srivenkateshwaraa
College of Engineering & Technology
ASPIRE TO EXCEL
Ariyur, Puducherry-605 102.

NAAC A

ISO 21001

DEPARTMENT OF SCIENCE AND HUMANITIES

Course File

SECOND SEMESTER

Batch (2023-2027)

Subject Code/Name: CSBS202 / MATHEMATICS - II

DEPARTMENT: CSE (A)

Submitted By

Dr. C. JOHN SUNDAR

ASSISTANT PROFESSOR/MATHEMATICS

srivenkateshwaraa
College of Engineering & Technology
ASPIRE TO EXCEL
Ariyur, Puducherry-605 102.

NAAC A

ISO 21001

DEPARTMENT OF SCIENCE AND HUMANITIES

COURSE FILE

STAFF NAME	:	Dr. C. JOHN SUNDAR
DESIGNATION	:	AP/MATHEMATICS
DEPT. HANDLED	:	CSE (A)
YEAR/SEM	:	I/II
SUBJECT CODE	:	CSBS202
SUBJECT	:	MATHEMATICS-II
DATE OF SUBMISSION	:	

Faculty In-charge HOD DEAN PRINCIPAL

CONTENT

2023-2024 EVEN SEM

- TIME TABLE
- SYLLABUS
- LESSON PLAN
- NAME LIST
- UNIT WISE QUESTION BANK
- CIA-I, CIA-II, MODEL QUESTION PAPERS
- ANSWER SCRIPTS
- MARK LIST WITH RESULT ANALYSIS
- CURRENT YEAR QUESTION PAPER
- ASSIGNMENT

srivenkateshwaraa
College of Engineering & Technology
ASPIRE TO EXCEL
Ariyur, Puducherry-605 102.
NAAC A
ISO 21001

DEPARTMENT OF SCIENCE AND HUMANITIES

INDIVIDUAL TIME TABLE

STAFF NAME: Dr. C. JOHN SUNDAR

DESIGNATION: AP/MATHEMATICS

TIME TABLE (2023-2024)

PERIOD /DAY	1	2	3	4	5	6	7	8
MONDAY				M-II	M-II			
TUESDAY		M-II						
WEDNESDAY	M-II							
THRUSDAY				M-II				
FRIDAY			M-II					

SUBJECT CODE	SUBJECT NAME	DEPT/YEAR	T/P	NO. OF PERIODS
CSBS202	MATHEMATICS-1	CSE/I	T	6

Head of the Department
Sciences & Humanities
Sri Venkateshwaraa College of Engineering
& Technology,
Ariyur, Puducherry - 605 102.

CSBS202 MATHEMATICS – II

UNIT I (12 Hrs) PARTIAL DIFFERENTIAL EQUATIONS

Formation of partial differential equations, Solutions of standard types of first order partial differential equations, Lagrange s linear equation, Linear partial differential equations of second and higher order with constant coefficients of both homogeneous and non-homogeneous types.

UNIT II (12 Hrs) LAPLACE TRANSFORM

Existence conditions, Transforms of elementary functions , Properties, Transform of unit step function and unit impulse function , Transforms of derivatives and integrals , Transforms of Periodic Functions , Initial and final value theorems.

UNIT III (12 Hrs) INVERSE LAPLACE TRANSFORM

Inverse Laplace Transforms Properties, Convolution theorem, Application - Solution of ordinary differential equations with constant coefficients - Solution of simultaneous ordinary differential equations.

UNIT IV (12 Hrs) FOURIER TRANSFORM

Fourier Integral theorem (statement only) , Fourier transform and its inverse, Properties: Fourier sine and cosine transforms, Properties, Convolution and Parseval's identity.

UNIT V (12 Hrs) FOURIER SERIES

Dirichlet s conditions , Expansion of periodic functions into Fourier series- Change of interval, Half-range Fourier series, Root mean square value - Parseval's theorem on Fourier coefficients, Harmonic analysis.

Text Books:

1. Grewal B.S, "Higher Engineering Mathematics", Khanna Publishers, New Delhi, 43rd Edition, 2015.
2. Veerarajan T, "Transforms and Partial Differential Equations", Tata McGraw-Hill, New Delhi, 2012.

Faculty In-charge HOD DEAN PRINCIPAL

srivenkateshwaraa
College of Engineering & Technology
ASPIRE TO EXCEL
Ariyur, Puducherry-605 102.

A NAAC

ISO 21001

DEPARTMENT OF SCIENCE AND HUMANITIES

STAFF NAME : **Dr. C. JOHN SUNDAR** **DEPT. HANDLE** : **CSE(SEC A)**

DESIGNATION : **ASST. PROFESSOR** **YEAR/SEM** : **I / II**

SUBJECT : **MATHEMATICS-II** **SUBJECT CODE** : **CSBS202**

No of periods required : 60

Sl. No	Unit Description	Duration		Test / Exam	
		From	To		
1	UNIT II : LAPLACE TRANSFORM	13.03.2024	25.03.2024	CIA – I 24.04.2024 to 27.04.2024	Model Exam 11.06.2024 to 15.06.2024
2	UNIT III: INVERSE LAPLACE TRANSFORM	16.04.2024	10.04.2024		
3	UNIT IV: FOURIER TRANSFORM	9.04.2024	18.04.2024	CIA – II 22.05.2024 to 25.05.2024	
4	UNIT V: FOURIER SERIES	22.04.2024	10.05.2024		
5	UNIT I: PARTIAL DIFFERENTIAL EQUATIONS	10.05.2024	27.05.2024		

Faculty In-charge HOD DEAN PRINCIPAL

LESSON PLAN

SL.NO	Topic	Teaching Method	Location	No. Of Hours	Date (Sec:A)	Faculty sign
UNIT II : LAPLACE TRANSFORM						
1	Existence conditions	BOARD	CLASS	1	13.03.2024	
2	Transforms of elementary function	BOARD	CLASS	1	14.03.2024	
3	Transforms of elementary function	BOARD	CLASS	1	15.03.2024	
4	Properties	BOARD	CLASS	1	15.03.2024	
5	Properties	BOARD	CLASS	1	18.03.2024	
6	Unit step function	BOARD	CLASS	1	18.03.2024	
7	Unit step function	BOARD	CLASS	1	19.03.2024	
8	Derivatives and integral	BOARD	CLASS	1	19.03.2024	
9	Derivatives and integral	BOARD	CLASS	1	20.03.2024	
10	Periodic function	BOARD	CLASS	1	20.03.2024	
11	I.V.T and F.V.T.	BOARD	CLASS	1	22.03.2024	
12	I.V.T and F.V.T.	BOARD	CLASS	1	25.03.2024	
UNIT III: INVERSE LAPLACE TRANSFORM						
1	Inverse Laplace transform	BOARD	CLASS	1	26.03.2024	
2	Inverse Laplace transform	BOARD	CLASS	1	27.03.2024	
3	Convolution theorem	BOARD	CLASS	1	28.03.2024	
4	Convolution theorem	BOARD	CLASS	1	01.04.2024	
5	Convolution theorem	BOARD	CLASS	1	01.04.2024	
6	Convolution theorem	BOARD	CLASS	1	02.04.2024	
7	Solution of differential equation	BOARD	CLASS	1	02.04.2024	

2	Partial derivatives of first order and higher order	BOARD	CLASS	1	22.4.2024	
3	Partial derivatives of first order and higher order	BOARD	CLASS	1	23.4.2024	
4	Partial differentiation of implicit functions	BOARD	CLASS	1	23.4.2024	
5	Euler's theorem on homogeneous functions	BOARD	CLASS	1	24.4.2024	
6	Total derivative	BOARD	CLASS	1	25.4.2024	
7	Jacobian Properties	BOARD	CLASS	1	26.4.2024	
8	Jacobian Properties	BOARD	CLASS	1	26.4.2024	
9	Taylor's series	BOARD	CLASS	1	7.5.2024	
10	Taylor's series	BOARD	CLASS	1	7.5.2024	
11	Maxima and minima of functions of two variables	BOARD	CLASS	1	8.5.2024	
12	Maxima and minima of functions of two variables	BOARD	CLASS	1	10.5.2024	
UNIT I: PARTIAL DIFFERENTIAL EQUATIONS						
1	Formation of Partial differential equation	BOARD	CLASS	1	10.5.2024	
2	Formation of Partial differential equation	BOARD	CLASS	1	12.5.2024	
3	Formation of Partial differential equation	BOARD	CLASS	1	15.5.2024	
4	Standard types first order of pde	BOARD	CLASS	1	16.5.2024	
5	Standard types first order of pde	BOARD	CLASS	1	17.5.2024	

srivenkateshwaraa
College of Engineering & Technology
ASPIRE TO EXCEL
Ariyur, Puducherry-605 102.

NAAC A

ISO 21001

NAME LIST

S.NO	REG.NO	NAME OF THE STUDENT
1	23TD0651	AARTHI .A
2	23TD0654	ABINAYA . V
3	23TD0657	ANISHA . V
4	23TD0658	ARISHKUMAR . K
5	23TD0659	ASHWIN . C
6	23TD0660	ASWIN T
7	23TD0666	DEVASRI . S
8	23TD0667	DHANUSH . V
9	23TD0668	DHARSHANI . S
10	23TD0670	DULASI KRISHNA . P
11	23TD0674	GOKUL . S
12	23TD0680	HEMACHANDRAN . G
13	23TD0681	JANITHAA K R
14	23TD0683	JAYA SRINIVASAN . A
15	23TD0684	JEEVARAAJAN S
16	23TD0685	JEEVITHA . E
17	23TD0686	KALAIVANAN . L
18	23TD0687	KAMARAJ . M
19	23TD0688	KARTHIKEYAN . R
20	23TD0690	KIROUBAKARAN . V
21	23TD0692	MADHAN . R
22	23TD0694	MAHESH V
23	23TD0696	MOHAMED ASIF . M
24	23TD0697	MOHAMED IBRAHIM
25	23TD0699	MOHAMMED AAQIL . M
26	23TD0703	NIKILESHYOGAN . G
27	23TD0706	PAVITHRA . K
28	23TD0707	PRAKASHRAJ . S
29	23TD0710	RAMA . S
30	23TD0711	RANI .P
31	23TD0712	RANJITH . S
32	23TD0715	SABREEN . S
33	23TD0716	SANIYASRI . J
34	23TD0719	SATHYA J
35	23TD0721	SHALINI . C
36	23TD0722	SHARAN SHANTH . R
37	23TD0723	SIVARANJANI K
38	23TD0725	SOWKANTHINI . Y
39	23TD0727	SRIVARDHINI .D
40	23TD0731	VIJAYAVEL .R. S
41	23TD0733	VITHYASAKAR S
42	23TD0735	YUVASREE . G

Head of the Department
Sciences & Humanities
Sri Venkateshwaraa College of Engineering
& Technology,
Ariyur, Puducherry - 605 102.

CHAPTER 3

Laplace Transforms

3.1 Introduction

In the elementary calculus, the transforms are differentiation and then integration. This means that, naturally the operations of transform is one function into another function. For example, the function $f(x) = x^2$ is transformed into a linear function or a family of cubic polynomials by the operations of differentiation or the integration respectively.

In Mathematics, Laplace transform is a powerful technique to obtain the solutions directly from the linear differential equations (both ordinary and partial), without finding the general solution then putting particular values. The Laplace transform has many important applications in probability theory, electrical engineering, control engineering, signal processing and so on.

3.1.1 Existence of Laplace Transforms (Nov.'10, Jan.'15)

Let $f(t)$ be a function of t defined for $t \geq 0$, then the Laplace transforms of $f(t)$, denoted by $L[f(t)]$ is defined by

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\ dt = F(s)$$

provided the integral exists, s is a parameter which may be a real or complex. Where $L[f(t)]$ is clearly a function of s and the symbol L, transforms $f(t)$ into $F(s)$, is called Laplace transformation operator.

Note: 1. A function $f(t)$ is said to be a piece wise (or) sectionally continuous in a finite interval $[a, b]$, if the interval can be divided into a finite number of subintervals such that in each $f(t)$ is continuous.

2. A function $f(t)$ is said to be of the ***exponential order*** if $\lim_{t\to\infty} e^{-at}f(t) =$ a finite quantity.

Existence Conditions of Laplace Transforms

If $f(t)$ be the function defined on $t \geq 0$ is,

1. a piecewise continuous in every finite interval in the range $t \geq 0$.
2. of the exponential order [i.e., $\lim_{t\to\infty} e^{-at}f(t)$ is finite] then $L[f(t)]$ exists.

Note **:** (1) The above conditions are sufficient and not necessary. For example, $L[1/\sqrt{t}]$ exists, even though $1/\sqrt{t}$ is infinite at $t = 0$.

(2) Laplace transforms of all functions do not exist. For example $L[e^{t^2}]$ does not exist, because $f(t) = e^{t^2}$ is not of exponential order.

3.2 Laplace Transforms of Elementary Functions

1. $L[k] = \dfrac{k}{s}$; if $s > 0$

Proof:

$$L[k] = \int_0^\infty e^{-st}.k\ dt = k\left[\frac{e^{-st}}{-s}\right]_0^\infty = k\left[0 - \frac{1}{-s}\right] = \frac{k}{s}.$$

2. $L[t] = \dfrac{1}{s^2}$; if $s > 0$

Proof:

$$L[t] = \int_0^\infty e^{-st}t\ dt = \left[\frac{e^{-st}}{-s}t - 1.\frac{e^{-st}}{s^2}\right]_0^\infty = \left[0 - \left(0 - \frac{1}{s^2}\right)\right] = \frac{1}{s^2}.$$

3. $L[t^n] = \frac{n!}{s^{n+1}}$; when $n = 0, 1, 2, 3,$ ... and $s > 0$.

Proof:

$$\begin{aligned} L[t^n] &= \int_0^\infty e^{-st}.t^n\ dt = \int_0^\infty e^{-u}.\left(\frac{u}{s}\right)^n \frac{du}{s} \quad \text{[putting } st = u\text{]} \\ &= \frac{1}{s^{n+1}}\int_0^\infty e^{-u}.u^n\ du = \frac{\Gamma(n+1)}{s^{n+1}}. \end{aligned}$$

4. $L[e^{at}] = \dfrac{1}{s-a}$, if $s - a > 0$.

Proof:

$$\begin{aligned} L[e^{at}] &= \int_0^\infty e^{-st}.e^{at}\ dt = \int_0^\infty e^{-(s-a)t}\ dt \\ &= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = \frac{1}{s-a}. \qquad [\because\ e^{-\infty} = 0] \end{aligned}$$

5. $L[e^{-at}] = \dfrac{1}{s+a}$, if $s + a > 0$. **(Nov.'11)**

Proof:

$$\begin{aligned} L[e^{at}] &= \int_0^\infty e^{-st}.e^{-at}\ dt = \int_0^\infty e^{-(s+a)t}\ dt \\ &= \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty = \frac{1}{s+a}. \end{aligned}$$

6. $L[\sin at] = \dfrac{a}{s^2+a^2}$, if $s > 0$.

Proof:

$$\begin{aligned} L[\sin at] &= \int_0^\infty e^{-st}.\sin at\ dt \\ &= \left[\frac{e^{-st}}{s^2+a^2}(-s\sin at - a.\cos at)\right]_0^\infty = \frac{a}{s^2+a^2}. \end{aligned}$$

7. $L[\cos at] = \frac{s}{s^2+a^2}$, if $s > 0$.

Proof:

$$\begin{aligned} L[\cos at] &= \int_0^\infty e^{-st}.\cos at\ dt \\ &= \left[\frac{e^{-st}}{s^2+a^2}(-s\cos at + a.\sin at)\right]_0^\infty \\ &= \frac{s}{s^2+a^2}. \end{aligned}$$

Aliter :

$$\begin{aligned} L[\cos at + i\sin at] &= L[e^{iat}] = \frac{1}{s-ia}, \qquad \text{[using result (3)]} \\ &= \frac{s+ia}{(s-ia)(s+ia)} \\ &= \frac{s+ia}{s^2+a^2} \\ \therefore\ L[\cos at] + iL[\sin at] &= \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2} \end{aligned}$$

Equating real and imaginary parts then we get

$$\text{R.P.: } L[\cos at] = \frac{s}{s^2+a^2}, \qquad \text{I.P.: } L[\sin at] = \frac{a}{s^2+a^2}.$$

8. $L[\sinh at] = \dfrac{a}{s^2-a^2}$, if $s > |a|$.

Proof:

$$\begin{aligned} L[\sinh at] &= L\left[\frac{e^{at}-e^{-at}}{2}\right] = \frac{1}{2}\left[L(e^{at}) - L(e^{-at})\right] \\ &= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{a}{s^2-a^2}. \end{aligned}$$

9. $L[\cosh at] = \dfrac{s}{s^2-a^2}$, if $s > |a|$.

Proof:

$$\begin{aligned} L[\cosh at] &= L\left[\frac{e^{at}+e^{-at}}{2}\right] = \frac{1}{2}\left[L(e^{at}) + L(e^{-at})\right] \\ &= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \frac{s}{s^2-a^2}. \end{aligned}$$

Simple Problems on Laplace Transforms

Example 3.2.1. Find Laplace transforms of following functions

(i) $3e^{2t}$ (ii) $e^{-(3/2)t}$ (iii) $\frac{t^4}{12}$ (iv) $t^{3/2t}$ (v) $2\sin\frac{3}{2}t$ (vi) $3\cos 2t$ (vii) $4\sinh\frac{t}{2}$ (viii) $\cosh\frac{2}{3}t$.

