UNIT I

OPERATION RESEARCH

1.1 Meaning and Definition of Operation Research:

Operations research also called operational research, application of scientific methods to the management and administration of organized military, governmental, commercial, and industrial processes. Operations research (OR) is an analytical method of problem-solving and decision-making that is useful in the management of organizations. In operations research, problems are broken down into basic components and then solved in defined steps by mathematical analysis.

1.2 Characteristics of OR:

There are three essential characteristics of operations research

- **Systems orientation**

The systems approach problems recognize that the behavior of any part of a system has some effect on the behavior of the system as a whole. Even if the individual components are performing well, however, the system as a whole is not necessarily performing equally well.

Operations research attempts to evaluate the effect of changes in any part of a system on the performance of the system as a whole and to search for causes of a problem that arises in one part of a system in other parts or in the interrelationships between parts. In industry, a production problem may be approached by a change in marketing policy.

For example, if a factory fabricates a few profitable products in large quantities and many less profitable items in small quantities, long efficient production runs of high-volume, high-profit items may have to be interrupted for short runs of low-volume, low-profit items.

An operations researcher might propose reducing the sales of the less profitable items and increasing those of the profitable items by placing salesmen on an incentive system that especially compensates them for selling particular items.

- **The interdisciplinary team**

Scientific and technological disciplines have proliferated rapidly in the last 100 years. The proliferation, resulting from the enormous increase in scientific knowledge, has provided science with a filing system that permits a systematic classification of knowledge. This classification system is helpful in solving many problems by identifying the proper discipline to appeal to for a solution. Difficulties arise when more complex problems, such as those arising in large organized systems, are encountered. It is then necessary to find a means of bringing together diverse disciplinary points of view. Furthermore, since methods differ among disciplines, the use of interdisciplinary teams makes available a much larger arsenal of research techniques and tools than would otherwise be available. Hence, operations research may be characterized by rather unusual combinations of disciplines on research teams and by the use of varied research procedures.

- **Methodology**

Operations research cannot be brought into laboratories. Furthermore, even if systems could be brought into the laboratory, what would be learned would not necessarily apply to their behavior in their natural environment, as shown by early experience with Radar Experiments on systems and subsystems conducted in their natural environment (“operational experiments”) are possible as a result of the experimental methods developed by the British statistician R.A. Fisher in 1923–24. For practical or even ethical reasons, however, it is seldom possible to experiment on large organized systems as a whole in their natural environments. This results in an apparent dilemma: to gain an understanding of complex systems experimentation seems to be necessary, but it cannot usually be carried out. This difficulty is solved by the use of models, and representations of the system under study. Provided the model is good, experiments (called “simulations”) can be conducted on it, or other methods can be used to obtain useful results.

1.3 Operations Research Models and Applications:

1. **Resource allocation**

Allocation problems involve the distribution of resources among competing alternatives in order to minimize total costs or maximize total return. Such problems have the following components:

- a set of resources available in given amounts;
- a set of jobs to be done, each consuming a specified amount of resources;
- a set of costs or returns for each job and resource.

The problem is to determine how much of each resource to allocate to each job.If more resources are available than needed, the solution should indicate which resources are not to be used, taking associated costs into account. Similarly, if there are more jobs than can be done with available resources, the solution should indicate which jobs are not to be done, again taking into account the associated costs.

If resources are divisible, and if both jobs and resources are expressed in units on the same scale, it is termed a transportation or distribution problem. If jobs and resources are not expressed in the same units, it is a general allocation problem.

An assignment problem may consist of assigning workers to offices or jobs, trucks to delivery routes, drivers to trucks, or classes to rooms.

A typical transportation problem involves the distribution of empty railroad freight cars where needed or the assignment of orders to factories for production.

The general allocation problem may consist of determining which machines should be employed to make a given product or what set of products should be manufactured in a plant during a particular period.

2. Linear programming

Linear programming (LP) refers to a family of mathematical optimization techniques that have proved effective in solving resource allocation problems, particularly those found in industrial production systems.

Linear programming methods are algebraic techniques based on a series of equations or inequalities that limit a problem and are used to optimize a mathematical expression called an objective function.

The objective function and the constraints placed upon the problem must be deterministic and able to be expressed in linear form.

Since linear programming is probably the most widely used mathematical optimization technique, numerous computer programs are available for solving LP problems.

3. Inventory control

Inventories include raw materials, component parts, work in process, finished goods, packing and packaging materials, and general supplies. The control of inventories, vital to the financial strength of a firm, in general, involves deciding at what points in the production system stocks shall be held and what their form and size are to be.

As some unit costs increase with inventory size—including storage, obsolescence, deterioration, insurance, and investment—and other unit costs decrease with inventory size—including setup or preparation costs, delays because of shortages, and so forth—a good part of inventory management consists of determining optimal purchase or production lot sizes and base stock levels that will balance the opposing cost influences. Another part of the general inventory problem is deciding the levels (reorder points) at which orders for replenishment of inventories are to be initiated.

The classic inventory problem involves determining how much of a resource to acquire, either by purchasing or producing it, and whether or when to acquire it to minimize the sum of the costs that increase with the size of inventory and those that decrease with increases in inventory. Costs of the first type include the cost of the capital invested in inventory, handling, storage, insurance, taxes, depreciation, deterioration, and obsolescence.

4. Japanese approaches

In the 1970s several Japanese firms, led by the Toyota Motor Corporation, developed radically different approaches to the management of inventories. Coined the “just-in-time” approach, the basic element of the new systems was the dramatic reduction of inventories throughout the total production system. The Japanese ensured that parts and supplies were available in the right quantity, with proper quality, at the exact time they were needed in the manufacturing or assembly process.

A second Japanese technique, called kanban (“card”), also permits Japanese firms to schedule production and manage inventories more effectively. In the kanban system, cards or tickets are attached to batches, racks, or pallet loads of parts in the manufacturing process. When a batch is depleted in the assembly process, its kanban is returned to the manufacturing department and another batch is shipped immediately.

5. Replacement and maintenance

Replacement problems involve items that degenerate with use or with the passage of time and those that fail after a certain amount of use or time. Items that deteriorate are likely to be large and costly (e.g., machine tools, trucks, ships, and home appliances). Non deteriorating items tend to be small and relatively inexpensive (e.g., light bulbs, vacuum tubes, ink cartridges).

The longer a deteriorating item is operated the more maintenance it requires to maintain efficiency. The longer such an item is kept the less is its resale value and the more likely it is to be made obsolete by new equipment. If the item is replaced frequently, however, investment costs increase. Thus the problem is to determine when to replace such items and how much maintenance (particularly preventive) to perform so that the sum of the operating, maintenance, and investment costs is minimized.

Non deteriorating items the problem involves determining whether to replace them as a group or to replace individuals as they fail. Though group replacement is wasteful, labour cost of replacements is greater when done singly.

6. Queuing

A queue is a waiting line, and queuing involves dealing with items or people in sequence. Thus, a queuing problem consists either of determining what facilities to provide or scheduling the use of them. The cost of providing service and the waiting time of users are minimized.

The maintenance problems can be treated as queuing problems; items requiring repair are like users of a service. Some inventory problems may also be formulated as queuing problems in which orders are like users and stocks are like service facilities.

7. Job shop sequencing

In queuing problems, the order in which users waiting for service are served is always specified. Selection of that order so as to minimize some function of the time to perform all the tasks is a sequencing problem. The performance measure may account for total elapsed time, total tardiness in meeting deadlines or due dates, and the cost of in-process inventories. The most common context for sequencing problems is a batch, or job shop, production facility that processes many different products with many combinations of machines.

8. Manufacturing progress function

With the enormous complexity of a typical mass production line and the almost infinite number of changes that can be made and alternatives that can be pursued, a body of quantitative theory of mass production manufacturing systems has not yet been developed. The volume of available observational data is, however, growing, and qualitative facts are emerging that may eventually serve as a basis for the quantitative theory.

Manufacturing progress functions can be of great value to the manufacturer, serving as a useful tool in estimating future costs. Furthermore, the failure of costs to follow a well-established progress function may be a sign that more attention should be given to the operation in order to bring its cost performance in line with expectations.

Though manufacturing progress functions are sometimes called “learning curves,” they reflect much more than the improved training of the manufacturing operators.

9. Network routing

A network may be defined by a set of points, or “nodes,” that are connected by lines, or “links.” A way of going from one node (the “origin”) to another (the “destination”) is called a “route” or “path.” Links, which may be one-way or two-way, are usually characterized by the time, cost, or distance required to traverse them. The time or cost of traveling in different directions on the same link may differ.

A network routing problem consists of finding an optimum route between two or more nodes in relation to total time, cost, or distance. Various constraints may exist, such as a prohibition on returning to a node already visited or a stipulation of passing through every node only once.

10. Competitive problems

Competitive problems deal with choice in interactive situations where the outcome of one decision maker’s choice depends on the choice, either helpful or harmful, of one or more others. Under conditions of certainty, it is easy to maximize gain or minimize loss. Competitive problems of the risk type require the use of statistical analysis for their solution; the most difficult aspect of solving such problems usually lies in estimating the probabilities of the competitor’s choices.

The theory of games was developed to deal with a large class of competitive situations of the uncertainty type in which each participant knows what choices he and each other participant have. There is a well-defined “end state” that terminates the interaction (e.g., win, lose, or draw), and the payoffs associated with each end state are specified in advance and are known to each participant.

11. Search problems

Search problems involve finding the best way to obtain information needed for a decision. Though every problem contains a search problem in one sense, situations exist in which the search itself is the essential process.

Two kinds of error are involved in the search: that observation and sampling. Observational errors, in turn, are of two general types: commission, seeing something that is not there; and omission, not seeing something that is there. In general, as the chance of making one of these errors is decreased, the chance of making the other is increased. Furthermore, if fixed resources are available for search, the larger the sample (and hence the smaller the sampling error), the fewer resources available per observation (and hence the larger the observational error).

The cost of search is composed of setup or design cost, cost of observations, cost of analyzing the data obtained, and cost of error. The objective is to minimize these costs by manipulating the sample size (amount of observation), the sample design (how the things or places to be observed are selected), and the way of analyzing the data (the inferential procedure).

A "reversed-search" problem arises when the search procedure is not under control but the object of the search is. Most retailers, for example, cannot control the manner in which customers search for goods in their stores, but they can control the location of the goods.

1.4 Limitations of Operations Research:

OR has some limitations however, these are related to the problem of model building and the time and money factors involved in the application rather than its practical utility. Some of them are as follows:

(i) Magnitude of Computation

Operations research models try to find out optimal solutions by taking into account all the factors. These factors are enormous and expressing them in quantity and establishing relationships among these require voluminous calculations which can be handled by computers.

(ii) Non-Quantifiable Factors

An OR provides a solution only when all elements related to a problem can be quantified. All relevant variables do not lend themselves to quantification. Factors that cannot be quantified find no place in OR study. Models in OR do not take into account qualitative factors or emotional factors which may be quite important.

(iii) Distance between User and Analyst

OR being a specialist's job requires a mathematician or statistician, who might not be aware of the business problems. Similarly, a manager fails to understand the complex working of OR. Thus there is a gap between the two. Management itself may offer a lot of resistance due to conventional thinking.

(iv) Time and Money Costs

When basic data are subjected to frequent changes, incorporating them into the OR models is a costly proposition. Moreover, a fairly good solution at present may be more desirable than a perfect OR solution available after some time. The computational time increases depending on the size of the problem and the accuracy of the results desired.

(v) Implementation

Implementation of any decision is a delicate task. It must take into account the complexities of human relations and behavior. Sometimes, resistance is offered due to psychological factors which may not have any bearing on the problem as well as its solution.

1.5 Introduction to Liner Programming:

Linear Programming is a mathematical technique for optimum allocation of limited or scarce resources, such as labor, material, machine, money, energy and so on , to several competing activities such as products, services, jobs and so on, on the basis of a given criteria of optimality.

The term 'Linear' is used to describe the proportionate relationship of two or more variables in a model. The given change in one variable will always cause a resulting proportional change in another variable.

The word , 'programming' is used to specify a sort of planning that involves the economic allocation of limited resources by adopting a particular course of action or strategy among various alternatives strategies to achieve the desired objective.

Hence, Linear Programming is a mathematical technique for the optimum allocation of limited or scarce resources, such as labor, material, machine, money energy, etc.

