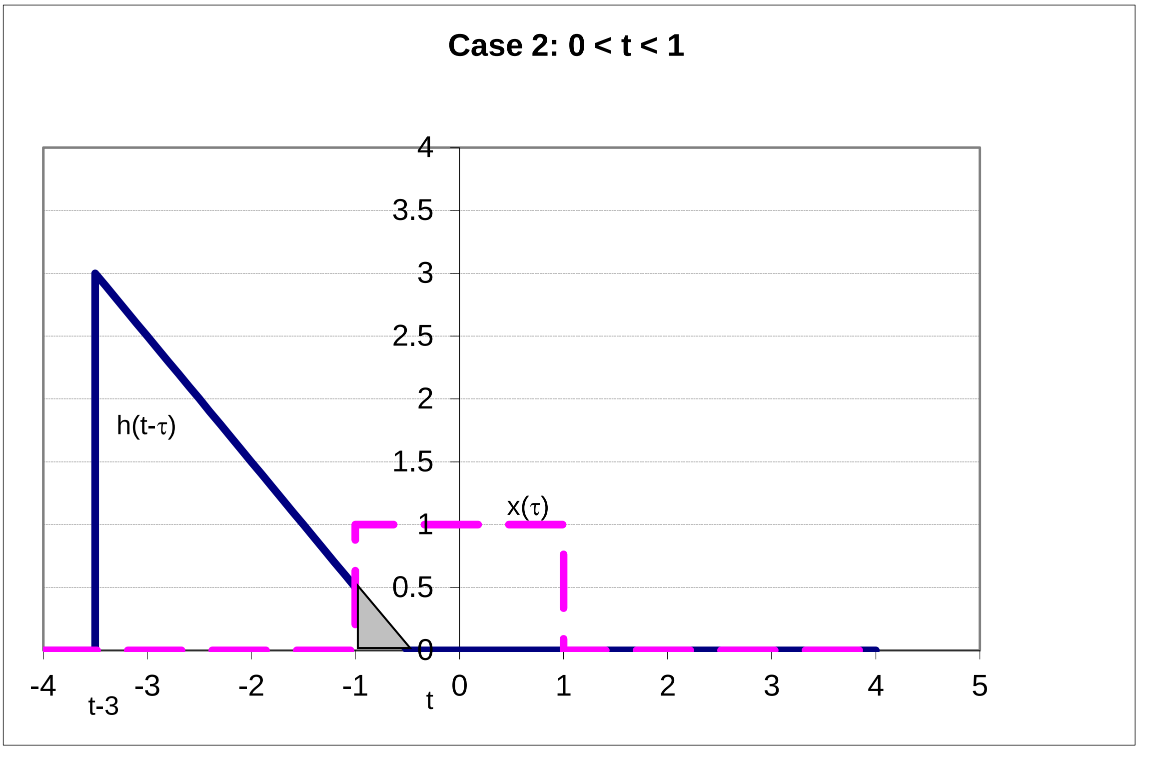
First flip the Impulse Response by substituting t = -τ

Now slide it back to start at a value of t ≤ -1 and plot the signal on the same chart

There is no overlap so y = 0 for t < -1



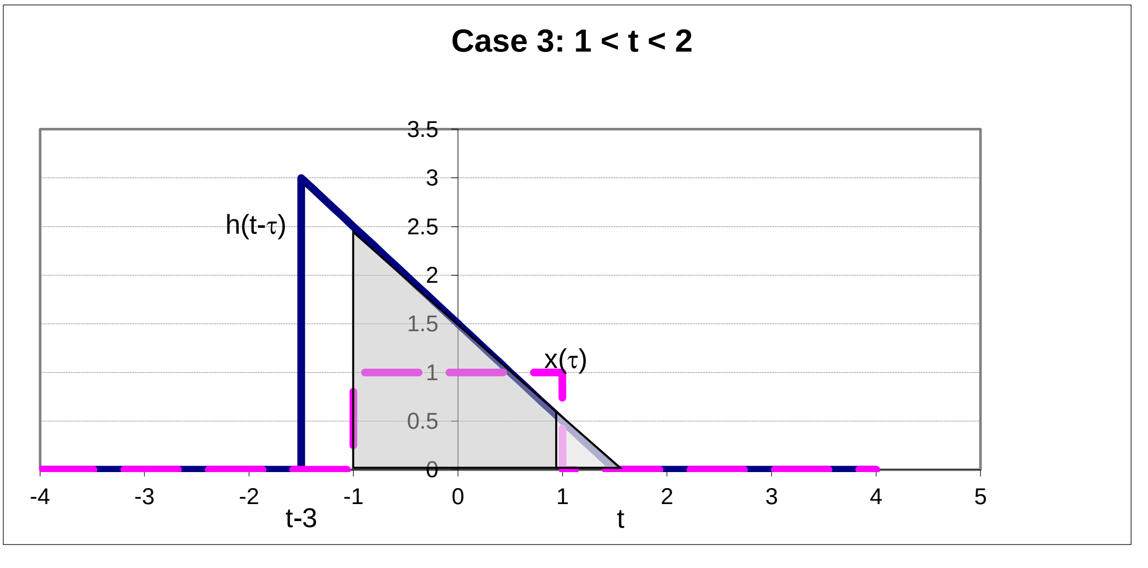
Now slide the tip of h(t-τ) just past t = -1 to set up Case 2



The shaded area is the integral of the product of the two functions. And:

$$y(t) = \int_{\tau=-1}^{t} (t-\tau)*1d\tau = \left[t*\tau - \frac{\tau^2}{2} \right]_{\tau=-1}^{t} = \frac{1}{2}t^2 + t + \frac{1}{2} \text{ for } 0 < t < 1$$

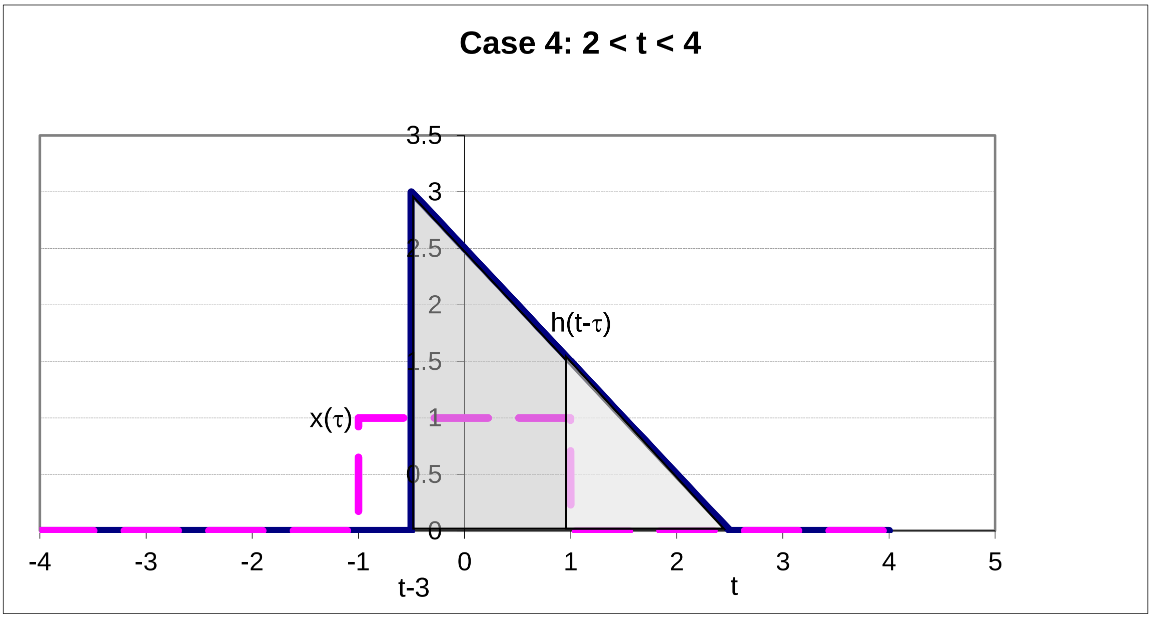
Case 3 is set up by sliding t to just past t = 1. Now the complete signal lies within the memory of the



system.

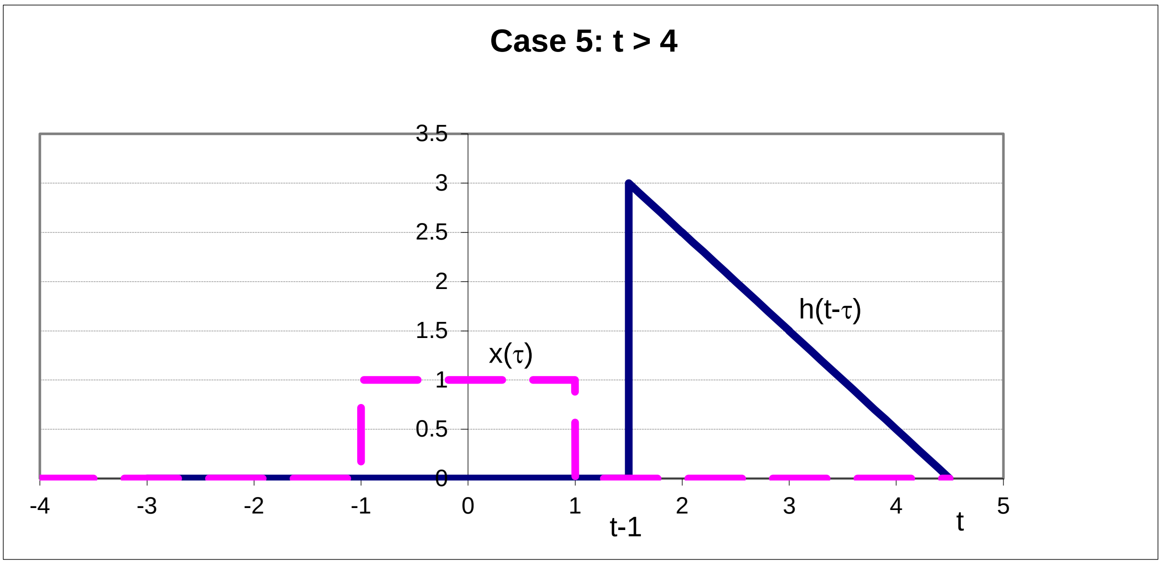
Now

$$y(t) = \int_{\tau=-1}^{1} (t-\tau)*1d\tau = \left[t*\tau - \frac{\tau^2}{2}\right]_{\tau=-1}^{1} = 2t \text{ for } 1 < t < 2$$



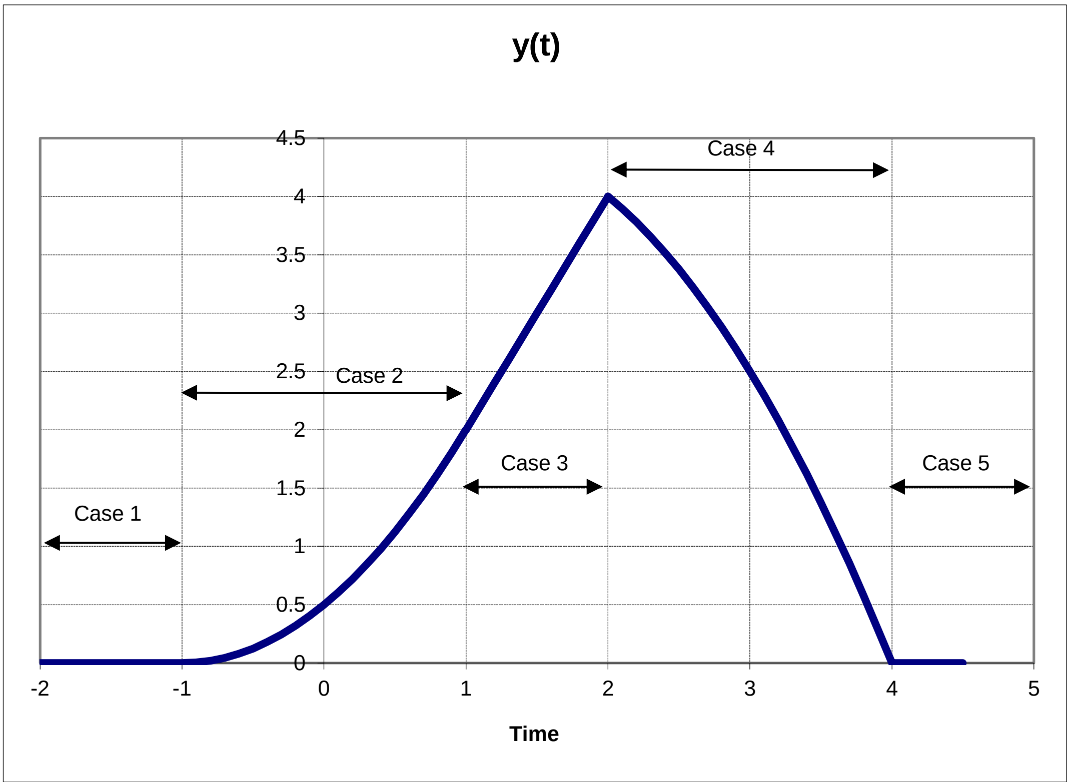
The fourth Case occurs when the back edge of h(t-τ) crosses τ = -1. This is when t = 2.