Solution .

$$\text{(i)} \quad L[3e^{2t}] = 3L[e^{2t}] = \frac{3}{s-2}.$$

$$\text{(ii)} \quad L[e^{-(3/2)t}] = \frac{1}{s+\frac{3}{2}} = \frac{2}{2s+3}.$$

$$\text{(iii)} \quad L\left[\frac{t^4}{12}\right] = \frac{1}{12}L\left[t^4\right] = \frac{1}{12}.\frac{4!}{s^{4+1}} = \frac{1}{s^5}$$

$$\text{(iv)} \quad L[t^{3/2}] = \frac{\Gamma(3/2+1)}{s^{(3/2)+1}} = \frac{\frac{3}{2}.\frac{1}{2}\Gamma(1/2)}{s^{5/2}} = \frac{3\sqrt{\pi}}{4s^{5/2}}.$$

$$\text{(v)} \quad L\left[2\sin\frac{3}{2}t\right] = 2\left[\frac{\frac{3}{2}}{s^2+\left(\frac{3}{2}\right)^2}\right] = \frac{3}{s^2+\frac{9}{4}} = \frac{12}{4s^2+9}.$$

$$\text{(vi)} \quad L\left[3\cos 2t\right] = 3L\left[\cos 2t\right] = \frac{3s}{s^2+4}.$$

$$\text{(vii)} \quad L\left[4\sinh\frac{t}{2}\right] = 4\left[\frac{\frac{1}{2}}{s^2-\left(\frac{1}{2}\right)^2}\right] = \frac{1}{2(4s^2-1)}.$$

$$\text{(viii)} \quad L\left[\cosh\frac{2}{3}t\right] = \frac{s}{s^2-\left(\frac{2}{3}\right)^2} = \frac{9s}{9s^2-4}.$$

Example 3.2.2. Find the Laplace transforms of $f(t)$ defined as

$$\text{(i) } f(t) = \begin{cases} e^{-t}, & 0 < t < 4; \\ 0, & t > 4. \end{cases} \quad \text{(ii) } f(t) = \begin{cases} t, & \text{for } 0 \le t \le 4; \\ 5, & \text{for } t \ge 4. \end{cases}$$

$$\text{(iii) } f(t) = \begin{cases} t/\tau, & \text{if } 0 < t < \tau; \\ 1, & \text{if } t > \tau. \end{cases} \quad \text{(iv) } f(t) = \begin{cases} 1, & 0 < t \le 1; \\ t, & 1 < t \le 2; \\ 0, & t > 2. \end{cases}$$

Solution . (i) Given $f(t) = \begin{cases} e^{-t}, & 0 < t < 4; \\ 0, & t > 4. \end{cases}$ **(Apr'11)**

$$\begin{aligned} L[f(t)] &= \int_0^\infty e^{-st} f(t)\, dt \qquad \text{[by the definition]} \\ &= \int_0^4 e^{-st}.e^{-t}\, dt + \int_4^\infty e^{-st}.0\, dt \\ &= \left[\frac{e^{-(s+1)t}}{(s+1)}\right]_0^4 \\ &= \frac{e^{-4(s+1)} - 1}{(s+1)}. \end{aligned}$$

(ii) Given $f(t) = \begin{cases} t, & \text{for } 0 \le t \le 4; \\ 5, & \text{for } t \ge 4. \end{cases}$ **(Apr.'06)**

$$\begin{aligned} L[f(t)] &= \int_0^4 e^{-st}.t\, dt + \int_4^\infty e^{-st}.5\, dt \\ &= \left[\frac{te^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_0^4 + 5\left[\frac{e^{-st}}{-s}\right]_4^\infty \\ &= \left[\frac{4e^{-4s}}{-s} - \frac{e^{-4s}}{s^2} + \frac{1}{s^2}\right] + 5\left[0 - \frac{e^{-4s}}{-s}\right] \\ &= \frac{e^{-4s}}{s} + \frac{1 - e^{-4s}}{s^2} \\ &= \frac{1 + e^{-4s}(s-1)}{s^2} \end{aligned}$$

(iii) Given $f(t) = \begin{cases} t/\tau, & \text{if } 0 < t < \tau; \\ 1, & \text{if } t > \tau. \end{cases}$

$$\begin{aligned} L[f(t)] &= \int_0^\tau e^{-st}.\frac{t}{\tau}\, dt + \int_\tau^\infty e^{-st}.1\, dt \\ &= \frac{1}{\tau}\left[\frac{te^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_0^\tau + \left[\frac{e^{-st}}{-s}\right]_\tau^\infty \\ &= \frac{1}{\tau}\left[\frac{\tau e^{-s\tau}}{-s} - \frac{e^{-s\tau}}{s^2} + \frac{1}{s^2}\right] + \left[0 - \frac{e^{-s\tau}}{-s}\right] \\ &= \frac{1 - e^{-s\tau}}{\tau s^2}. \end{aligned}$$

(iv) Given $f(t) = \begin{cases} 1, & 0 < t \le 1; \\ t, & 1 < t \le 2; \\ 0, & t > 2. \end{cases}$

$$\begin{aligned} L[f(t)] &= \int_0^1 e^{-st}.1\ dt + \int_1^2 e^{-st}.t\ dt + \int_2^\infty e^{-st}.0\ dt \\ &= \left[\frac{e^{-st}}{-s}\right]_0^1 + \left[\frac{te^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_1^2 + 0 \\ &= \frac{1-e^{-s}}{s} + \left[\frac{2e^{-2s}}{-s} - \frac{e^{-2s}}{s^2} - \left(\frac{e^{-s}}{-s} - \frac{e^{-s}}{s^2}\right)\right] \\ &= \frac{1}{s} - \frac{2e^{-2s}}{s} + \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2}. \end{aligned}$$

Exercise

Find Laplace transform of the following functions

(i) $f(t) = \begin{cases} e^t, & 0 < t < 1; \\ 0, & t > 1. \end{cases}$

(ii) $f(t) = \begin{cases} t & 0 \le t < 1; \\ 2-t & 1 \le t < 2; \\ 0, & t \ge 2. \end{cases}$

(iii) $f(t) = \begin{cases} \cos t & 0 \le t < \pi; \\ \sin t & \pi \le t < 2\pi. \end{cases}$

(iv) $f(t) = \begin{cases} \sin t, & 0 < t \le \pi; \\ 0, & t > \pi. \end{cases}$

Answers

(i) $\dfrac{1}{1-s}\left[1 - e^{1-s}\right]$

(ii) $\dfrac{1}{s^2}(1 - 2e^{-s} + e^{-2s})$

(iii) $\dfrac{1}{s^2+1}\left[s + e^{-s\pi}(s-1)\right]$

(iv) $\dfrac{2}{s^2+4}(1 - e^{-\pi s})$

3.3 Laplace Transforms of Standard Functions

3.3.1 Unit Step Function (or) Heaviside's Unit Function:

In engineering problems, one frequently encounters functions that are either "on" or "off". For example, an external force acting on a mechanical system or a voltage impressed on a circuit can be turned off after a period of time. Then it is convenient to define a special function that is the number 0 (off) up to a certain time $t = a$ and then the number 1 (on) after that time. This function is called the unit step function or Heaviside function.

Example 3.3.1. Find Laplace transform of unit step function.

(Apr.'11, Apr.'15)

The unit step function denoted by $u(t-a)$ or $u_a(t)$ is defined as

$$u(t-a) = \begin{cases} 0, & \text{for } t < a; \\ 1, & \text{for } t \geq a. \end{cases} \quad \text{Where } a \text{ is always positive.}$$

The Laplace transform of unit step function is

$$\begin{aligned} L\left[u(t-a)\right] &= \int_0^\infty e^{-st} u(t-a)\, dt \\ &= \int_0^a e^{-st}.0\, dt + \int_a^\infty e^{-st}.1\, dt \\ &= 0 + \left[\frac{e^{-st}}{-s}\right]_a^\infty = \frac{e^{-as}}{s}. \end{aligned}$$

3.3.2 Dirac Delta Function (or) Unit Impulse Function

Mechanical systems are often acted on by an external force (or emf in an electrical circuit) of large magnitude that acts only for a very short period of time. For example, in the study of bending beams, we have point loads which is equivalent to large pressure acting over a very small area. To deal with such and similar ideas, we introduce the unit impulse function also called Dirac delta function.

Example 3.3.2. Find Laplace transform of unit impulse function. (**Nov.'05**)

Thus the unit impulse function is considered as the limiting form of the function $\delta_\epsilon(t-a) = \begin{cases} 1/\epsilon, & a < t < a+\epsilon; \\ 0, & \text{otherwise.} \end{cases}$

This can also be represented interms of two unit step functions as

$$\delta_\epsilon(t-a) = \frac{1}{\epsilon}\left\{u(t-a) - u[t-(a+\epsilon)]\right\} \qquad ...(1)$$

Note that, $\displaystyle\int_0^\infty \delta_\epsilon(t-a) = \int_0^a 0.dt + \int_a^{a+\epsilon} \frac{1}{\epsilon}\, dt + \int_{a+\epsilon}^\infty 0.dt = 1.$

Taking Laplace transforms on both sides of (1), we get

$$\begin{aligned} L[\delta_\epsilon(t-a)] &= \frac{1}{\epsilon}\, L\left\{u(t-a) - u[t-(a+\epsilon)]\right\} \\ &= \frac{1}{\epsilon}\left[\frac{e^{-as}}{s} - \frac{e^{-(a+\epsilon)s}}{s}\right] \\ &= e^{-as}\left[\frac{1-e^{-\epsilon s}}{\epsilon s}\right] \qquad ...(2) \end{aligned}$$

The Dirac delta function denoted by $\delta(t-a)$ is defined as

$$\delta(t-a) = \lim_{\epsilon\to 0} \delta_\epsilon(t-a).$$

Taking Laplace transform on both sides, we get

$$\begin{aligned} L[\delta(t-a)] &= \lim_{\epsilon\to 0} L[\delta_\epsilon(t-a)] \quad \text{[using (2)]} \\ &= e^{-as} \lim_{\epsilon\to 0}\left[\frac{1-e^{-\epsilon s}}{\epsilon s}\right] \\ &= e^{-as.} \end{aligned}$$

Thus, in general the limiting form of $\delta_\epsilon(t-a)$ as $\epsilon \to 0$ is expressed as unit impulse function denoted by $\delta(t-a)$ is defined as follows

$$\delta(t-a) = \begin{cases} \infty, & \text{for } t = a; \\ 0, & \text{for } t \neq a. \end{cases} \quad \text{such that} \quad \int_{-\infty}^{\infty} \delta(t-a)\; dt = 1.$$

3.4 Properties of Laplace Transforms

3.4.1 Linear Property

If c_1 and c_2 be any two constants and f and g any two functions of t with Laplace transforms $F(s)$ and $G(s)$ respectively, then we have

$$L\left[c_1 f(t) \pm c_2 g(t)\right] = c_1 L[f(t)] \pm c_2 L[g(t)] = c_1 F(s) + c_2 G(s)$$

Proof:

$$\begin{aligned} L\left[c_1 f(t) \pm c_2 g(t)\right] &= \int_0^\infty e^{-st}\left[c_1 f(t) \ \pm \ c_2 g(t)\right] \, dt \\ &= c_1 \int_0^\infty e^{-st} f(t) \ dt \ \pm \ c_2 \int_0^\infty e^{-st} g(t) \ dt \\ &= c_1 L[f(t)] \ \pm \ c_2 L[g(t)] \\ &= c_1 F(s) \ \pm \ c_2 G(s). \end{aligned}$$

Hence the operator L is a linear.

Problems Based on Linear Property

Example 3.4.1. Find Laplace transforms of the following functions:

(i) $t^3 - 3t^2 + 2$, (ii) $(e^t + e^{-t})^2$, (iii) $\frac{1+2t}{\sqrt{t}}$, (iv) $\sin\sqrt{t}$, (v) $\frac{1}{1-t}$ (vi) $\cos^2 3t$, (vii) $\cos^3 2t$, (viii) $\sin 2t \cos 3t$, (ix) $\sin t \sin 2t$, (x) $\cos(2t+3)$

Solution .

(i)
$$\begin{aligned} L\left[t^3 - 3t^2 + 2\right] &= L\left[t^3\right] - 3L\left[t^2\right] + L[2] \\ &= \frac{3!}{s^4} - 3\,\frac{2!}{s^3} + \frac{2}{s} \\ &= \frac{6}{s^4} - \frac{6}{s^3} + \frac{2}{s}. \end{aligned}$$

(ii)
$$\begin{aligned} L\left[(e^t + e^{-t})^2\right] &= L\left[e^{2t} + 2 + e^{-2t}\right] \\ &= \frac{1}{s-2} + \frac{2}{s} + \frac{1}{s+2}. \end{aligned}$$

(iii)
$$\begin{aligned} L\left[\frac{1+2t}{\sqrt{t}}\right] &= L\left[t^{-1/2}+2t^{1/2}\right] \\ &= \frac{\Gamma(1/2)}{s^{1/2}}+2\frac{\Gamma(3/2)}{s^{3/2}} \qquad \left[\because\ \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}\right] \\ &= \frac{\sqrt{\pi}}{\sqrt{s}}+\frac{2.\frac{1}{2}\sqrt{\pi}}{s\sqrt{s}} \qquad [\because\ \Gamma(n+1)=n\Gamma n] \\ &= \sqrt{\frac{\pi}{s}}\left[1+\frac{1}{s}\right]. \end{aligned}$$

(iv)
$$\begin{aligned} \text{W.k.t. } \sin x &= \frac{x}{1!}-\frac{x^3}{3!}+\frac{x^5}{5!}-\ ... \\ \text{Thus, } \sin\sqrt{t} &= \frac{(\sqrt{t})}{1!}-\frac{(\sqrt{t})^3}{3!}+\frac{(\sqrt{t})^5}{5!}-\ ... \\ &= \frac{t^{1/2}}{1!}-\frac{t^{3/2}}{3!}+\frac{t^{5/2}}{5!}-\ ... \end{aligned}$$

$$\begin{aligned} \therefore\ L[\sin\sqrt{t}] &= \frac{1}{1!}L[t^{1/2}]-\frac{1}{3!}L[t^{3/2}]+\frac{1}{5!}L[t^{5/2}]-\ ... \\ &= \frac{\Gamma(3/2)}{s^{3/2}}-\frac{1}{3!}\frac{\Gamma(5/2)}{s^{5/2}}+\frac{1}{5!}\frac{\Gamma(7/2)}{s^{7/2}}-\ ... \\ &= \frac{\Gamma(3/2)}{s^{3/2}}-\frac{1}{3!}\frac{3}{2}\frac{\Gamma(3/2)}{s^{3/2}.s}+\frac{1}{5!}\frac{5}{2}\frac{3}{2}\frac{\Gamma(3/2)}{s^{3/2}.s^2}-... \\ &= \frac{\Gamma(3/2)}{s^{3/2}}\left[1-\frac{1}{4s}+\frac{1}{4.2.1}.\frac{1}{4s^2}-\ ...\right] \\ &= \frac{1}{2}\frac{\Gamma(1/2)}{s^{3/2}}\left[1-\frac{1}{1!}\left(\frac{1}{4s}\right)+\frac{1}{2!}\left(\frac{1}{4s}\right)^2-...\right] \\ &= \frac{\sqrt{\pi}}{2s^{3/2}}\ e^{-1/4s}. \end{aligned}$$

(v)
$$\begin{aligned} \frac{1}{1-t} &= [1-t]^{-1} = 1+t+t^2+t^3+t^4+\ ... \\ \therefore\ L\left[\frac{1}{1-t}\right] &= L[1]+L[t]+L[t^2]+L[t^3]+L[t^4]+\ ... \\ &= \frac{1}{s}+\frac{1!}{s^2}+\frac{2!}{s^3}+\frac{3!}{s^4}+\ ...\ = \sum_{n=0}^{\infty}\frac{n!}{s^{n+1}}. \end{aligned}$$

(vi) **Jan.'15**

$$\begin{aligned} L[\cos^2 3t] &= L\left[\frac{1+\cos 6t}{2}\right] \\ &= \frac{1}{2}\left[\frac{1}{s}+\frac{s}{s^2+36}\right] \\ &= \frac{s^2+18}{s(s^2+36)}. \end{aligned}$$

(vii)

$$\begin{aligned} L[\cos^3 2t] &= L\left[\frac{3\cos 2t+\cos 3(2t)}{4}\right] \\ &= \frac{1}{4}\left[3L(\cos 2t)+L(\cos 6t)\right] \\ &= \frac{1}{4}\left[\frac{3s}{s^2+4}+\frac{s}{s^2+36}\right] \\ &= \frac{s(s^2+28)}{(s^2+4)(s^2+36)}. \end{aligned}$$

(viii)

$$\begin{aligned} L[\sin 2t\cos 3t] &= \frac{1}{2}L\left[\sin 5t+\sin(-t)\right] \\ &= \frac{1}{2}\left[L(\sin 5t)-L(\sin t)\right] \\ &= \frac{1}{2}\left[\frac{5}{s^2+25}-\frac{1}{s^2+1}\right] \\ &= \frac{2(s^2-5)}{(s^2+25)(s^2+1)}. \end{aligned}$$

(ix)

$$\begin{aligned} L[\sin t\sin 2t] &= \frac{1}{2}L\left[\cos t-\cos 3t\right] \\ &= \frac{1}{2}\left[\frac{s}{s^2+1}-\frac{s}{s^2+9}\right] \\ &= \frac{4s}{(s^2+1)(s^2+9)}. \end{aligned}$$

(x)

$$\begin{aligned} L[\cos(at+b)] &= L\left[\cos at\cos b-\sin at\sin b\right] \\ &= \cos b\ L\left[\cos at\right]-\sin b\ L\left[\sin at\right] \\ &= \cos b\frac{s}{s^2+a^2}-\sin b\frac{s}{s^2+a^2}. \end{aligned}$$

Exercise

Find Laplace transform of the following functions

(i) $2t^3-6t+8$, (ii) $2e^{3t}-e^{-3t}$, (iii) $(t^2+1)^2$ (iv) $4e^{-3t}+5\cos 2t$, (v) $\cosh 3t+2e^{-3t}+\sin 2t$, (vi) $\sin(at+b)$, (vii) $\cosh(5t+2)$, (viii) $\cos^2 4t$, (ix) $\sin^3 2t$, (x) $\cosh^3 2t$, (xi) $\sin 2t\cos 4t$, (xii) $\sin 2t \sin 3t$,

Answers

(i) $\dfrac{12}{s^4}-\dfrac{6}{s^2}+\dfrac{8}{s}$ (ii) $\dfrac{s+9}{s^2-9}$ (iii) $\dfrac{s^4+4s^2+24}{s^5}$ (iv) $\dfrac{4}{s+3}+\dfrac{5s}{s^2+4}$

(v) $\dfrac{s}{s^2-9}+\dfrac{2}{s+3}+\dfrac{2}{s^2+4}$ (vi) $\dfrac{a\cos b+s\sin b}{s^2+a^2}$ (vii) $\dfrac{e^2}{2(s-5)}+\dfrac{e^{-2}}{2(s+5)}$

(viii) $\dfrac{1}{2s}+\dfrac{s}{2(s^2+64)}$ (ix) $\dfrac{48}{(s^2+4)(s^2+36)}$ (x) $\dfrac{s(s^2-28)}{(s^2-4)(s^2-36)}$

(xi) $\dfrac{3}{s^2+36}-\dfrac{1}{s^2+4}$ (xii) $\dfrac{12s}{(s^2+1)(s^2+25)}$

3.4.2 First Shifting Property

If $L[f(t)]=F(s)$ then

$$(i)\ L\left[e^{at}f(t)\right]=F\left(s-a\right) \qquad (ii)\ L\left[e^{-at}f(t)\right]=F\left(s+a\right)$$

Proof:

$$\begin{aligned}
\text{By definition,}\qquad L[f(t)] &= \int_0^\infty e^{-st}f(t)\ dt \ = \ F(s)\\
\therefore\quad L[e^{at}f(t)] &= \int_0^\infty e^{-st}.e^{at}f(t)\ dt\\
&= \int_0^\infty e^{-(s-a)t}f(t)\ dt\\
&= F(s-a).
\end{aligned}$$

Hence proved (i). Similarly to prove (ii) by replacing a by $-a$.

Note: The above property can be rewritten as a working rule for solving this type problems in the following way.

$$\begin{aligned} (i) \quad L[e^{at}f(t)] &= L[f(t)]_{s\to s-a} = [F(s)]_{s\to s-a} = F(s-a). \\ (ii) \quad L[e^{-at}f(t)] &= L[f(t)]_{s\to s+a} = [F(s)]_{s\to s+a} = F(s+a). \end{aligned}$$

Where $s \to (s-a)$ means that s is replaced by $(s-a)$ and $s \to (s+a)$ means that s is replaced by $(s+a)$ in $F(s)$.