1.5.1 Structure of Linear Programming model:

The general structure of the Linear Programming model essentially consists of three components.

i) The activities (variables) and their relationships

ii) The objective function and

iii) The constraints

1.5.2 Linear Programming Applications:

A real-time example would be considering the limitations of labor and materials and finding the best production levels for maximum profit in particular circumstances. It is part of a vital area of mathematics known as optimization techniques.

The applications of LP in some other fields are

- Engineering – It solves design and manufacturing problems as it is helpful for doing shape optimization
- Efficient Manufacturing – To maximize profit, companies use linear expressions
- Energy Industry – It provides methods to optimize the electric power system.
- Transportation Optimization – For cost and time efficiency.

Example:

An electronics company produces three types of parts for automatic washing machines .It purchases castings of the parts from a local foundry and then finishes the part on drilling, shaping and polishing machines. The selling prices of parts A, B, and C respectively are Rs 8, Rs.10 and Rs.14.All parts made can be sold. Castings for parts A, B and C respectively cost Rs.5, Rs.6 and Rs.10.
The shop possesses only one of each type of machine. Cost per hour to run each of the three machines are Rs.20 for drilling, Rs.30 for shaping and Rs.30 for polishing. The capacities (parts per hour) for each part on each machine are shown in the following table.

Machine	Capacities Per Hour		
	Part A	Part B	Part C
Drilling	25	40	25
Shaping	25	20	20
Polishing	40	30	40

The management of the shop wants to know how many parts of each type it should produce per hour in order to maximize profit for an hour's run. Formulate this problem as an LP model so as to maximize total profit to the company.

Sol:

i) Identify and define the decision variable of the problem
Let X1 and X2 and X3 be the number of types A, B and C parts produced per hour respectively .
ii) Define the objective function
With the information given, the hourly profit for part A, B, and C would be as follows
Profit per type A part = (8 – 5) – (20/25 +30/25 + 30/40) = 0.25
Profit per type B part = (10 – 6) – (20/40 + 30/20 + 30/30) = 1
Profit per type C part = (14 – 10) – (20/25 + 30/20 + 30/40) = 0.95
Then,
Maximize Z = 0.25 X1 + 1X2 + 0.95X3

:

iii) State the constraints to which the objective function should be optimized.
The above objective function is subjected to following constraints.

i) The drilling machine constraint
$X1/25 + X2/40 + X3/24 \leq 1$
ii) The shaping machine constraint
$X1/25 + X2/20 + X3/20 \leq 1$
iii) The polishing machine constraint
$X1/40 + X2/30 + X3/40 \leq 1$
$X1, X2, X3 \geq 0$

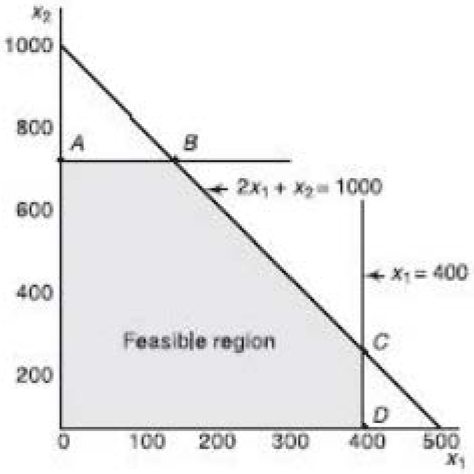
Finally we have,
Maximize $Z = 0.25 X1 + 1X2 + 0.95X3$
Subject to constraints
$X1/25 + X2/40 + X3/24 \leq 1$
ii) The shaping machine constraint
$X1/25 + X2/20 + X3/20 \leq 1$
iii) The polishing machine constraint
$X1/40 + X2/30 + X3/40 \leq 1$
$X1, X2, X3 \geq 0$

1.5.3 LLP IN GRAPHICAL MODEL SOLUTION

Maximize model;

Solve the following LPP by graphical method
Maximize $Z = 5X1 + 3X2$
Subject to constraints
$2X1 + X2 \leq 1000$
$X1 \leq 400$
$X1 \leq 700$
$X1, X2 \geq 0$

Solution:
The first constraint $2X1 + X2 \leq 1000$ can be represented as follows.
We set $2X1 + X2 = 1000$
When $X1 = 0$ in the above constraint, we get,
$2 \times 0 + X2 = 1000$
$X2 = 1000$
Similarly when $X2 = 0$ in the above constraint, we get,
$2X1 + 0 = 1000$
$X1 = 1000/2 = 500$
The second constraint $X1 \leq 400$ can be represented as follows,
We set $X1 = 400$
The third constraint $X2 \leq 700$ can be represented as follows,
We set $X2 = 700$



The constraints are shown plotted in the above figure

Point	X1	X2	Z = 5X1 +3X2
0	0	0	0
A	0	700	Z = 5 x 0 + 3 x 700 = 2,100
B	150	700	Z = 5 x 150 + 3 x 700 = 2,850* Maximum
C	400	200	Z = 5 x 400 + 3 x 200 = 2,600
D	400	0	Z = 5 x 400 + 3 x 0 = 2,000

The Maximum profit is at point B
When X1 = 150 and X2 = 700
Z = <u>2850</u>

Minimize Model;

Example:

Solve the following LPP by graphical method

Minimize $Z = 20X1 + 40X2$

Subject to constraints

$36X1 + 6X2 \geq 108$

$3X1 + 12X2 \geq 36$

$20X1 + 10X2 \geq 100$

$X1\ X2 \geq 0$

Solution:

The first constraint $36X1 + 6X2 \geq 108$ can be represented as follows.

We set $36X1 + 6X2 = 108$

When $X1 = 0$ in the above constraint, we get,

$36 \times 0 + 6X2 = 108$

$X2 = 108/6 = 18$

Similarly when $X2 = 0$ in the above constraint, we get,

$36X1 + 6 \times 0 = 108$

$X1 = 108/36 = 3$

The second constraint $3X1 + 12X2 \geq 36$ can be represented as follows,

We set $3X1 + 12X2 = 36$

When $X1 = 0$ in the above constraint, we get,

$3 \times 0 + 12X2 = 36$

$X2 = 36/12 = 3$

Similarly when $X2 = 0$ in the above constraint, we get,

$3X1 + 12 \times 0 = 36$

$X1 = 36/3 = 12$

The third constraint $20X1 + 10X2 \geq 100$ can be represented as follows,

We set $20X1 + 10X2 = 100$

When $X1 = 0$ in the above constraint, we get,

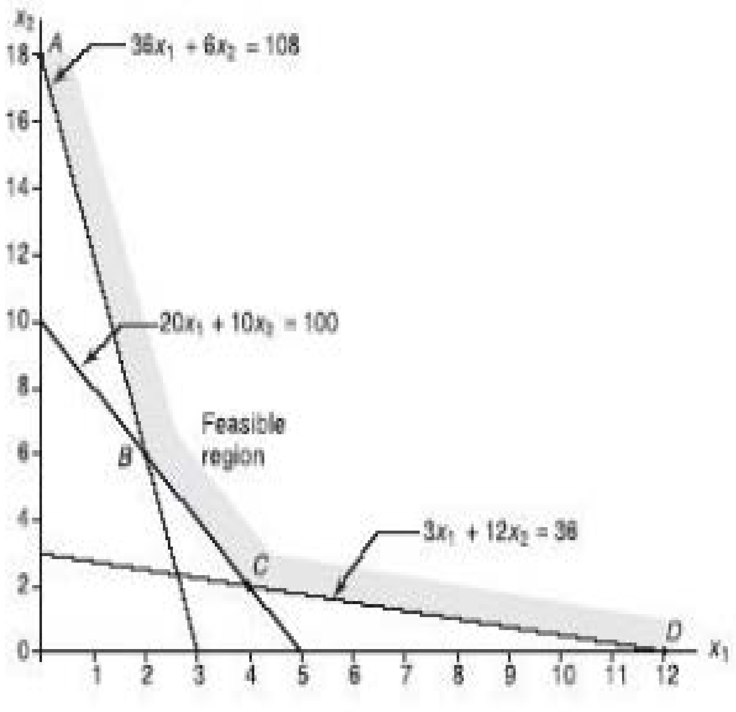
$20 \times 0 + 10X2 = 100$

$X2 = 100/10 = 10$

Similarly when $X2 = 0$ in the above constraint, we get,

$20X1 + 10 \times 0 = 100$

$X1 = 100/20 = 5$



Point	X1	X2	Z = 20X1 + 40X2
0	0	0	0
A	0	18	Z = 20 x 0 + 40 x 18 = 720
B	2	6	Z = 20 x2 + 40 x 6 = 280
C	4	2	Z = 20 x 4 + 40 x 2 = 160* Minimum
D	12	0	Z = 20 x 12 + 40 x 0 = 240

1.6 LLP Simplex Method:

Under 'Graphical solutions' to LP, the objective function obviously should have not more than two decision variables. If the decision variables are more than two, the 'Cartesian Plane' cannot accommodate them. And hence, a most popular and widely used analysis called 'SIMPLEX METHOD', is used. This method of analysis was developed by one American Mathematician byname George B. Dantzig, during 1947.

This method provides an algorithm (a procedure which is iterative) which is based on fundamental theorems of Linear Programming. It helps in moving from one basic feasible solution to another in a prescribed manner such that the value of the objective function is improved. This procedure of jumping from one vertex to another vertex is repeated.

Steps:

1. Convert the inequalities into equalities by adding slack variables, surplus variables or artificial variables, as the case may be.

2. Identify the coefficient of equalities and put them into a matrix form AX = B

   Where "A" represents a matrix of coefficient, "X" represents a vector of unknown quantities and B represents a vector of constants, leads to AX = B [This is according to system of equations].

3. Tabulate the data into the first iteration of Simplex Method.

		Cj					
Basic Variable (BV)	CB	XB	X1	X2	S1	S2	Ratio
S1							
S2							
		Zj Cj					
		Zj- Cj					

(a) Cj is the coefficient of unknown quantities in the objective function.

Zj = CB* Xs (Multiples and additions of coefficients in the table, i.e., CB1 × X1 + CB2 ×X2)

(b) Identify the Key or Pivotal column with the minimum element of Zj-Cj denoted as 'KC' throughout to the problems in the chapter.

(c) Find the 'Minimum Ratio' i.e., XBi/Xij.

(d) Identify the key row with the minimum element in a minimum ratio column. Key row is denoted as 'KP'.

(e) Identify the key element at the intersecting point of key column and key row, which is put into a box throughout to the problems in the chapter.

4. Reinstate the entries to the next iteration of the simplex method.

   (a) The pivotal or key row is to be adjusted by making the key element as '1' and dividing the other elements in the row by the same number.

   (b) The key column must be adjusted such that the other elements other than key elements should be made zero.

   (c) The same multiple should be used to other elements in the row to adjust the rest of the elements. But, the adjusted key row elements should be used for deducting out of the earlier iteration row.

   (d) The same iteration is continued until the values of $Z_j - C_j$ become either '0' or positive.

5. Find the 'Z' value given by C_B, X_B.

1.7 Duality Linear Program:

The dual of a given linear program (LP) is another LP that is derived from the original (the primal) LP in the following schematic way:

Each variable in the primal LP becomes a constraint in the dual LP;

Each constraint in the primal LP becomes a variable in the dual LP;

The objective direction is inversed – maximum in the primal become minimum in the dual and vice versa.

The weak duality theorem states that the objective value of the dual LP at any feasible solution is always a bound on the objective of the primal LP at any feasible solution (upper or lower bound, depending on whether it is a maximization or minimization problem). In fact, this bounding property holds for the optimal values of the dual and primal LPs.

The strong duality theorem states that, moreover, if the primal has an optimal solution then the dual has an optimal solution too, and the two optima are equal.

These theorems belong to a larger class of duality theorems in optimization. The strong duality theorem is one of the cases in which the duality gap (the gap between the optimum of the primal and the optimum of the dual) is 0.

1.7.1 Form of the dual LP:

Maximize $c^T x$ subject to $Ax \leq b$, $x \geq 0$.

We would like to construct an upper bound on the solution. So we create a linear combination of the constraints, with positive coefficients, such that the coefficients of x in the constraints are at least c^T. this linear combination gives us an upper bound on the objective.