Now $y(t) = \int_{\tau=t-3}^{1} (t-\tau)*1d\tau = \left[t*\tau - \frac{\tau^2}{2}\right]_{\tau=t-3}^{1} = -\frac{1}{2}t^2 + t + 4$ for 2 > t > 4



The last Case is when t > 4. Now there is no overlap and the output remains at zero.

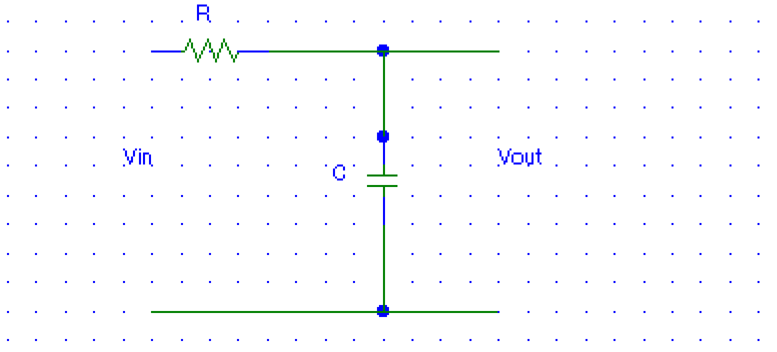
Now we can plot y(t). Note that it is a continuous function. This is the normal case (the exception is when there are Impulse Functions in either the signal or h(t)). Use this fact to check your work by comparing the values at the boundary conditions between cases.



Convolution Example 3: The RC Low Pass Filter

The input signal x(t) is a unit rectangular pulse from t = 0 to t = t_0

The circuit is:

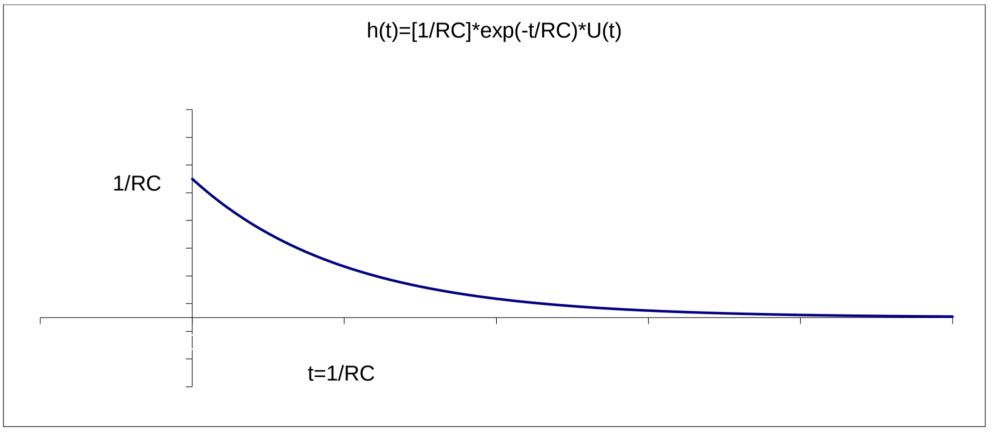


*Find the **ImpulseResponse** of the circuit using the **TransferFunction**:*

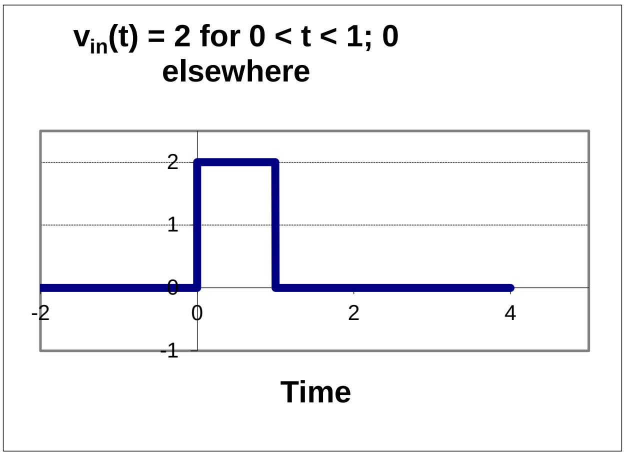
$$H(j\omega) = \frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C} + R} = \frac{\frac{1}{RC}}{\frac{1}{RC} + j\omega}$$ Since it is a simple AC voltage divider

From Example 1 of the section on Fourier Transforms the inverse transform of this Transfer function is:

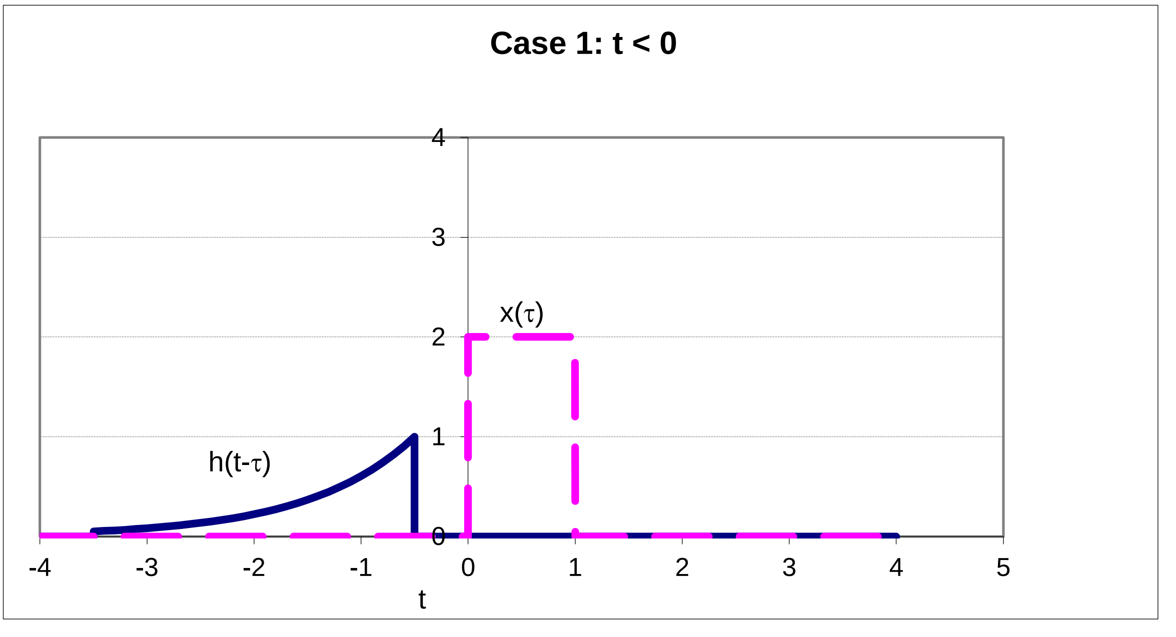
$h(t) = \frac{1}{RC}\varepsilon^{-\frac{t}{RC}} * U(t)$ which looks like:



The input, v_{in}, is:

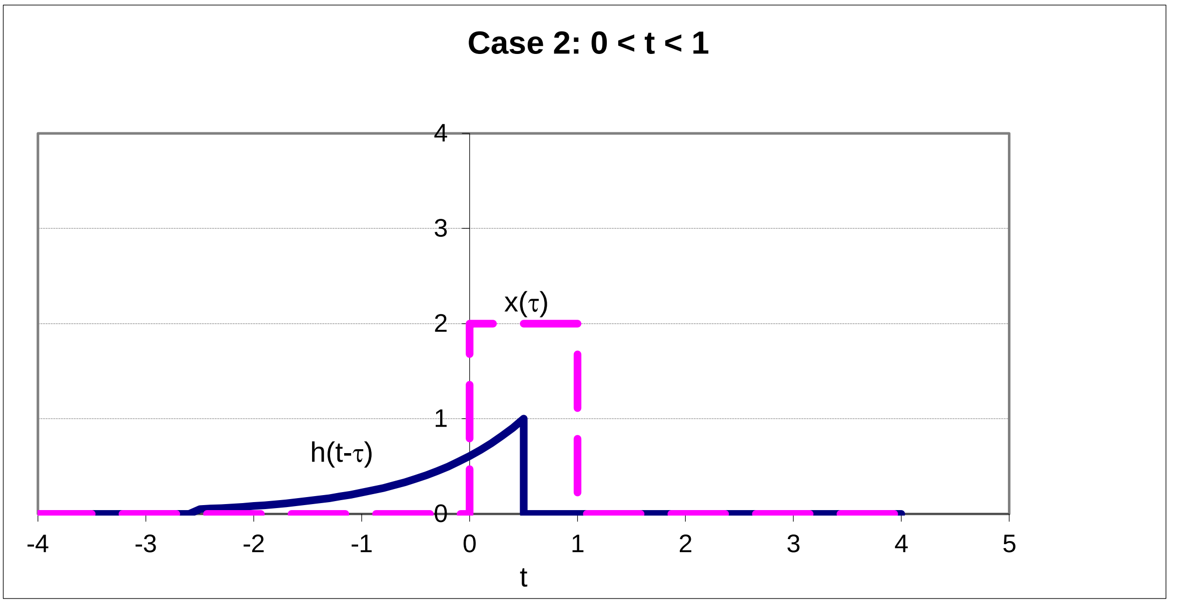


*Now using **Convolution**, find the output:*
Flipping h(t), sliding it to the left, t < 0, we have Case 1:



And of course y(t) = 0 for t < 0 since there is no overlap.