Example 3.4.2. Find Laplace transform of the following functions

(i) $e^{at}t$ (ii) $e^{-at}t$ (iii) $e^{-at}t^2$ (iv) $e^{at}t^n$ (v) $e^{at}\cos bt$ (vi) $e^{at}\sin bt$ (vii) $e^{-at}\cos bt$ (viii) $e^{-at}\sin bt$ (ix) $e^{at}\sinh bt$ (x) $e^{at}\cosh bt$ (xi) $e^{-at}\cosh bt$ (xii) $e^{-at}\sinh bt$

Solution .

$$\text{(i)} \quad L\left[e^{at}t\right] = L\left[t\right]_{s\to s-a} = \left[\frac{1}{s^2}\right]_{s\to s-a} = \frac{1}{(s-a)^2}.$$

$$\text{(ii)} \quad L\left[e^{-at}t\right] = L\left[t\right]_{s\to s+a} = \left[\frac{1}{s^2}\right]_{s\to s+a} = \frac{1}{(s+a)^2}.$$

$$\text{(iii)} \quad L\left[e^{-at}t^2\right] = L\left[t^2\right]_{s\to s+a} = \left[\frac{2!}{s^3}\right]_{s\to s+a} = \frac{2}{(s+a)^3}.$$

$$\text{(iv)} \quad L\left[e^{at}t^n\right] = L\left[t^n\right]_{s\to s-a} = \left[\frac{n!}{s^{n+1}}\right]_{s\to s-a} \qquad \textbf{(Apr.'14)}$$

$$= \frac{n!}{(s-a)^{n+1}}.$$

$$\text{(v)} \quad L\left[e^{at}\cos bt\right] = L\left[\cos bt\right]_{s\to s-a} = \left[\frac{s}{s^2+b^2}\right]_{s\to s-a} \qquad \textbf{(Apr.'14)}$$

$$= \frac{s-a}{(s-a)^2+b^2}.$$

$$\text{(vi)} \quad L\left[e^{at}\sin bt\right] = L\left[\sin bt\right]_{s\to s-a} = \left[\frac{b}{s^2+b^2}\right]_{s\to s-a}$$

$$= \frac{b}{(s-a)^2+b^2}.$$

$$\text{(vii)}\quad L\left[e^{-at}\cos bt\right] = L\left[\cos bt\right]_{s\to s+a} = \left[\frac{s}{s^2+b^2}\right]_{s\to s+a} = \frac{s+a}{(s+a)^2+b^2}.$$

$$\text{(viii)}\quad L\left[e^{-at}\sin bt\right] = L\left[\sin bt\right]_{s\to s+a} = \left[\frac{b}{s^2+b^2}\right]_{s\to s+a} = \frac{b}{(s+a)^2+b^2}.$$

$$\text{(ix)}\quad L\left[e^{at}\cosh bt\right] = L\left[\cosh bt\right]_{s\to s-a} = \left[\frac{s}{s^2-b^2}\right]_{s\to s-a} = \frac{s-a}{(s-a)^2-b^2}.$$

$$\text{(x)}\quad L\left[e^{at}\sinh bt\right] = L\left[\sinh bt\right]_{s\to s-a} = \left[\frac{b}{s^2-b^2}\right]_{s\to s-a} = \frac{b}{(s-a)^2-b^2}.$$

$$\text{(xi)}\quad L\left[e^{-at}\cosh bt\right] = L\left[\cosh bt\right]_{s\to s+a} = \left[\frac{s}{s^2-b^2}\right]_{s\to s+a} = \frac{s+a}{(s+a)^2-b^2}.$$

$$\text{(xii)}\quad L\left[e^{-at}\sinh bt\right] = L\left[\sinh bt\right]_{s\to s+a} = \left[\frac{b}{s^2-b^2}\right]_{s\to s+a} = \frac{b}{(s+a)^2-b^2}.$$

Problems Based on First Shifting Property

Example 3.4.3. Find Laplace transforms of the following functions (i) $t^{7/2}e^{3t}$ (ii) $(1-te^{-t})^3$ (iii) $\left(\frac{e^{-t}}{1-t}\right)^2$ (iv) $e^{-t}\sin^2 3t$ (v) $e^{-2t}\cos^3 2t$ (vi) $t^2\sinh 3t$ (vii) $e^{-t}\sin 2t\cos 3t$ (viii) $e^{3t}\sin t\sin 2t$ (ix) $\sinh\frac{t}{2}\sin\frac{\sqrt{3}}{2}t$ (x) $\cosh t\sin 2t$ **(Nov.'14)** (xi) $\sinh t\cos 2t$

Solution .

$$\text{(i)}\quad L[t^{7/2}e^{3t}] = L[t^{7/2}]_{s\to s-3} = \left[\frac{\Gamma\left(\frac{7}{2}+1\right)}{s^{\frac{7}{2}+1}}\right]_{s\to s-3}$$

$$= \left[\frac{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma(\frac{1}{2})}{s^{9/2}}\right]_{s\to s-3} = \frac{105\sqrt{\pi}}{16(s-3)^{9/2}}.$$

(ii)
$$\begin{aligned} L\left[(1-te^{-t})^3\right] &= L\left[1-3te^{-t}+3t^2e^{-2t}-t^3e^{-3t}\right] \\ &= L[1]-3L\left[te^{-t}\right]+3L\left[t^2e^{-2t}\right]-L\left[t^3e^{-3t}\right] \\ &= \frac{1}{s}-3L\left[t\right]_{s\to s+1}+3L\left[t^2\right]_{s\to s+2}-L\left[t^3\right]_{s\to s+3} \\ &= \frac{1}{s}-3\left[\frac{1}{s^2}\right]_{s\to s+1}+3\left[\frac{2!}{s^3}\right]_{s\to s+2}-\left[\frac{3!}{s^4}\right]_{s\to s+3} \\ &= \frac{1}{s}-\frac{3}{(s+1)^2}+\frac{6}{(s+2)^3}-\frac{6}{(s+3)^4}. \end{aligned}$$

(iii)
$$\begin{aligned} \left(\frac{e^{-t}}{1-t}\right)^2 &= e^{-2t}[1-t]^{-2} \\ &= e^{-2t}\left[1+2t+3t^2+4t^3+...\right] \\ \therefore\ L\left[\frac{e^{-t}}{1-t}\right]^2 &= L\left\{e^{-2t}\left[1+2t+3t^2+4t^3+\ ...\right]\right\} \\ &= \left[L(1)+2L(t)+3L(t^2)+4L(t^3)+\ ...\right]_{s\to s+2} \\ &= \left[\frac{1}{s}+2.\frac{1!}{s^2}+3.\frac{2!}{s^3}+4.\frac{3!}{s^4}+\ ...\right]_{s\to s+2} \\ &= \frac{1}{(s+2)}+\frac{2!}{(s+2)^2}+\frac{3!}{(s+2)^3}+... \\ &= \sum_{n=0}^{\infty}\frac{n!}{(s+2)^n}. \end{aligned}$$

(iv)
$$\begin{aligned} L[e^{-t}\sin^2 3t] &= L[\sin^2 3t]_{s\to s+1} = L\left[\frac{1-\cos 6t}{2}\right]_{s\to s+1} \\ &= \frac{1}{2}\left[\frac{1}{s}-\frac{s}{s^2+36}\right]_{s\to s+1} \\ &= \left[\frac{18}{s(s^2+36)}\right]_{s\to s+1} \\ &= \frac{18}{(s+1)[(s+1)^2+36]}. \end{aligned}$$

$$\begin{aligned}
\text{(v)}\quad L[e^{-2t}\cos^3 2t] &= L[\cos^3 2t]_{s\to s+2}\\
&= L\left[\frac{3\cos 2t+\cos 6t}{4}\right]_{s\to s+2}\\
&= \frac{1}{4}\left[\frac{3s}{s^2+4}+\frac{s}{s^2+36}\right]_{s\to s+2}\\
&= \left[\frac{s(s^2+28)}{(s^2+4)(s^2+36)}\right]_{s\to s+2}\\
&= \frac{(s+2)[(s+2)^2+28]}{[(s+2)^2+4][(s+2)^2+36]}.
\end{aligned}$$

$$\begin{aligned}
\text{(vi)}\qquad t^2\sinh 3t &= t^2\left[\frac{e^{3t}-e^{-3t}}{2}\right] \;=\; \frac{1}{2}\left[e^{3t}t^2-e^{-3t}t^2\right]\\
\therefore\; L[t^3\sinh 3t] &= \frac{1}{2}\left[L(e^{3t}t^2)-L(e^{-3t}t^2)\right]\\
&= \frac{1}{2}\left\{L\left[t^2\right]_{s\to s-3}-L\left[t^2\right]_{s\to s+3}\right\}\\
&= \frac{1}{2}\left\{\left[\frac{2!}{s^3}\right]_{s\to s-3}-\left[\frac{2!}{s^3}\right]_{s\to s+3}\right\}\\
&= \frac{1}{(s-3)^3}-\frac{1}{(s+3)^3} \;=\; \frac{18(s^2+3)}{(s^2-9)^3}.
\end{aligned}$$

$$\begin{aligned}
\text{(vii)}\quad L[e^{-t}\sin 2t\cos 3t] &= L[\sin 2t\cos 3t]_{s\to s+1}\\
&= L\left[\frac{1}{2}\left\{\sin 5t+\sin(-t)\right\}\right]_{s\to s+1}\\
&= \frac{1}{2}\left[L(\sin 5t)-L(\sin t)\right]_{s\to s+1}\\
&= \frac{1}{2}\left[\frac{5}{s^2+25}-\frac{1}{s^2+1}\right]_{s\to s+1}\\
&= \left[\frac{2(s^2-5)}{(s^2+25)(s^2+1)}\right]_{s\to s+1}\\
&= \frac{2[(s+1)^2-5]}{[(s+1)^2+25][(s+1)^2+1]}.
\end{aligned}$$

(viii)
$$\begin{aligned} L[e^{3t}\sin t\sin 2t] &= L[\sin t\sin 2t]_{s\to s-3}\\ &= \frac{1}{2}L\left[\cos t-\cos 3t\right]_{s\to s-3}\\ &= \frac{1}{2}\left[\frac{s}{s^2+1}-\frac{s}{s^2+9}\right]_{s\to s-3}\\ &= \left[\frac{4s}{(s^2+1)(s^2+9)}\right]_{s\to s-3}\\ &= \frac{4(s-3)}{[(s-3)^2+1][(s-3)^2+9]}. \end{aligned}$$

(ix)
$$\begin{aligned} \sinh\frac{t}{2}\sin\frac{\sqrt{3}}{2}\,t &= \frac{1}{2}\left[e^{t/2}-e^{-t/2}\right]\sin\frac{\sqrt{3}}{2}\,t\\ &= \frac{1}{2}\left[e^{t/2}\sin\frac{\sqrt{3}}{2}\,t-e^{-t/2}\sin\frac{\sqrt{3}}{2}\,t\right] \end{aligned}$$

$$\therefore\ L\left[\sinh\frac{t}{2}\sin\frac{\sqrt{3}}{2}\,t\right]$$

$$\begin{aligned} &= \frac{1}{2}\left\{L\left[e^{t/2}\sin\frac{\sqrt{3}}{2}\,t\right]-L\left[e^{-t/2}\sin\frac{\sqrt{3}}{2}\,t\right]\right\}\\ &= \frac{1}{2}\left\{L\left[\sin\frac{\sqrt{3}}{2}\,t\right]_{s\to s-\frac{1}{2}}-L\left[\sin\frac{\sqrt{3}}{2}\,t\right]_{s\to s+\frac{1}{2}}\right\}\\ &= \frac{1}{2}\left\{\left[\frac{\frac{\sqrt{3}}{2}}{s^2+\frac{3}{4}}\right]_{s\to s-\frac{1}{2}}-\left[\frac{\frac{\sqrt{3}}{2}}{s^2+\frac{3}{4}}\right]_{s\to s+\frac{1}{2}}\right\}\\ &= \frac{\sqrt{3}}{4}\left\{\frac{1}{\left[s-\frac{1}{2}\right]^2+\frac{3}{4}}-\frac{1}{\left[s+\frac{1}{2}\right]^2+\frac{3}{4}}\right\}\\ &= \frac{\sqrt{3}}{4}\left\{\frac{1}{s^2-s+1}-\frac{1}{s^2+s+1}\right\}\\ &= \frac{\sqrt{3}}{4}\left[\frac{2s}{(s^2-s+1)(s^2+s+1)}\right]\\ &= \frac{\sqrt{3}}{2}\left[\frac{s}{s^4+s^2+1}\right]. \end{aligned}$$

$$\begin{aligned}\text{(x)}\quad L[\cosh t\sin 2t] &= L\left[\left(\frac{e^t+e^{-t}}{2}\right)\sin 2t\right] \qquad \textbf{(Nov.'14)}\\ &= \frac{1}{2}\left\{L[e^t\sin 2t]+L[e^{-t}\sin 2t]\right\}\\ &= \frac{1}{2}\left\{L[\sin 2t]_{s\to(s-1)}+L[\sin 2t]_{s\to(s+1)}\right\}\\ &= \frac{1}{2}\left\{\left[\frac{2}{s^2+4}\right]_{s\to(s-1)}+\left[\frac{2}{s^2+4}\right]_{s\to(s+1)}\right\}\\ &= \frac{1}{(s-1)^2+4}+\frac{1}{(s+1)^2+4}.\end{aligned}$$

$$\begin{aligned}\text{(xi)}\quad L[\sinh t\cos 2t] &= L\left[\left(\frac{e^t-e^{-t}}{2}\right)\cos 2t\right]\\ &= \frac{1}{2}\left\{L[e^t\cos 2t]-L[e^{-t}\cos 2t]\right\}\\ &= \frac{1}{2}\left\{L[\cos 2t]_{s\to(s-1)}-L[\cos 2t]_{s\to(s+1)}\right\}\\ &= \frac{1}{2}\left\{\left[\frac{s}{s^2+4}\right]_{s\to(s-1)}+\left[\frac{s}{s^2+4}\right]_{s\to(s+1)}\right\}\\ &= \frac{1}{2}\left[\frac{(s-1)}{(s-1)^2+4}-\frac{(s+1)}{(s+1)^2+4}\right].\end{aligned}$$

Example 3.4.4. If $L[f(t)] = F(s)$, show that

$$\begin{aligned}&(a)\quad L[(\sinh at)f(t)] = \frac{1}{2}[F(s-a)-F(s+a)],\\ &(b)\quad L[(\cosh at)f(t)] = \frac{1}{2}[F(s-a)+F(s+a)]\end{aligned}$$

Hence evaluate (i) $\sinh 2t\sin 3t$, (ii) $\cosh 3t\cos 2t$.

Solution .

$$\begin{aligned}(a)\quad L[(\sinh at)f(t)] &= L\left[\frac{1}{2}(e^{at}-e^{-at})f(t)\right]\\ &= \frac{1}{2}\left[L(e^{at}f(t))-L(e^{-at})f(t)\right]\\ &= \frac{1}{2}\left[F(s-a)-F(s+a)\right].\end{aligned}$$

$$(b)\quad L[(\cosh at)f(t)] = L\left[\frac{1}{2}(e^{at}+e^{-at})f(t)\right]$$
$$= \frac{1}{2}\left[L(e^{at}f(t))+L(e^{-at})f(t)\right]$$
$$= \frac{1}{2}\left[F(s-a)+F(s+a)\right].$$

$$\text{(i)}\quad L[(\sinh 2t)\sin 3t] = L\left[\frac{1}{2}(e^{2t}-e^{-2t})\sin 3t\right]$$
$$= \frac{1}{2}\left[L(e^{2t}\sin 3t)-L(e^{-2t})\sin 3t)\right]$$
$$= \frac{1}{2}\left[\frac{3}{(s-2)^2+9}-\frac{3}{(s+2)^2+9}\right],$$
$$= \frac{12s}{s^4+10s^2+169}.$$

$$\text{(ii)}\quad L[(\cosh 3t)\cos 2t] = L\left[\frac{1}{2}(e^{3t}+e^{-3t})\cos 2t\right]$$
$$= \frac{1}{2}\left[L(e^{3t}\cos 2t)+L(e^{-3t})\cos 2t\right]$$
$$= \frac{1}{2}\left[\frac{s-2}{(s-3)^2+4^2}+\frac{s-2}{(s-3)^2+4^2}\right]$$
$$= \frac{2s(s^2-5)}{s^4-10s^2+169}.$$

Exercise

Find Laplace transform of the following functions

(i) t^3e^{5t} (ii) $e^{-3t}\sin 2t$ (iii) $(t+2)^2e^{-3t}$ (iv) $\cosh t\cos 2t$
(v) $e^{2t}\sin t\cos 2t$ (vi) $e^{-t}[3\sin 2t-5\cosh 2t]$ (vii) $e^{-t}\sin^2 t$

Answers

(i) $\dfrac{6}{(s-5)^4}$ (ii) $\dfrac{2}{s^2+6s+13}$ (iii) $\dfrac{4s^2+30s+50}{(s+3)^2}$

(iv) $\dfrac{1}{2}\dfrac{s-1}{s^2-2s+5}+\dfrac{1}{2}\dfrac{s+1}{s^2+2s+5}$ (v) $\dfrac{s^2-4s+1}{(s^2-4s+5)(s^2-4s+13)}$

(vi) $\dfrac{6}{s^2+2s+5}-\dfrac{5s+5}{s^2+2s-3}$ (vii) $\dfrac{2}{(s+1)(s^2+2s+5)}$

3.4.3 Second Shifting Property

If $L[f(t)] = F(s)$, then $L[f(t-a).u(t-a)] = e^{-as}F(s)$.

Proof:

$$\begin{aligned}
L[f(t-a).u(t-a)] &= \int_0^{\infty} e^{-st} f(t-a)u(t-a)\ dt \\
&= \int_0^{a} e^{-st} f(t-a).(0)\ dt + \int_a^{\infty} e^{-st} f(t-a)\ dt \\
&= \int_0^{\infty} e^{-s(u+a)} f(u)\ du \quad [\text{Put } t-a=u] \\
&= e^{-as}\int_0^{\infty} e^{-su} f(u)\ du \\
&= e^{-as}F(s).
\end{aligned}$$

***Note* :**

W.k.t the unit step function (put $f(t) = 1$) is, $L[u(t-a)] = e^{-as}/s$.

Various discontinuous functions can often be expressed in terms of Heaviside unit step functions as follows:

If $f(t) = \begin{cases} f_1(t), & \text{if } 0 < t < a; \\ f_2(t), & \text{if } t > a. \end{cases}$

Then $f(t)$ can be written as

$$f(t) = f_1(t) + [f_2(t) - f_1(t)]u(t-a).$$

In general, $f(t) = \begin{cases} f_1(t), & \text{if } 0 < t < a_1; \\ f_2(t), & \text{if } a_1 < t < a_2; \\ f_3(t), & \text{if } a_2 < t < a_3; \\ \dots & \\ f_{n-1}(t), & \text{if } a_{n-2} < t < a_{n-1}; \\ f_n(t), & \text{if } t > a_{n-1}. \end{cases}$

Then $f(t)$ can be written as

$$\begin{aligned}
f(t) &= f_1(t) + [f_2(t) - f_1(t)]u(t-a_1) + [f_3(t) - f_2(t)]u(t-a_2) + \\
&\quad .... + [f_n(t) - f_{n-1}(t)]u(t-a_{n-1}).
\end{aligned}$$

3.4.3.1 Problems Based on Second Shifting Property

Example 3.4.5. Find the Laplace transforms of given functions

(i) $f(t) = \begin{cases} \cos\left[t - \frac{2\pi}{3}\right], & t > \frac{2\pi}{3}; \\ 0, & t < \frac{2\pi}{3}. \end{cases}$ (ii) $f(t) = \begin{cases} (t-2)^2, & t > 2; \\ 0, & t < 2. \end{cases}$

Solution . (i) Given $f(t) = \begin{cases} 0, & 0 < t < \frac{2\pi}{3}; \\ \cos\left[t - \frac{2\pi}{3}\right], & t > \frac{2\pi}{3}. \end{cases}$

Let $f_1(t) = 0$, $f_2(t) = \cos\left[t - \frac{2\pi}{3})\right]$ then $f(t)$ is written as

$$\begin{aligned} f(t) &= f_1(t) + [f_2(t) - f_1(t)]u(t-a) \\ &= 0 + \left[\cos\left(t - \frac{2\pi}{3}\right)\right] u\left(t - \frac{2\pi}{3}\right) \\ \therefore\ L[f(t)] &= L\left[\cos\left(t - \frac{2\pi}{3}\right) u\left(t - \frac{2\pi}{3}\right)\right] \\ &\quad \text{by using second shifting theorem,} \\ &= e^{-as}L[\cos t] \quad \text{where } a = \frac{2\pi}{3} \\ &= \frac{se^{-2\pi s/3}}{s^2+1}. \end{aligned}$$

(ii) Given $f(t) = \begin{cases} 0, & \text{if } 0 < t < 2; \\ (t-2)^2, & \text{if } t > 2. \end{cases}$

Let $f_1(t) = 0$, $f_2(t) = (t-2)^2$ then $f(t)$ is written as

$$\begin{aligned} f(t) &= f_1(t) + [f_2(t) - f_1(t)]u(t-a) \\ &= 0 + (t-2)^2 u\,(t-2) \\ \therefore\ L[f(t)] &= L\left[(t-2)^2 u\,(t-2)\right] \\ &\quad \text{by using second shifting theorem,} \\ &= e^{-as}L[t^2] \quad \text{where } a = 2 \\ &= \frac{2e^{-2s}}{s^3}. \end{aligned}$$