The variables y of the dual LP are the coefficients of this linear combination. The dual LP tries to find such coefficients that *minimize* the resulting upper bound.

This gives the following LP:

Minimize $b^T y$ subject to $A^T y \geq c$, $y \geq 0$

This LP is called the dual of the original LP.

1.7.2 Constructing the dual LP:

In general, given a primal LP, the following algorithm can be used to construct its dual LP. The primal LP is defined by:

- A set of *n* variables: $x_1, \ldots, x_n$.
- For each variable x_i, a *sign constraint* – it should be either non-negative ($x_i \geq 0$), or non-positive ($x_i \leq 0$), or unconstrained ($x_i \in \mathbb{R}$).
- An objective function: $\text{maximize} \quad c_1 x_1 + \cdots + c_n x_n$
- A list of *m* constraints. Each constraint *j* is: $a_{j1} x_1 + \cdots + a_{jn} x_n \lesseqqgtr b_j$ where the symbol before the b_j can be one of $\geq$ or $\leq$ or $=$.

The dual LP is constructed as follows.

- Each primal constraint becomes a dual variable. So there are *m* variables: $y_1, \ldots, y_m$.
- The sign constraint of each dual variable is "opposite" to the sign of its primal constraint. So "$\geq b_j$" becomes $y_j \leq 0$ and "$\leq b_j$" becomes $y_j \geq 0$ and "$= b_j$" becomes $y_j \in \mathbb{R}$.
- The dual objective function is $\text{minimize} \quad b_1 y_1 + \cdots + b_m y_m$
- Each primal variable becomes a dual constraint. So there are *n* constraints. The coefficient of a dual variable in the dual constraint is the coefficient of its primal variable in its primal constraint. So each constraint *i* is: $a_{1i} y_1 + \cdots + a_{mi} y_m \lesseqqgtr c_i$, where the symbol before the c_i is similar to the sign constraint on variable *i* in the primal LP. So $x_i \leq 0$ becomes "$\leq c_i$" and $x_i \geq 0$ becomes "$\geq c_i$" and $x_i \in \mathbb{R}$ becomes "$= c_i$".

From this algorithm, it is easy to see that the dual of the dual is the primal.

1.7.3 The duality theorems:

The primal LP is "maximize $c^T x$ subject to [constraints]" and the dual LP is "minimize $b^T y$ subject to [constraints]

Weak duality

The weak duality theorem says that, for each feasible solution x of the primal and each feasible solution y of the dual: $c^T x \leq b^T y$. In other words, the objective value in each feasible solution of the dual is an upper-bound on the objective value of the primal, and objective value in each feasible solution of the primal is a lower-bound on the objective value of the dual. Here is a proof for the primal LP "Maximize $c^T x$ subject to $Ax \leq b$, $x \geq 0$":

- $c^T x$
- $= x^T c$ [since this just a scalar product of the two vectors]
- $\leq x^T(A^T y)$ [since $A^T y \geq c$ by the dual constraints, and $x \geq 0$]
- $= (x^T A^T)y$ [by associability]
- $= (Ax)^T y$ [by properties of transpose]
- $\leq b^T y$ [since $Ax \leq b$ by the primal constraints]

Weak duality implies:

$\max_x c^T x \leq \min_y b^T y$

In particular, if the primal is unbounded (from above) then the dual has no feasible solution, and if the dual is unbounded (from below) then the primal has no feasible solution

Strong duality

The strong duality theorem says that if one of the two problems has an optimal solution, so does the other one and that the bounds given by the weak duality theorem are tight, i.e.:

$\max_x c^T x = \min_y b^T y$

The strong duality theorem is harder to prove; the proofs usually use the weak duality theorem as a sub-routine.

One proof uses the simplex algorithm and relies on the proof that, with the suitable pivot rule, it provides a correct solution. The proof establishes that, once the simplex algorithm finishes with a solution to the primal LP, it is possible to read from the final tableau, a solution to the dual LP. So, by running the simplex algorithm, we obtain solutions to both the primal and the dual simultaneously.

1.7.4 Theoretical implications

The weak duality theorem implies that finding a single feasible solution is as hard as finding an optimal feasible solution. Suppose we have an oracle that, given an LP, finds an arbitrary feasible solution (if one exists). Given the LP "Maximize $\mathbf{c}^T\mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq 0$", we can construct another LP by combining this LP with its dual. The combined LP has both **x** and **y** as variables

Maximize **1**

Subject to $A\mathbf{x} \leq \mathbf{b}$, $A^T\mathbf{y} \geq \mathbf{c}$, $\mathbf{c}^T\mathbf{x} \geq \mathbf{b}^T\mathbf{y}$, $\mathbf{x} \geq 0$, $\mathbf{y} \geq 0$

If the combined LP has a feasible solution (**x**,**y**), then by weak duality, $\mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{y}$. So **x** must be a maximal solution of the primal LP and **y** must be a minimal solution of the dual LP. If the combined LP has no feasible solution, then the primal LP has no feasible solution either.

The strong duality theorem provides a "good characterization" of the optimal value of an LP in that it allows us to easily prove that some value t is the optimum of some LP. The proof proceeds in two steps:

- Show a feasible solution to the primal LP with value t; this proves that the optimum is at least t.
- Show a feasible solution to the dual LP with value t; this proves that the optimum is at most t.

Unit II

Transportation Problem, Assignment Problem, Crew Assignment, Inventory Control

Transportation problem

Introduction

A special class of linear programming problem is the Transportation Problem, where the objective is to minimize the cost of distributing a product from a number of sources (e.g. factories) to a number of destinations (e.g. warehouses) while satisfying both the supply limits and the demand requirement. Because of the special structure of the Transportation Problem, the Simplex Method of solving is unsuitable for the Transportation Problem. The model assumes that the distributing cost on a given route is directly proportional to the number of units distributed on that route. Generally, the transportation model can be extended to areas other than the direct transportation of a commodity, including among others, inventory control, employment scheduling, and personnel assignment.

Transportation Algorithm

The steps of the transportation algorithm are exact parallels of the simplex algorithm, they are:

Step 1: Determine a starting basic feasible solution, using any one of the following three methods

- North West Corner Method
- Least Cost Method
- Vogel Approximation Method

Step 2: Determine the optimal solution using the following method

- MODI (Modified Distribution Method) or UV Method.

Various Types of Methods:

1. **North West Corner Method:**

North West Corner Method: The method starts at the North West (upper left) corner cell of the tableau (variable x11).

Step -1: Allocate as much as possible to the selected cell, and adjust the associated amounts of capacity (supply) and requirement (demand) by subtracting the allocated amount.

Step 2: Cross out the row (column) with zero supply or demand to indicate that no further assignments can be made in that row (column). If both the row and column become zero simultaneously, cross out one of them only, and leave a zero supply or demand in the uncrossed-out row (column).

Step -3: If exactly one row (column) is left uncrossed out, then stop. Otherwise, move to the cell to the right if a column has just been crossed or the one below if a row has been crossed out.

Go to step -1.

2. **Least Cost Method**

The least cost method is also known as the matrix minimum method in the sense we look for the row and the column corresponding to which Cij is minimum. This method finds a better initial basic feasible solution by concentrating on the cheapest routes. Instead of starting the allocation with the northwest cell as in the North West Corner Method, we start by allocating as much as possible to the cell with the smallest unit cost. If there are two or more minimum costs then we should select the row and the column corresponding to the lower-numbered row. If they appear in the same row we should select the lower numbered column. We then cross out the satisfied row or column and adjust the amounts of capacity and requirement accordingly. If both a row and a column are satisfied simultaneously, only one is crossed out. Next, we look for the uncrossed-out cell with the smallest unit cost and repeat the process until we are left at the end with exactly one uncrossed-out row or column.

3. **Vogel Approximation Method (VAM):**

VAM is an improved version of the least cost method that generally produces better solutions. The steps involved in this method are:

Step 1: For each row (column) with strictly positive capacity (requirement), determine a penalty by subtracting the smallest unit cost element in the row (column) from the next smallest unit cost element in the same row (column).

Step 2: Identify the row or column with the largest penalty among all the rows and columns. If the penalties corresponding to two or more rows or columns are equal we select the topmost row and the extreme left column

Step 3: We select Xij as a basic variable if Cij is the minimum cost in the row or column with the largest penalty. We choose the numerical value of Xij as high as possible subject to the row and the column constraints. Depending upon whether ai or bj is the smaller of the two ith row or jth column is crossed out.

Step 4: Step 2 is now performed on the uncrossed-out rows and columns until all the basic variables have been satisfied.

Example:

Consider the following transportation problem

Origin	Destination				a_i
	1	2	3	4	
1	20	22	17	4	120
2	24	37	9	7	70
3	32	37	20	15	50
b_j	60	40	30	110	240

Note: a_i=capacity (supply)
b_j=requirement (demand)

Solution:

Now, compute the penalty for various rows and columns which is shown in the following table:

Origin	Destination				a_i	Column Penalty
	1	2	3	4		
1	20	22	17	4	120	13
2	24	37	9	7	70	2
3	32	37	20	15	50	5
b_j	60	40	30	110	240	
Row Penalty	4	15	8	3		

Look for the highest penalty in the row or column, the highest penalty occurs in the second column and the minimum unit cost i.e. cij in this column is c12=22. Hence assign 40 to this cell i.e. x12=40 and cross out the second column (since the second column was satisfied. This is shown in the following table:

Origin	Destination				a_i	Column Penalty
	1	2	3	4		
1	20	22 **40**	17	4	80	13
2	24	37	9	7	70	2
3	32	37	20	15	50	5
b_j	60	40	30	110	240	
Row Penalty	4	15	8	3		

The next highest penalty in the uncrossed-out rows and columns is 13 which occur in the first row and the minimum unit cost in this row is c14=4, hence x14=80 and cross out the first row. The modified table is as follows

Origin	Destination 1	2	3	4	a_i	Column Penalty
1	20	22 **40**	17	4 **80**	0	13
2	24	37	9	7	70	2
3	32	37	20	15	50	5
b_j	60	40	30	110	240	
Row Penalty	4	15	8	3		

The next highest penalty in the uncrossed-out rows and columns is 8 which occurs in the third column and the minimum cost in this column is c23=9, hence x23=30 and cross out the third column with adjusted capacity, requirement and penalty values. The modified table is as follows:

Origin	Destination 1	2	3	4	a_i	Column Penalty
1	20	22 **40**	17	4 **80**	0	13
2	24	37	9 **30**	7	40	17
3	32	37	20	15	50	17
b_j	60	40	30	110	240	
Row Penalty	8	15	8	8		

The next highest penalty in the uncrossed-out rows and columns is 17 which occurs in the second row and the smallest cost in this row is c24=15, hence x24=30 and cross out the fourth column with the adjusted capacity, requirement and penalty values. The modified table is as follows:

Origin	Destination 1	2	3	4	a_i	Column Penalty
1	20	22 **40**	17	4 **80**	0	13
2	24	37	9 **30**	7 **30**	10	17
3	32	37	20	15	50	17
b_j	60	40	30	110	240	
Row Penalty	8	15	8	8		

The next highest penalty in the uncrossed-out rows and columns is 17 which occurs in the second row and the smallest cost in this row is c21=24, hence xi21=10 and cross out the second row with the adjusted capacity, requirement and penalty values. The modified table is as follows:

Origin	Destination				a_i	Column Penalty
	1	2	3	4		
1	20	22 **40**	17	4 **80**	0	13
2	24 **10**	37	9 **30**	7 **30**	0	17
3	32	37	20	15	50	17
b_j	60	40	30	110	240	
Row Penalty	8	15	8	8		

The next highest penalty in the uncrossed-out rows and columns is 17 which occurs in the third row and the smallest cost in this row is c31=32, hence xi31=50 and cross out the third row or first column. The modified table is as follows:

Origin	Destination				a_i	Column Penalty
	1	2	3	4		
1	20	22 **40**	17	4 **80**	0	13
2	24 **10**	37	9 **30**	7 **30**	0	17
3	32 **50**	37	20	15	0	17
b_j	60	40	30	110	240	
Row Penalty	8	15	8	8		

The transportation cost corresponding to this choice of basic variables is

$22 * 40 + 4 * 80 + 9 * 30 + 7 * 30 + 24 * 10 + 32 * 50 = 3520$

4. Modified Distribution Method

The Modified Distribution Method, also known as the MODI method or u-v method, provides a minimum-cost solution (optimal solution) to the transportation problem.