Case 2 is while the leading edge of h(t-τ) is within the square pulse or when 0 < t < 1



Now the integral becomes:

$$v_{out}(t) = x(t) \otimes h(t) = \int_{\tau=-\infty}^{\infty} x(\tau) * h(t-\tau)\, d\tau$$

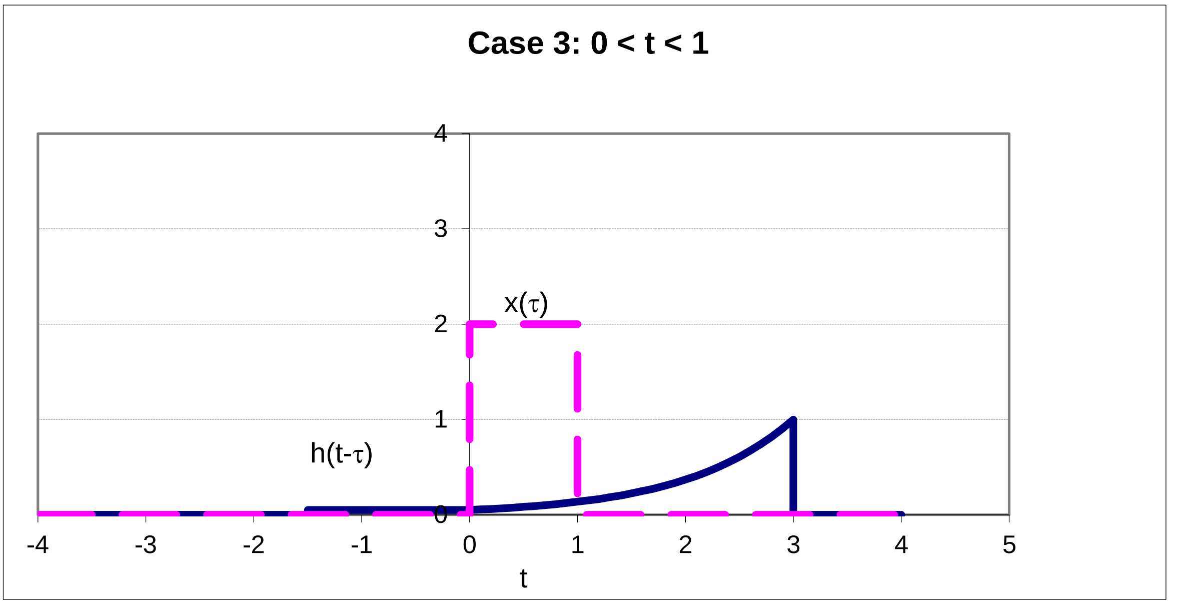
$$v_{out}(t) = \int_{\tau=0}^{t} 2 * \frac{1}{RC} \varepsilon^{-\frac{(t-\tau)}{RC}} d\tau$$

$$v_{out}(t) = \frac{2}{RC} * \varepsilon^{-\frac{t}{RC}} \int_{\tau=0}^{t} \varepsilon^{\frac{\tau}{RC}} d\tau$$

$$v_{out}(t) = \frac{2}{RC} * \varepsilon^{-\frac{t}{RC}} \left[RC * \varepsilon^{\frac{\tau}{RC}} \right]_{\tau=0}^{t}$$

$$v_{out}(t) = 2 * \varepsilon^{-\frac{t}{RC}} \left[\varepsilon^{\frac{\tau}{RC}} \right]_{\tau=0}^{t} = 2 * \varepsilon^{-\frac{t}{RC}} \left[\varepsilon^{\frac{t}{RC}} - 1 \right] = 2 * \left[1 - \varepsilon^{-\frac{t}{RC}} \right] \text{for } 0 < t < 1$$

Case 3 is the final case and it is good for t > 1



Now the integral becomes:

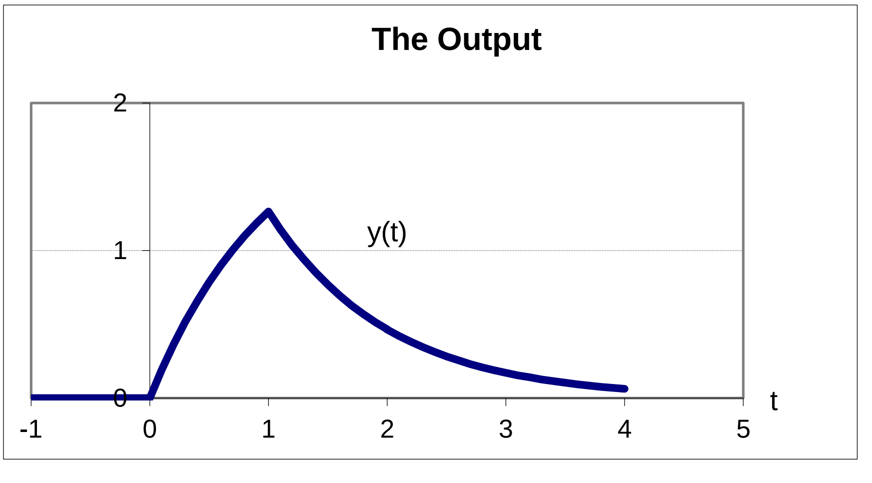
$$v_{out}(t) = \int_{\tau=0}^{1} 2 * \frac{1}{RC} \varepsilon^{-\frac{(t-\tau)}{RC}} d\tau$$

$$v_{out}(t) = \frac{2}{RC} * \varepsilon^{-\frac{t}{RC}} \int_{\tau=0}^{1} \varepsilon^{\frac{\tau}{RC}} d\tau$$

$$v_{out}(t) = \frac{2}{RC} * \varepsilon^{-\frac{t}{RC}} \left[RC * \varepsilon^{\frac{\tau}{RC}} \right]_{\tau=0}^{1}$$

$$v_{out}(t) = 2 * \varepsilon^{-\frac{t}{RC}} \left[\varepsilon^{\frac{\tau}{RC}} \right]_{\tau=0}^{1} = 2 * \varepsilon^{-\frac{t}{RC}} \left[\varepsilon^{\frac{1}{RC}} - 1 \right]$$

Moreover, we can now plot the output:



Convolution Sum

Example : Convolve {1,3,1) and (1,2,2,).

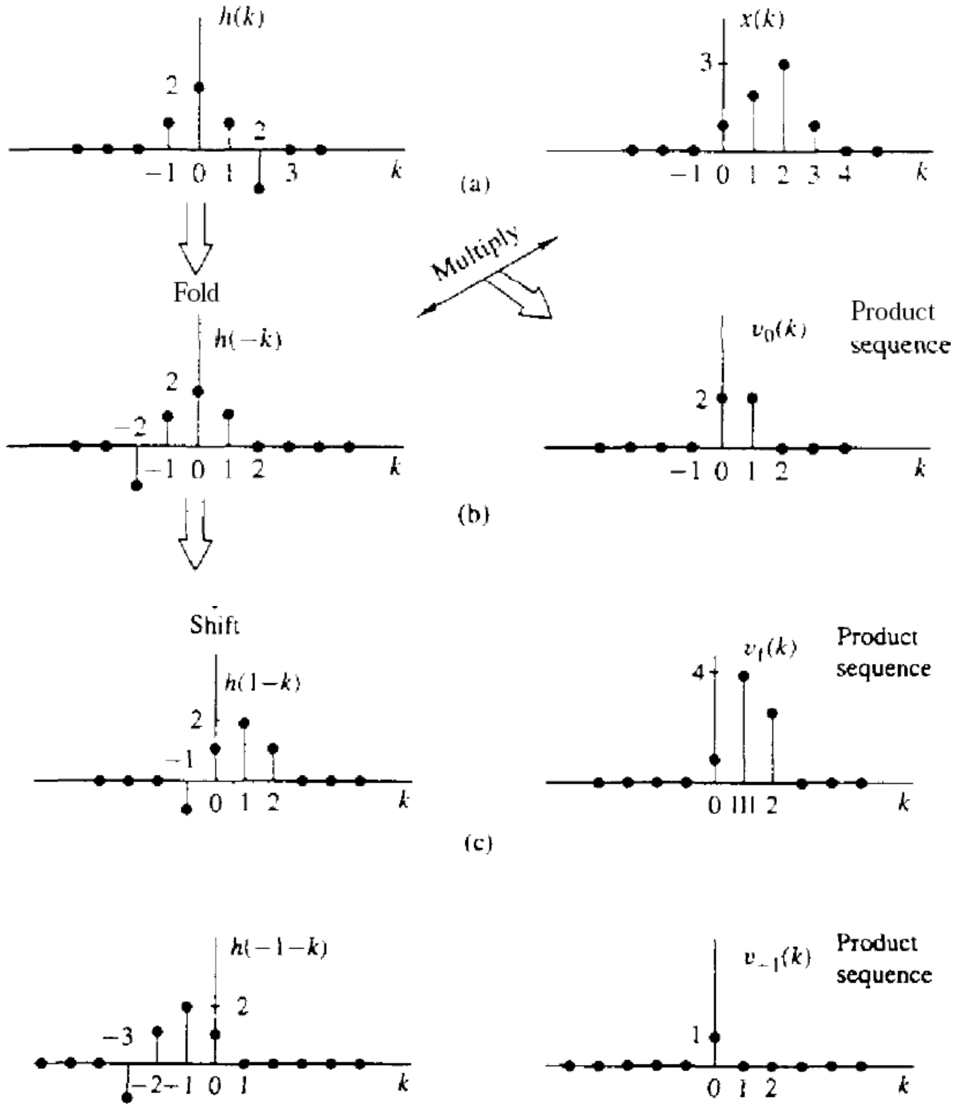
Ans.

Example . The impulse response of a linear time-invariant system is

$$h(n) = \{1, \underset{\uparrow}{2}, 1, -1\}$$

Determine the response of the system to the input signal

$$x(n) = \{\underset{\uparrow}{1}, 2, 3, 1\}$$



(d)

$$y(1) = \sum_{k=-\infty}^{\infty} v_1(k) = 8$$

$$y(-1) = 1$$

$$y(n) = \{\ldots, 0, 0, 1, \underset{\uparrow}{4}, 8, 8, 3, -2, -1, 0, 0, \ldots\}$$

Block diagram representation and reduction

Block diagram representation

State variable techniques

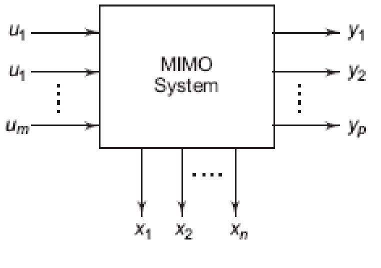
The main tools of analysis of single input and single output (SISO) systems are transfer function and frequency response methods where the systems must be linear time invariant. These tools cannot be applied for time varying and non-linear systems. In conventional control theory the main theme is to formulate the transfer function putting all initial conditions to zero. The state variable analysis takes care of initial conditions automatically and it is also possible to analyze time varying or time-invariant, linear or non-linear, single or multiple input-output systems. The main target of this chapter is to introduce state variable analysis for continuous systems.

State: The state of a dynamic system is the smallest set of variables and the knowledge of these variables at t = t0 together with inputs for t ≥ t0 completely determines the behaviour of the system at t ≥ t0. A compact and concise representation of the past history of the system can be termed as the state of the system.
State Variables: The smallest set of variables that determine the state of the system are known as state variables.
The knowledge of capacitor voltage at t = 0 i.e., the initial voltage of the capacitor is a history dependent term and it forms a state variable. Similarly, initial current in an inductor is treated as state variable.
State Vector: The 'n' state variables that completely describe the behaviour of a given system are said to be 'n' components of a vector.
State Space: The n dimensional space whose co-ordinate axes consist of the x1 axis, x2 axis,….,xn axis are known as a state space.

Advantages
It is possible to analyze time-varying or time-invariant, linear or non-linear, single or multiple input-output systems.
It is possible to confirm the state of the system parameters also and not merely input-output relations.
It is possible to optimize the systems and useful for optimal design.
It is possible to include initial conditions.
Disadvantages
Complex techniques
Many computations are required.

STATE MODEL

Figure shows an nth order system having multiple input multiple output.



$$\boldsymbol{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix}_{m\times 1} \quad \boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}_{n\times 1} \quad \boldsymbol{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{bmatrix}_{p\times 1}$$

For time invariant system, the functional equations can be written in the form as below:

$$\dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}u(t)$$

$$\boldsymbol{y}(t) = \mathbf{C}x(t) + \mathbf{D}u(t)$$

where A, B, C and D are constant matrices.

The order of the above matrices is given below:

A $n \times n$ known as the evolution matrix,
B $n \times m$ known as the control matrix,
C $p \times n$ known as the observation matrix and
D $p \times m$ known as the direct transmission matrix.

This is known as the State Model of the given system.

TRANSFER FUNCTION DERIVATION FROM THE STATE MODEL

From the state model equations, we can derive the transfer function of the system. From definition of transfer function, we can write

$$\text{Transfer function} = \frac{\text{Laplace transform of output}}{\text{Laplace transform of input}}$$

$$\text{Transfer function} = \frac{Y(s)}{U(s)} = \mathbf{C}\,[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} + \mathbf{D}$$

Formula

The characteristic equation is given by $|sI - A| = 0$

***Example* 1.** **Find the transfer function when** $\mathbf{A} = \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 1 & 1 \end{bmatrix}$

Solution:

$$s\,\mathbf{I} - \mathbf{A} = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} s+2 & -1 \\ 0 & s+3 \end{bmatrix}$$

$$|s\,\mathbf{I} - \mathbf{A}| = \begin{vmatrix} s+2 & -1 \\ 0 & s+3 \end{vmatrix} = (s+2)\,(s+3) \neq 0$$

Therefore, $(s\mathbf{I} - \mathbf{A})^{-1}$ exists.