Example 3.4.6. Express the following functions in terms of unit step function and find its Laplace transform

(i) $f(t) = \begin{cases} 0, & \text{if } 0 < t < \frac{\pi}{2}; \\ \sin t, & \text{if } t > \frac{\pi}{2}. \end{cases}$ (ii) $f(t) = \begin{cases} 0, & \text{if } 0 < t < 5; \\ t-3, & \text{if } t > 5. \end{cases}$

(iii) $f(t) = \begin{cases} \sin t, & \text{if } 0 \le t < \pi; \\ \sin 2t, & \text{if } \ \pi \le t < 2\pi; \\ \sin 3t, & t \ge 2\pi. \end{cases}$

Solution . (i) Given $f(t) = \begin{cases} 0, & \text{if } 0 < t < \frac{\pi}{2}; \\ \sin t, & \text{if } t > \frac{\pi}{2}. \end{cases}$

To apply the second shifting theorem, express the functional values $\sin t$ for $t > \pi/2$ in terms of $t - (\pi/2)$ then we have

$$g(t) = \begin{cases} 0, & \text{if } 0 < t < \frac{\pi}{2}; \\ \sin\left(t - \frac{\pi}{2}\right), & \text{if } t > \frac{\pi}{2}. \end{cases}$$

Let $g_1(t) = 0$, $g_2(t) = \sin\left(t - \frac{\pi}{2}\right)$ then $g(t)$ can be written as

$$\begin{aligned} g(t) &= g_1(t) + [g_2(t) - g_1(t)]u(t-a) \\ &= 0 + \left[\sin\left(t - \frac{\pi}{2}\right)\right] u\left(t - \frac{\pi}{2}\right) \\ \therefore\ L[g(t)] &= L\left[\sin\left(t - \frac{\pi}{2}\right) u\left(t - \frac{\pi}{2}\right)\right] \\ &\quad \text{by using second shifting theorem,} \\ &= e^{-as} L[\sin t] \quad \text{where } a = \frac{\pi}{2} \\ &= \frac{e^{-\pi s/2}}{s^2+1}. \end{aligned}$$

(ii) Given $f(t) = \begin{cases} 0, & \text{if } 0 < t < 5; \\ t-3, & \text{if } t > 5. \end{cases}$

To apply the second shifting theorem, express the functional values $t-3$ for $t > 5$ in terms of $t-5$ i.e., express $t-3$ as $(t-5)+2$ then we rewrite as

$$g(t) = \begin{cases} 0, & \text{if } 0 < t < 5; \\ (t-5)+2, & \text{if } t > 5. \end{cases}$$

Let $g_1(t) = 0$, $g_2(t) = (t-5) + 2$ then $g(t)$ can be written as

$$\begin{aligned} g(t) &= g_1(t) + [g_2(t) - g_1(t)]u(t-a) \\ &= 0 + [(t-5) + 2]u\,(t-5) \\ \therefore\ L[g(t)] &= L\left\{[(t-5)+2]u\,(t-5)\right\} \\ &\quad \text{by using second shifting theorem,} \\ &= e^{-as}L[t+2] \quad \text{where } a = 5 \\ &= e^{-5s}\left[\frac{1}{s^2} + \frac{2}{s}\right]. \end{aligned}$$

(iii) Given $f(t) = \begin{cases} \sin t, & \text{if } 0 \le t < \pi; \\ \sin 2t, & \text{if } \ \pi \le t < 2\pi; \\ \sin 3t, & t \ge 2\pi. \end{cases}$

Let $f_1(t) = \sin t$, $f_2(t) = \sin 2t$ and $f_3(t) = \sin 3t$ then,

$$\begin{aligned} f(t) &= f_1(t) + [f_2(t) - f_1(t)]u(t-\pi) + [f_3(t) - f_2(t)]u(t-2\pi) \\ &= \sin t + [\sin 2t - \sin t]u(t-\pi) + [\sin 3t - \sin 2t]u(t-2\pi). \end{aligned}$$

$$\begin{aligned} \therefore\ L[f(t)] &= L[\sin t] + L[(\sin 2t - \sin t)u(t-\pi)] \\ &\qquad + [\sin 3t - \sin 2t]u(t-2\pi) \\ &\quad \text{by using second shifting theorem and } L[\sin at] = \frac{a}{s^2+a^2} \\ &= \frac{1}{s^2+1} + e^{-\pi s}L[\sin 2t - \sin t] - e^{-2\pi s}L[\sin 3t - \sin 2t] \\ &= \frac{1}{s^2+1} + e^{-\pi s}\left[\frac{2}{s^2+4} - \frac{1}{s^2+1}\right] - e^{-2\pi s}\left[\frac{3}{s^2+9} - \frac{2}{s^2+4}\right]. \end{aligned}$$

Note: On the other way, $f(t)$ in the above problem is written as,

$$\begin{aligned} f(t) &= \sin t.[u(t-0) - u(t-\pi)] + \sin 2t.[u(t-\pi) - u(t-2\pi)] \\ &\qquad + \sin 3t.u(t-2\pi) \\ &= \sin t + [\sin 2t - \sin t].u(t-\pi) + [\sin 3t - \sin 2t].u(t-2\pi). \end{aligned}$$

Example 3.4.7. Obtain Laplace transform of following functions

(i) $(t-1)^3u_1(t)$, (ii) $4\sin(t-3)u(t-3)$, (iii) $4u(t-\pi)\cos t$, (iv) $e^{-t}[1-u(t-2)]$

Solution .

(i)
$$\begin{aligned} L[(t-1)^3u_1(t)] &= L[(t-1)^3.u(t-1)] \\ &= L[(t-1)^3.u(t-1)], \quad \text{using second shifting} \\ &= e^{-as}L[t^3], \quad \text{where } a=1 \\ &= e^{-s}\frac{3!}{s^4} = \frac{6e^{-s}}{s^4}. \end{aligned}$$

(*ii*)
$$\begin{aligned} L[4\sin(t-3)u(t-3)] &= 4L[\sin(t-3)u(t-3)] \\ &\qquad \text{using second shifting property} \\ &= 4e^{-as}L[\sin t], \quad \text{where } a=3 \\ &= \frac{4e^{-3s}}{s^2+1}. \end{aligned}$$

(iii)
$$\begin{aligned} L[4u(t-\pi)\cos t] &= 4L\{u(t-\pi)\cos[(t-\pi)+\pi]\} \\ &\qquad \text{using second shifting property} \\ &= 4e^{-as}L[\cos(t+\pi)], \quad \text{where } a=\pi \\ &= -4e^{-\pi s}L[\cos t] \\ &= -4e^{-\pi s}\left[\frac{s}{s^2+1}\right]. \end{aligned}$$

(iv)
$$L\left\{e^{-t}[1-u(t-2)]\right\} = L[e^{-t}] - L[e^{-t}.u(t-2)]$$
$$\begin{aligned} &= \frac{1}{s+1} - L[e^{-[(t-2)+2]}.u(t-2)] \quad \text{by second shifting} \\ &= \frac{1}{s+1} - e^{-as}L[e^{-[t+2]}], \text{ where } a=2 \\ &= \frac{1}{s+1} - e^{-2s}e^{-2}L[e^{-t}] \\ &= \frac{1}{s+1} - \frac{e^{-2(s+1)}}{s+1}. \end{aligned}$$

Exercise

Find Laplace transforms of the following functions

(i)$f(t) = \begin{cases} \cos\left[t - \frac{\pi}{6}\right], & t > \frac{\pi}{6}; \\ 0, & t < \frac{\pi}{6}. \end{cases}$ (ii) $f(t) = \begin{cases} \sin\left[t - \frac{2\pi}{3}\right], & t > \frac{2\pi}{3}; \\ 0, & t < \frac{2\pi}{3}. \end{cases}$

Express $f(t)$ in terms of unit step function then find its Laplace transform

(iii) $f(t) = \begin{cases} e^{-t}, & \text{if } 0 < t < 3; \\ 0, & \text{if } t > 3. \end{cases}$ (iv) $f(t) = \begin{cases} t, & \text{if } 0 < t < 3; \\ 3, & t > 3. \end{cases}$

(v) $f(t) = \begin{cases} \sin t, & \text{if } t > \pi; \\ \cos t, & \text{if } t < \pi. \end{cases}$ (vi) $f(t) = \begin{cases} 0, & \text{if } 0 < t < 1; \\ t - 1, & \text{if } 1 < t < 2; \\ 1, & t > 2. \end{cases}$

Obtain Laplace transforms for the following functions

(vii) $4\sin(t-3)u(t-3)$, (viii) $\sin t.u(t-\pi)$, (ix) $e^{-3t}u(t-2)$.

Answers

(i) $\dfrac{se^{-\pi s/6}}{s^2+1}$, (ii) $\dfrac{e^{-2\pi s/3}}{s^2+1}$,

(iii) $\dfrac{1-e^{-3(s+1)}}{s+1}$, (iv) $\dfrac{1-e^{-3s}}{s^2}$,

(v) $\dfrac{s(1-e^{-s\pi})}{s^2+1} + \dfrac{e^{-s\pi}}{s^2+1}$, (vi) $\dfrac{e^{-s}-e^{-2s}}{s^2}$

(vii) $\dfrac{4e^{-3s}}{s^2+1}$, (viii) $\dfrac{e^{-\pi s}}{s^2+1}$,

(ix) $\dfrac{e^{-(2s+6)}}{s+3}$.

3.4.4 Change of Scale Property

If $L[f(t)] = F(s)$ then

$$\text{(i) } L\left[f(at)\right] = \frac{1}{a}\ F\left(\frac{s}{a}\right) \qquad \text{(ii) } L\left[f\left(\frac{t}{a}\right)\right] = a\ F\left(as\right)$$

Proof:

$$\begin{aligned}\text{(i) } L[f(at)] &= \int_0^\infty e^{-st} f(at)\ dt \\ &= \int_0^\infty e^{-su/a} f(u)\ \frac{du}{a} \quad [\text{Put } at = u \quad \therefore\ dt = du/a] \\ &= \frac{1}{a}\int_0^\infty e^{-(s/a)u} f(u)\ du \\ &= \frac{1}{a} F\left(\frac{s}{a}\right).\end{aligned}$$

$$\begin{aligned}\text{(ii) } L\left[f\left(\frac{t}{a}\right)\right] &= \int_0^\infty e^{-st} f\left(\frac{t}{a}\right)\ dt \\ &= \int_0^\infty e^{-s(au)} f(u)\, adu \ \left[\text{Put } \frac{t}{a} = u\ \therefore\ dt = adu\right] \\ &= a\int_0^\infty e^{-(as)u} f(u)\ du \\ &= aF\left(as\right).\end{aligned}$$

Problems Based on Change of Scale Property

Example 3.4.8. Find $L\left[\frac{\sin at}{at}\right]$, given that $L\left[\frac{\sin t}{t}\right] = \tan^{-1}\left(\frac{1}{s}\right)$

Solution . Since given result is

$$L\left[\frac{\sin t}{t}\right] = \tan^{-1}\left(\frac{1}{s}\right)$$

By the change scale of property,

$$L\left[\frac{\sin at}{at}\right] = \frac{1}{a}\tan^{-1}\left(\frac{1}{s/a}\right) = \frac{1}{a}\tan^{-1}\left(\frac{a}{s}\right).$$

3.5 Laplace Transform of $t^n f(t)$

If $L[f(t)] = F(s)$ then

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s); \quad n = 1, 2, 3, ..$$

Proof:

By the definition $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = F(s)$

Differentiating both sides w.r.to s, we get

$$\begin{aligned}
\frac{d}{ds}\left[\int_0^\infty e^{-st} f(t)\, dt\right] &= \frac{d}{ds} F(s) \\
\int_0^\infty \frac{\partial}{\partial s} e^{-st} f(t)\, dt &= \frac{d}{ds} F(s) \\
\int_0^\infty (-t) e^{-st} f(t)\, dt &= \frac{d}{ds} F(s) \\
\text{or} \quad \int_0^\infty e^{-st} [t f(t)]\, dt &= -\frac{d}{ds} F(s).
\end{aligned}$$

Which proves the theorem to be true for $n = 1$.

Now we assume that the theorem to be true for $n = m$, so that

$$\begin{aligned}
\int_0^\infty e^{-st} [t^m f(t)]\, dt &= (-1)^m \frac{d^m}{ds^m} F(s) \\
\therefore \frac{d}{ds} \int_0^\infty e^{-st} [t^m f(t)]\, dt &= (-1)^m \frac{d^{m+1}}{ds^{m+1}} F(s) \, \text{[by Leibnitz's rule]} \\
\int_0^\infty \frac{\partial}{\partial s} e^{-st} [t^m f(t)]\, dt &= (-1)^m \frac{d^{m+1}}{ds^{m+1}} F(s) \\
\int_0^\infty (-t) e^{-st} [t^m f(t)]\, dt &= (-1)^m \frac{d^{m+1}}{ds^{m+1}} F(s) \\
\text{or} \int_0^\infty e^{-st} [t^{m+1} f(t)]\, dt &= (-1)^{m+1} \frac{d^{m+1}}{ds^{m+1}} F(s).
\end{aligned}$$

If the theorem is true for $n = m$, it is also true for $n = m + 1$. But it is true for $n = 1$. Hence it is true for $n = 1 + 1 = 2$ and $n = 2 + 1 = 3$ and so on. Thus the theorem is true for all positive integers of n.

Example 3.5.1. Find the Laplace transforms of following functions

(i) $t\cos at$ **(May'14)**, (ii) $t\sin at$ **(May'14)**, (iii $t^2\sin at$ **(Jan.'15**), (iv) $t^2\cos at$

Solution .

$$\begin{aligned}
\text{(i)}\quad L[t\cos at] &= -\frac{d}{ds}L[\cos at] = -\frac{d}{ds}\left[\frac{s}{s^2+a^2}\right] \qquad \textbf{(May'14)}\\
&= -\left[\frac{(s^2+a^2)-s.2s}{(s^2+a^2)^2}\right] = \frac{s^2-a^2}{(s^2+a^2)^2}.\\
\text{(ii)}\quad L[t\sin at] &= -\frac{d}{ds}L[\sin at] = -\frac{d}{ds}\left[\frac{a}{s^2+a^2}\right] \qquad \textbf{(May'14)}\\
&= -\left[\frac{-a.2s}{(s^2+a^2)^2}\right] = \frac{2as}{(s^2+a^2)^2}.\\
\text{(iii)}\quad L[t^2\sin at] &= (-1)^2\frac{d^2}{ds^2}\,L[\sin at] \qquad \textbf{(Jan.'15})\\
&= \frac{d}{ds}\,\frac{d}{ds}\left[\frac{a}{s^2+a^2}\right]\\
&= \frac{d}{ds}\left[\frac{-2as}{(s^2+a^2)^2}\right]\\
&= -2a\left[\frac{\left[s^2+a^2\right]^2.1-s.[2(s^2+a^2).2s]}{(s^2+a^2)^4}\right]\\
&= \frac{2a(3s^2-a^2)}{(s^2+a^2)^3}.\\
\text{(iv)}\quad L[t^2\cos at] &= (-1)^2\frac{d^2}{ds^2}\,L[\cos at] = \frac{d}{ds}\,\frac{d}{ds}\left[\frac{s}{s^2+a^2}\right]\\
&= \frac{d}{ds}\left[\frac{(s^2+a^2).1-s.2s}{(s^2+a^2)^2}\right] = \frac{d}{ds}\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]\\
&= \frac{\left[s^2+a^2\right]^2.(-2s)-(a^2-s^2).[2(s^2+a^2).2s]}{(s^2+a^2)^4}\\
&= \frac{-2s(s^2+a^2)-4s.(a^2-s^2)}{(s^2+a^2)^3}\\
&= \frac{2s(s^2-3a^2)}{(s^2+a^2)^3}.
\end{aligned}$$

Problems Based on $L[t^n f(t)]$

Example 3.5.2. Find the Laplace transforms of following functions

(i) $t\cos^2 t$, (ii) $t\sin^3 t$, (iii) $t\sinh^3 t$, (iv) $t\sin 3t\cos 2t$ **(Apr'11)**, (v) $t^2\cos 3t$ **(Apr'03)** (vi) $(t\sin at)^2$, (vii) $t^3 e^{-3t}$

Solution .

$$\begin{aligned}\text{(i)}\ \ L[t\cos^2 t] &= (-1)\frac{d}{ds}L\left[\frac{1+\cos 2t}{2}\right] = -\frac{1}{2}\frac{d}{ds}\left[\frac{1}{s}+\frac{s}{s^2+4}\right]\\ &= -\frac{1}{2}\left[\frac{-1}{s^2}+\frac{(s^2+4)-s.2s}{(s^2+4)^2}\right]\\ &= \frac{1}{2}\left[\frac{1}{s^2}+\frac{s^2-4}{(s^2+4)^2}\right].\end{aligned}$$

$$\text{(ii)}\ L[t\sin^3 t] = (-1)\frac{d}{ds}L\left[\frac{3\sin t-\sin 3t}{4}\right] = -\frac{1}{4}\frac{d}{ds}\left[\frac{3}{s^2+1}-\frac{3}{s^2+9}\right]$$

$$= -\frac{3}{4}\left[-\frac{2s}{(s^2+1)^2}+\frac{2s}{(s^2+9)^2}\right] = \frac{3s}{2}\left[\frac{1}{(s^2+1)^2}-\frac{1}{(s^2+9)^2}\right].$$

$$\begin{aligned}\text{(iii)}\ \ L[t\sinh^3 t] &= (-1)\frac{d}{ds}L\left[\frac{e^t-e^{-t}}{2}\right]^3\\ &= -\frac{1}{8}\frac{d}{ds}L\left[e^{3t}-3e^t+3e^{-t}-e^{-3t}\right]\\ &= -\frac{1}{8}\frac{d}{ds}\left[\frac{1}{s-3}-\frac{3}{s-1}+\frac{3}{s+1}-\frac{1}{s+3}\right]\\ &= \frac{1}{8}\left[\frac{1}{(s-3)^2}-\frac{3}{(s-1)^2}+\frac{3}{(s+1)^2}-\frac{1}{(s+3)^2}\right].\end{aligned}$$

Aliter

$$\begin{aligned}L[t\sinh^3 t] &= L\left[t\left(\frac{e^t-e^{-t}}{2}\right)^3\right]\\ &= \frac{1}{8}L\left[t\left(e^{3t}-3e^t+3e^{-t}-e^{-3t}\right)\right]\ \text{[by first shifting]}\\ &= \frac{1}{8}\left[\frac{1}{(s-3)^2}-\frac{3}{(s-1)^2}+\frac{3}{(s+1)^2}-\frac{1}{(s+3)^2}\right].\end{aligned}$$

(iv) **(Apr'11)**

$$\begin{aligned}
L\left[t\sin 3t\cos 2t\right] &= (-1)\frac{d}{ds}\ L\left[\sin 3t\ \ \cos 2t\right]\\
&= (-1)\frac{d}{ds}\ L\left\{\frac{1}{2}\left[\sin(3t+2t)+\sin(3t-2t)\right]\right\}\\
&= -\frac{1}{2}\ \frac{d}{ds}\ \left\{\frac{5}{s^2+25}+\frac{1}{s^2+1}\right\}\\
&= -\frac{1}{2}\ \left\{\frac{-10s}{(s^2+25)^2}+\frac{-2s}{(s^2+1)^2}\right\}\\
&= \frac{5s}{(s^2+25)^2}+\frac{s}{(s^2+1)^2}.
\end{aligned}$$

(v) **(Apr'03)**

$$\begin{aligned}
L[t^2\cos 3t] &= (-1)^2\frac{d^2}{ds^2}\ L[\cos 3t]\\
&= \frac{d}{ds}\ \frac{d}{ds}\ \left[\frac{s}{s^2+9}\right]\\
&= \frac{d}{ds}\ \left[\frac{(s^2+9).1-s.2s}{(s^2+9)^2}\right]\ =\ \frac{d}{ds}\ \left[\frac{9-s^2}{(s^2+9)^2}\right]\\
&= \frac{\left[s^2+9\right]^2.(-2s)-(9-s^2).[2(s^2+9).2s]}{(s^2+9)^4}\\
&= \frac{-2s(s^2+9)-4s.(9-s^2)}{(s^2+9)^3}\ =\ \frac{2s^3-54s}{(s^2+9)^3}.
\end{aligned}$$

(vi)

$$\begin{aligned}
L[(t\sin at)^2] &= L\left[t^2\left(\frac{1-\cos 2at}{2}\right)\right]\\
&= \frac{(-1)^2}{2}\ \frac{d^2}{ds^2}L\left[1-\cos 2at\right]=\frac{1}{2}\frac{d^2}{ds^2}\left[\frac{1}{s}-\frac{s}{s^2+4a^2}\right]\\
&= \frac{1}{2}\frac{d}{ds}\left\{-\frac{1}{s^2}-\left[\frac{(s^2+4a^2)-s.2s}{s^2+4a^2}\right]\right\}\\
&= \frac{1}{2}\frac{d}{ds}\left\{-\frac{1}{s^2}+\left[\frac{s^2-4a^2}{(s^2+4a^2)^2}\right]\right\}\\
&= \frac{1}{2}\left[\frac{2}{s^3}+\frac{2s(12a^2-s^2)}{(s^2+4a^2)^3}\right]\\
&= \frac{1}{s^3}+\frac{s(12a^2-s^2)}{(s^2+4a^2)^3}.
\end{aligned}$$

(vii) $$L[t^3e^{-3t}] = (-1)^3\frac{d^3}{ds^3}L[e^{-3t}] = (-1)^3\frac{d^3}{ds^3}\left[\frac{1}{s+3}\right]$$
$$= -\frac{(-1)^3.3!}{(s+3)^{3+1}} = \frac{6}{(s+3)^4}.$$

Aliter: $$L[t^3e^{-3t}] = L[t^3]_{s\to s+3} = \left[\frac{3!}{s^4}\right]_{s\to s+3} = \frac{6}{(s+3)^4}.$$

Problems Based on $L[e^{at}t^nf(t)]$

Example 3.5.3. Find Laplace transforms of the following functions

(i) $te^{-t}\sin 3t$, (ii) $te^{-2t}\cos t$ **(Nov.'06)**, (iii) $te^{-2t}\cos 3t$ **(Nov.'15)**, (iv) $\frac{t\sin t}{e^{2t}}$ **(Nov.'01)** (v) $te^{-2t}\sinh 3t$, (vi) $t\cosh t\cos t$, (vii) $t^2e^t\sin t$ **(Apr.'11)** (viii) $t^2e^{2t}\cos 3t$ **(Apr.'09)**

Solution .