Unbalanced Transportation Problem

In the previous section, we discussed the balanced transportation problem i.e. the total supply (capacity) at the origins is equal to the total demand (requirement) at the destination. In this section, we are going to discuss the unbalanced transportation problems i.e. when the total supply is not equal to the total demand, which is called as unbalanced transportation problem.

In the unbalanced transportation problem if the total supply is more than the total demand then we introduce an additional column which will indicate the surplus supply with transportation cost zero. Similarly, if the total demand is more than the total supply an additional row is introduced in the transportation table which indicates unsatisfied demand with zero transportation cost

Example:

Consider the following unbalanced transportation problem

Warehouses

Plant	w_1	w_2	w_3	Supply
X	20	17	25	400
Y	10	10	20	500
Demand	400	400	500	

Solution:

In this problem the demand is 1300 whereas the total supply is 900. Thus, we now introduce an additional row with zero transportation cost denoting the unsatisfied demand. So that the modified transportation problem table is as follows:

		Warehouses		
Plant	w_1	w_2	w_3	**Supply**
X	20	17	25	400
Y	10	10	20	500
Unsatisfied Demand	0	0	0	400
Demand	400	400	500	1300

Now we can solve as balanced problem discussed as in the previous sections.

The Assignment Problem

The Assignment Problem can be defined as follows: Given n facilities, n jobs, and the effectiveness of each facility to each job, the problem is to assign each facility to one and only one job so that the measure of effectiveness is optimized. Here the optimization means Maximized or Minimized. There are many management problems that have an assignment problem structure.

For example, the head of the department may have 6 people available for assignment and 6 jobs to fill. Here the head may like to know which job should be assigned to which person so that all tasks can be accomplished in the shortest time possible.

Another example is a container company may have an empty container in each of locations 1, 2,3,4,5 and requires an empty container in each of locations 6, 7, 8,9,10. It would like to ascertain the assignments of containers to various locations so as to minimize the total distance.

Hungarian Method

The Hungarian Method is discussed in the form of a series of computational steps as follows, when the objective function is that of minimization type.

Example:

Solve the following assignment problem. Cell values represent the cost of assigning jobs A, B, C and D to machines I, II, III, and IV.

		machines			
		I	II	III	IV
jobs	*A*	10	12	19	11
	B	5	10	7	8
	C	12	14	13	11
	D	8	15	11	9

Here the number of rows and columns are equal.

∴ The given assignment problem is balanced. Now let us find the solution.

Step 1: Select the smallest element in each row and subtract this from all the elements in its row.

	I	II	III	IV
A	0	2	9	1
B	0	5	2	3
C	1	3	2	0
D	0	7	3	1

Look for at least one zero in each row and each column. Otherwise, go to step 2.

Step 2: Select the smallest element in each column and subtract this from all the elements in its column.

	I	II	III	IV
A	0	0	7	1
B	0	3	0	3
C	1	1	0	0
D	0	5	1	1

Since each row and column contains atleast one zero, assignments can be made.

Step 3 (Assignment):

Examine the rows with exactly one zero. First three rows contain more than one zero. Go to row D. There is exactly one zero. Mark that zero by □ (i.e) job D is assigned to machine I. Mark other zeros in its column by × .

	I	II	III	IV
A	~~0~~	0	7	1
B	~~0~~	3	0	3
C	1	1	0	0
D	[0]	5	1	2

Step 4: Now examine the columns with exactly one zero. Already there is an assignment in column I. Go to the column II. There is exactly one zero. Mark that zero by □. Mark other zeros in its rowby × .

	I	II	III	IV
A	~~0~~	[0]	7	1
B	~~0~~	3	0	3
C	1	1	0	0
D	[0]	5	1	2

Column III contains more than one zero. Therefore proceed to Column IV, there is exactly one zero. Mark that zero by □. Mark other zeros in its row by × .

Column III contains more than one zero. Therefore proceed to Column IV, there is exactly one zero. Mark that zero by □. Mark other zeros in its row by × .

	I	II	III	IV
A	~~0~~	[0]	7	1
B	~~0~~	3	0	3
C	1	1	~~0~~	[0]
D	[0]	5	1	2

Step 5: Again examine the rows. Row B contains exactly one zero. Mark that zero by □.

	I	II	III	IV
A	~~0~~	[0]	7	1
B	~~0~~	3	[0]	3
C	1	1	~~0~~	[0]
D	[0]	5	1	2

Thus all the four assignments have been made. The optimal assignment schedule and total cost is

Job	Machine	cost
A	II	12
B	III	7
C	IV	11
D	I	8
Total cost		38

The optimal assignment (minimum) cost

= ₹ 38

Crew Assignment Problem

The Crew Assignment Problem (CAP) refers to a class of optimization problems that typically arise in the airline industry, especially when assigning flight legs to crew members such that total costs (or other metrics) are minimized while adhering to legal, contractual, and logistical constraints. These problems can be quite complex due to the multitude of constraints involved, including work-hour restrictions, rest requirements, flight connections, skills, crew base locations, and more.

CAP is related to, but distinct from, the Aircraft Routing Problem (ARP). While ARP determines the routes for aircraft, CAP deals with the assignment of crew members to the flight legs determined by ARP. Both are important for the efficient operation of airlines.

Inventory Control:

Inventory control, also known as inventory management, refers to the process of overseeing and managing the ordering, storage, and usage of components that a company uses in the production of the items it sells. It also involves managing finished products that are ready for sale. Effective inventory control is crucial for ensuring the smooth operation of the supply chain, meeting customer demands, and maintaining optimal inventory levels to reduce costs.

Key aspects of inventory control include:

1. **Stock Tracking**: Monitor the stock levels of both raw materials and finished products. With advancements in technology, businesses use inventory management software, barcoding, and RFID systems to track inventory levels efficiently.
2. **Ordering**: Determine when to reorder products or materials and in what quantity. This is often based on forecasted demand, historical sales, or economic order quantity (EOQ) calculations.
3. **Storage**: Efficiently store items to ensure easy access, optimal storage conditions, and efficient space utilization. This often involves warehouse management.
4. **Demand Forecasting**: Predict the quantity of a particular product that customers will want to purchase in a given timeframe. This helps in deciding how much inventory to hold.
5. **Safety Stock**: Maintain a buffer stock to handle uncertainties in demand and supply. Safety stock ensures that there's stock available if there's a sudden surge in demand or a delay in a shipment.
6. **Inventory Turnover**: Monitor how often inventory is sold and replaced over a certain period. High inventory turnover typically indicates good sales but can also mean there's too little stock. Low turnover may indicate slow sales or too much inventory.
7. **ABC Analysis**: Categorize inventory items into three categories:
   - A: High-value items with low sales frequency.
   - B: Moderate value and frequency.
   - C: Low-value items with high sales frequency. This helps prioritize inventory management efforts.
8. **Cycle Counting**: Regularly count a subset of your inventory items so that each item is counted several times a year. This is an alternative to doing a full inventory count.
9. **Just-In-Time (JIT)**: A strategy that aims to improve a business's return on investment by reducing in-process inventory and associated carrying costs. Companies only produce or order goods as required.

10. **Shrinkage Control**: Monitor and reduce inventory shrinkage due to theft, damage, miscounting, or supplier fraud.

11. **Bulk Shipments and Discounts**: Order in bulk or during discount periods to reduce costs, but only if storage costs or the risk of obsolescence are low.

Benefits of Effective Inventory Control:

- Reduces carrying costs.
- Minimizes stockouts or overstock situations.
- Enhances cash flow.
- Ensures products are available when customers need them, thereby improving customer service.
- Reduces losses due to perishability, obsolescence, or theft.

Challenges:

- Predicting market demand.
- Balancing between overstock and stockout situations.
- Ensuring accurate data recording and tracking.
- Managing suppliers and lead times.

Objectives of Inventory

Inventory has the following main objectives:

- To supply the raw material, sub-assemblies, semi-finished goods, finished goods, etc. to its users as per their requirements at the right time and at the right price.
- To maintain the minimum level of waste, surplus, inactive, scrap, and obsolete items.
- To minimize the inventory costs such as holding cost, replacement cost, breakdown cost, and shortage cost.
- To maximize the efficiency in production and distribution.
- To maintain the overall inventory investment at the lowest level.
- To treat inventory as an investment that is risky? For some items, an investment may lead to higher profits, and for others less profit

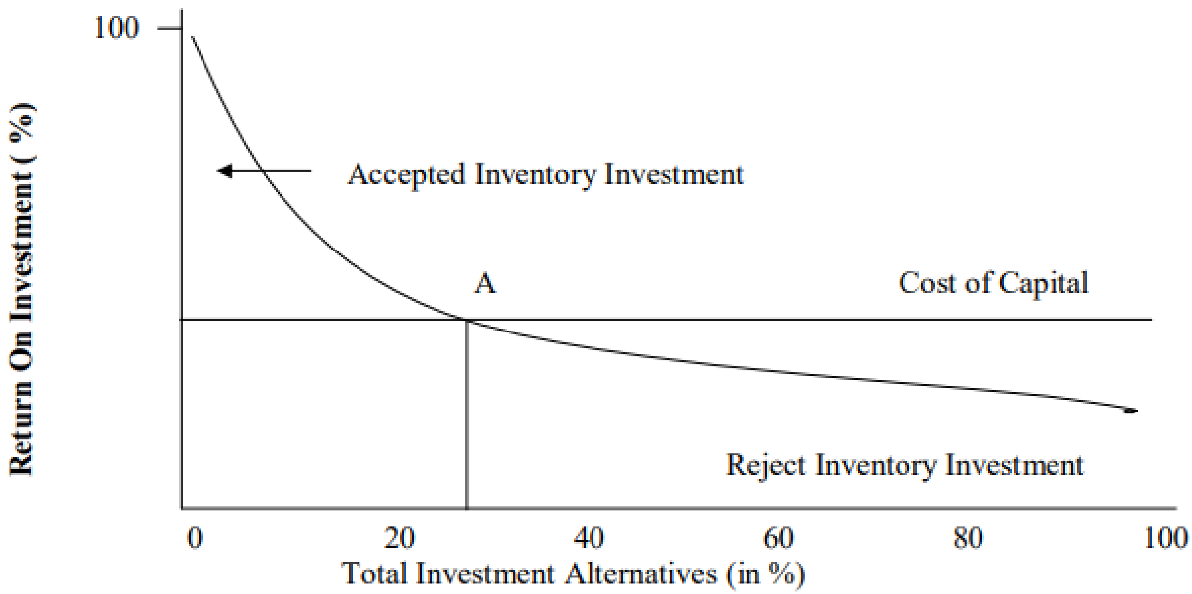
Basic Functions of Inventory

The important basic function of inventory is

- Increase the profitability- through manufacturing and marketing support. But a zero inventory manufacturing-distribution system is not practically possible, so it is important to remember that each rupee invested in inventory should achieve a specific goal. The other inventory basic functions are
- Geographical Specialization
- Decoupling
- Balancing supply and demand
- Safety stock

Inventory Investment Alternative

Investment is most important and major part of asset, which should be required to produce a minimum investment return. The MEC (Marginal Efficiency of Capital) concept holds that an organization should invest in those alternatives that produce a higher investment return than capital to borrow.