Now
$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{\text{Adj}(s\mathbf{I} - \mathbf{A})}{|s\mathbf{I} - \mathbf{A}|} = \frac{\begin{bmatrix} s+3 & -1 \\ 0 & s+2 \end{bmatrix}}{(s+2)(s+3)}$$

$$\therefore \quad C(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = [1 \;\; 1] \frac{\begin{bmatrix} s+3 & -1 \\ 0 & s+2 \end{bmatrix}}{(s+2)(s+3)} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{[1 \;\; 1]\begin{bmatrix} 1 \\ s+2 \end{bmatrix}}{(s+2)(s+3)} = \frac{(s+3)}{(s+2)(s+3)} = \frac{1}{s+2}$$

Example 2. **Find Trnasfer Function of the following system**

$$\dot{x}(t) = \begin{bmatrix} -2.2 & 0.4 \\ -0.6 & -0.8 \end{bmatrix} x(t) + \begin{bmatrix} 2 \\ -1 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} -1 & 3 \end{bmatrix} x(t) + 2u(t).$$

Solution :

First, we compute the eigenvalues of the A matrix by solving det(λI – A) = 0 to obtain λ1 = –1, λ2 = –2 Thus, the system is BIBO stable since the poles of the transfer function are negative, being equal to the eigenvalues.

$$H(s) = \begin{bmatrix} -1 & 3 \end{bmatrix} \begin{bmatrix} s+2.2 & -0.4 \\ 0.6 & s+0.8 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ -1 \end{bmatrix} + 2$$

$$= \frac{1}{(s+2.2)(s+0.8)-(-0.4)\cdot 0.6} \begin{bmatrix} -1 & 3 \end{bmatrix} \begin{bmatrix} s+0.8 & 0.4 \\ -0.6 & s+2.2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} + 2$$

$$= \frac{1}{s^2+3s+2} \begin{bmatrix} -1 & 3 \end{bmatrix} \begin{bmatrix} 2s+1.2 \\ -s-3.4 \end{bmatrix} + 2 = \frac{-5s-11.4+2(s^2+3s+2)}{s^2+3s+2}$$

$$= \frac{2s^2+s-7.4}{s^2+3s+2} = 2\frac{s^2+0.5s-3.7}{s^2+3s+2} = 2\frac{(s+2.19)(s-1.69)}{(s+1)(s+2)}, \quad \mathrm{Re}\{s\} > -1$$

STATE EQUATIONS FOR DISCRETE TIME SYSTEMS

State variable methods for continuous time systems have already been introduced. In this chapter we are interested to discuss the following:

i. to represent a given z–transfer function by state variable equation and output equation of the form

$$\left.\begin{aligned} \mathbf{x}(k+1) &= \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k) && : \text{State equation} \\ y(k) &= \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k) && : \text{Output equation} \end{aligned}\right\}$$

ii. to get a relation between state equations, output equation and transfer function and

iii. finally the solution of state equation.

With the help of state equations we can calculate the next value of state variable from the given value of state variables and inputs.

Example 1 Find the SV model for the system with

$$H(z) = \frac{1}{z^4 + 4z^3 + 5z^2 + 6z + 3}$$

The given transfer function is

$$H(z) = \frac{Y(z)}{U(z)} = \frac{1}{z^4 + 4z^3 + 5z^2 + 6z + 3}$$

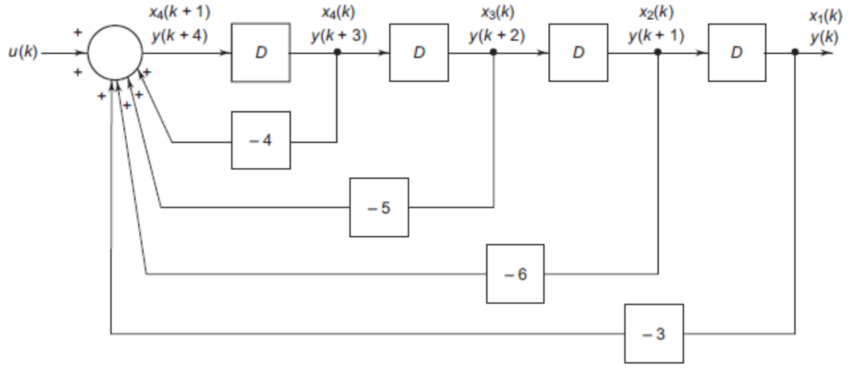
i.e., $(z^4 + 4z^3 + 5z^2 + 6z + 3)Y(z) = U(z)$

i.e., $z^4Y(z) + 4z^3Y(z) + 5z^2Y(z) + 6zY(z) + 3Y(z) = U(z) \quad (1)$

The inverse transform of Eq. (1) gives

$$y(k+4) + 4y(k+3) + 5y(k+2) + 6y(k+1) + 3y(k) = u(k)$$

i.e., $y(k+4) = u(k) - 4y(k+3) - 5y(k+2) - 6y(k+1) - 3y(k)$



$$x_1(k+1) = x_2(k) \quad (2)$$

$$x_2(k+1) = x_3(k) \quad (3)$$

$$x_3(k+1) = x_4(k) \quad (4)$$

$$x_4(k+1) = -3x_1(k) - 6x_2(k) - 5x_3(k) - 4x_4(k) + u(k) \quad (5)$$

From Eqs. (2), (3), (4) and (5) we get

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & -6 & -5 & -4 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(k) \quad (6)$$

The output equation is

$$y(k) = x_1(k) = x_1(k) + 0.\, x_2(k) + 0.\, x_3(k) + 0.\, x_4(k) + 0.\, u(k) \quad (7)$$

i.e., $$y(k) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix} + [0]u(k) \quad (8)$$

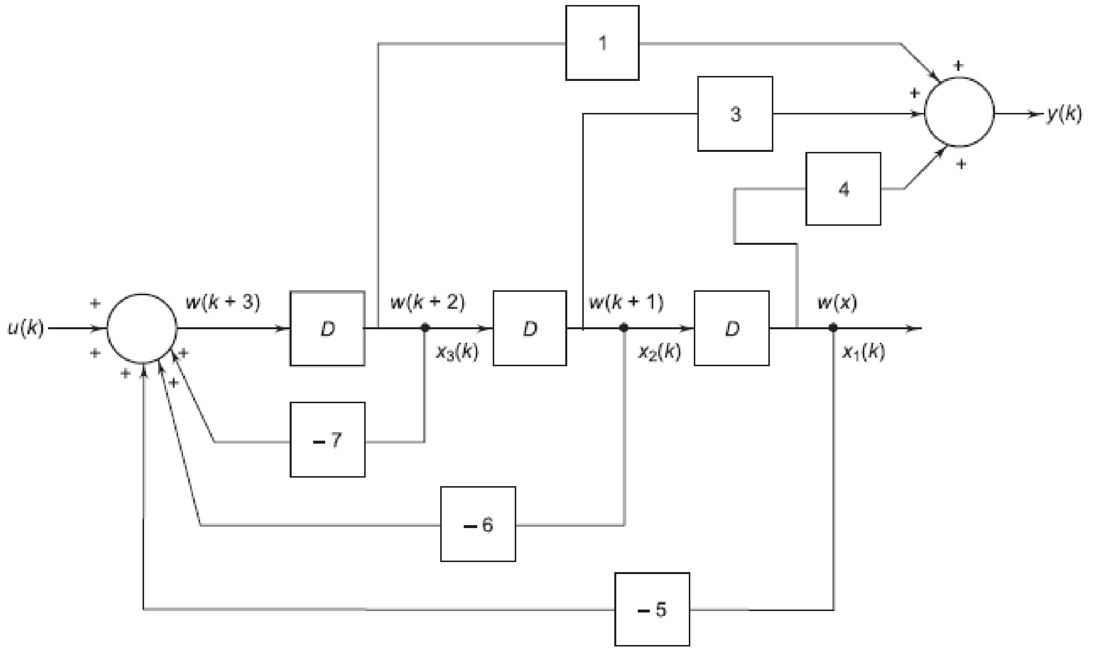
$$H(z) = \frac{z^2 + 3z + 4}{z^3 + 7z^2 + 6z + 5}$$

Example 2.

$$\frac{Y(z)}{U(z)} = \frac{Y(z)}{W(z)} \times \frac{W(z)}{U(z)} = (z^2 + 3z + 4) \times \left(\frac{1}{z^3 + 7z^2 + 6z + 5} \right)$$

$$\frac{W(z)}{U(z)} = \frac{1}{z^3 + 7z^2 + 6z + 5}$$

$$\frac{Y(z)}{W(z)} = z^2 + 3z + 4$$



$x_1(k + 1) = x_2(k)$

$x_2(k + 1) = x_3(k)$

$x_3(k + 1) = (-5)x_1(k) + (-6)\, x_2(k) + (-6)\, x_3(k) + u(k)$

From the above we get

$$\begin{bmatrix} x_1\,(k+1) \\ x_2\,(k+1) \\ x_3\,(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -6 & -7 \end{bmatrix} \begin{bmatrix} x_1\,(k) \\ x_2\,(k) \\ x_3\,(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k)$$

The Out put equation is

$y\,(k) = 4\,x_1(k) + 3x_2(k) + x_3(k)$

Therefore

$$y\,(k) = \begin{bmatrix} 4 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

Example 3. If the state model of a system is given by

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$\mathbf{Y}(k) = [-2 \quad -1] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + [1]\, U(k)$, find the transfer function of the system.

Solution :

Here $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{C} = [-2 \quad -1]$ and $\mathbf{D} = [1]$

We know that

$$H(z) = \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B} + \mathbf{D}$$

$$\therefore \quad (z\mathbf{I} - \mathbf{A}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} z & -1 \\ 2 & z+3 \end{bmatrix}$$

Therefore, $|(z\mathbf{I} - \mathbf{A})| = z(z+3) + 2 = z^2 + 3z + 2$

$$\therefore \quad (z\mathbf{I} - \mathbf{A})^{-1} = \left(\frac{1}{z^2+3z+2}\right) \begin{bmatrix} z+3 & 1 \\ -2 & z \end{bmatrix} = \begin{bmatrix} \frac{z+3}{z^2+3z+2} & \frac{1}{z^2+3z+2} \\ \frac{-2}{z^2+3z+2} & \frac{z}{z^2+3z+2} \end{bmatrix}$$

$$\therefore \quad H(z) = \mathbf{C}\,(z\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B} + \mathbf{D} = [-2 \quad -1] \begin{bmatrix} \frac{z+3}{z^2+3z+2} & \frac{1}{z^2+3z+2} \\ \frac{-2}{z^2+3z+2} & \frac{z}{z^2+3z+2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + [1]$$

$$= [-2 \quad -1] \begin{bmatrix} \frac{1}{z^2+3z+2} \\ \frac{z}{z^2+3z+2} \end{bmatrix} + [1] = \frac{-2-z}{z^2+3z+2} + 1 = \frac{-2-z+z^2+3z+2}{z^2+3z+2}$$

$$= \frac{z^2+2z}{z^2+3z+2} = \frac{1+2z^{-1}}{1+3z^{-1}+2z^{-2}}$$

UNIT V

DISCRETE FOURIER TRANSFORM

DISCRETE FOURIER TRANSFORM

One of the main advantages of discrete-time signals is that they can be processed and represented in digital computers. However, when we examine the definition of the Fourier transform of discrete-time signal

$$X\left(e^{j\omega}\right)=\sum_{n=-\infty}^{\infty}x(n)e^{-j\omega n} \qquad \text{DTFT}$$

We notice that such a characterization in frequency domain depends on the **continuous variable** ω **.**

This implies that the Fourier transform is not suitable for the processing of discrete-time signals in digital computers.