(i) $$L[te^{-t}\sin 3t] = L[t\sin 3t]_{s\to(s+1)} \quad ...(1)$$
$$L[t\sin 3t] = (-1)\frac{d}{ds}\,L[\sin 3t] = (-1)\frac{d}{ds}\left[\frac{3}{s^2+9}\right]$$
$$= \frac{6s}{(s^2+9)^2}. \quad ...(2)$$
Using (2) in (1), we have
$$L[te^{-t}\sin 3t] = \left[\frac{6s}{(s^2+9)^2}\right]_{s\to(s+1)} = \frac{6(s+1)}{(s^2+2s+10)^2}.$$

(ii) $$L[te^{-2t}\cos t] = L[t\cos t]_{s\to(s+2)} \quad ...(1)$$ (**Nov.'06**)
$$L[t\cos t] = (-1)\frac{d}{ds}\,L[\cos t] = (-1)\frac{d}{ds}\left[\frac{s}{s^2+1}\right]$$
$$= (-1)\left[\frac{(s^2+1)-s.2s}{(s^2+1)^2}\right] = \frac{s^2-1}{(s^2+1)^2} \quad ...(2)$$
Using (2) in (1), we have
$$L[te^{-2t}\cos t] = \left[\frac{s^2-1}{(s^2+1)^2}\right]_{s\to(s+2)} = \frac{(s+2)^2-1}{[(s+2)^2+1]^2} = \frac{s^2+4s+3}{[s^2+4s+5]^2}.$$

(iii) $$L[te^{-2t}\cos 3t] = L[t\cos 3t]_{s\to(s+2)} \quad ...(1)$$ (**Nov.'15**)

$$L[t\cos 3t] = (-1)\frac{d}{ds}\ L[\cos 3t] = (-1)\frac{d}{ds}\ \left[\frac{s}{s^2+9}\right]$$
$$= (-1)\ \left[\frac{\left(s^2+9\right).1-s.2s}{(s^2+9)^2}\right] = \frac{s^2-9}{(s^2+9)^2} \qquad ...(2)$$

Using (2) in (1), we have

$$L[te^{-2t}\cos 3t] = \left[\frac{s^2-9}{(s^2+9)^2}\right]_{s\to(s+2)} = \frac{(s+2)^2-9}{\left[(s+2)^2+9\right]^2}$$
$$= \frac{s^2+4s-5}{\left[s^2+4s+13\right]^2}.$$

(iv) $L\left[\frac{t\sin t}{e^{2t}}\right] = L\left[e^{-2t}\ t\sin t\right] = L[t\sin t]_{s\to(s+2)}...(1)$ (**Nov'01**)

$$L[t\sin t] = (-1)\frac{d}{ds}\ \left[\frac{1}{s^2+1}\right] = \frac{2s}{(s^2+1)^2}. \qquad ...(2)$$

Using (2) in (1), we have

$$L\left[\frac{t\sin t}{e^{2t}}\right] = \left[\frac{2s}{(s^2+1)^2}\right]_{s\to(s+2)} = \frac{2(s+2)}{\left[s^2+4s+5\right]^2}.$$

(v) $L[te^{-2t}\sinh 3t] = L[t\sinh 3t]_{s\to s+2}$ ...(1)

$$L[t\sinh 3t] = (-1)\frac{d}{ds}\ \left[\frac{3}{s^2-9}\right] = \frac{6s}{(s^2-9)^2} \qquad ...(2)$$

Using (2) in (1), we have

$$L\left[te^{-2t}\sinh 3t\right] = \left[\frac{6s}{(s^2-9)^2}\right]_{s\to(s+2)} = \frac{6(s+2)}{\left[s2+4s-5\right]^2}.$$

Aliter:

$$L[te^{-2t}\sinh 3t] = L\left\{te^{-2t}\frac{\left(e^{3t}-e^{-3t}\right)}{2}\right\} = \frac{1}{2}\left\{L(te^{t})-L(te^{-5t})\right\}$$
$$= \frac{1}{2}\left\{\left[\frac{1}{s^2}\right]_{s\to s-1} - \left[\frac{1}{s^2}\right]_{s\to s+5}\right\}$$
$$= \frac{1}{2}\left\{\frac{1}{(s-1)^2}-\frac{1}{(s+5)^2}\right\} = \frac{1}{2}\left\{\frac{12s+24}{(s-1)^2(s+5)^2}\right\}$$
$$= \frac{6(s+2)}{(s-1)^2(s+5)^2}.$$

$$\begin{aligned}
\text{(vi) } L[t\cosh t\cos t] &= L\left[t\left(\frac{e^t-e^{-t}}{2}\right)\cos t\right]\\
&= \frac{1}{2}\left[L(te^t\cos t)-L(te^{-t}\cos t)\right]\\
&= \frac{1}{2}\left[L(t\cos t)_{s\to s-1}-L(t\cos t)_{s\to s+1}\right]\\
&= \frac{1}{2}\left[\left\{\frac{s^2-1}{(s^2+1)^2}\right\}_{s\to s-1}-\left\{\frac{s^2-1}{(s^2+1)^2}\right\}_{s\to s+1}\right]\\
&= \frac{1}{2}\left[\frac{s^2-2s}{(s^2-2s+2)^2}-\frac{s^2+2s}{(s^2+2s+2)^2}\right].
\end{aligned}$$

$$\text{(vii)}\qquad L\left[t^2e^t\sin t\right] = L[t^2\sin t]_{s\to(s-1)} \qquad ...(1)\quad (\textbf{Apr.'11})$$

$$\begin{aligned}
L[t^2\sin t] &= (-1)^2\frac{d^2}{ds^2}\,L[\sin t] = \frac{d}{ds}\,\frac{d}{ds}\left[\frac{1}{s^2+1}\right]\\
&= \frac{d}{ds}\left[-\frac{2s}{(s^2+1)^2}\right]\\
&= -2\left\{\frac{(s^2+1)^2-s.[2(s^2+1).2s]}{(s^2+1)^4}\right\}\\
&= -2\left\{\frac{1-3s^2}{(s^2+1)^3}\right\} = \frac{2(3s^2-1)}{(s^2+1)^3} \qquad ...(2)
\end{aligned}$$

Using (2) in (1), we have

$$\begin{aligned}
L\left[t^2e^t\sin t\right] &= \left[\frac{2(3s^2-1)}{(s^2+1)^3}\right]_{s\to(s-1)} = \frac{2\left[3(s-1)^2-1\right]}{\left[(s-1)^2+1\right]^3}\\
&= \frac{2\left[3s^2-6s+2\right]}{\left[s^2-2s+2\right]^3}.
\end{aligned}$$

$$\text{(viii)}\qquad L\left[t^2e^{2t}\cos 3t\right] = L[t^2\cos 3t]_{s\to(s-2)} \qquad ...(1)\quad (\textbf{Apr'09})$$

$$\begin{aligned}
L[t^2\cos 3t] &= (-1)^2\frac{d^2}{ds^2}\,L[\cos 3t]\\
&= \frac{d}{ds}\,\frac{d}{ds}\left[\frac{s}{s^2+9}\right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{d}{ds}\left[\frac{(s^2+9).1-s.2s}{(s^2+9)^2}\right] = \frac{d}{ds}\left[\frac{(s^2+9)-2s^2}{(s^2+9)^2}\right] \\
&= \frac{d}{ds}\left[\frac{9-s^2}{(s^2+9)^2}\right] \\
&= \frac{\left[s^2+9\right]^2.(-2s)-(9-s^2).[2(s^2+9).2s]}{(s^2+9)^4} \\
&= \frac{-2s(s^2+9)-4s.(9-s^2)}{(s^2+9)^3} \\
&= \frac{2s^3-54s}{(s^2+9)^3} \qquad \text{...(2)}
\end{aligned}$$

Using in (2) in (1), we have

$$\begin{aligned}
L\left[t^2e^{2t}\cos 3t\right] &= \left[\frac{2s^3-54s}{(s^2+9)^3}\right]_{s\to(s-2)} = \frac{2(s-2)^3-54(s-2)}{\left[(s-2)^2+9\right]^3} \\
&= \frac{2[s^3-6s^2+12s-8]-54(s-2)}{\left[s^2-4s+4+9\right]^3} \\
&= \frac{2s^3-12s^2-30s+92}{\left[s^2-4s+13\right]^3}.
\end{aligned}$$

Exercise

Example 3.5.4. Find the Laplace transforms of given functions

(i) $t\cosh 3t$ (ii) $t(3\sin 2t-2\cos 2t)$ (iii) $t\sin 3t\cos 2t$ (iv) $t\cos^3 t$
(v) $t^2\sin 2t$ (vi) $t^3\cos t$ (vii) $e^{-4t}t\sin 3t$ (viii) $te^{-2t}\cos 2t$
(ix) $te^{-2t}\sinh 2t$ (x) $te^{-t}\cosh t$ (xi) $te^{-2t}\cosh 2t$ (xii) $t^2e^{-t}\sin 2t$
(xiii) $t^2e^{-t}\cos t$.

Answers

(i) $\frac{s^2+9}{(s^2-9)^2}$ (ii) $\frac{8+12s-2s^2}{(s^2+4)^2}$ (iii) $\frac{5s}{(s^2+25)^2}+\frac{s}{(s^2+1)^2}$

(iv) $\frac{1}{4}\left[\frac{3s^2-12}{(s^2+4)^2}+\frac{s^2-36}{(s^2+36)^2}\right]$ (v) $\frac{4(3s^2-4)}{(s^2+4)^3}$ (vi) $\frac{6s^4-36s^2+6}{(s^2+1)^4}$

(vii) $\frac{6(s+4)}{(s^2+8s+25)^2}$ (viii) $\frac{s(s+4)}{(s^2+4s+8)^2}$ (ix) $\frac{4(s+2)}{s^2(s+4)^2}$ (x) $\frac{s^2+2s+2}{(s^2+2s)^2}$

(xi) $\frac{s^2+4s+8}{s^2(s+4)^2}$ (xii) $\frac{4(3s^2+6s-1)}{(s^2+2s+5)^3}$ (xiii) $\frac{2(s+1)(s^2+2s-3)}{(s^2+2s+2)^3}$.

3.6 Laplace Transform of $\frac{f(t)}{t}$

If $L[f(t)] = F(s)$ then **(Nov'03)**

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\ ds, \qquad \text{provided } \lim_{t\to 0}\frac{f(t)}{t} \text{ exists.}$$

$$\text{We have}\quad F(s) = \int_0^\infty e^{-st} f(t)\ dt$$

Integrating both sides w.r.to s from s to ∞.

$$\int_s^\infty F(s)\ ds = \int_s^\infty \int_0^\infty e^{-st} f(t)\ dt\, ds$$

Where s and t are independent variables so that the order in the double integration can be interchanged.

$$\begin{aligned}
\therefore\ \int_s^\infty F(s)\ ds &= \int_0^\infty \int_s^\infty e^{-st} f(t)\ ds\, dt \\
&= \int_0^\infty f(t) \int_s^\infty e^{-st}\ ds\, dt \\
&= \int_0^\infty f(t) \left[\frac{e^{-st}}{-t}\right]_s^\infty dt \\
&= \int_0^\infty f(t) \left[0 - \frac{e^{-st}}{-t}\right] dt \\
&= \int_0^\infty e^{-st} \left[\frac{f(t)}{t}\right] dt \qquad \left[e^{-\infty} = 0\right] \\
&= L\left[\frac{f(t)}{t}\right].
\end{aligned}$$

$$\therefore\ L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\ ds. \qquad \text{Where}\ F[s] = L[f(t)].$$

Note: Extending the above rule, we get

$$\begin{aligned}
L\left[\frac{f(t)}{t^2}\right] &= \int_s^\infty \int_s^\infty F(s)\ dsds \\
L\left[\frac{f(t)}{t^n}\right] &= \int_s^\infty \int_s^\infty \ldots \int_s^\infty F(s)\ (ds)^n
\end{aligned}$$

Problems based on $\dfrac{f(t)}{t}$

Example 3.6.1. Evaluate: (i) $L\left[\frac{1-e^t}{t}\right]$, (ii) $L\left[\frac{e^{-at}-e^{-bt}}{t}\right]$ (iii) $L\left[\frac{e^t-1}{te^{2t}}\right]$ **(Apr.'07)**, (iv) $L\left[\frac{e^{at}-\cos bt}{t}\right]$ **(Apr.'06, Apr.'13)**, (v) $L\left[\frac{1-\cos at}{t}\right]$ **(Nov.'04)**, (vi) $L\left[\frac{\cos at-\cos bt}{t}\right]$ (vii) $L\left[\frac{\sinh at}{t}\right]$ (viii) $L\left[\frac{\sin^2 t}{t}\right]$ **(Nov.'05, Nov.'07)**, (ix) $L\left[\frac{2\sin t\sin 2t}{t}\right]$.

Solution .

$$\begin{aligned}
\text{(i)} \quad L\left[\frac{1-e^t}{t}\right] &= \int_s^\infty L\left[1-e^t\right]\, ds \;=\; \int_s^\infty \left[\frac{1}{s}-\frac{1}{s-1}\right]\, ds \\
&= \left[\log(s)-\log(s-1)\right]_s^\infty \;=\; \left[\log\left(\frac{s}{s-1}\right)\right]_s^\infty \\
&= \left[\log\left(\frac{1}{1-1/s}\right)\right]_{s\to\infty} - \log\left(\frac{s}{s-1}\right) \\
&= \log(1)+\log\left(\frac{s}{s-1}\right)^{-1} \qquad [\because\ \log 1 = 0] \\
&= \log\left(\frac{s-1}{s}\right).
\end{aligned}$$

$$\begin{aligned}
\text{(ii)} \quad L\left[\frac{e^{-at}-e^{-bt}}{t}\right] &= \int_s^\infty L\left[e^{-at}-e^{-bt}\right]\, ds \\
&= \int_s^\infty \left[\frac{1}{s+a}-\frac{1}{s+b}\right]\, ds \\
&= \left[\log(s+a)-\log(s+b)\right]_s^\infty = \left[\log\left(\frac{s+a}{s+b}\right)\right]_s^\infty \\
&= \left[\log\left(\frac{1+a/s}{1+b/s}\right)\right]_{s\to\infty} - \log\left(\frac{s+a}{s+b}\right) \\
&= \log(1)+\log\left(\frac{s+a}{s+b}\right)^{-1} \;=\; \log\left(\frac{s+b}{s+a}\right).
\end{aligned}$$

Evaluate : $L\left[\dfrac{e^{-2t}-e^{-3t}}{t}\right]$. **(Apr'03)**

Replace $a = 2$ and $b = 3$ in the above problem and proceeding the steps as in the above, we get $\log\left(\frac{s+3}{s+2}\right)$.