The curve shows that about 20% of the inventory investment alternatives will produce a return on investment above the capital cost

Geographical Specialization

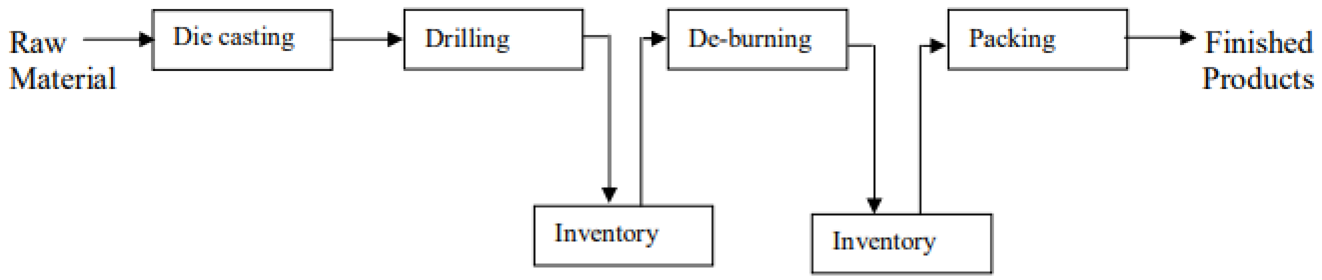
Another basic inventory function is to allow the geographical specialization of individual operating units. There is a considerable distance between the economical manufacturing location and demand areas due to factors of production such as raw material, labor, water, and power. So that the goods from various manufacturing locations are collected at a simple warehouse or plant to assemble in a final product or to offer consumers a single mixed product shipment. This also provides economic specialization between manufacturing and distribution units/locations of an organization

Decoupling

The provision of maximum efficiency of operations within a single facility is also one of the important basic functions of the inventory. This is achieved by decoupling, which is done by breaking operations apart so that one operation(s) supply is independent of another(s) supply

The decoupling function serves in two ways of purposes, they are

1. Inventories are needed to reduce the dependencies among successive stages of operations so that shortage of materials, breakdowns or other production fluctuations at one stage do not cause later stage to shut down.



The figure shows that the de-burning, and packing could continue to operate from inventories should die-casting and drilling be shut down or they can be decoupled from the production processes that precede them.

2. One organizational unit schedules its operations independently of another organizational unit.

For example: Consider an automobile organization, here assembly process can be scheduled separately from the engine built-up operation, and each can be decoupled from the final automobile assembly operations through in-process inventories.

Supply and Demand Balancing

The function of Balancing concerns the elapsed time between manufacturing and using the product. Balancing inventories exists to reconcile supply with demand. The most noticeable example of balancing is seasonal production and year-round usage like sugar, rice, woolen textiles, etc. Thus the investment of balancing inventories links the economies of manufacturing with variations of usage.

Safety Stock

The safety stock is also called buffer stock. The function of safety stock concerns short-range variations in either replacement or demand. Determination of the safety stock size requires a great deal of inventory planning. Safety stock provides protection against two types of uncertainty, which are:

1. Sales in excess of forecast during the replenishment period 2. Delays in replenishment

Thus, the inventories committed to safety stocks denote the greatest potential for improved performance. There are different techniques are available to develop safety stocks

Types of Inventory

Here's a breakdown of the different types:

1. **Raw Materials**: These are basic materials that are used in the production process but have not yet been transformed into finished products. For instance, leather used to make shoes or steel used to produce car parts.
2. **Work-in-Progress (WIP)**: These are items that are in the process of being converted from raw materials to finished goods. For a car manufacturer, this could be a car that's partially assembled but not yet completed.
3. **Finished Goods**: Products that have completed the manufacturing process and are ready to be sold to customers. Using the previous example, a car ready to be shipped to a dealer would be considered a finished good.
4. **MRO (Maintenance, Repair, and Operations) Inventory**: Items that aren't directly incorporated into a product but are used in the production process and operations. This might include lubricants, gloves, tools, etc.
5. **Packing Materials**: Items used to pack finished goods for shipment, like boxes, bubble wrap, and tape.
6. **Safety Stock**: This is a buffer stock to prevent stockouts due to unpredictable variations in supply or demand.
7. **Cycle Stock**: This is the inventory that's used to satisfy regular demand. As this inventory is sold or used, it's replenished.
8. **Decoupling Inventory**: Inventory held to decouple or separate various parts of the production process. This helps ensure that a hiccup or slowdown in one part of the process doesn't halt the entire operation.
9. **Anticipation/Seasonal Inventory**: Stock accumulated in anticipation of a seasonal increase in demand or an expected event, such as a big sale or a factory shutdown.
10. **Transit/In-Transit Inventory**: These are items that are on the move from one location to another, such as goods being shipped from a manufacturer to a retailer.
11. **Obsolete Inventory**: Items that have become outdated or are no longer in demand. These items often have to be written off or sold at a significant discount.
12. **Consignment Inventory**: Goods that are in the possession of a retailer (or another third party) but are still owned by the supplier. Ownership is transferred only when the item is sold.

13. **Dropshipping Inventory**: In this model, retailers don't keep the products they sell in stock. Instead, when a store sells a product, it purchases the item from a third party and has it shipped directly to the customer.

Understanding the different types of inventory and their roles is crucial for businesses to manage their resources efficiently, meet customer demand, and optimize their cash flow. Properly managing these various types can lead to increased efficiency, reduced costs, and improved profitability.

Factors Affecting Inventory

1. **Demand Forecasting Accuracy**: Predicting the future demand for a product can be tricky. If forecasts are inaccurate, this can result in stockouts or overstock situations.

2. **Lead Time**: The time taken from placing an order for inventory to its actual receipt can impact how much stock a company needs to hold.

3. **Economic Order Quantity (EOQ)**: This is the order quantity that minimizes the total holding costs and ordering costs. It is influenced by demand, ordering cost, and holding cost.

4. **Supplier Reliability**: If suppliers are unreliable or inconsistent with deliveries, a business might need to hold more safety stock.

5. **Product Life Cycle**: Products at different stages of their life cycle will have different inventory needs.

6. **Storage and Holding Costs**: The cost of storing inventory can influence how much inventory a business holds. This includes rent, utilities, and the cost of deterioration or obsolescence.

7. **Inventory Turnover**: This is a measure of how frequently a business sells its inventory. High turnover indicates efficient inventory management, while low turnover might indicate overstocking or issues with product desirability.

8. **Safety Stock Levels**: The buffer stock that companies maintain to account for uncertainties in supply and demand.

9. **Shrinkage**: Losses due to theft, mismanagement, or damage can affect inventory levels and costs.

10. **Economic Factors**: Economic downturns or upswings can affect consumer purchasing behavior, impacting inventory requirements.

11. **Political and Environmental Factors**: Tariffs, trade restrictions, or environmental calamities can impact inventory, either through supply chain disruption or changes in demand.

For any business, understanding and optimizing these factors can lead to a more efficient and cost-effective inventory management system.

Inventory Control Model

Economic Order Quantity (EOQ) Model I

A manufacturer uses Rs.20, 000 worth of an item during the year. Manufacturer estimated the ordering cost as Rs.50 per order and holding costs as 12.5% of average inventory value. Find the optimal order size, number of orders per year, time period per order and total cost.

Solution

Given that:

D = Rs.20, 000
Co = Rs.50
Ch = 12.5% of average inventory value / unit

Total Cost = TC = $\frac{25D}{Q} + (0.125)\frac{Q}{D}$, where Q is order size in Rs.

By applying the equations (eq.6) to (eq.9), we will get Q*, t*, N
Ñ

$$Q^* = \sqrt{\frac{2C_oD}{C_h}}$$

$$= \sqrt{\frac{2 * 50 * 20000}{0.125}} \quad = \text{Rs.}4000$$

$$t^* = \sqrt{\frac{2C_o}{C_hD}}$$

$$= \sqrt{\frac{2 * 50}{(0.125) * (20000)}} \quad = \frac{1}{5} \text{ years} = 73 \text{ days}$$

$$N = \frac{1}{t^*} = 5$$

Note: TC means in this case variable cost only

$$TC^* = \sqrt{2 * 50 * 0.125 * 20000} \quad = \text{Rs.}500$$

Order Size = Q = Rs.4000
Number of order / year = N = 5
Time period / order = t* = 73 days
Total Cost = TC* = Rs.500

Economic Order Quantity (EOQ) Model II

Example

The demand for an inventory item each costing Re5, is 20000 units per year. The ordering cost is Rs.10. The inventory carrying cost is 30% based on the average inventory per year.

Stock out cost is Rs.5 per unit of shortage incurred. Find out various parameters.

Solution

Given that

Demand = D = 20000

Ordering Cost = Co = Rs.10

Carrying Cost = Ch = 30% of Re 5 = $\frac{30}{100} * 5 = 1.5$

Stock out Cost = Cs = Rs.5

Now we have to determine the various parameter of EOQ Model II such as EOQ, Inventory Level, Shortage Level, Cycle Period, number of orders/year and Total Cost.

$$EOQ = Q^* = \sqrt{\left(\frac{2CoD}{C_h}\right)\left(\frac{C_o+C_h}{C_s}\right)}$$

$$= \sqrt{\left(\frac{2*10*20000}{0.30}\right)\left(\frac{0.30+10}{5}\right)} = 1657 \text{ units}$$

$$\text{Inventory Level} = I^* = \sqrt{\left(\frac{2CoD}{C_h}\right)\left(\frac{Cs}{C_o+C_h}\right)}$$

$$\text{Inventory Level} = I^* = \sqrt{\left(\frac{2*10*20000}{0.30}\right)\left(\frac{5}{10+0.30}\right)} = 804 \text{ units}$$

Shortage Level = Q* - I* = 1657 – 804 = 853 units

$$\text{Cycle Period} = t^* = \frac{Q^*}{D} = \frac{1657}{20000} = 30.24 \text{ days} = 30 \text{ days}$$

$$\text{Number of Orders/Year} = \frac{1}{t^*} = \frac{1}{Q^*/D} = \frac{D}{Q^*} = \frac{20000}{1657} = 12 \text{ Orders/year}$$

$$\text{Total Cost} = \sqrt{\frac{2C_oC_hC_sD}{(C_h + C_s)}}$$

$$= \sqrt{\left(\frac{2*10*0.30*5*20000}{0.30+5}\right)} = RS.336.4$$

UNIT III

3.1 A minimum spanning tree (MST)

A minimum spanning tree (MST) or minimum weight spanning tree is a subset of the edges of a connected, edge-weighted undirected graph that connects all the vertices, without any cycles, and with the minimum possible total edge weight. That is, it is a spanning tree whose sum of edge weights is as small as possible. More generally, any edge-weighted undirected graph (not necessarily connected) has a minimum-spanning forest, which is a union of the minimum-spanning trees for its connected components.

3.2 Applications of MST

Minimum spanning trees have direct applications in the design of networks, including computer networks, telecommunications networks, transportation networks, water supply networks, and electrical grids (which they were first invented for, as mentioned above). They are invoked as subroutines in algorithms for other problems, including the Christofides algorithm for approximating the traveling salesman problem, approximating the multi-terminal minimum cut problem (which is equivalent in the single-terminal case to the maximum flow problem), and approximating the minimum-cost weighted perfect matching.

Other practical applications based on minimal spanning trees include:

- Taxonomy.
- Cluster analysis: clustering points in the plane, single-linkage clustering (a method of hierarchical clustering), graph-theoretic clustering, and clustering gene expression data.
- Constructing trees for broadcasting in computer networks.
- Image registration and segmentation – see minimum spanning tree-based segmentation.
- Curvilinear feature extraction in computer vision.
- Handwriting recognition of mathematical expressions.
- Circuit design: implementing efficient multiple constant multiplications, as used in finite impulse response filters.
- Regionalization of socio-geographic areas, the grouping of areas into homogeneous, contiguous regions.
- Comparing ecotoxicology data.
- Topological observability in power systems.
- Measuring homogeneity of two-dimensional materials.
- Minimax process control.
- Minimum spanning trees can also be used to describe financial markets. A correlation matrix can be created by calculating a coefficient of correlation between any two stocks. This matrix can be represented topologically as a complex network and a minimum spanning tree can be constructed to visualize relationships.

3.3 Floyd-Warshall Algorithm

Floyd-Warshall Algorithm is an algorithm for finding the shortest path between all the pairs of vertices in a weighted graph. This algorithm works for both the directed and undirected weighted graphs. But, it does not work for the graphs with negative cycles (where the sum of the edges in a cycle is negative).

3.3.1 Follow the steps below to find the shortest path between all the pairs of vertices.

1. Create a matrix A^0 of dimension n*n where n is the number of vertices. The row and the column are indexed as i and j respectively. i and j are the vertices of the graph.

Each cell A [i] [j] is filled with the distance from the i_{th} vertex to the j_{th} vertex. If there is no path from the i_{th} vertex to the j_{th} vertex, the cell is left as infinity.

$$A^0 = \begin{bmatrix} 0 & 3 & \infty & 5 \\ 2 & 0 & \infty & 4 \\ \infty & 1 & 0 & \infty \\ \infty & \infty & 2 & 0 \end{bmatrix}$$

2. Now, create a matrix A^1 using matrix A^0. The elements in the first column and the first row are left as they are. The remaining cells are filled in the following way.