As a consequence we need a transform depending on a discrete-frequency variable. This can be obtaining from the Fourier transform itself in very simple way, by sampling uniformly the continuous-frequency variable ω . In this way, we obtain a mapping of a signal depending on a discrete-time variable n to a transform depending a discrete-frequency variable k, such a mapping is referred to as the discrete Fourier transform (DFT).

DFT is given by

$$X(k)=\sum_{n=0}^{N-1}x(n)e^{-j\frac{2\pi}{N}kn}=\sum_{n=0}^{N-1}x(n)W_N^{nk}, \quad \text{for } 0\le k\le N-1$$

IDFT

$$x(n)=\sum_{k=0}^{N-1}X(k)e^{j\frac{2\pi}{N}kn}=\sum_{k=0}^{N-1}X(k)W_N^{-nk}, \quad \text{for } 0\le n\le N-1$$

Twiddle factor

Most approaches for improving the efficiency of computation of DFT, exploits

the symmetry and periodicity property of W_N^{kn} i.e.

$$W_N^{\left(k+\frac{N}{2}\right)} = -W_N^k \qquad \textbf{[Symmetry property]}$$

$$W_N^{k+N} = W_N^k \qquad \textbf{[Periodicity property]}$$

Properties of the DFT

1. Linearity : $Ax(n) + By(n) \leftrightarrow AX(k) + BX(k)$

2. Time Shift: $x(n-m) \leftrightarrow X(k)e^{-j2\pi km/N} = X(k)W_N{}^{k-m}$

3. Frequency Shift: $x(n)e^{j2\pi km/N} \leftrightarrow X(k-m)$

4. Duality : $N^{-1}x(n) \leftrightarrow X(-k)$

why? $X(k) = \sum_{m=0}^{N-1} x(m)e^{-j2\pi mk/N}$

$$DFT(X(n)) = \sum_{n=0}^{N-1} X(n)e^{-j2\pi nk/N}$$

DFT of x(m)

$$x(-n) = x(N-n)$$
$$= \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j2\pi k(N-n)/N}$$
$$e^{j2\pi k(N-n)/N} = e^{j2\pi kN/N} e^{-j2\pi kn/N}$$
$$x(-n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{-j2\pi kn/N}$$

$$\Rightarrow DFT(N^{-1}X(n)) = \frac{1}{N}\sum_{n=0}^{N-1} X(n) e^{-j2\pi nk/N} = x(-k)$$

5. Circular convolution

$$\sum_{m=0}^{N-1} x(m)y(n-m) = x(n)\mathrm{O}y(n) \leftrightarrow X(k)Y(k)$$ circular convolution

6. Multiplication

$$\underbrace{x(n)y(n)}_{\substack{new \quad sequence \\ z(n)=x(n)y(n)}} \leftrightarrow N^{-1}\sum_{m=0}^{N-1} X(m)Y(k-m) = N^{-1}X(k)\mathrm{O}Y(k)$$

7. Parseval's Theorem

$$\sum_{n=0}^{N-1} | x(n) |^2 = N^{-1}\sum_{k=0}^{N-1} | X(k) |^2$$

<u>Summary of Properties of the Discrete Fourier Transform</u>

Property	**Periodic signal**	**Fourier Series Coefficients**
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$

Time Shifting	$x[n-n_0]$	$a_k \cdot e^{-jk\left(\frac{2\pi}{N}\right)n_0}$
Conjugation	$x^*[n]$	a_{-k}^*
Time Reversal	$x[-n]$	a_{-k}
Frequency Shifting	$e^{jMw_0 n}x[n]$	a_{k-M}
First Difference	$x[n]-x[n-1]$	$\left(1-e^{-jk\left(2\pi/N\right)}\right)a_k$
Conjugate Symmetry for Real Signals	x[n] real	$a_k = a_{-k}^*$
Real & Even Signals	x[n] real and even	a_k real and even
Real & Odd signals	x[n] real and odd	a_k purely imaginary and odd
Even-Odd Decomposition Of Real Signals	$x_e[n] = Ev\{x[n]\} \quad [x[n] real]$ $x_o[n] = Od\{x[n]\} \quad [x[n] real]$	$\mathrm{Re}\{a_k\}$ $j\,\mathrm{Im}\{a_k\}$
Parseval's Relation	$\frac{1}{N}\sum_{n=\langle N\rangle}\lvert x[n]\rvert^2 = \sum_{k=\langle N\rangle}\lvert a_k\rvert^2$	

PROBLEMS

Example 1. : Find DFT of { 1 0 0 1}.

The DFT of the sequence { 1, 0, 0, 1} will be evaluated

$x(0) = 1, \quad x(T)= 0, \quad x(2T) = 0, \quad x(3T) = 1 \;, \qquad N = 4$

We desire to find X(k) for k = 0,1,2,3.

For k = 0

$$X(0) = \sum_{n=0}^{3} x(nT)e^{j0} = \sum_{n=0}^{3} x(nT) = x(0) + x(T) + x(2T) + x(3T)$$
$$= 1 + 0 + 0 + 1 = 2$$

k = 1

$$X(1) = \sum_{n=0}^{3} x(nT)e^{-j\Omega nT} = \sum_{n=0}^{3} x(nT)e^{-j2\pi n/N} = 1 + 0 + 0 + 1e^{-j\frac{6\pi}{4}} = 1 + j$$

k = 2

$$X(2) = \sum_{n=0}^{3} x(nT)e^{-j2\pi n2/N} = 1 + 0 + 0 + 1e^{-j3\pi} = 1 - 1 = 0$$

k = 3

$$X(3) = \sum_{n=0}^{3} x(nT)e^{-j2\pi n3/N} = 1 + 0 + 0 + 1e^{-j\frac{9\pi}{2}} = 1 - j$$

Ans: **X(k) = { 2 , (1+j) , 0, (1-j) }**

Example 2: What is the DFT of the signal δ[n]?

$x[n] = \delta[n]$

$$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} \delta[n]\, \varphi_k[-n] = \frac{1}{N}\, \varphi_k[0] = \frac{1}{N}.$$

This means that all frequencies are equally strong present in the impulse signal. So its frequency spectrum is flat.

Example 3: What is the DFT of a shifted impulse δ[n-1]?

$$x[n] = \delta[n-1]$$

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1}\delta[n-1]\,\varphi_k[-n] = \frac{1}{N}\,\varphi_k[-1] = \frac{1}{N}\,e^{-ik\Omega_0} \text{ with } \Omega_0 = \frac{2\pi}{N}.$$

In this case again all frequencies X[k] are equally strong (they have the same modulus $\frac{1}{N}$), but now the frequency spectrum consists of complex numbers.

Example 4: What is the DFT of $\cos(5\Omega_0)$ with $\Omega_0 = \frac{2\pi}{N}$?

$$x[n] = \cos(n\,5\Omega_0).$$

We could calculate X[k] in the same way as in the previous examples, but we can also directly find X[k] because we may write $\cos(5\Omega_0)$ as:

$$x[n] = \cos(n5\Omega_0) = \frac{1}{2}\,e^{in5\Omega_0} + \frac{1}{2}\,e^{-in5\Omega_0} = \frac{1}{2}\,\varphi_5[n] + \frac{1}{2}\,\varphi_{-5}[n].$$

As $\varphi_{k+N}[n] = \varphi_k[n]$ we obtain:

$$x[n] = \frac{1}{2}\,\varphi_5[n] + \frac{1}{2}\,\varphi_{N-5}[n].$$

So $X[5] = \frac{1}{2}$ and $X[N-5] = \frac{1}{2}$ and all other frequencies are zero:

$$X[k] = 0, \text{ for } 0 \le k < N \text{ and } k \ne 5 \text{ and } k \ne N-5.$$

Example 5: What is the DFT of a shifted signal $x[n-n_0]$?

Denote the DFT of the shifted signal by X'[k] and of the original signal by X[k].

$$X'[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n-n_0]\,\varphi_k[-n] = \frac{1}{N}\sum_{n=0}^{N-1} x[n-n_0]\,\varphi_k[-n+n_0]\,e^{-ik\Omega_0 n_0} =$$

$$= X[k]\,e^{-ik\Omega_0 n_0} = X[k]\,\varphi_k[-n_0] \quad \text{with } \Omega_0 = \frac{2\pi}{N}.$$

Example 6 : Compute the DFT of sequence $x(n) = \{-2, 2, 1, -1\}$.

Ans.

$$X(k) = \sum_{n=0}^{N-1} x(n)W_N^{kn} = \sum_{n=0}^{N-1} x(n)W_4^{kn} \quad k = 0, 1, 2, 3$$

$$X(k) = -2 + 2\ W_4^k + W_4^{2k} - W_4^{3k}$$

$$X(0) = -2 + 2 + 1 - 1 = 0$$

$$X(1) = -2 - 2j - 1 - j = -3 - 3j$$

$$X(2) = -2 - 2 + 1 + 1 = -2$$

$$X(3) = -2 + 2j - 1 + j = -3 + 3j$$

$$X(k) = \{0, -3, -3j, -2, -3 + 3j\}$$

Example 7: What is the relationship between Z transform and the Discrete Fourier transform?

Ans. Let us consider a sequence x(n) having z-transforrn with ROC that includes the

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \qquad ...(1)$$

unit circle. If X(z) is sampled at the N equally spaced points on the unit circle. If X(z) is

sampled at N equally spaced pomts on the unit circle.

$$Z_k = e^{j\frac{2\pi k}{N}},\ k = 0, 1, 2, 3 \ .... \ N-1.$$

We obtain

$$X(k) = X(z)\Big|_{z=e^{\frac{j2\pi k}{N}}} \ ; k = 0,1, \ ...., N-1$$

$$= \sum_{n=-\infty}^{\infty} x(n)\ e^{\frac{-j2\pi nk}{N}} \qquad ...(2)$$

Expression is (2) identical to the Fourier transform X(w) evaluated at the N. equally spaced. Frequencies

$$w_k = \frac{2\pi k}{N},\ k = 0,1, \ .... N-1.$$

If the sequence x(n) has a finite duration of length N or less, the sequence can be

recovered from its N-point DFT. Hence its Z-transform is uniquely determined by its N-point DFI'. Consequently, X(z) can be expressed as a function of the DFT {X(k)} as follows

$$X(z) = \sum_{n=0}^{N-1} x(n)\, z^{-n}$$

$$X(z) = \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{\frac{j2\pi kn}{N}}\right] z^{-n} = \frac{1}{N}\sum_{k=0}^{N-1} X(k) \sum_{n=0}^{N-1}\left(e^{\frac{j2\pi k}{N}} z^{-1}\right)^{n}$$

$$= \frac{1-z^{-N}}{N}\sum_{k=0}^{N-1}\frac{X(k)}{1-e^{\frac{j2\pi k}{N}} z^{-1}} \qquad \ldots(3)$$

When evaluated on the unit circle (3) yields the Fourier transform of the finite duration sequence in terms of its DFT in the form:

$$X(w) = \frac{1-e^{-jwN}}{N}\sum_{k=0}^{N-1}\frac{X(k)}{1-e^{-j\left(w-\frac{2\pi k}{N}\right)}}$$

This expression for Fourier transform is a polynomial interpolation formula for X(w)

expressed in terms of the, values {x(k)) of the polynomial at a set of equally spaced

discrete frequencies

$$\omega_k = \frac{2\pi k}{N},\ k = 0, 1, \ldots\ldots\ldots\ldots N-1.$$

Example 8 : Perform circular, convolution of two sequences

$$x_1(n) = \{0.2, 0.4, 0.6, 0.8, 1, 1.2, 1.4, 1.6\}$$
$$x_2(n) = \{0.1, 0.3, 0.5, 0.7, 0.9. 1.1, 1, 3, 1\} \qquad \textbf{(Dec. 2006)}$$