(iii)
$$\begin{aligned} \frac{e^t - 1}{te^{2t}} &= \frac{(e^t - 1)e^{-2t}}{t} = \frac{e^{-t} - e^{-2t}}{t} \\ L\left[\frac{e^t - 1}{te^{2t}}\right] &= L\left[\frac{e^{-t} - e^{-2t}}{t}\right] \end{aligned}$$

(**Apr'07**)

[The steps are as in (ii)]

$$= \log\left(\frac{s+2}{s+1}\right).$$

(iv)
$$\begin{aligned} L\left[\frac{e^{at} - \cos bt}{t}\right] &= \int_s^\infty L\left[e^{at} - \cos bt\right]\, ds \\ &= \int_s^\infty \left[\frac{1}{s-a} - \frac{s}{s^2+b^2}\right] ds \\ &= \left[\log(s-a) - \frac{1}{2}\log(s^2+b^2)\right]_s^\infty \\ &= \left[\log\left(\frac{s-a}{\sqrt{s^2+b^2}}\right)\right]_s^\infty \\ &= \left[\log\left(\frac{1-a/s}{\sqrt{1+b^2/s^2}}\right)\right]_{s\to\infty} - \log\left(\frac{s-a}{\sqrt{s^2+b^2}}\right) \\ &= \log\left(\frac{\sqrt{s^2+b^2}}{s-a}\right). \end{aligned}$$

(**Apr'06, Apr.'13**)

(v)
$$\begin{aligned} L\left[\frac{1-\cos at}{t}\right] &= \int_s^\infty L\left[1 - \cos at\right]\, ds \\ &= \int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+a^2}\right] ds = \left[\log(s) - \frac{1}{2}\log(s^2+a^2)\right]_s^\infty \\ &= \left[\log\left(\frac{s}{\sqrt{s^2+a^2}}\right)\right]_s^\infty \\ &= \left[\log\left(\frac{1}{\sqrt{1+a^2/s^2}}\right)\right]_{s\to\infty} - \log\left(\frac{s}{\sqrt{s^2+a^2}}\right) \\ &= \log\left(\frac{\sqrt{s^2+a^2}}{s}\right). \end{aligned}$$

(**Nov.'04**)

$$\text{(vi)}\quad L\left[\frac{\cos at-\cos bt}{t}\right] = \int_s^\infty L\left[\cos at-\cos bt\right]\ ds$$
$$= \int_s^\infty \left[\frac{s}{s^2+a^2}-\frac{s}{s^2+b^2}\right]\ ds$$
$$= \frac{1}{2}\left[\log\left(s^2+a^2\right)-\log\left(s^2+b^2\right)\right]_s^\infty$$
$$= \frac{1}{2}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]_s^\infty$$
$$= \frac{1}{2}\left\{\left[\log\left(\frac{1+a^2/s^2}{1+b^2/s^2}\right)\right]_{s\to\infty}-\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right\}$$
$$= \log\left(\sqrt{\frac{s^2+b^2}{s^2+a^2}}\right).$$

Note: Evaluate : $L\left[\frac{\cos 2t-\cos 3t}{t}\right]$. Replace $a=2$ and $b=3$ in the above problem and proceeding then we get, $\log\sqrt{\frac{s^2+9}{s^2+4}}$.

$$\text{(vii)}\ L\left[\frac{\sinh at}{t}\right] = \int_s^\infty L\left[\sinh at\right]\ ds = \int_s^\infty \frac{a}{s^2-a^2}\ ds$$
$$= a\left[\frac{1}{2a}\log\left(\frac{s-a}{s+a}\right)\right]_s^\infty$$
$$= \left[\frac{1}{2}\log\left(\frac{1-a/s}{1+a/s}\right)\right]_{s\to\infty}-\frac{1}{2}\log\left(\frac{s-a}{s+a}\right)$$
$$= \frac{1}{2}\log\left(\frac{s+a}{s-a}\right).$$

$$\text{(viii)}\ L\left(\frac{\sin^2 t}{t}\right) = \frac{1}{2}\int_s^\infty L\left[1-\cos 2t\right]\ ds \qquad (\textbf{Nov'05, Nov'07})$$
$$= \frac{1}{2}\int_s^\infty\left[\frac{1}{s}-\frac{s}{s^2+4}\right]\ ds = \frac{1}{2}\left[\log s-\frac{1}{2}\log(s^2+4)\right]_s^\infty$$
$$= \frac{1}{2}\left[\log s-\log\sqrt{s^2+4}\right]_s^\infty$$
$$= \frac{1}{2}\left\{\left[\log\left(\frac{1}{\sqrt{1+4/s^2}}\right)\right]_{s\to\infty}-\left[\log\left(\frac{s}{\sqrt{s^2+4}}\right)\right]\right\}$$
$$= \frac{1}{2}\log\left(\frac{\sqrt{s^2+4}}{s}\right).$$

(ix)
$$\begin{aligned}
L\left[\frac{2\sin t\sin 2t}{t}\right] &= \int_s^\infty L[2\sin t\sin 2t]\ ds \\
&= \int_s^\infty L\left[\cos t-\cos 3t\right]\ ds \\
&= \int_s^\infty \left[\frac{s}{s^2+1}-\frac{s}{s^2+9}\right]\ ds\ =\ \left[\frac{1}{2}\log\left(\frac{s^2+1}{s^2+9}\right)\right]_s^\infty \\
&= \left[\frac{1}{2}\log\left(\frac{1+1/s^2}{1+9/s^2}\right)\right]_{s\to\infty} - \left[\frac{1}{2}\log\left(\frac{s^2+1}{s^2+9}\right)\right] \\
&= \frac{1}{2}\log\left(\frac{s^2+9}{s^2+1}\right).
\end{aligned}$$

Example 3.6.2. Does $L\left[\frac{\cos at}{t}\right]$ exist.

Solution . Let $f(t) = \cos at$ then $\lim_{t\to 0}\frac{\cos at}{t} = \frac{1}{0} = \infty$. Since $\lim_{t\to 0}\frac{f(t)}{t}$ does not exist. So that $L\left[\frac{\cos at}{t}\right]$ does not exist.

Example 3.6.3. Find $L\left[\frac{\sin at}{t}\right]$. Hence show that $\int_0^\infty \frac{\sin t}{t}\ dt = \frac{\pi}{2}$.

Solution .

$$\begin{aligned}
L\left[\frac{\sin at}{t}\right] &= \int_s^\infty L\left[\sin at\right]\ ds\ =\ \int_s^\infty \frac{a}{s^2+a^2}\ ds \\
&= a\ \left[\frac{1}{a}\ \tan^{-1}(s/a)\right]_s^\infty \\
&= \tan^{-1}(\infty)-\tan^{-1}(s/a) \\
&= \frac{\pi}{2}-\tan^{-1}(s/a) \\
&= \cot^{-1}(s/a)\ =\ \tan^{-1}(a/s).
\end{aligned}$$

Now
$$\begin{aligned}
\int_0^\infty \frac{f(t)}{t}\ dt &= \lim_{s\to 0}\int_0^\infty e^{-st}.\frac{\sin t}{t}\ dt \\
&= \lim_{s\to 0} L\left[\frac{\sin t}{t}\right] \\
&= \lim_{s\to 0}\tan^{-1}(1/s)\ \text{ where } a=1 \\
&= \tan^{-1}(\infty)\ =\ \frac{\pi}{2}.
\end{aligned}$$

Exercise

Find the Laplace transforms of the following functions

(i) $\dfrac{\sin at}{t}$ (ii) $\dfrac{e^{-t}\sin t}{t}$ (iii) $\dfrac{e^{-t}\sin t}{t}$ (iv) $\dfrac{1-\cos 2t}{t}$ (v) $\dfrac{1-\cosh t}{t}$
(vi) $\dfrac{\cos 4t \sin 2t}{t}$.

Answers

(i) $\cot^{-1}\left(\dfrac{s}{a}\right)$ (ii) $\cot^{-1}(s+1)$ (iii) $\cot^{-1}(s+1)$ (iv) $\log\left(\dfrac{\sqrt{s^2+4}}{s}\right)$
(v) $\log\left(\dfrac{\sqrt{s^2-1}}{s}\right)$ (vi) $\dfrac{1}{2}\left[\tan^{-1}\left(\dfrac{s}{2}\right)-\tan^{-1}\left(\dfrac{s}{6}\right)\right]$.

Example 3.6.4. Find the Laplace transform $f(t) = te^{-t}\cos t + \frac{\sin 2t}{t}$
(**May '02, Nov'08, Nov'09**)

Solution .

$$\begin{aligned}
\text{Let } L[f(t)] &= L\left[te^{-t}\cos t + \frac{\sin 2t}{t}\right] \\
&= L\left[te^{-t}\cos t\right] + L\left[\frac{\sin 2t}{t}\right] \\
&= I_1 + I_2 \qquad ...(1)
\end{aligned}$$

To Find I_1:

$$I_1 = L\left[te^{-t}cost\right] = L[t\cos t]_{s\to(s+1)} \qquad ...(2)$$

$$\begin{aligned}
\text{Now } L[t\cos t] &= (-1)\frac{d}{ds}\,L[\cos t] = (-1)\frac{d}{ds}\left[\frac{s}{s^2+1}\right] \\
&= (-1)\left[\frac{(s^2+1)-s.2s}{(s^2+1)^2}\right] \\
&= \frac{s^2-1}{(s^2+1)^2} \qquad ...(3)
\end{aligned}$$

Using (3) in (2), we have

$$\begin{aligned}
L[te^{-t}\cos t] &= \left[\frac{s^2-1}{(s^2+1)^2}\right]_{s\to(s+1)} = \frac{(s+1)^2-1}{\left[(s+1)^2+1\right]^2} \\
\therefore I_1 &= \frac{s^2+2s}{\left[s^2+2s+2\right]^2}.
\end{aligned}$$

To find I_2:

$$\begin{aligned}
I_2 &= L\left[\frac{\sin 2t}{t}\right] = \int_s^\infty L\left[\sin 2t\right]\ ds \\
&= \int_s^\infty \frac{2}{s^2+2^2}\ ds \\
&= 2\left[\frac{1}{2}\ \tan^{-1}\frac{s}{2}\right]_s^\infty = \left[\tan^{-1}\frac{s}{2}\right]_s^\infty \\
&= \frac{\pi}{2} - \tan^{-1}\frac{s}{2} = \cot^{-1}\left(\frac{s}{2}\right).
\end{aligned}$$

Substitute I_1 and I_2 values in equation (1) we get

$$L[f(t)] = \frac{s^2+2s}{\left[s^2+2s+2\right]^2} + \left(\frac{s}{2}\right).$$

3.7 Initial and Final Value Theorems

[State and prove initial and final value theorems.] (**Apr'12**)

Statement: If $L[f(t)]$, $L[f'(t)]$ are exist and $L[f(t)] = F(s)$ then

$$\begin{aligned}
\text{(i)}\ \lim_{t\to\ 0} f(t) &= \lim_{s\to\ \infty} sF(s)\ \ [I.V.T] \\
\text{(ii)}\ \lim_{t\to\ \infty} f(t) &= \lim_{s\to\ 0} sF(s)\ \ [F.V.T]
\end{aligned}$$

Proof: **To prove (i) [I.V.T]:**

$$\begin{aligned}
\text{We know that}\quad L[f'(t)] &= sF(s) - f(0) \\
\therefore\quad sF(s) &= L[f'(t)] + f(0) \\
&= \int_0^\infty e^{-st} f'(t)\ dt + f(0)
\end{aligned}$$

Taking limit as $s\to\infty$ on both sides, we get

$$\begin{aligned}
\lim_{s\to\infty} sF(s) &= \lim_{s\to\infty}\int_0^\infty e^{-st} f'(t)\ dt + f(0) \\
&= 0 + f(0) \\
\text{i.e.}\quad \lim_{s\to\infty} sF(s) &= \lim_{t\to 0} f(t). \quad \text{Hence proved I.V.T.}
\end{aligned}$$

To prove (ii) [F.V.T]:

$$\begin{aligned} \text{W.k.t. } L[f'(t)] &= sF(s) - f(0) \\ \therefore\ \ sF(s) &= L[f'(t)] + f(0) \ = \ \int_0^\infty e^{-st} f'(t)\ dt + f(0) \end{aligned}$$

Taking limit as $s \to 0$ on both sides, we get

$$\begin{aligned} \lim_{s\to 0} sF(s) &= \lim_{s\to 0} \int_0^\infty e^{-st} f'(t)\ dt + f(0) \\ &= \int_0^\infty f'(t)\ dt + f(0) \ = \ [f(t)]_0^\infty + f(0) \\ &= \lim_{t\to\infty} f(t) - f(0) + f(0) \\ \text{Thus} \quad \lim_{s\to 0} sF(s) &= \lim_{t\to\infty} f(t). \quad \text{Hence proved F.V.T.} \end{aligned}$$

Example 3.7.1. Verify Initial and final value theorems for the functions (i) $f(t) = 1+e^{-t}[\sin t+\cos t]$, (ii) $f(t) = e^{-t}(t+2)^2$, (iii) $f(t) = e^{-t}\cos^2 t$, (iv) $f(t) = L^{-1}\left[\frac{1}{s(s+2)^2}\right]$

Solution . (i) Given $f(t) = 1 + e^{-t}[\sin t + \cos t]$. (**Apr'08**)

W.k.t. the Initial and final value theorems of Laplace transform,

$$\text{(i) } \lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s). \qquad \text{(ii) } \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s).$$

$$\begin{aligned} F(s) &= L[f(t)] \ = \ L[1] + L\left[e^{-t}(\sin t + \cos t)\right] \\ &= \frac{1}{s} + L\left[\sin t + \cos t\right]_{s\to s+1} \\ &= \frac{1}{s} + \left[\frac{1}{s^2+1} + \frac{s}{s^2+1}\right]_{s\to s+1} \\ &= \frac{1}{s} + \left[\frac{1+s}{s^2+1}\right]_{s\to s+1} \\ \therefore\ F(s) &= \frac{1}{s} + \left[\frac{s+2}{s^2+2s+2}\right] \end{aligned}$$

$$\text{Thus} \quad sF(s) \ = \ 1 + \left[\frac{s^2+2s}{s^2+2s+2}\right].$$

$$\therefore\ \lim_{t\to 0} f(t) \ = \ \lim_{t\to 0}\left[1 + e^{-t}(\sin t + \cos t)\right] \ = \ 2 \qquad ...(1)$$

$$\begin{aligned}\lim_{s\to\infty} sF(s) &= \lim_{s\to\infty}\left[1+\left(\frac{s^2+2s}{s^2+2s+2}\right)\right]\\ &= 1+\lim_{s\to\infty}\left[\frac{s^2[1+(2/s)]}{s^2[1+(2/s)+(2/s^2)]}\right]\\ &= 1+\left[\frac{1+0}{1+0+0}\right] \;=\; 2. \qquad ...(2)\end{aligned}$$

From (1) and (2) we have $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s) = 2$.
Hence Initial value theorem is verified.

To verify F.V.T:

$$\lim_{t\to\infty} f(t) \;=\; \lim_{t\to\infty}[1+e^{-t}(\sin t+\cos t)] \;=\; 1. \qquad ...(3)$$

When t tends to ∞ before that e^{-t} will reach to ∞. $\therefore\ e^{-\infty}=0$.

$$\lim_{s\to 0} sF[s] \;=\; \lim_{s\to 0}\left[1+\left(\frac{s^2+2s}{s^2+2s+2}\right)\right] \;=\; 1. \qquad ...(4)$$

From (3)& (4) $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) = 1$ $\therefore$ F.V.T. is verified.

(ii) Given $f(t) = e^{-t}(t+2)^2$.
(**Apr.'05, Nov.'10, Nov.'13, Nov.'15**)

W.k.t. the Initial and final value theorems of Laplace transform,

$$\text{(i)}\ \lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s). \qquad \text{(ii)}\ \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s).$$

$$\begin{aligned}F[s] &= L[f(t)] \;=\; L[e^{-t}(t^2+4t+4)]\\ &= L[t^2+4t+4]_{s\to(s+1)}\\ &= \left[\frac{2!}{s^3}+4\ \frac{1!}{s^2}+4\ \frac{1}{s}\right]_{s\to(s+1)}\\ \therefore\ F(s) &= \frac{2}{(s+1)^3}+\frac{4}{(s+1)^2}+\frac{4}{s+1}\\ \text{Thus}\quad sF[s] &= \frac{2s}{(s+1)^3}+\frac{4s}{(s+1)^2}+\frac{4s}{s+1}\end{aligned}$$

$$\therefore\ \lim_{t\to 0} f(t) \;=\; \lim_{t\to 0}[e^{-t}(t^2+4t+4)] \;=\; 4. \quad [\because\ e^{-0}=1] \qquad ...(1)$$

$$\begin{aligned}
\lim_{s\to\infty} sF[s] &= \lim_{s\to\infty}\left[\frac{2s}{s^3(1+1/s)^3}+\frac{4s}{s^2(1+1/s)^2}+\frac{4s}{s(1+1/s)}\right]\\
&= \lim_{s\to\infty}\left[\frac{2}{s^2(1+1/s)^3}+\frac{4}{s(1+1/s)^2}+\frac{4}{(1+1/s)}\right]\\
&= 0+0+4 \;=\; 4. \qquad ...(2)
\end{aligned}$$

From (1) and (2) $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s) = 4$.

Hence Initial value theorem is verified.

To verify F.V.T:

$$\therefore \quad \lim_{t\to\infty} f(t) \;=\; \lim_{t\to\infty}[e^{-t}(t^2+4t+4)] \;=\; 0. \qquad ...(3)$$

$$\lim_{s\to 0} sF[s] \;=\; \lim_{s\to 0}\left[\frac{2s}{(s+1)^3}+\frac{4s}{(s+1)^2}+\frac{4s}{s+1}\right] \;=\; 0. \qquad ...(4)$$

From equations (3) and (4) $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) = 0$.

Hence Final value theorem is verified.

(iii) Given $f(t) = e^{-t}\cos^2 t$.

W.k.t. the Initial and final value theorems of Laplace transform,

$$(i)\ \lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s). \qquad (ii)\ \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s).$$

$$\begin{aligned}
F(s) &= L[f(t)] = L\left[e^{-t}\cos^2 t\right]\\
&= L\left[e^{-t}\left(\frac{1+\cos 2t}{2}\right)\right]\\
&= \frac{1}{2}\left[\frac{1}{s}+\frac{s}{s^2+4}\right]_{s\to(s+1)}\\
&= \frac{1}{2}\left[\frac{1}{s+1}+\frac{s+1}{s^2+2s+5}\right]\\
\text{Thus}\quad sF(s) &= \frac{1}{2}\left[\frac{s}{s+1}+\frac{s^2+s}{s^2+2s+5}\right].
\end{aligned}$$

$$\therefore \quad \lim_{t\to 0} f(t) \;=\; \lim_{t\to 0}\left[e^{-t}\cos^2 t\right] \;=\; 1. \qquad ...(1)$$

$$\begin{aligned}\lim_{s\to\infty} sF(s) &= \lim_{s\to\infty}\frac{1}{2}\left[\frac{s}{s+1}+\frac{s^2+s}{s^2+2s+5}\right]\\ &= \lim_{s\to\infty}\frac{1}{2}\left[\frac{1}{1+1/s}+\frac{1+1/s}{1+2/s+5/s^2}\right]\\ &= \frac{1}{2}[1+1] \;=\; 1. \qquad \text{...(2)}\end{aligned}$$

From (1) and (2) we have $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s) = 1$.
Hence Initial value theorem is verified.

To verify F.V.T:

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty}[e^{-t}\cos^2 t] = 0. \qquad [\because\ e^{-\infty}=0] \qquad \text{...(3)}$$

$$\lim_{s\to 0} sF[s] = \lim_{s\to 0}\frac{1}{2}\left[\frac{s}{s+1}+\frac{s^2+s}{s^2+2s+5}\right] = 0. \qquad \text{...(4)}$$

From (3) & (4) $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) = 0$. $\therefore$ F.V.T. is verified.

(iv) Given
$$\begin{aligned} f(t) &= L^{-1}\left[\frac{1}{s(s+2)^2}\right] = \int_0^t L^{-1}\left[\frac{1}{(s+2)^2}\right] dt\\ &= \int_0^t te^{-2t}\,dt = \left[\frac{te^{-2t}}{-2}-\frac{e^{-2t}}{4}\right]_0^t\\ \therefore\ f(t) &= \frac{1}{4}[1-2te^{-2t}-e^{-2t}].\end{aligned}$$

W.k.t. the Initial and final value theorems of Laplace transform is

$$(i)\ \lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s). \qquad (ii)\ \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s).$$

$$F[s] = \frac{1}{s(s+2)^2} \quad \therefore\ sF[s] = \frac{1}{(s+2)^2}.$$

$$\begin{aligned}\therefore\ \lim_{t\to 0} f(t) &= \lim_{t\to 0}\frac{1}{4}\left[1-2te^{-2t}-e^{-2t}\right]\\ &= \frac{1}{4}[1-0-1] = 0 \qquad \text{...(1)}\end{aligned}$$

$$\lim_{s\to\infty} sF[s] = \lim_{s\to\infty}\frac{1}{(s+2)^2} = 0. \qquad \text{...(2)}$$

From (1) & (2) $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s) = 0$. $\therefore$ I.V.T. is verified.

To verify F.V.T.

$$\therefore \quad \lim_{t\to\infty} f(t) \;=\; \lim_{t\to\infty} \frac{1}{4}\left[1 - 2te^{-2t} - e^{-2t}\right] \;=\; \frac{1}{4}. \qquad ...(3)$$

$$\text{Now} \quad \lim_{s\to 0} sF[s] \;=\; \lim_{s\to 0} \frac{1}{(s+2)^2} \;=\; \frac{1}{4}. \qquad ...(4)$$

From equations (3) and (4) $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) = \frac{1}{4}$.
Hence Final value theorem is verified.