Let k be the intermediate vertex in the shortest path from source to destination. In this step, k is the first vertex. A[i][j] is filled with (A[i][k] + A[k][j]) if (A[i][j] > A[i][k] + A[k][j]).

That is, if the direct distance from the source to the destination is greater than the path through the vertex k, then the cell is filled with A[i][k] + A[k][j].

In this step, k is vertex 1. We calculate the distance from the source vertex to the destination vertex through this vertex k.

$$A^1 = \begin{bmatrix} 0 & 3 & \infty & 5 \\ 2 & 0 & & \\ \infty & & 0 & \\ \infty & & & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 3 & \infty & 5 \\ 2 & 0 & 9 & 4 \\ \infty & 1 & 0 & 8 \\ \infty & \infty & 2 & 0 \end{bmatrix}$$

Calculate the distance from the source vertex to the destination vertex through this vertex k

For example: For A^1[2, 4], the direct distance from vertex 2 to 4 is 4 and the sum of the distance from vertex 2 to 4 through vertex (ie. from vertex 2 to 1 and from vertex 1 to 4) is 7. Since 4 < 7, A^0 [2, 4] is filled with 4.

3. Similarly, A^2 is created using A^1. The elements in the second column and the second row are left as they are.

In this step, k is the second vertex (i.e. vertex 2). The remaining steps are the same as in step 2.

$$A^2 = \begin{bmatrix} 0 & 3 & & \\ 2 & 0 & 9 & 4 \\ & 1 & 0 & \\ & \infty & & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 3 & 9 & 5 \\ 2 & 0 & 9 & 4 \\ 3 & 1 & 0 & 5 \\ \infty & \infty & 2 & 0 \end{bmatrix}$$

Calculate the distance from the source vertex to the destination vertex through this vertex 2

4. Similarly, A^3 and A^4 is also created

$$A^3 = \begin{bmatrix} 0 & & \infty & \\ & 0 & 9 & \\ \infty & 1 & 0 & 8 \\ & & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 3 & 9 & 5 \\ 2 & 0 & 9 & 4 \\ 3 & 1 & 0 & 5 \\ 5 & 3 & 2 & 0 \end{bmatrix}$$

Calculate the distance from the source vertex to the destination vertex through these 3 vertexes

$$A^4 = \begin{bmatrix} 0 & & & 5 \\ & 0 & & 4 \\ & & 0 & 5 \\ 5 & 3 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 3 & 7 & 5 \\ 2 & 0 & 6 & 4 \\ 3 & 1 & 0 & 5 \\ 5 & 3 & 2 & 0 \end{bmatrix}$$

Calculate the distance from the source vertex to the destination vertex through this vertex 4

5. A^4 gives the shortest path between each pair of vertices.

3.4 Decision trees

Decision trees are useful whenever we have to evaluate interdependent decisions that must be made in sequence and when there is uncertainty about events. For that reason, they are especially useful for evaluating capacity expansion alternatives given that future demand is uncertain. Remember that our main decision is whether to purchase a large facility or a small one with the possibility of expansion later. You can see that the decision to expand later is dependent on choosing a small facility now. Which alternative ends up being best will depend on whether demand turns out to be high or low. Unfortunately, we can only forecast future demand and have to incur some risks.

Decision tree

Modeling tool used to evaluate independent decisions that must be made in sequence. A decision tree is a diagram that models the alternatives being considered and the possible outcomes. Decision trees help by giving structure to a series of decisions and providing an objective way of evaluating alternatives.

3.4.1 Decision trees contain the following information:

- Decision points. These are the points in time when decisions, such as whether or not to expand, are made. They are represented by squares, called "nodes."
- Decision alternatives. Buying a large facility and buying a small facility are two decision alternatives. They are represented by "branches" or arrows leaving a decision point.

3.5 Network Analysis

The network is a technique used for planning and scheduling of large projects in the fields of construction, maintenance, fabrication, purchasing, computer system instantiation, research and development planning etc. There is multitude of operations research situations that can be modeled and solved as network. Some recent surveys reports that as much as 70% of the real-world mathematical programming problems can be represented by network related models. Network analysis is known by many names _PERT (Programme Evaluation and Review Technique), CPM (Critical Path Method), PEP (Programme Evaluation Procedure), LCES (Least Cost Estimating and Scheduling), SCANS (Scheduling and Control by Automated Network System), etc

3.5.1 Terminologies Used in Network Analysis

Network

It is a graphical representation of logical and sequentially connected activities and events of a project. The network is also called an arrow diagram. PERT (Program Evolution Review Technique) and (Critical Path Method) are the two most widely applied techniques.

Project

A project is defined as a combination of interrelated activities which must be executed in a certain order for its completion. Project Management Process Network analysis is the general name given to certain specific techniques which can be used for the planning, management, and control of projects

Activity

Any individual operation, which utilizes resources and has an end and a beginning, is called an activity.

- A task or a certain amount of work required in the project
- Requires time to complete
- Represented by an arrow

These are usually classified into four categories:

- Predecessor activity
- Successor activity
- Concurrent activity
- Dummy activity

Dummy Activity

It Indicates only precedence relationships and does not require any time or effort PERT(Program Evaluation and Review Technique) is a method to analyze the involved tasks in completing a given project, especially the time needed to complete each task, and identifying the minimum time needed to complete the total project. PERT is based on the assumption that an activity's duration follows a probability distribution instead of being a single value.

Three-time estimates are required to compute the parameters of an activity's duration distribution:

1. Pessimistic time (t_p) - the time the activity would take if things did not go well

2. Most likely time (t_m) - the consensus best estimate of the activity's duration

3. Optimistic time (t_o) - the time the activity would take if things did go well.

$$\text{Mean (expected time)} = \frac{(t_p - 4t_m + t_o)}{6}$$

$$\text{Variance } (\sigma^2) = \left(\frac{t_p - t_o}{6}\right)^2$$

Probability computation:

Determine the probability that the project is completed within the specified Time

$$Z = \frac{X - \mu}{\sigma}$$

μ = project mean time
σ = project standard mean time
x = (proposed) specified time

Float:

The float of activity represents the excess of available time over its duration.

Total Float (Ft)

The amount of time by which the completion of an activity could be delayed beyond the earliest expected completion time without affecting the overall project duration.

Tf= (Latest start-Earliest start) for activity (i-j), or, (Tf)ij=(LS)jj-(ES)ij

Free Float (Ff)

` The time by which the completion of an activity can be delayed beyond the earliest finish time without affecting the earliest start of subsequent (succeeding) activities.

3.5.2 Benefits of CPM/PERT

1) Useful at many stages of project management

2) Mathematically simple

3) Give critical path and slack time

4) Provide project documentation

5) Useful in monitoring costs

3.4 Critical Path Method and Project Evaluation and Review Technique

3.4.1. Critical Path Method (CPM):

CPM is a technique that is used for projects where the time needed for the completion of the project is already known. It is majorly used for determining the approximate time within which a project can be completed. The critical path is the largest path in project management which always provide minimum time taken for the completion of the project

3.4.2. Program Evaluation Review Technique (PERT):

PERT is an appropriate technique that is used for projects where the time required or needed to complete different activities is not known. PERT is majorly applied for scheduling, organization, and integration of different tasks within a project. It provides the blueprint of the project and is an efficient technique for project evaluation.

3.4.3 Difference between PERT and CPM:

Difference between PERT and CPM:

S.No.	PERT	CPM
1.	PERT is a technique of project management that is used to manage uncertain (i.e., time is not known) activities of any project.	CPM is a technique of project management that is used to manage only certain (i.e., time is known) activities of any project.
2.	It is event oriented technique which means that the network is constructed based on the event.	It is activity oriented technique which means that the network is constructed based on activities.
3.	It is a probability model.	It is a deterministic model.
4.	It majorly focuses on time as meeting time targets or estimation of percent completion is more important.	It majorly focuses on the Time-cost trade-off as minimizing cost is more important.
5.	It is appropriate for high-precision time estimation.	It is appropriate for reasonable time estimation.
6.	It has Non-repetitive nature to the job.	It has repetitive nature of the job.
7.	There is no chance of crashing as there is no certainty of time.	There may be crashing because of certain time boundation.
8.	It doesn’t use any dummy activities.	It uses dummy activities for representing a sequence of activities.
9.	It is suitable for projects which required research and development.	It is suitable for construction projects.

UNIT IV

Dynamic Programming:

Dynamic Programming (DP) is an important mathematical tool in operations research, particularly in the optimization of certain types of decision processes. It's instrumental in problems where decisions over time need to be optimized and where the situation at any given time is dependent on choices made in the past.

Nature of Dynamic Programming in OR:

- **Stages**: Problems are broken down into stages or periods, with each stage representing a point in time where a decision is made.
- **States**: The solution at each stage depends on some conditions or situations, called states.
- **Decision Variables**: At each stage, based on the current state, a decision is made from a set of available choices.
- **Recurrence Relations**: The relationship between stages is defined in terms of recurrence relations, which provide a formula or a rule to progress from one stage to the next.

Capital budgeting:

Capital budgeting involves mathematical optimization models to decide the best combination of projects or assets to invest in, given a limited budget and multiple constraints. These problems can be formulated using various optimization techniques like linear programming, integer programming, or dynamic programming, depending on the problem specifics.

Shortest path method:

The Shortest Path method is a technique primarily used in network optimization, most commonly in transportation and communication network problems. The aim is to find the shortest (or least cost) path between two given nodes (or points) in a graph.

Dijkstra's and the Floyd-Warshall algorithm are two of the most well-known algorithms for determining the shortest path.

Dijkstra's Algorithm:

- Start with a 'source' node.
- Assign to every node a tentative distance: set it to zero for our initial node and to infinity for all other nodes.
- Set the initial node as current.

- For the current node, consider all its neighbors and calculate their tentative distances. If the newly calculated tentative distance is less than the current assigned value, update the distance.
- Mark the current node as visited. A visited node will not be checked again.
- Select the unvisited node with the smallest tentative distance and set it as the new "current node". Go back to the previous step.
- When all nodes have been visited or the smallest tentative distance among the unvisited nodes is infinity (meaning there's no connection to the remaining unvisited nodes), the algorithm finishes.

Floyd-Warshall Algorithm:

- This algorithm works for all pairs of nodes.
- The graph is represented as a matrix. If there's no direct edge between nodes **i** and **j**, the matrix value (i,j) is infinity (∞).
- For each node, check if it works as a shorter intermediary between two other nodes. If it does, update the shortest path between those nodes.
- Repeat the process for all nodes.
- At the end, you'll have the shortest path between all pairs of nodes.

<u>Introduction of Game Theory:</u>

Game Theory is a mathematical tool used to analyze situations in which two or more players make decisions that potentially affect each other's welfare. In essence, it's the study of mathematical models of conflict and cooperation among intelligent rational decision-makers. Initially introduced by John von Neumann and Oskar Morgenstern in their 1944 book, "Theory of Games and Economic Behavior," it has since become an essential tool in various fields, especially operations research, economics, and computer science.

Key concepts in the Theory of Games

Players:

The competitors or decision makers in a game are called the players of the game.

Strategies:

The alternative courses of action available to a player are referred to as his strategies.

Pay off:

The outcome of playing a game is called the payoff to the concerned player.

Optimal Strategy:

A strategy by which a player can achieve the best pay off is called the optimal strategy for him.

Zero-sum game:

A game in which the total payoffs to all the players at the end of the game are zero is referred to as a zero-sum game.

Non-zero sum game:

Games with "less than complete conflict of interest" are called non-zero sum games. The problems faced by a large number of business organizations come under this category. In\ such games, the gain of one player in terms of his success need not be completely at the expense of the other player.

Payoff matrix:

The tabular display of the payoffs to players under various alternatives is called the payoff matrix of the game.

Pure strategy:

If the game is such that each player can identify one and only one strategy as the optimal strategy in each play of the game, then that strategy is referred to as the best strategy for that player and the game is referred to as a game of pure strategy or a pure game.

Mixed strategy:

If there is no one specific strategy as the ‘best strategy’ for any player in a game, then the game is referred to as a game of mixed strategy or a mixed game. In such a game, each player has to choose different alternative courses of action from time to time.

N-person game:

A game in which N-players take part is called an N-person game.