Ans. Circular convolution is

$$y(n) = x_1(n)\ \mathrm{N}\ x_2(n)$$

$$\therefore \quad \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \\ y(5) \\ y(6) \\ y(7) \end{bmatrix} = \begin{bmatrix} 0.2 & 1.6 & 1.4 & 1.2 & 1 & 0.8 & 0.6 & 0.4 \\ 0.4 & 0.2 & 1.6 & 1.4 & 1.2 & 1 & 0.8 & 0.6 \\ 0.6 & 0.4 & 0.2 & 1.6 & 1.4 & 1.2 & 1 & 0.8 \\ 0.8 & 0.6 & 0.4 & 0.2 & 1.6 & 1.4 & 1.2 & 1 \\ 1 & 0.8 & 0.6 & 0.4 & 0.2 & 1.6 & 1.4 & 1.2 \\ 1.2 & 1 & 0.8 & 0.6 & 0.4 & 0.2 & 1.6 & 1.4 \\ 1.4 & 1.2 & 1 & 0.8 & 0.6 & 0.4 & 0.2 & 1.6 \\ 1.6 & 1.4 & 1.2 & 1 & 0.8 & 0.6 & 0.4 & 0.2 \end{bmatrix} \begin{bmatrix} 0.1 \\ 0.3 \\ 0.5 \\ 0.7 \\ 0.9 \\ 1.1 \\ 1.3 \\ 1.5 \end{bmatrix}$$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \\ y(5) \\ y(6) \\ y(7) \end{bmatrix} = \begin{bmatrix} 0.02 & +0.18 & +0.7 & +0.64 & +0.9 & +0.88 & +0.78 & +0.6 \\ 0.04 & +0.06 & +0.8 & +0.98 & +1.08 & +1.1 & +1.04 & +0.9 \\ 0.06 & +0.12 & +0.1 & +1.12 & +1.26 & +1.32 & +1.3 & +1.2 \\ 0.08 & +0.18 & +0.2 & +0.14 & +1.44 & +1.54 & +1.56 & +1.5 \\ 0.1 & +0.24 & +0.30 & +0.28 & +0.18 & +1.76 & +1.82 & +1.8 \\ 0.12 & +0.3 & +0.4 & +0.42 & +0.36 & +0.22 & +2.08 & +2.1 \\ 0.14 & +0.36 & +0.5 & +0.56 & +0.54 & +0.44 & +0.26 & +2.4 \\ 0.16 & +0.42 & +0.6 & +0.7 & +0.72 & +0.66 & +0.52 & +0.3 \end{bmatrix} = \begin{bmatrix} 5.2 \\ 6 \\ 6.48 \\ 6.64 \\ 6.48 \\ 6 \\ 5.2 \\ 4.08 \end{bmatrix}$$

FFT algorithms –advantages over direct computation of DFT – radix 2 algorithms

What are the advantages of FFT algorithm?

Fast fourier transform reduces the computation time. In DFT computation, number of multiplication is N^2 and the number of addition is N(N-1). In FFT algorithm, number of multiplication is only $N/2(\log_2 N)$. Hence FFT reduces the number of elements (adder, multiplier Z &delay elements). This is achieved by effectively utilizing the symmetric and periodicity properties of Fourier transform.

(Preparation for Mathematical Derivation of FFT)

1. DFT Algorithm

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} = \sum_{n=0}^{N-1} x(n) \left(e^{-j2\pi/N}\right)^{nk}$$

Denote $W_N = e^{-j2\pi/N}$, then

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N{}^{nk}$$

Properties of $W_N{}^m$:

(1) $W_N{}^0 = (e^{-j2\pi/N})^0 = e^0 = 1, \qquad W_N{}^N = e^{-j2\pi} = 1$

(2) $W_N{}^{N+m} = W_N{}^m$

$$\begin{aligned} W_N{}^{N+m} &= (e^{-j2\pi/N})^{N+m} \\ &= (e^{-j2\pi/N})^N (e^{-j2\pi/N})^m \\ &= 1 \cdot (e^{-j2\pi/N})^m = W_N{}^m \end{aligned}$$

(3) $W_N{}^{N/2} = e^{-j2\pi/(N/2)/N} = e^{-j\pi} = -1$

$$W_N{}^{N/4} = e^{-j2\pi/(N/4)/N} = e^{-j\pi/2} = -j$$

$$W_N{}^{3N/4} = e^{-j2\pi/(3N/4)/N} = e^{-j3\pi/2} = j$$

RADIX 2

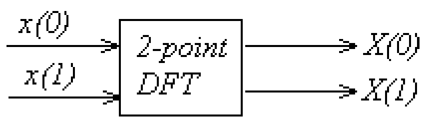
The FFT algorithm is most efficient in calculating N point DFT. If the number of point N can be expressed as a power of 2 ie N= 2^M where M is an integer , then this algorithm is known as radix-2 FFT algorithm.

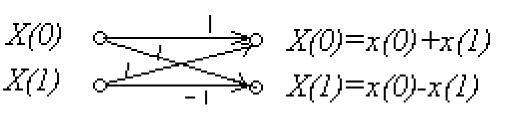
Two-Point DFT

$$x(0), x(1):\quad X(k) = \sum_{n=0}^{1} x(n) W_2^{nk} \qquad k = 0,1$$

$$X(0) = \sum_{n=0}^{1} x(n) W_2^{n0} = \sum_{n=0}^{1} x(n) = x(0) + x(1)$$

$$\begin{aligned} X(1) &= \sum_{n=0}^{1} x(n) W_2^{n1} = \sum_{n=0}^{1} x(n) W_2^{n} \\ &= x(0) W_2^{0} + x(1) W_2^{1} \\ &= x(0) + x(1) W_2^{(1/2)2} \\ &= x(0) + x(1)(-1) \\ &= x(0) - x(1) \end{aligned}$$





Four-point DFT

$x(0), x(1), x(2), x(3)$

$$X(k) = \sum_{n=0}^{3} x(n) W_4^{nk} \qquad k = 0,1,2,3,$$

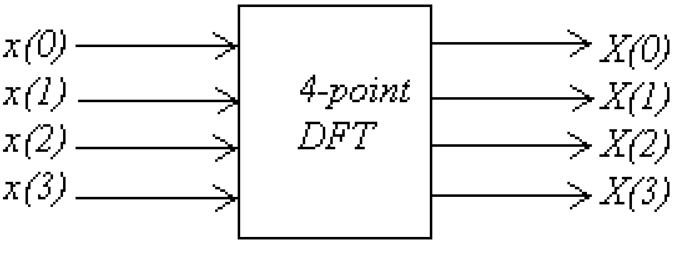
$$X(0) = \sum_{n=0}^{3} x(n) W_4^{n0} = \sum_{n=0}^{3} x(n) = x(0) + x(1) + x(2) + x(3)$$

$$X(1) = \sum_{n=0}^{3} x(n)W_4^{\,n} = x(0)W_4^{\,0} + x(1)W_4^{\,1} + x(2)W_4^{\,2} + x(3)W_4^{\,3}$$
$$= x(0) - jx(1) - x(2) + jx(3)$$

$$X(2) = \sum_{n=0}^{3} x(n)W_4^{\,2n} = x(0)W_4^{\,0} + x(1)W_4^{\,2} + x(2)W_4^{\,4} + x(3)W_4^{\,6}$$
$$= x(0) + x(1)(-1) + x(2)(1) + x(3)W_4^{\,2}$$
$$= x(0) - x(1) + x(2) - x(3)$$

$$X(3) = \sum_{n=0}^{3} x(n)W_4^{\,3n} = x(0)W_4^{\,0} + x(1)W_4^{\,3} + x(2)W_4^{\,6} + x(3)W_4^{\,9}$$
$$= x(0) + x(1)W_4^{\,3} + x(2)(1)W_4^{\,2} + x(3)W_4^{\,1}$$
$$= x(0) + jx(1) + (-1)x(2) + (-j)x(3)$$
$$= x(0) + jx(1) - x(2) - jx(3)$$

$$\Rightarrow \quad \begin{aligned} X(0) &= [x(0) + x(2)] + [x(1) + x(3)] \\ X(1) &= [x(0) - x(2)] + (-j)[x(1) - x(3)] \\ X(2) &= [x(0) + x(2)] - [x(1) + x(3)] \\ X(3) &= [x(0) - x(2)] + j[x(1) - x(3)] \end{aligned}$$



Two – point DFT

If we denote $z(0) = x(0)$, $z(1) = x(2) \Rightarrow Z(0) = z(0) + z(1) = x(0) + x(2)$

$Z(1) = z(0) - z(1) = x(0) - x(2)$

$v(0) = x(1)$, $v(1) = x(3) \Rightarrow V(0) = v(0) + v(1) = x(1) + x(3)$

$V(1) = v(0) - v(1) = x(1) - x(3)$

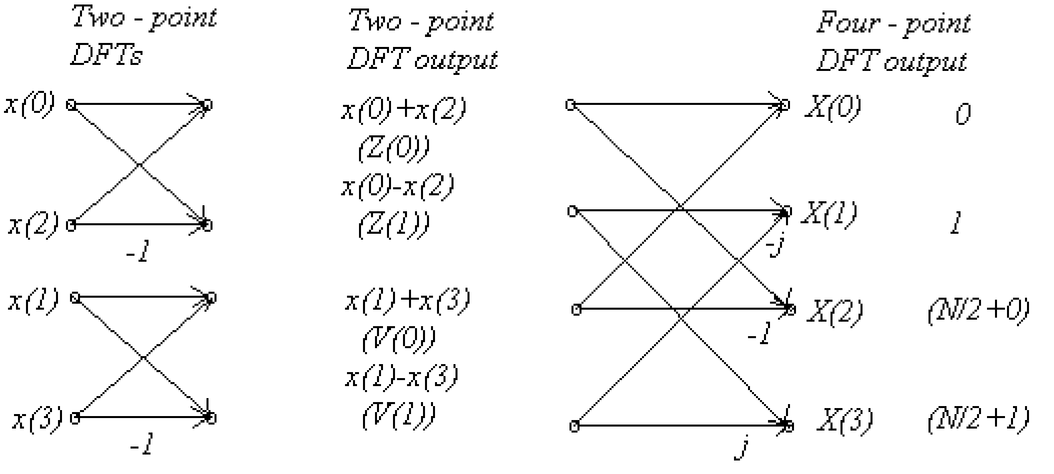
Four – point DFT   Two-point DFT

➔ $X(0) = Z(0) + V(0)$

$X(1) = Z(1) + (-j)V(1)$

$X(2) = Z(0) - V(0)$

$X(3) = Z(1) + jV(1)$



Decimation-in-Time FFT Algorithm

$x(0), x(1), \ldots, x(N-1)$ $N = 2^m$

$$=>\begin{cases} g(0), g(1), \cdots, g(\frac{N}{2}-1) & -enen \quad \frac{N}{2} \quad \text{po}\mathrm{int}\, s \\ ((x(0), x(2), \cdots, x(N-2)) & (g(r)=x(2r)) \\ h(0), h(1), \cdots, h(\frac{N}{2}-1) & -odd \quad \frac{N}{2} \quad \text{po}\mathrm{int}\, s \\ ((x(1), x(3), \cdots, x(N-1)) & (h(r)=x(2r+1)) \end{cases}$$

$$\begin{aligned} X(k) &= \sum_{n=0}^{N-1} x(n) W_N{}^{kn} \\ &= \sum_{r=0}^{N/2-1} g(r) W_N{}^{k(2r)} + \sum_{r=0}^{N/2-1} h(r) W_N{}^{k(2r+1)} \qquad (k=0,1,\ldots,N-1) \\ &= \sum_{r=0}^{N/2-1} g(r) W_N{}^{2kr} + W_N{}^{k} \sum_{r=0}^{N/2-1} h(r) W_N{}^{2kr} \end{aligned}$$

$$W_N{}^{2kr} = (e^{-j2\pi./N})^{2kr} = (e^{-j2\pi/(.N/2)})^{kr} = W_{\frac{N}{2}}{}^{kr}$$

$$\begin{aligned} \Rightarrow X(k) &= \sum_{r=0}^{N/2-1} g(r) W_{N/2}{}^{kr} + W_N{}^{k} \sum_{r=0}^{N/2-1} h(r) W_{N/2}{}^{kr} \\ &= G(k) + W_N{}^{k} H(k) \end{aligned}$$