Example 3.7.2. Verify initial and final value theorem for the function $f(t) = t^2e^{-4t}$ (**Apr.'07, Apr.'14, Apr.'15**)

Solution . Given $f(t) = t^2e^{-4t}$.W.k.t. the final value theorem of Laplace transform is, $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$.

$$F[s] \;=\; L[t^2e^{-4t}] \;=\; L[t^2]_{s\to(s+4)} \;=\; \left[\frac{2!}{s^3}\right]_{s\to(s+4)}$$

$$\therefore \quad F[s] \;=\; \frac{2}{(s+4)^3} \quad \text{and} \quad sF(s) \;=\; \frac{2s}{(s+4)^3}.$$

W.k.t. the Initial and final value theorems of Laplace transform is

$$(i)\ \lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s). \qquad (ii)\ \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s).$$

$$\therefore \quad \lim_{t\to 0} f(t) \;=\; \lim_{t\to 0}\ t^2e^{-4t} \;=\; 0 \qquad ...(1)$$

$$\lim_{s\to\infty} sF[s] \;=\; \lim_{s\to\infty} \frac{2s}{(s+4)^3} \;=\; \lim_{s\to\infty} \frac{2}{s^2\,(1+4/s)^3} \;=\; 0. \qquad ...(2)$$

From (1) & (2) $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s) = 0$. $\therefore$ I.V.T. is verified.

To verify F.V.T.

$$\text{Now} \quad \lim_{t\to\infty} f(t) \;=\; \lim_{t\to\infty} t^2e^{-4t} = 0 \qquad \left[\because\ e^{-\infty} = 0\right]$$

$$\lim_{s\to 0y} sF[s] \;=\; \lim_{s\to 0} \frac{2s}{(s+4)^3} \;=\; 0.$$

So that $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s) = 0$. $\therefore$ F.V.T. is verified.

Exercise

Verify initial and final value theorems of the functions
(i) $1-e^{-t}$ **(Jan.'16)** (ii) t^2e^{-3t}, (iii) $3e^{-2t}$, (iv) $2+t^2+\sin t$ **(Nov.'12)**

3.8 Laplace Transforms of Derivatives

Theorem : If $f'(t)$ be continuous and $L[f(t)] = F(s)$, then

$$L[f'(t)] = sF(s) - f(0).$$

Proof:

$$\begin{aligned}\text{By definition,}\quad L[f(t)] &= \int_0^\infty e^{-st} f(t)\ dt\\ \text{Then}\quad L[f'(t)] &= \int_0^\infty e^{-st} f'(t)\ dt\\ &= \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty (-s)e^{-st} f(t)\ dt\\ &= [0 - f(0)] + s\int_0^\infty e^{-st} f(t)\ dt\\ \therefore\ L[f'(t)] &= sF(s) - f(0). \qquad \text{...(1)}\end{aligned}$$

Note:

If $f'(t)$ and $f''(t)$ derivatives are continuous, then replace $f'(t)$ by $f''(t)$ in the equation (1), we get

$$\begin{aligned}L[f''(t)] &= sL[f'(t)] - f'(0)\\ . &= s\,[sF(s) - f(0)] - f'(0)\\ &= s^2F(s) - sf(0) - f'(0).\end{aligned}$$

Do this similar procedure, if $f'(t)$ and its first $(n-1)$ derivatives be continuous, then we get

$$L[f^n(t)] = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - ... - f^{n-1}(0).$$

3.9 Laplace Transforms of Integrals

3.9.1 Evaluation of Integrals by Using Laplace Transforms

In the case of evaluation of integrals using Laplace transform, first we compare the given integrals with Laplace transforms and make the suitable substitution for the parameter s then evaluate it.

Problems Based on Evaluation of Integrals

Example 3.9.1. Prove that $\int_0^\infty \frac{e^{-t}-e^{-3t}}{t}\,dt = \log 3$, using Laplace transform. **(Nov'05)**

Solution .

$$\text{Given } \int_0^\infty \frac{e^{-t}-e^{-3t}}{t}\,dt = L\left[\frac{e^{-t}-e^{-3t}}{t}\right]_{s=0} \qquad ...(1)$$

$$\begin{aligned} \therefore\ L\left[\frac{e^{-t}-e^{-3t}}{t}\right] &= \int_s^\infty \left[\frac{1}{s+1}-\frac{1}{s+3}\right] ds \\ &= \left[\log\left(\frac{s+1}{s+3}\right)\right]_s^\infty \\ &= \log\left(\frac{s+3}{s+1}\right). \qquad ...(2) \end{aligned}$$

Using (2) in (1), we have

$$\int_0^\infty \frac{e^{-t}-e^{-3t}}{t}\,dt = \left[\log\left(\frac{s+3}{s+1}\right)\right]_{s=0} = \log(3).$$

Example 3.9.2. Evaluate the integrals : (i) $\int_0^\infty \left[\frac{\cos at-\cos bt}{t}\right] dt$ (ii) $\int_0^\infty e^{-2t}\, t\sin 3t\,dt$ **(Nov'10, Jan.'15)**, (iii) $\int_0^\infty \frac{\sin^2 t}{te^t}\,dt$

Solution .

$$\text{(i) Given } \int_0^\infty \frac{\cos at-\cos bt}{t}\,dt = L\left[\frac{\cos at-\cos bt}{t}\right]_{s=0} \qquad ...(1)$$

$$\begin{aligned} \therefore\ L\left[\frac{\cos at-\cos bt}{t}\right] &= \int_s^\infty \left[\frac{s}{s^2+a^2}-\frac{s}{s^2+b^2}\right] ds \\ &= \frac{1}{2}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]_s^\infty \\ &= \log\left(\sqrt{\frac{s^2+b^2}{s^2+a^2}}\right) \qquad ...(2) \end{aligned}$$

Using (2) in (1), we get

$$\int_0^\infty \left[\frac{\cos at-\cos bt}{t}\right] dt = \left[\log\left(\sqrt{\frac{s^2+b^2}{s^2+a^2}}\right)\right]_{s=0} = \log\left(\frac{b}{a}\right).$$

(ii) Given $\int_0^\infty e^{-2t}\ t\ \sin 3t\ dt\ =\ L[t\ \sin 3t]_{s=2}$ ...(1)

$$\therefore\ \ L[t\sin 3t]\ =\ (-1)\frac{d}{ds}\left[\frac{3}{s^2+9}\right]\ =\ \frac{6s}{(s^2+9)^2} \qquad ...(2)$$

Using (2) in (1), we have

$$\int_0^\infty e^{-2t}\ t\ \sin 3t\ dt\ =\ \left[\frac{6s}{(s^2+9)^2}\right]_{s=2}\ =\ \frac{12}{169}.$$

(iii) Given $\displaystyle\int_0^\infty \frac{\sin^2 t}{te^t}\ dt\ =\ \left[\int_0^\infty e^{-t}\left(\frac{\sin^2 t}{t}\right)\ dt\right]_{s=1}$ ...(1)

$$\begin{aligned}\therefore\ \ L\left(\frac{\sin^2 t}{t}\right) &= \frac{1}{2}L\left[\frac{1-\cos 2t}{t}\right]\\ &= \frac{1}{2}\int_s^\infty\left[\frac{1}{s}-\frac{s}{s^2+4}\right]\ ds\\ &= \frac{1}{2}\left[\log\left(\frac{s}{\sqrt{s^2+4}}\right)\right]_s^\infty\\ &= \frac{1}{2}\log\left(\frac{\sqrt{s^2+4}}{s}\right) \qquad ...(2)\end{aligned}$$

Using (2) in (1), we get

$$\int_0^\infty \frac{\sin^2 t}{te^t}\ dt\ =\ \left[\frac{1}{2}\log\left(\frac{\sqrt{s^2+4}}{s}\right)\right]_{s=1}\ =\ \frac{1}{4}\ \log(5).$$

3.9.2 Laplace Transforms of Integrals

Theorem : If $L[f(t)] = F(s)$, then $L\left[\int_0^t f(u)\ du\right] = \frac{1}{s}F(s)$.

Proof:

$$\begin{aligned}\text{Let }\ \phi(t) &= \int_0^\infty f(u)\ du,\ \text{then } \phi'(t) = f(t) \text{ and } \phi(0) = 0\\ \text{W.k.t. }\ L[\phi'(t)] &= s\Phi(s) - \phi(0)\\ \text{or }\ \Phi(s) &= \frac{1}{s}L[\phi'(t)]\\ \text{i.e., } L\left[\int_0^t f(u)\ du\right] &= \frac{1}{s}F(s).\end{aligned}$$

Problems Based on Laplace Transforms of Integrals

Example 3.9.3. Evaluate :

(i) $L\left[\int_0^t \frac{\sin t}{t}dt\right]$ **(Nov'01, May'15)** (ii) $L\left[e^{-2t}\int_0^t t\sin t dt\right]$ **(May'02)**(iii) $L\left[t\int_0^t e^{4t}\sin 3t dt\right]$ (iv) $L\left[\frac{1}{t}\int_0^t e^{-t}\sin t\ dt\right]$ **(Apr.'08)** (v) $L\left[\int_0^t te^{-t}\sin t\, dt\right]$ **(Nov.'14)**

Solution . (i) By the theorem on Laplace transform of integral,

$$L\left[\int_0^t f(t)\ dt\right] = \frac{1}{s}\ F[s]$$

Given $$L\left[\int_0^t \frac{\sin t}{t}\ dt\right] = \frac{1}{s}L\left[\frac{\sin t}{t}\right] \qquad \textbf{(Nov'01, May'15)}$$

$$= \frac{1}{s}\int_s^\infty L\left[\sin t\right]\ ds$$

$$= \frac{1}{s}\int_s^\infty \frac{1}{s^2+1}\ ds$$

$$= \frac{1}{s}\ \left[\tan^{-1}(s)\right]_s^\infty = \frac{1}{s}\ \left[\frac{\pi}{2}-\tan^{-1}(s)\right]$$

$$= \frac{1}{s}\ \cot^{-1}s.$$

(ii) $$L\left[e^{-2t}\int_0^t t\sin t\ dt\right] = L\left[\int_0^t t\sin t\ dt\right]_{s\to(s+2)} \qquad (\textbf{May'02})$$

by the theorem of LT of integral

$$= \left\{\frac{1}{s}L\left[t\sin t\right]\right\}_{s\to(s+2)} \qquad ...(1)$$

$$L[t\sin t] = (-1)\frac{d}{ds}\left[\frac{1}{s^2+1}\right] = \frac{2s}{(s^2+1)^2} \qquad ...(2)$$

Using (2) in (1), we have

$$L\left[e^{-2t}\int_0^t t\sin t\ dt\right] = \left\{\frac{1}{s}\left[\frac{2s}{(s^2+1)^2}\right]\right\}_{s\to(s+2)} = \frac{2}{[s^2+4s+5]^2}.$$

(iii) $$L\left[t\int_0^t e^{4t}\sin 3t\ dt\right] = (-1)\frac{d}{ds}L\left[\int_0^t e^{4t}\sin 3t\ dt\right] \qquad ...(1)$$

$$\text{Now } L\left[\int_0^t e^{4t}\,\sin 3t\ dt\right] = \frac{1}{s}\,L[e^{4t}\sin 3t]$$

by the theorem on LT of integral

$$= \frac{1}{s}L[\sin 3t]_{s\to(s-4)}$$

$$= \frac{1}{s}\left[\frac{3}{s^2+9}\right]_{s\to(s-4)}$$

$$= \frac{1}{s}\left[\frac{3}{(s-4)^2+9}\right]$$

$$= \frac{3}{s^3-8s^2+25s} \qquad ...(2)$$

Using (2) in (1), we get

$$L\left[t\int_0^t e^{4t}\sin 3t\ dt\right] = (-1)\frac{d}{ds}\left[\frac{3}{s^3-8s^2+25s}\right]$$

$$= \frac{9s^2-48s+75}{(s^3-8s^2+25s)^2}$$

(iv) $L\left[\frac{1}{t}\int_0^t e^{-t}\sin t\ dt\right] = \int_s^\infty L\left[\int_0^t e^{-t}\sin t\ dt\right] ds$ ...(1) (**Apr'08**)

$$\therefore\ L\left[\int_0^t e^{-t}\ \sin t\ dt\right] = \frac{1}{s}\ L[e^{-t}\sin t]$$

by the theorem on LT of integral

$$= \frac{1}{s}L[\sin t]_{s\to(s+1)} = \frac{1}{s}\left[\frac{1}{s^2+1}\right]_{s\to(s+1)}$$

$$= \frac{1}{s}\left[\frac{1}{(s+1)^2+1}\right] = \frac{1}{s.(s^2+2s+2)} \qquad ...(2)$$

On resolving the equation (2) into partial fractions, we get

$$\frac{1}{s.(s^2+2s+2)} = \frac{A}{s}+\frac{Bs+C}{(s^2+2s+2)}$$

Multiplying both sides by $s.(s^2+2s+2)$, we get

$$\therefore\ 1 = A(s^2+2s+2)+(Bs+C)s \qquad ...(3)$$

Put $s = 0$ in (3)	Comparing s^2 coefficients :	Comparing s coeff. :
$1 = 2A$	$0 = A + B$	$0 = 2A + C$
$\Rightarrow\ A = \frac{1}{2}$	Substitute A value we get	Substitute A value
	$B = -\frac{1}{2}$	$C = -1$

$$L\left[\frac{1}{t}\int_0^t e^{-t}\sin t\, dt\right] = \int_s^\infty \left[\frac{A}{s} + \frac{Bs + C}{(s^2+2s+2)}\right] ds$$

substitute A, B and C values, we get

$$= \int_s^\infty \left[\frac{\frac{1}{2}}{s} + \frac{-\frac{1}{2}s - 1}{(s^2+2s+2)}\right] ds = \frac{1}{2}\int_s^\infty \left[\frac{1}{s} - \frac{s+2}{(s^2+2s+2)}\right] ds$$

on resolving the integrand into partial fractions,

$$= \frac{1}{2}\int_s^\infty \left[\frac{1}{s} - \frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}\right] ds$$

$$= \frac{1}{2}\left[\log s - \frac{1}{2}\log[(s+1)^2+1] + \cot^{-1}(s+1)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{s}{\sqrt{s^2+2s+2}}\right) + \cot^{-1}(s+1)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{\sqrt{s^2+2s+2}}{s}\right) - \cot^{-1}(s+1)\right].$$

(v) $L\left[\int_0^t te^{-t}\sin t\, dt\right] = \frac{1}{s}L\left[te^{-t}\sin t\right]$ **(May'15)**

by the theorem of LT of integral

$$= \frac{1}{s}\left\{L\left[t\sin t\right]_{s\to(s+1)}\right\}$$

$$= \frac{1}{s}\left\{\left[(-1)\frac{d}{ds}.\frac{1}{s^2+1}\right]_{s\to(s+1)}\right\}$$

$$= \frac{1}{s}\left\{\left[-\frac{2s}{(s^2+1)^2}\right]_{s\to(s+1)}\right\}$$

$$= \frac{1}{s}\left[-\frac{2(s+1)}{[(s+1)^2+1]^2}\right]$$

$$= -\frac{2(s+1)}{s(s^2+2s+2)^2}.$$

Exercise

Evaluate the following integrals using Laplace transforms

(i) $\int_0^\infty te^{-3t}\cos 2t\ dt$ (ii) $\int_0^\infty t^2e^{-t}\sin t\ dt$ (iii) $\int_0^\infty \frac{e^{-at}-\cos bt}{t}\ dt$

(iv) $\int_0^\infty \frac{e^{-t}(1-\cos t)}{t}\ dt$ (v) $\int_0^\infty \frac{e^{-t}\sin\sqrt{3}t}{t}\ dt$

(vi) $\int_0^\infty \frac{e^{-2t}-e^{-4t}}{t}\ dt$ (vii) $\int_0^\infty \frac{e^{-t}-e^{-3t}}{t}\ dt$

Answers

(i) $\frac{5}{169}$ (ii) $\frac{1}{2}$ (iii) $\log\left(\frac{b}{a}\right)$ (iv) $\frac{1}{2}\log 2$ (v) $\frac{\pi}{3}$ (vi) $\log 2$, (vii) $\log 3$.

Exercise

Evaluate the following Laplace transforms of integrals

(i) $\int_0^t te^{-t}\,dt$ (ii) $\int_0^t e^{-2t}\sin t\,dt$ (iii) $\int_0^t t\sin t\,dt$ (iv) $\int_0^t te^{-4t}\sin 3t\,dt$

(v) $\int_0^t \frac{1-e^{-t}}{t}\ dt$ (vi) $\int_0^t \frac{e^t\sin t}{t}\ dt$ (vii) $e^{-2t}\int_0^t t\sin 2t\ dt$

(viii) $e^{-t}\int_0^t t^2\cos t\,dt$ (ix) $e^{-2t}\int_0^t \left(\frac{1-\cos t}{t}\right)dt$ (x) $e^{-t}\int_0^t \frac{\sin t}{t}\ dt$

(xi) $t\int_0^t e^{-2t}\sin 2t\ dt$ (xii) $t\int_0^t e^{-4t}\sin 3t\ dt.$

Answers

(i) $\frac{1}{s(s+1)^2}$ (ii) $\frac{1}{s(s^2+2s+2)}$ (iii) $\frac{2}{(s^2+1)^2}$ (iv) $\frac{6(s+4)}{s(s^2+8s+25)^2}$

(v) $\frac{1}{s}\log\left(\frac{s-1}{s}\right)$ (vi) $\frac{1}{s}\cot^{-1}(s-1)$ (vii) $\frac{4}{(s^2+4s+8)^2}$

(viii) $\frac{2(s^2+2s-2)}{(s^2+2s+2)^3}$ (ix) $\frac{1}{2s+4}\log\left(\frac{s^2+4s+5}{s^2+4s+4}\right)$ (x) $\frac{1}{s+1}\cot^{-1}(s+1)$

(xi) $\frac{2(3s^2+8s+8)}{s^2(s^2+4s+8)^2}$ (xii) $\frac{3(3s^2+16s+25)}{s^2(s^2+8s+25)^2}$.

3.10 Laplace Transforms of Periodic Functions

Periodic Functions:

A function $f(t)$ is said to be periodic with period T if for all values of t, $f(t+T) = f(t)$. The least value of $T > 0$ is called period.

For example, $\sin t$, $\cos t$ are periodic functions with period 2π and $\tan t$ is also a periodic function with period π.