Maximin-Minimax Principle:

The maximum of the minimum gains is called the maximin value of the game and the corresponding strategy is called the maximin strategy. Similarly the minimum of the maximum losses is called the minimax value of the game and the corresponding strategy is called the minimax strategy. If both the values are equal, then that would guarantee the best of the worst results.

Negotiable or cooperative game:

If the game is such that the players are taken to cooperate on any or every action which may increase the payoff of either player, then we call it a negotiable or cooperative game.

Non-negotiable or non-cooperative game:

If the players are not permitted for coalition then we refer to the game as a non-negotiable or non-cooperative game.

Saddle point:

A saddle point of a game is that place in the payoff matrix where the maximum of the row minima is equal to the minimum of the column maxima. The payoff at the saddle point is called **the value of the game** and the corresponding strategies are called the **pure strategies.**

Dominance:

One of the strategies of either player may be inferior to at least one of the remaining ones. The superior strategies are said to dominate the inferior ones.

Managerial Applications of the Theory of Games

The techniques of game theory can be effectively applied to various managerial problems as detailed below:

1) Analysis of the market strategies of a business organization in the long run.

2) Evaluation of the responses of the consumers to a new product.

3) Resolving the conflict between two groups in a business organization.

4) Decision making on the techniques to increase market share.

5) Material procurement process.

6) Decision making for transportation problem.

7) Evaluation of the distribution system.

8) Evaluation of the location of the facilities.

9) Examination of new business ventures and

10) Competitive economic environment.

Types of Games:

Two person games and n-person games:

In two person games, there are exactly two players and each competitor will have a finite number of strategies. If the number of players in a game exceeds two, then we refer to the game as n-person game.

Zero sum game and non-zero sum game:

If the sum of the payments to all the players in a game is zero for every possible outcome of the game, then we refer to the game as a zero sum game. If the sum of the payoffs from any play of the game is either positive or negative but not zero, then the game is called a non-zero sum game

2x2 two person game and 2xn and mx2 games:

When the number of players in a game is two and each player has exactly two strategies, the game is referred to as 2x2 two person game.

A game in which the first player has precisely two strategies and the second player has three or more strategies is called a 2xn game.

A game in which the first player has three or more strategies and the second player has exactly two strategies is called an mx2 game.

Dominance Property:

In game theory, the concept of dominance refers to one strategy being consistently better than another, regardless of what the other players do. The dominance property can be used to simplify the analysis of games by eliminating dominated strategies**.**

Formula for dominance Property:

Probability that Player 1

First player strategy 1

$$p = \frac{d - c}{(a + d) - (b + c)}$$

First player strategy 2

$1\text{-}p$

Probability that Player 2

Second player strategy 1

$$r = \frac{d - b}{(a + d) - (b + c)}$$

Second player strategy 2

$(1\text{-}r)$.

Value of the Game

$$\text{Value of the game V} = \frac{ad - bc}{(a + d) - (b + c)}$$

UNIT – V

Queueing Theory

A flow of customers from finite or infinite population towards the service facility forms a queue (waiting line) an account of lack of capability to serve them all at a time. In the absence of a perfect balance between the service facilities and the customers, waiting time is required either for the service facilities or for the customer's arrival. In general, the queueing system consists of one or more queues and one or more servers and operates under a set of procedures. Depending upon the server status, the incoming customer either waits at the queue or gets the turn to be served. If the server is free at the time of arrival of a customer, the customer can directly enter into the counter for getting service and then leave the system. In this process, over a period of time, the system may experience "Customer waiting" and /or "Server idle time".

List of Variables- The list of variable used in queueing models is give below:

- n - No of customers in the system
- C - No of servers in the system
- Pn (t) – Probability of having n customers in the system at time t
- Pn - Steady state probability of having customers in the system
- P_0 - Probability of having zero customer in the system
- Lq - Average number of customers waiting in the queue.
- Ls - Average number of customers waiting in the system (in the queue and in the service counters)
- Wq - Average waiting time of customers in the queue.
- Ws - Average waiting time of customers in the system (in the queue and in the service counters)
- δ- Arrival rate of customers
- μ- Service rate of server
- ϕ- Utilization factor of the server
- δeff - Effective rate of arrival of customers
- M - Poisson distribution
- N - Maximum numbers of customers permitted in the system. Also, it denotes the size of the calling source of the customers.
- GD - General discipline for service. This may be first in first – serve (FIFS), last-in-first serve (LIFS) random order (Ro) etc

Classification of Queueing models

Generally, queueing models can be classified into six categories using Kendall's notation with six parameters to define a model. The parameters of this notation are:

- P- Arrival rate distribution ie probability law for the arrival /inter – arrival time.
- Q - Service rate distribution, ie probability law according to
- which the customers are being served.
- R - Number of Servers (ie number of service stations)
- X - Service discipline
- Y - Maximum number of customers permitted in the system.
- Z - Size of the calling source of the customers.
- A queuing model with the above parameters is written as - (P/Q/R : X/Y/Z)

Model 1: (M/M/1): (GD/∞ /∞) Model

In this model

(i) The arrival rate follows poisson (M) distribution.

(ii) Service rate follows poisson distribution (M)

(iii) Number of servers is 1

(iv) Service discipline is general disciple (ie GD)

(v) Maximum number of customers permitted in the system is infinite (∞)

(vi) Size of the calling source is infinite (∞)

Formula Used in Model 1:

Given arrival rate= δ

Given service rate= μ

The probability= $P_0 = \phi^{0} (1- \phi)$

$$L_s = \frac{\phi}{1 - \phi}$$

$$L_q = \frac{\phi^2}{1 - \phi}$$

$$W_s = \frac{1}{\mu - \delta}$$

$$W_q = \frac{\phi}{\mu - \delta}$$

Example:

The arrival rate of customers at a banking counter follows a poisson distibution with a mean of 30 per hours. The service rate of the counter clerk also follows poisson distribution with mean of 45 per hour.

a) What is the probability of having zero customers in the system?

b) What is the probability of having 8 customers in the system?

c) What is the probability of having 12 customers in the system?

d) Find Ls, Lq, Ws and Wq.

Solution:

Given arrival rate follows poisson distribution with mean =30

$\therefore \delta = 30$ per hour

Given service rate follows poisson distribution with mean = 45

$\therefore \mu = 45$ Per hour

$\therefore$ Utilization factor $\phi = \delta/\mu$

$= 30/45$

$= 2/3$

$= 0.67$

a) The probability of having zero customer in the system

$P_0 = \phi^0 (1-\phi)$

$= 1-\phi$

$= 1-0.67$

$= 0.33$

b) The probability of having 8 customers in the system

$P_8 = \phi^8 (1-\phi)$

$= (0.67)^8 (1-0.67)$

$= 0.0406 \times 0.33$

$= 0.0134$

Probability of having 12 customers in the system is

$P_{12} = \phi^{12} (1-\phi)$

$= (0.67)^{12} (1-0.67)$

$= 0.0082 \times 0.33$ $=$ **0.002706**

$$L_s = \frac{\phi}{1-\phi} = \frac{0.67}{1-0.67}$$

$$= \frac{0.67}{0.33} = 2.03$$

= **2 customers**

$$L_q = \frac{\phi^2}{1-\phi} = \frac{(0.67)^2}{1-0.67} = \frac{0.4489}{0.33}$$

$= 1.36$

= **1 Customer**

$$W_s = \frac{1}{\mu - \delta} = \frac{1}{45-30} = \frac{1}{15}$$

= **0.0666 hour**

$$W_q = \frac{\phi}{\mu - \delta} = \frac{0.67}{45-30} = \frac{0.67}{15}$$

= **0.4467 hour**

Second Model- (M/M/C): (GD/∞/ ∞)Model

The parameters of this model are as follows:

- Arrival rate follows poisson distribution
- Service rate follows poisson distribution
- No of servers is C'.
- Service discipline is general discipline.
- Maximum number of customers permitted in the system is infinite

Formula used in Model 2:

δ=Arrival rate

μ=Unloading rate

C = No. of unloading crews, C-1

$$P_0 = \{[\Sigma \phi^n/n!] + \phi^c / (c! [1 - \phi/c])\}^{-1}$$

$$P_n = (\phi^n / n!) P_o \quad \text{for} \quad 0 \leq n \leq c$$

$$L_q = \frac{\phi^{C+1} \text{ X } P_0}{}$$

$$L_s = L_q + \phi$$

$$W_q = L_q / \delta$$

$$W_s = W_q + 1 / \mu$$

Example:

A supermarket has two girls ringing up sales at the counters. If the service time for each customer is exponential with mean 4 minutes and if the people arrive in poisson fashion at the rate of 10 per hour

a) What is the probability of having to wait for service?

b) What is the expected percentage of idle time for each girl?

c) If a customer has to wait, what is the expected length of his waiting time?

Solution:

$$P_0 = \{[\sum_{n=0}^{C-1} \phi^n/n!] + \phi^c / (c! [1 - \phi/c])\}^{-1}$$

Where $\phi = \delta / \mu$ ∴ given arrival rate = 10 per hour

$\delta = 10 / 60 = 1 / 6$ per minute

Service rate = 4 minutes

∴ $\mu = 1 / 4$ person per minute

Hence $\phi = \delta / \mu = (1 / 6) \times 4 = 2 / 3$

$= \mathbf{0.67}$

$$P_0 = \{[\sum_{n=0}^{1} \phi^n/n!] + (0.67)^2 / (2! [1 - 0.67/2])\}^{-1}$$

$=[1 + (\phi / 1!)] + 0.4489 / (2 – 0.67)]^{-1}$

$=[1 + 0.67 + 0.4489 / (1.33)]^{-1}$

$=[1 + 0.67 + 0.34]^{-1}$

$=[2.01]^{-1}$

$= \mathbf{1 / 2}$

The Probability of having to wait for the service is

P (w > 0)

$$= \frac{\phi^c}{c! [1 - \phi / c]} \times P_0$$

$$= \frac{0.67^2 \times (1 / 2)}{2! [1 – 0.67 / 2]}$$

$= 0.4489 / 2.66$

$= \mathbf{0.168}$

b) The probability of idle time for each girl is

$$= 1 - P(w > 0)$$

$$= 1 - 1/3$$

$$\mathbf{= 2/3}$$

∴ Percentage of time the service remains idle = 67% approximately

c) The expected length of waiting time (w/w>0)

$$= 1 / (c\mu - \delta)$$

$$= 1 / [(1/2) - (1/6)]$$

$$= \mathbf{3\ minutes}$$

Simulation:

The representation of reality in some physical form or in some form of Mathematical equations are called Simulations

Need for simulation:

Consider an example of the queueing system, namely the reservation system of a transport corporation. The elements of the system are booking counters (servers) and waiting customers (queue). Generally the arrival rate of customers follow a Poisson distribution and the service time follows exponential distribution.

Some advantage of simulation:

1. Simulation is mathematically less complicated

2. Simulation is flexible

3. It can be modified to suit the changing environments.

4. It can be used for training purpose

5. It may be less expensive and less time consuming in a quite a few real world situations

Some Limitations of Simulation:

1. Quantification or Enlarging of the variables maybe difficult.

2. Large number of variables make simulations unwieldy and more difficult.

3. Simulation may not. Yield optimum or accurate results.

4. Simulation are most expensive and time consuming model.

5. We cannot relay too much on the results obtained from simulation models.

Replacement models

The replacement problems are concerned with the situations that arise when some items such as men, machines and usable things etc need replacement due to their decreased efficiency, failure or breakdown. Such decreased efficiency or complete breakdown may either be gradual or all of a sudden.

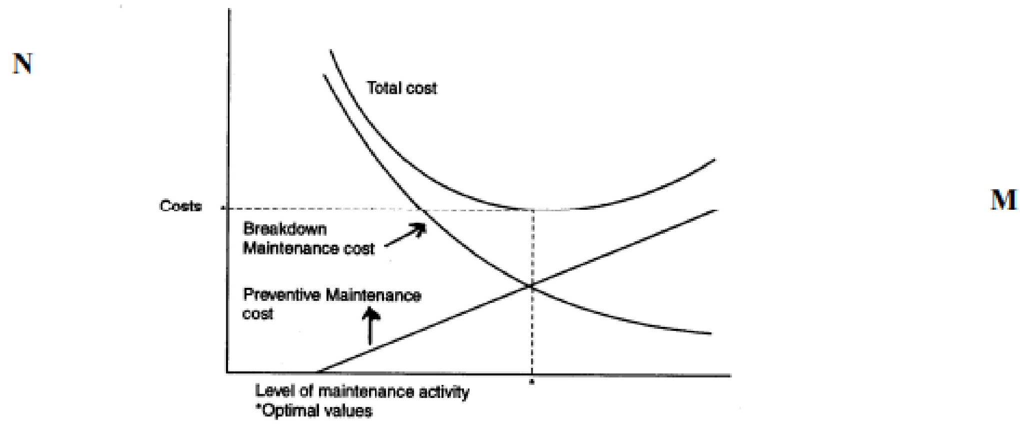
There are two basic reasons for considering the replacement of equipment.