(G(k): N/2 point DFT output (even indexed), H(k) : N/2 point DFT output (odd indexed))

$$X(k) = G(k) + W_N{}^{k} H(k) \qquad k = 0,1,\ldots,N-1$$

$$G(k) = \sum_{r=0}^{N/2-1} g(r) W_{N/2}{}^{kr} = \sum_{r=0}^{N/2-1} x(2r) W_{N/2}{}^{kr}$$

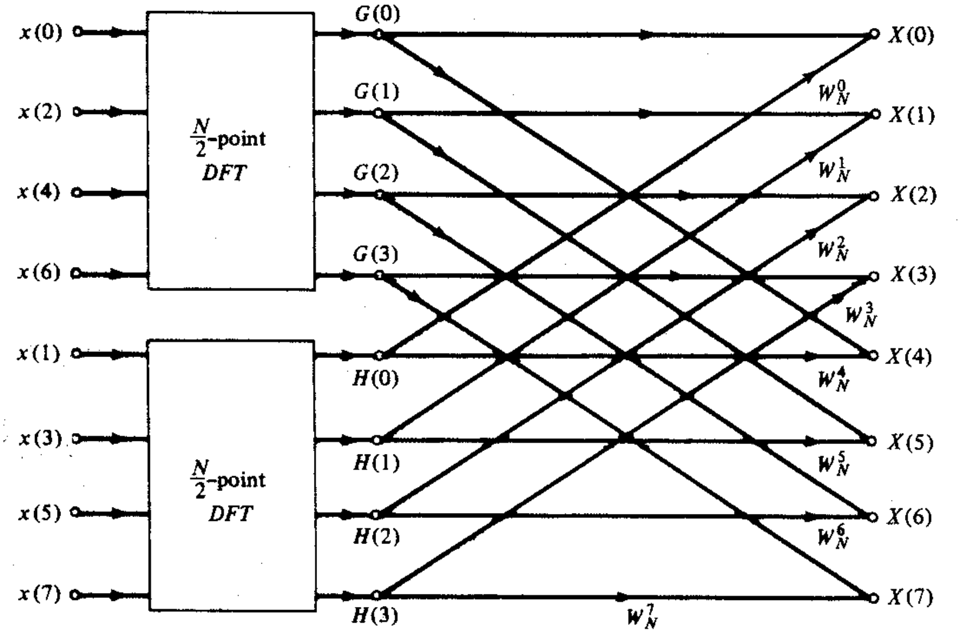
$$H(k) = \sum_{r=0}^{N/2-1} h(r) W_{N/2}{}^{kr} = \sum_{r=0}^{N/2-1} x(2r+1) W_{N/2}{}^{kr}$$

Question: $X(k)$ needs $G(k)$, $H(k)$, k=… N-1

How do we obtain $G(k)$, $H(k)$, for $k > N/2$-1 ?

$G(k) = G(N/2+k)$ $\qquad$ $k <= N/2$-1

$H(k) = H(N/2+k)$ $\quad\quad\quad\quad$ $k <= N/2-1$



(a) Result of one decimation of the time samples

Future Decimation

g(0), g(1), …, g(N/2-1) $\quad\quad$ *G(k)*

h(0), h(1), …, h(N/2-1) $\quad\quad$ *H(k)*

$$g(0), g(2), \cdots, g(\frac{N}{2}-2)$$

$$ge(0), ge(1), \ldots ge(\frac{N}{4}-1)$$

$$g(1), g(3), \cdots, g(\frac{N}{2}-1)$$

$$go(0), go(1), \ldots go(\frac{N}{4}-1)$$

$$G(k) = \sum_{r=0}^{N/2-1} g(r) W_{(N/2)}{}^{kr}$$

$$= \sum_{m=0}^{N/4-1} ge(m) W_{(N/4)}{}^{km}$$

$$+ W_{(N/2)}{}^{k} \sum_{m=0}^{N/4-1} go(m) W_{(N/4)}{}^{km}$$

$$= GE(k) + W_{(N/2)}{}^{k} Go(k)$$

even indexed g $\quad$ odd indexed g

(N/4 point) $\quad\quad$ (N/4 point)

$$W_{N/2}{}^{k} = W_N{}^{2k} \, ?$$

$$\begin{aligned} W_{N/2}{}^{k} &= (e^{-j2\pi/(N/2)})^{k} \\ &= (e^{-j2\pi 2/N})^{k} = (e^{-j2\pi/N})^{2k} \\ &= W_N{}^{2k} \end{aligned}$$

=> $G(k) = GE(k) + W_N{}^{2k} Go(k)$

Similarly,

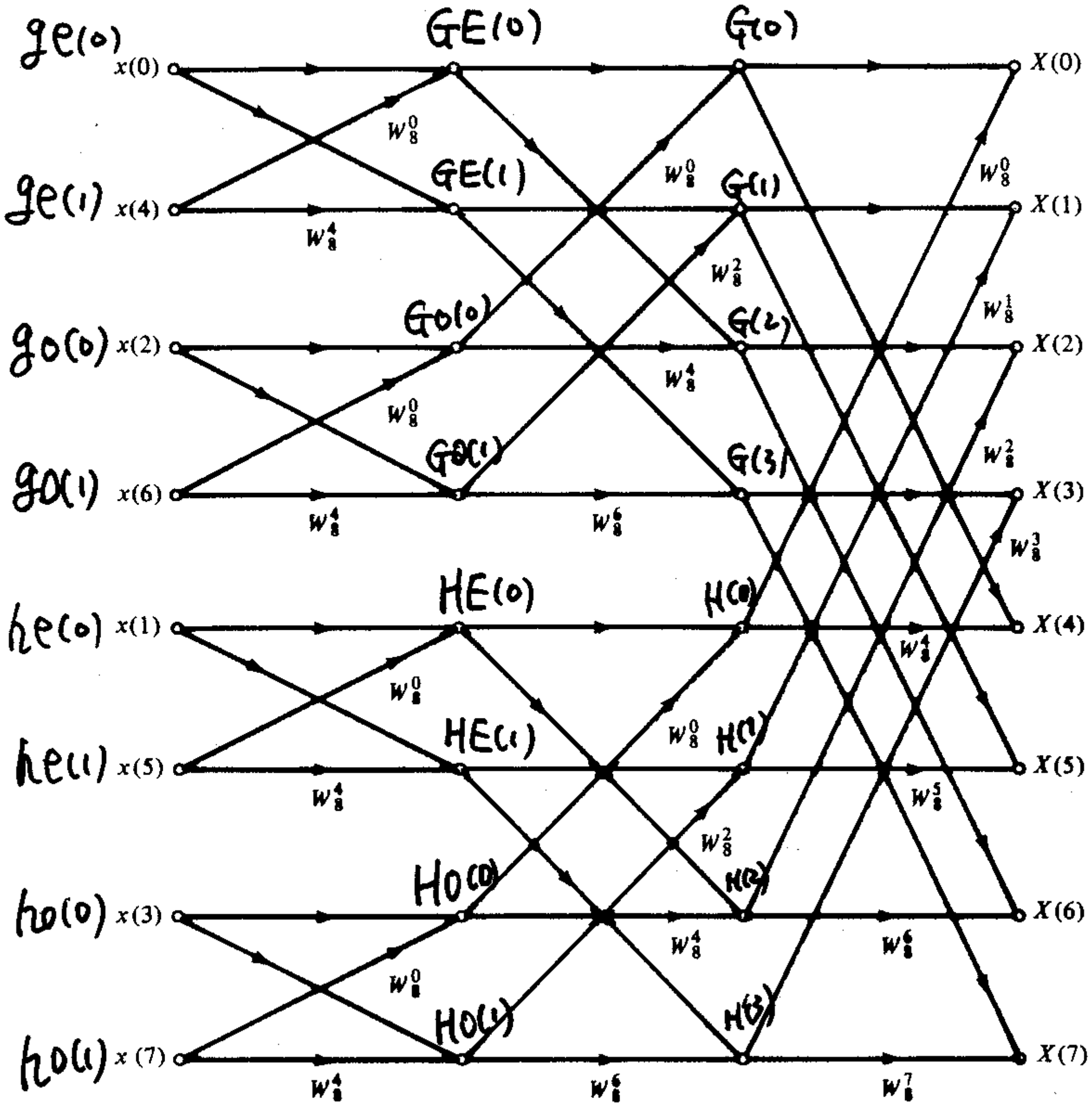
$$H(k) = HE(k) + W_N{}^{2k} Ho(k)$$

even indexed odd indexed

h (N/4 point) *h* (N/4 point)

For 8 – point

$$\begin{array}{cccccccc}
g(0) & g(1) & g(2) & g(3) & h(0) & h(1) & h(2) & h(3) \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
x(0) & x(2) & x(4) & x(6) & x(1) & x(3) & x(5) & x(7) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
ge(0) & \downarrow & ge(1) & \downarrow & he(0) & \downarrow & he(1) & \downarrow \\
 & go(0) & & go(1) & & ho(0) & & ho(1)
\end{array}$$



Decimation-in-Frequency FFT Algorithm

$x(0), x(1), \ldots, x(N\text{-}1)$ $\qquad N = 2^m$

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{\;nk}$$

$$= \sum_{n=0}^{N/2-1} x(n) W_N^{\;nk} + \sum_{n=N/2}^{N-1} x(n) W_N^{\;nk}$$

let $m = n\text{-}N/2$ ($n = N/2+m$) $\qquad n = N/2 \Rightarrow m = N/2\text{-}N/2 = 0$

n = N-1 => m = N-1-N/2 = N/2-1

$$\Rightarrow X(k) = \sum_{n=0}^{N/2-1} x(n) W_N^{nk} + \sum_{m=0}^{N/2-1} x(N/2+m) W_N^{(N/2+m)k}$$

$$= \sum_{n=0}^{N/2-1} x(n) W_N^{nk} + \sum_{m=0}^{N/2-1} x(N/2+m) W_N^{mk} W_N^{\frac{N}{2}k}$$

$$W_N^{\frac{N}{2}} = -1 \Rightarrow W_N^{\frac{N}{2}k} = (-1)^k$$

$$X(k) = \sum_{n=0}^{N/2-1} x(n) W_N^{nk} + \sum_{m=0}^{N/2-1} (-1)^k x(N/2+m) W_N^{mk}$$

$$= \sum_{n=0}^{N/2-1} [x(n) + (-1)^k x(N/2+n)] W_N^{nk}$$

$$k : even\ (k = 2r) \Rightarrow X(k) = X(2r) = \sum_{n=0}^{N/2-1} [x(n) + x(N/2+n)] W_N^{2rn}$$

$$W_N^{2rn} = (e^{-j2\pi/N})^{2rn} = (e^{-j2\pi/(N/2)})^{rn} = W_{N/2}^{rn}$$

N/2 point DFT

$$\Rightarrow X(k) = X(2r) = \sum_{n=0}^{N/2-1} [\underbrace{x(n) + x(N/2+n)}_{y(n)}] W_{N/2}^{rn}$$

$$Y(r) = \sum_{n=0}^{N/2-1} y(n) W_{N/2}^{rn}$$

$Z(r)$

$$k : odd \Rightarrow k = 2r+1$$

$$\Rightarrow X(k) = X(2r+1)$$

$$= \sum_{n=0}^{N/2-1} [x(n) - x(N/2+n)] W_N^{n(2r+1)}$$

$$= \sum_{n=0}^{N/2-1} \underbrace{[x(n) - x(N/2+n)] W_N^{n}}_{z(n)} W_N^{2rn}$$

$$= \sum_{n=0}^{N/2-1} z(n) W_N^{2rn}$$

$$= \sum_{n=0}^{N/2-1} z(n) W_{N/2}^{rn}$$

$$Z(r)=\sum_{n=0}^{N/2-1} z(n)W_{N/2}{}^{rn} \leftarrow \frac{N}{2}\ \text{point} \quad DFT \quad of \quad z(0), \quad \cdots, \quad z(\frac{N}{2}-1)$$

X(*k*) : N-point DFT of *x*(0), …, *x*(*N*) ➔ two *N*/2 point DFT

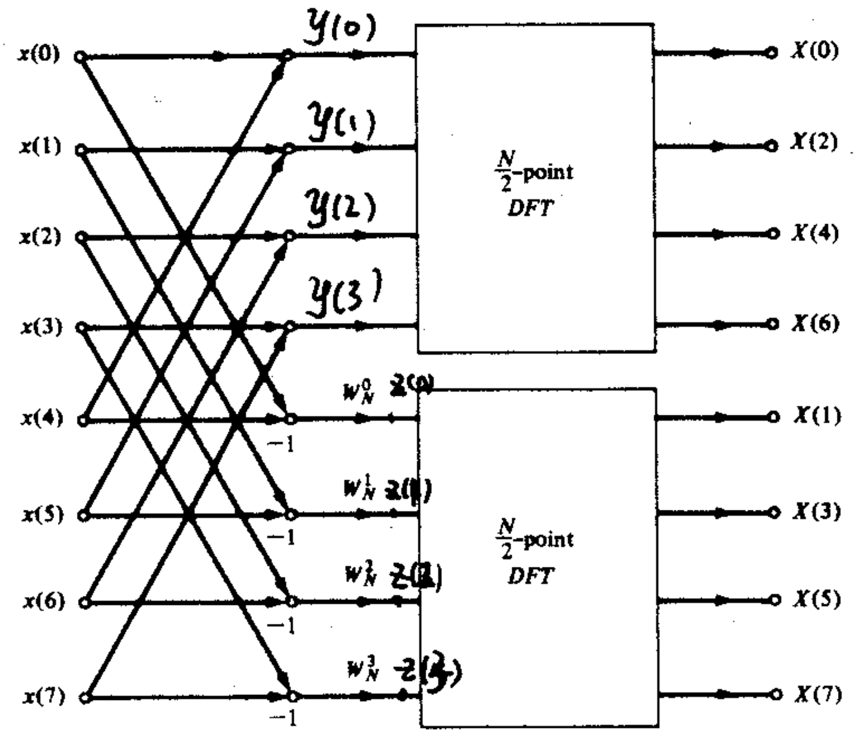


FIGURE 10-10. Flow graph after a single decimation.

One N/2 point DFT => two N/4 point DFT

… two point DFTs

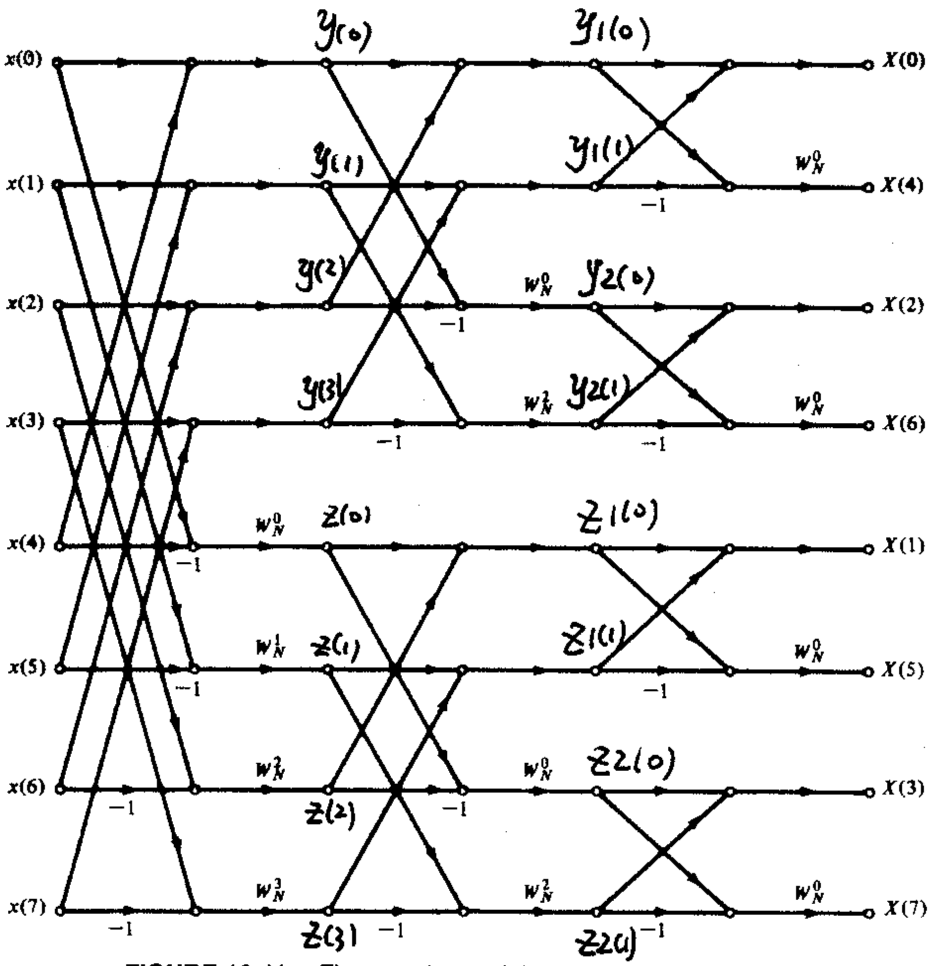


FIGURE 10-11. Flow graph pertaining to decimation-in-frequency FFT (N = 8).

Efficiency of FFT

N – point DFT : *4N(N*-1*)* real multiplications

4N(N-1*)* real additions

N – point FFT : $2N\log_2 N$ real multiplications

(N = 2^m^) — $(N = 2^m)$ **3N*log*$_2$N *real additions***

Computation ration

$$\frac{FFT's \quad computations}{DFT's \quad computations} = \frac{5\log_2 N}{8(N-1)}$$

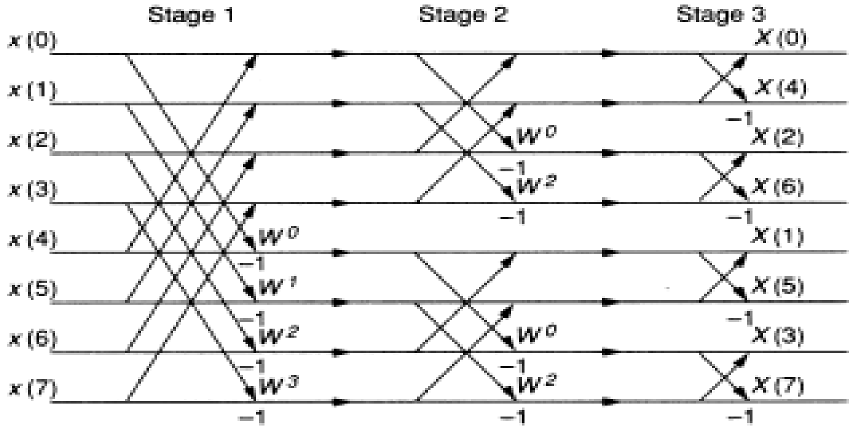
$$\stackrel{N=2^{12}=4096}{\Rightarrow} \frac{5\times 12}{8\times 4095} = 0.18\%$$

Example 1 Eight-Point FFT Using Decimation-in-Frequency of

x(n) = {1 1 1 1 0 0 0 0}

$x(0) = x(1) = x(2) = x(3) = 1,$ and $x(4) = x(5) = x(6) = x(7) = 0.$

$$W^0 = 1$$
$$W^1 = e^{-j2\pi/8} = \cos(\pi/4) - j\sin(\pi/4) = 0.707 - j0.707$$
$$W^2 = e^{-j4\pi/8} = -j$$
$$W^3 = e^{-j6\pi/8} = -0.707 - j0.707$$



Eight-point FFT flow graph using decimation-in-frequency.

1. **At stage 1:**

$$x(0) + x(4) = 1 \longrightarrow x'(0)$$
$$x(1) + x(5) = 1 \longrightarrow x'(1)$$
$$x(2) + x(6) = 1 \longrightarrow x'(2)$$
$$x(3) + x(7) = 1 \longrightarrow x'(3)$$
$$[x(0) - x(4)]W^0 = 1 \longrightarrow x'(4)$$
$$[x(1) - x(5)]W^1 = 0.707 - j0.707 \longrightarrow x'(5)$$
$$[x(2) - x(6)]W^2 = -j \longrightarrow x'(6)$$
$$[x(3) - x(7)]W^3 = -0.707 - j0.707 \longrightarrow x'(7)$$

where $x'(0), x'(1), \ldots, x'(7)$ represent the intermediate output sequence afterthe first iteration that becomes the input to the second stage.

2. **At stage 2:**

$$x'(0) + x'(2) = 2 \rightarrow x''(0)$$
$$x'(1) + x'(3) = 2 \rightarrow x''(1)$$
$$[x'(0) - x'(2)]W^0 = 0 \rightarrow x''(2)$$
$$[x'(1) - x'(3)]W^2 = 0 \rightarrow x''(3)$$
$$x'(4) + x'(6) = 1 - j \rightarrow x''(4)$$
$$x'(5) + x'(7) = (0.707 - j0.707) + (-0.707 - j0.707) = -j1.41 \rightarrow x''(5)$$
$$[x'(4) - x'(6)]W^0 = 1 + j \rightarrow x''(6)$$
$$[x'(5) - x'(7)]W^2 = -j1.41 \rightarrow x''(7)$$

The resulting intermediate, second-stage output sequence $x''(0), x''(1), \ldots$

$X''(7)$ becomes the input sequence to the third stage.

3. At stage 3:

$$X(0) = x''(0) + x''(1) = 4$$
$$X(4) = x''(0) - x''(1) = 0$$
$$X(2) = x''(2) + x''(3) = 0$$
$$X(6) = x''(2) - x''(3) = 0$$
$$X(1) = x''(4) + x''(5) = (1 - j) + (-j1.41) = 1 - j2.41$$
$$X(5) = x''(4) - x''(5) = 1 + j0.41$$
$$X(3) = x''(6) + x''(7) = (1 + j) + (-j1.41) = 1 - j0.41$$
$$X(7) = x''(6) - x''(7) = 1 + j2.41$$

Answer

X(k) = { 4 , *1-j2.41* , 0 , *1 – j0.41* , 0 , *1 + j0.41* , *0* , *1 + j2. 41* }

DIT radix-2 FFT	DIF radix-2 FFT

1.When the input is bit reversed order, the output will be in normal order .	1.When the input is normal order, the output will be in bit reversed order .
2.In each stage of computation the phase factor are multiplied before add and subtract operation	2.In each stage of computation the phase factor are multiplied after add and subtract operation
3.The value of N should be expressed such that $N=2^{m}$ and this algorithm consists of m stage of computation.	3.The value of N should be expressed such that $N=2^{m}$ and this algorithm consists of m stage of computation.
4.Total number of arthemetric operations is N log N complex addition and N/2logN complex multiplications.	4.Total number of arithmetic operations is N log N complex addition and N/2logN complex multiplications

COMPUTATION OF IDFT USING FFT

The inverse DFT of an N point sequence X (K); K=0, 1…N-1 is defined as

$$x(n) = 1/N \sum_{K=0}^{N-1} X(K)\, e^{+j2\pi nk/N} \quad \text{for } n=0, 1,2,\ldots N-1$$

Take complex conjugate and multiply by N, we get

$$Nx^*(n) = \sum_{K=0}^{N-1} X^*(K)\, e^{+j2\pi nk/N} \quad \text{for } n=0, 1, 2 \ldots N-1$$

The desired output sequence x (n) can then be obtained by complex conjugating

the DFT and divided by N

$$x(n) = 1/N \left[\sum_{K=0}^{N-1} X^*(K)\, e^{+j2\pi nk/N} \right]^*$$