- The **physical impairment** refers only to changes in the physical condition of the equipment itself. This will lead to decline in the value of service rendered by the equipment, increased operating cost of the equipments, and increased maintenance cost of the equipment or the combination of these costs.
- **Obsolescence of the equipment** is caused due to improvement in the existing Tools and machinery mainly when the technology becomes advanced therefore; it becomes uneconomical to continue production with the same equipment under any of the above situations.

Type of Maintenance:

Breakdown maintenance is maintenance performed on a piece of equipment that has broken down, faulted, or otherwise cannot be operated. The goal of breakdown-maintenance is to fix something that has malfunctioned.

Preventive maintenance is the act of performing regularly scheduled maintenance activities to help prevent unexpected failures in the future. Put simply, it’s about fixing things before they break.



The points M and N denote optimal level of maintenance and optimal cost respectively

Types of replacement problem -The replacement problem can be classified into two categories.

1) Replacement of assets that deteriorate with time (replacement due to gradual failure, due to wear and tear of the components of the machines)
    a) Determination of economic type of an asset.
    b) Replacement of an existing asset with a new asset.

2) Simple probabilistic model for assets which will fail completely (replacement due to sudden failure).

<u>Determination of Economic Life of an asset</u>

Asset’s cost component always includes-

- **Capital Recovery Cost**, which reduces over the period owing to depreciation. This cost is derived from the Purchase Cost of the asset.
- **Operating and Maintenance Cost of an asset.** This type of cost increases over the period as the efficiency of the asset reduces, and thus the cost of operating and maintaining increases.
- Total cost refers to the sum of Capital Recovery costs and Operating and Maintenance costs. It typically refers to the sum of all the **fixed costs** and all the variables required operating it. Further total costs help in defining average costs and marginal costs.

A typical shape of each of the above cost with respect to life of the asset is shown below

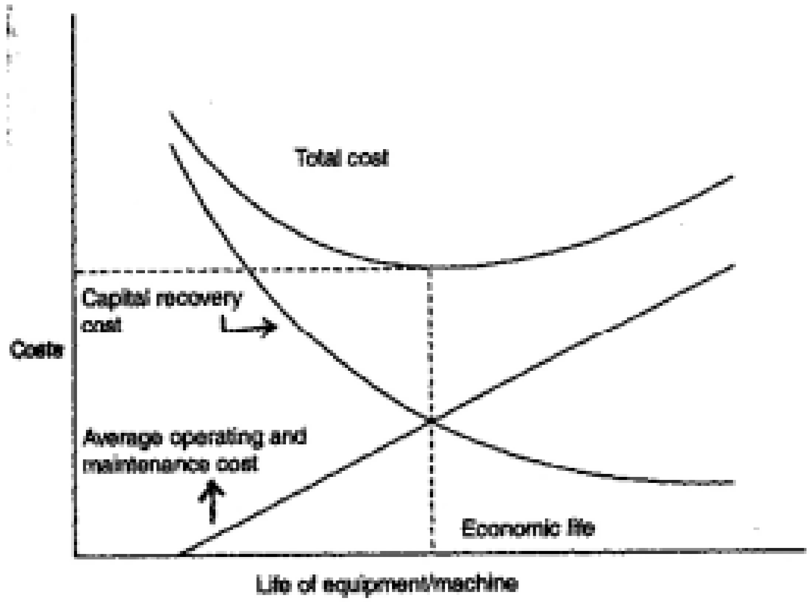


Fig. 13.2 Chart showing economic life.

How Is Economic Life Used?

Asset operators manage the assets of the company where the salvage value of equipment, machinery, and various other equipments has to be estimated. The efficiency of the assets is necessary to reduce the operational cost of a company or its subsidiaries.

Managers at the highest of hierarchies in a company have to make decisions related to the investment or divestment of the assets of a company. For this purpose, the salvage value of the assets of a company like its subsidiaries or divisions.

Only after the salvage value of an asset is estimated can the company move forward with the decision of investment or divestment, as it affects not only the company accounts but also the shareholders or the external investors in a business.

Companies estimate the useful life of equipment before buying them, considering the normal usage level and maintenance of the asset over a while.

These businesses also need to appropriate the amount to be set aside to replace the asset bought after its physical life ends.

Example:

A firm is considering replacement of equipment whose first cost is Rs. 1750 and the scrap value is negligible at any year. Based on experience, it is found that maintenance cost is zero during the first year and it increases by Rs. 100 every year thereafter.

(i) When should be the equipment replaced if

a) i = 0%

b) i = 12%

Solution:

Given the first cost = Rs 1750 and the maintenance cost is Rs. Zero during the first years and then increases by Rs. 100 every year thereafter. Then the following table shows the calculation.

Calculations to determine Economic life

(a) First cost Rs. 1750 Interest rate = 0%

End of year (n)	**Maintenance cost at end of year**	**Summation of maintenance Cost**	**Average cost of maintenance through the given year**	**Average first cost if replaced at the given year and**	**Average total cost through the given year**
A	**B (Rs)**	**C (Rs)**	**D (in Rs)**	**E (Rs)**	**F (Rs)**
		C = ΣB	**C/A**	$\frac{1750}{A}$	**D + E**
1	0	0	0	1750	1750
2	100	100	50	875	925
3	200	300	100	583	683
4	300	600	150	438	588
5	400	1000	200	350	550
6	**500**	**1500**	**250**	**292**	**542**
7	600	2100	300	250	550
8	700	2800	350	219	569

The value corresponding to any end-of-year (n) in Column F represents the average total cost of using the equipment till the end of that particulars year.

In this problem, the average total cost decreases till the end of the year 6 and then it increases. Hence the optimal replacement period is 6 years

ie the economic life of the equipment is 6 years.

(b) First cost Rs. 1750 Interest rate = 12%

End of year (n)	Maintenance cost at end of years	(P/F,12v,n)	Present worth as beginning of years 1 of maintenance costs	Summation of present worth of maintenance costs through the given year	Present simulator maintenance cost and first cost	(A/P, 12%,n) $=\frac{i(1+i)^n}{(1+i)^n-1}$ G	Annual equipment total cost through the giver year
A	B	C	D	E	F	G	H
	B (iR)	C = $\frac{1}{(1+12/100)^n}$	BxC	ΣD	E+ Rs. 1750		FxG
1	0	0.8929	0	0	1750	1.1200	1960
2	100	0.7972	79.72	79.72	1829.72	0.5917	1082.6
3	200	0.7118	142.36	222.08	1972.08	0.4163	820.9
4	300	0.6355	190.65	412.73	2162.73	0.3292	711.9
5	400	0.5674	226.96	639.69	2389.69	0.2774	662.9
6	500	0.5066	253.30	892.99	2642.99	0.2432	642.7
7	600	0.4524	271.44	1164.43	2914.430	0.2191	638.5
8	700	0.4039	282.73	1447.16	3197.16	0.2013	680.7

Identify the end of year for which the annual equivalent total cost is minimum in column. In this problem the annual equivalent total cost is minimum at the end of year hence the economics life of the equipment is 7 years.

12% compound interest table reference for Column G in the above table for

(A/P, 12%, n)

$= i(1+i)^n/(1+i)^{n-}1$

12% **Compound Interest Factors** 12

	Single Payment		Uniform Payment Series				Arithmetic Gradient	
	Compound Amount Factor	Present Worth Factor	Sinking Fund Factor	Capital Recovery Factor	Compound Amount Factor	Present Worth Factor	Gradient Uniform Series	Gradient Present Worth
	Find *F* Given *P*	Find *P* Given *F*	Find *A* Given *F*	Find *A* Given *P*	Find *F* Given *A*	Find *P* Given *A*	Find *A* Given *G*	Find *P* Given *G*
n	*F/P*	*P/F*	*A/F*	*A/P*	*F/A*	*P/A*	*A/G*	*P/G*
1	1.120	.8929	1.0000	1.1200	1.000	0.893	0	0
2	1.254	.7972	.4717	.5917	2.120	1.690	0.472	0.797
3	1.405	.7118	.2963	.4163	3.374	2.402	0.925	2.221
4	1.574	.6355	.2092	.3292	4.779	3.037	1.359	4.127
5	1.762	.5674	.1574	.2774	6.353	3.605	1.775	6.397
6	1.974	.5066	.1232	.2432	8.115	4.111	2.172	8.930
7	2.211	.4523	.0991	.2191	10.089	4.564	2.551	11.644
8	2.476	.4039	.0813	.2013	12.300	4.968	2.913	14.471
9	2.773	.3606	.0677	.1877	14.776	5.328	3.257	17.356
10	3.106	.3220	.0570	.1770	17.549	5.650	3.585	20.254
11	3.479	.2875	.0484	.1684	20.655	5.938	3.895	23.129
12	3.896	.2567	.0414	.1614	24.133	6.194	4.190	25.952
13	4.363	.2292	.0357	.1557	28.029	6.424	4.468	28.702
14	4.887	.2046	.0309	.1509	32.393	6.628	4.732	31.362
15	5.474	.1827	.0268	.1468	37.280	6.811	4.980	33.920
16	6.130	.1631	.0234	.1434	42.753	6.974	5.215	36.367
17	6.866	.1456	.0205	.1405	48.884	7.120	5.435	38.697
18	7.690	.1300	.0179	.1379	55.750	7.250	5.643	40.908
19	8.613	.1161	.0158	.1358	63.440	7.366	5.838	42.998
20	9.646	.1037	.0139	.1339	72.052	7.469	6.020	44.968
21	10.804	.0926	.0122	.1322	81.699	7.562	6.191	46.819
22	12.100	.0826	.0108	.1308	92.503	7.645	6.351	48.554
23	13.552	.0738	.00956	.1296	104.603	7.718	6.501	50.178
24	15.179	.0659	.00846	.1285	118.155	7.784	6.641	51.693
25	17.000	.0588	.00750	.1275	133.334	7.843	6.771	53.105
26	19.040	.0525	.00665	.1267	150.334	7.896	6.892	54.418
27	21.325	.0469	.00590	.1259	169.374	7.943	7.005	55.637
28	23.884	.0419	.00524	.1252	190.699	7.984	7.110	56.767
29	26.750	.0374	.00466	.1247	214.583	8.022	7.207	57.814
30	29.960	.0334	.00414	.1241	241.333	8.055	7.297	58.782
31	33.555	.0298	.00369	.1237	271.293	8.085	7.381	59.676
32	37.582	.0266	.00328	.1233	304.848	8.112	7.459	60.501
33	42.092	.0238	.00292	.1229	342.429	8.135	7.530	61.261
34	47.143	.0212	.00260	.1226	384.521	8.157	7.596	61.961
35	52.800	.0189	.00232	.1223	431.663	8.176	7.658	62.605
40	93.051	.0107	.00130	.1213	767.091	8.244	7.899	65.116
45	163.988	.00610	.00074	.1207	1 358.2	8.283	8.057	66.734
50	289.002	.00346	.00042	.1204	2 400.0	8.304	8.160	67.762
55	509.321	.00196	.00024	.1202	4 236.0	8.317	8.225	68.408
60	897.597	.00111	.00013	.1201	7 471.6	8.324	8.266	68.810
65	1 581.9	.00063	.00008	.1201	13 173.9	8.328	8.292	69.058
70	2 787.8	.00036	.00004	.1200	23 223.3	8.330	8.308	69.210
75	4 913.1	.00020	.00002	.1200	40 933.8	8.332	8.318	69.303
80	8 658.5	.00012	.00001	.1200	72 145.7	8.332	8.324	69.359
85	15 259.2	.00007	.00001	.1200	127 151.7	8.333	8.328	69.393