Theorem: If $f(t)$ is a periodic function with period T, then

$$L[f(t)] \;=\; \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\;dt \qquad \textbf{(Nov.'07)}$$

Proof: By the definition of Laplace transform,

$$L[f(t)] \;=\; \int_0^\infty e^{-st}f(t)\;dt \;=\; \int_0^T e^{-st}f(t)\;dt + \int_T^\infty e^{-st}f(t)\;dt$$

In the second integral Putting $t = x + T$ and differentiating $dt = dx$. When $t = T$, $x = 0$ and $t = \infty$, $x = \infty$ then the second integral is

$$\begin{aligned}
L[f(t)] &= \int_0^T e^{-st}f(t)\;dt + \int_0^\infty e^{-s(T+x)}f(T+x)\;dx \\
&= \int_0^T e^{-st}f(t)\;dt + e^{-sT}\int_0^\infty e^{-sx}f(x)\;dx \\
&\qquad\qquad\qquad [\because\; f(T+x) = f(x)] \\
&= \int_0^T e^{-st}f(t)\;dt + e^{-sT}\int_0^\infty e^{-st}f(t)\;dt \\
\therefore\; L[f(t)] &= \int_0^T e^{-st}f(t)\;dt + e^{-sT}L[f(t)]
\end{aligned}$$

$$\begin{aligned}
\text{i.e., } L[f(t)] - e^{-sT}L[f(t)] &= \int_0^T e^{-st}f(t)\;dt \\
\Rightarrow\; L[f(t)] &= \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\;dt.
\end{aligned}$$

Example 3.10.1. Find the Laplace transform of

$$f(t) = \begin{cases} t, & \text{for } 0 \le t \le a; \\ 2a - t, & \text{for } a \le t \le 2a. \end{cases} \quad \text{and } f(t+2a) = f(t) \text{ for all t.}$$

(**Nov.'05, Nov.'07, Nov.'10, Nov.'12, Apr.'14, Jan.'15**)

Solution . Given function $f(t)$ has a period $T = 2a$. W.k.t Laplace transform for the the periodic function is,

$$\begin{aligned} L[f(t)] &= \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\ dt = \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)\ dt \\ &= \frac{1}{1-e^{-2as}} \left[\int_0^a e^{-st}\ t\ dt + \int_a^{2a} e^{-st}(2a-t)\ dt \right] \end{aligned}$$

$$\begin{array}{r|c|c} u = t & dv = e^{-st}\ dt & U = (2a-t) \\ u' = 1 & v = e^{-st}/(-s) & U' = -1 \\ u'' = 0 & v_1 = e^{-st}/s^2 & U'' = 0 \end{array}$$

$$\begin{aligned} &= \frac{1}{1-e^{-2as}} \left\{ \left[uv - u'v_1\right]_0^a + \left[Uv - U'v_1\right]_a^{2a} \right\} \\ &= \frac{1}{1-e^{-2as}} \left\{ \left[\frac{t\ e^{-st}}{-s} - \frac{e^{-st}}{s^2} \right]_0^a + \left[\frac{(2a-t)\ e^{-st}}{-s} + \frac{e^{-st}}{s^2} \right]_a^{2a} \right\} \\ &= \frac{1}{1-e^{-2as}} \left\{ \left[\frac{ae^{-as}}{-s} - \frac{e^{-as}}{s^2} + \frac{1}{s^2} \right] + \left[\frac{e^{-2as}}{s^2} - \frac{ae^{-as}}{-s} - \frac{e^{-as}}{s^2} \right] \right\} \\ &= \frac{1}{1-(e^{-as})^2} \left[\frac{e^{-2as} - 2e^{-as} + 1}{s^2} \right] \\ &= \frac{1}{(1+e^{-as})(1-e^{-as})} \left[\frac{(1-e^{-as})^2}{s^2} \right] = \frac{1}{s^2} \left[\frac{1-e^{-as}}{1+e^{-as}} \right] \\ &= \frac{1}{s^2} \left[\frac{e^{as/2} - e^{-as/2}}{e^{as/2} + e^{-as/2}} \right] = \frac{1}{s^2} \tanh\left(\frac{as}{2}\right). \end{aligned}$$

Example 3.10.2. Find the Laplace transform of

$$f(t) = \begin{cases} t, & \text{for } 0 \le t \le 1; \\ 2 - t, & \text{for } 1 \le t \le 2. \end{cases} \quad \text{and } f(t+2) = f(t) \text{ for all t.}$$

(**Nov.'06, Nov.'10**)

Solution . Given function $f(t)$ has a period $T = 2$. Replace $a = 1$ in the above worked example and proceed the steps , we get

$$L[f(t)] \;=\; \frac{1}{s^2}\,\tanh\left(\frac{s}{2}\right).$$

Example 3.10.3. Find the Laplace transform of

$$f(t) = \begin{cases} t, & \text{for } 0 \le t \le 2; \\ 4-t, & \text{for } 2 \le t \le 4. \end{cases} \quad \text{and} \quad f(t+4) = f(t).$$

(**Apr.'03**)

Solution . Given function $f(t)$ has a period $T = 4$. Replace $a = 2$ in the above worked example and proceed the steps as in the above, we get

$$L[f(t)] \;=\; \frac{1}{s^2}\,\tanh(s).$$

Example 3.10.4. Find the Laplace transform of

$$f(t) = \begin{cases} K, & \text{for } 0 < t < a; \\ -K, & \text{for } a < t < 2a. \end{cases} \quad \text{and} \quad f(t+2a) = f(t).$$

(**Apr.'14**)

Solution . Given function $f(t)$ has a period $T = 2a$. We know that the Laplace transform for the the periodic function is,

$$\begin{aligned}
L[f(t)] &= \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt \;=\; \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st}f(t)\,dt \\
&= \frac{1}{1-e^{-2as}}\left[\int_0^a e^{-st}(K)\,dt + \int_a^{2a} e^{-st}(-K)\,dt\right] \\
&= \frac{K}{1-e^{-2as}}\left\{\left[-\frac{e^{-st}}{s}\right]_0^a - \left[-\frac{e^{-st}}{s}\right]_a^{2a}\right\} \\
&= \frac{K}{1-e^{-2as}}\left\{\left[-\frac{e^{-as}}{s}+\frac{1}{s}\right] - \left[-\frac{e^{-2as}}{s}+\frac{e^{-as}}{s}\right]\right\} \\
&= \frac{K}{1-e^{-2as}}\left[\frac{1-2e^{-2as}+e^{-2as}}{s}\right] \\
&= \frac{K}{s}\left[\frac{(1-e^{-as})^2}{1-(e^{-as})^2}\right] \;=\; \frac{K}{s}\left[\frac{1-e^{-as}}{1+e^{-as}}\right] \\
&= \frac{K}{s}\left[\frac{e^{as/2}-e^{-as/2}}{e^{as/2}+e^{-as/2}}\right] \;=\; \frac{K}{s}\,\tanh\left[\frac{as}{2}\right]
\end{aligned}$$

Example 3.10.5. Find the Laplace transform of the square wave function $f(t) = \begin{cases} E, & \text{for } 0 \le t \le \frac{a}{2}; \\ -E, & \text{for } \frac{a}{2} \le t \le a. \end{cases}$ with period a.

(**May'02 , Nov.'04, Nov.'14**)

Solution . Given function $f(t)$ has a period $T = a$. W.k.t Laplace transform for the the periodic function is,

$$\begin{aligned}
L[f(t)] &= \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\ dt \ = \ \frac{1}{1-e^{-as}} \int_0^a e^{-st} f(t)\ dt \\
&= \frac{1}{1-e^{-as}} \left[\int_0^{a/2} e^{-st}\ E\ dt + \int_{a/2}^{a} e^{-st}(-E)\ dt \right] \\
&= \frac{E}{1-e^{-as}} \left\{ \left[-\frac{e^{-st}}{s} \right]_0^{a/2} - \left[-\frac{e^{-st}}{s} \right]_{a/2}^{a} \right\} \\
&= \frac{E}{1-e^{-as}} \left\{ \left[-\frac{e^{-as/2}}{s} + \frac{1}{s} \right] - \left[-\frac{e^{-as}}{s} + \frac{e^{-as/2}}{s} \right] \right\} \\
&= \frac{E}{1-e^{-as}} \left[\frac{1 - 2e^{-as/2} + e^{-as}}{s} \right] = \frac{E}{s} \left[\frac{\left(1-e^{-as/2}\right)^2}{1-\left(e^{-as/2}\right)^2} \right] \\
&= \frac{E}{s} \left[\frac{1-e^{-as/2}}{1+e^{-as/2}} \right] \\
&= \frac{E}{s} \left[\frac{e^{as/4} - e^{-as/4}}{e^{as/4} + e^{-as/4}} \right] \\
&= \frac{E}{s} \tanh \left[\frac{as}{4} \right].
\end{aligned}$$

Example 3.10.6. Find the Laplace transform of the function $f(t) = \begin{cases} 1, & \text{for } 0 \le t \le \frac{a}{2}; \\ -1, & \text{for } \frac{a}{2} \le t \le a. \end{cases}$ and $f(t+a) = f(t)$ for $t > 0$.

(**May.'02, Apr.'08, Nov.'09**)

Solution . The given function $f(t)$ has a period $T = a$. Replace $E = 1$ in the above worked example and proceed the steps as in the above, we get

$$L[f(t)] = \frac{1}{s} \tanh \left[\frac{as}{4} \right]$$

Example 3.10.7. Find the Laplace transform of

$$f(t) = \begin{cases} -E, & \text{for } 0 < t < \pi; \\ E, & \text{for } \pi < t < 2\pi. \end{cases} \quad \text{and } f(t + 2\pi) = f(t).$$

(Apr.'12, May'15)

Solution . Given function $f(t)$ has a period $T = 2\pi$. We know that the Laplace transform for the the periodic function is,

$$\begin{aligned}
L[f(t)] &= \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)\, dt = \frac{1}{1 - e^{-2\pi s}} \int_0^{2\pi} e^{-st} f(t)\, dt \\
&= \frac{1}{1 - e^{-2\pi s}} \left[\int_0^{\pi} e^{-st}(-E)\, dt + \int_{\pi}^{2\pi} e^{-st} E\, dt \right] \\
&= \frac{E}{1 - e^{-2\pi s}} \left\{ -\left[-\frac{e^{-st}}{s} \right]_0^{\pi} + \left[-\frac{e^{-st}}{s} \right]_{\pi}^{2\pi} \right\} \\
&= \frac{E}{1 - e^{-2\pi s}} \left\{ -\left[-\frac{e^{-\pi s}}{s} + \frac{1}{s} \right] + \left[-\frac{e^{-2\pi s}}{s} + \frac{e^{-\pi s}}{s} \right] \right\} \\
&= \frac{E}{1 - e^{-2\pi s}} \left[\frac{2e^{-\pi s} - 1 - e^{-2\pi s}}{s} \right] \\
&= \frac{-E}{1 - e^{-2\pi s}} \left[\frac{1 - 2e^{-\pi s} + e^{-2\pi s}}{s} \right] \\
&= \frac{-E}{s} \left[\frac{(1 - e^{-\pi s})^2}{1 - (e^{-\pi s})^2} \right] = \frac{-E}{s} \left[\frac{1 - e^{-\pi s}}{1 + e^{-\pi s}} \right] \\
&= \frac{-E}{s} \left[\frac{e^{\pi s/2} - e^{-\pi s/2}}{e^{\pi s/2} + e^{-\pi s/2}} \right] = \frac{-E}{s} \tanh\left[\frac{\pi s}{2} \right]
\end{aligned}$$

Example 3.10.8. Find the Laplace transform of

$$f(t) = \begin{cases} t, & \text{for } 0 \le t \le \pi; \\ 2\pi - t, & \text{for } \pi \le t \le 2\pi \end{cases} \quad \text{and } f(t + 2a) = f(t) \text{ for all t.}$$

(Jan.'16)

Solution . Given function $f(t)$ has a period $T = 2\pi$. W.k.t Laplace transform for the the periodic function is,

$$\begin{aligned}
L[f(t)] &= \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)\, dt = \frac{1}{1 - e^{-2\pi s}} \int_0^{2\pi} e^{-st} f(t)\, dt \\
&= \frac{1}{1 - e^{-2\pi s}} \left[\int_0^{\pi} e^{-st}\, t\, dt + \int_{\pi}^{2\pi} e^{-st}(2\pi - t)\, dt \right]
\end{aligned}$$

$$\begin{array}{r|c|c} u = t & dv = e^{-st}\ dt & U = (2\pi - t) \\ u' = 1 & v = e^{-st}/(-s) & U' = -1 \\ u'' = 0 & v_1 = e^{-st}/s^2 & U'' = 0 \end{array}$$

$$\begin{aligned}
&= \frac{1}{1-e^{-2\pi s}} \left\{ \left[uv - u'v_1\right]_0^{\pi} + \left[Uv - U'v_1\right]_{\pi}^{2\pi} \right\} \\
&= \frac{1}{1-e^{-2\pi s}} \left\{ \left[\frac{t\ e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_0^{\pi} + \left[\frac{(2\pi - t)\ e^{-st}}{-s} + \frac{e^{-st}}{s^2}\right]_{\pi}^{2\pi} \right\} \\
&= \frac{1}{1-e^{-2\pi s}} \left\{ \left[\frac{\pi e^{-\pi s}}{-s} - \frac{e^{-\pi s}}{s^2} + \frac{1}{s^2}\right] + \left[\frac{e^{-2\pi s}}{s^2} - \frac{\pi e^{-\pi s}}{-s} - \frac{e^{-\pi s}}{s^2}\right] \right\} \\
&= \frac{1}{1-(e^{-\pi s})^2} \left[\frac{e^{-2\pi s} - 2e^{-\pi s} + 1}{s^2}\right] \\
&= \frac{1}{(1+e^{-\pi s})(1-e^{-\pi s})} \left[\frac{(1-e^{-\pi s})^2}{s^2}\right] = \frac{1}{s^2}\left[\frac{1-e^{-\pi s}}{1+e^{-\pi s}}\right] \\
&= \frac{1}{s^2}\left[\frac{e^{\pi s/2} - e^{-\pi s/2}}{e^{\pi s/2} + e^{-\pi s/2}}\right] = \frac{1}{s^2}\ \tanh\left(\frac{\pi s}{2}\right).
\end{aligned}$$

Example 3.10.9. Find the Laplace transform of

$$f(t) = \begin{cases} \sin wt, & \text{for } 0 < t < \frac{\pi}{w}; \\ 0, & \text{for } \frac{\pi}{w} < t < \frac{2\pi}{w}. \end{cases} \quad \text{with period } \frac{2\pi}{w}.$$ (**Apr.'03**)

Solution . Given function $f(t)$ has a period $T = \frac{2\pi}{w}$. W.k.t Laplace transform for the the periodic function is,

$$\begin{aligned}
L[f(t)] &= \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\ dt \\
&= \frac{1}{1-e^{-2s\pi/w}} \int_0^{2\pi/w} e^{-st} f(t)\ dt \\
&= \frac{1}{1-e^{-2s\pi/w}} \int_0^{\pi/w} e^{-st} \sin wt\ dt + 0 \\
&= \frac{1}{1-e^{-2s\pi/w}} \left[\frac{e^{-st}}{s^2+w^2}\left(-s\sin wt - w\cos wt\right)\right]_0^{\pi/w} \\
&= \frac{1}{1-e^{-2s\pi/w}} \left[\frac{e^{-s\pi/w}}{s^2+w^2}(0+w) - \frac{1}{s^2+w^2}(0-w)\right]
\end{aligned}$$

$$\therefore\ L[f(t)] \ = \ \frac{1}{1-e^{-2s\pi/w}}\ \frac{[1+e^{-s\pi/w}]w}{[s^2+w^2]} \ = \ \frac{w}{\left[1-e^{-s\pi/w}\right][s^2+w^2]}.$$

Example 3.10.10. Find the Laplace transform of

$$f(t) = \begin{cases} \sin t, & \text{for } 0 < t < \pi; \\ 0, & \text{for } \pi < t < 2\pi. \end{cases} \text{ with period } 2\pi. \qquad \textbf{(Nov.'11)}$$

Solution . The given function $f(t)$ has a period $T = 2\pi$. Replace $w = 1$ in the above worked example and proceed the steps as in the above, we get $$L[f(t)] \ = \ \frac{1}{[1-e^{-s\pi}]\,[s^2+1]}.$$

Exercise

Find the Laplace transform of the function

(i) $f(t) = \begin{cases} 1, & \text{for } 0 < t < \pi; \\ -1, & \text{for } \pi < t < 2\pi. \end{cases}$ and $f(t+2\pi) = f(t)$.

(ii) $f(t) = \begin{cases} t, & \text{for } 0 < t < \pi; \\ 0, & \text{for } \pi < t < 2\pi. \end{cases}$ and $f(t+2\pi) = f(t)$.

(iii) $f(t) = \begin{cases} Et/a, & 0 < t < a; \\ \frac{Et}{a}(2a-t), & a < t < 2a. \end{cases}$ and $f(t+2a) = f(t)$.

(iv) $f(t) = \begin{cases} \cos t, & \text{for } 0 < t < \pi; \\ 0, & \text{for } \pi < t < 2\pi. \end{cases}$ and $f(t+2\pi) = f(t)$.

(v) saw-tooth wave function $f(t)$, which is periodic with period 1 and defined as $f(t) = kt$, in $0 < t < 1$.

(vi) full-sine wave rectifier function $f(t) = |\sin \omega t|$, $t \geq 0$ with period $\frac{\pi}{\omega}$.

(vi) $f(t) = |\cos \omega t|$, $t \geq 0$ with period $\frac{\pi}{\omega}$.

Answers

(i) $\frac{1}{s}\tanh\left(\frac{s\pi}{2}\right)$ (ii) $\frac{1}{1-e^{-2\pi s}}\left[\frac{1-e^{-\pi s}}{s^2} - \frac{\pi e^{-s\pi}}{s}\right]$ (iii) $\frac{E}{as^2}\tanh\left(\frac{sa}{2}\right)$

(iv) $\frac{s}{(s^2+1)(1-e^{-\pi s})}$ (v) $\frac{k}{s^2} - \frac{ke^{-s}}{s(1-e^{-s})}$

(vi) $\frac{\omega}{s^2+\omega^2}\coth\left[\frac{\pi s}{2\omega}\right]$ (vii) $\frac{1}{s^2+\omega^2}\left[s + \omega\text{cosech}\left(\frac{\pi s}{2\omega}\right)\right]$.

3.11 Laplace Transform - Two Marks

Question 3.11.1. Define Laplace transform. **(Nov.'10, Jan.'15)**

Solution . Let $f(t)$ is a function of t defined for $t \geq 0$ then the Laplace transform of $f(t)$, denoted by $L[f(t)]$ or $F(s)$, defined as

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\ dt = F(s), \quad \text{provided the integral exists.}$$

Question 3.11.2. State existence conditions of Laplace transforms.

Statement : If $f(t)$ be the function defined on $t \geq 0$ is,
(i) a piecewise continuous in every finite interval in the range $t \geq 0$.
(ii) of the exponential order [i.e., $\lim_{t\to\infty} e^{-at} f(t)$ is finite]
then $L[f(t)]$ exists.

Question 3.11.3. State the conditions under which $L[f(t)]$ exists.

Solution . If $f(t)$ is a piecewise continuous on $t \geq 0$ and is of exponential order, i.e., $\lim_{t\to\infty} f(t) = a$ finite quantity.

Question 3.11.4. Give any two examples that the functions have no Laplace transform.

Solution . The functions e^{t^2} and $\tan t$ have no Laplace transform. Since the exponential order of these functions do not exist.

$$\begin{aligned} \lim_{t\to\infty} e^{-st} e^{t^2} &= \lim_{t\to\infty} \frac{e^{t^2}}{e^{st}} = \infty \\ \lim_{t\to\infty} e^{-st} \tan t &= \lim_{t\to\infty} \frac{\tan t}{e^{st}} = \frac{\infty}{\infty}. \end{aligned}$$

Question 3.11.5. Give an example of a function such that if it has Laplace transform but does not satisfy the continuity condition.

Solution . The function $f(t) = t^{-1/2}$ then $L[t^{-1/2}] = \sqrt{\frac{\pi}{s}}$. But the function $t^{-1/2} \to \infty$ as $t \to 0$ from the right hand side. So that $t^{-1/2}$ is not piecewise continuous on every finite interval in $(0, \infty)$.

Question 3.11.6. Can the Laplace transform of $\frac{\cos at}{t}$ exist?

Solution .

$$\text{No. Since } \lim_{t\to\infty} e^{-st}\frac{\cos at}{t} \;=\; \lim_{t\to\infty}\frac{\cos 2t}{te^{st}} \;=\; \infty.$$

Question 3.11.7. Prove that: $L[1] = \frac{1}{s};\ s > 0$ **(May'12)**

Proof:

$$L[1] \;=\; \int_0^\infty e^{-st}.1\ dt \;=\; \left[\frac{e^{-st}}{-s}\right]_0^\infty \;=\; \left[0 - \frac{1}{-s}\right] \;=\; \frac{1}{s}.$$

Question 3.11.8. Prove that: $L[e^{-at}] = \frac{1}{s+a}$, if $s > -a$. **(Nov'11)**

Proof:

$$L[e^{at}] \;=\; \int_0^\infty e^{-(s+a)t}\,dt \;=\; \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty \;=\; \frac{1}{s+a}.$$

Question 3.11.9. $L[\cos at] = \frac{s}{s^2+a^2}$, if $s > 0$.

Proof:

$$\begin{aligned} L[\cos at] &= \int_0^\infty e^{-st}.\cos at\,dt \\ &= \left[\frac{e^{-st}}{s^2+a^2}(-s\cos at + a.\sin at)\right]_0^\infty \\ &= \frac{s}{s^2+a^2}. \end{aligned}$$

Question 3.11.10. Evaluate $\int_0^\infty e^{-2t}\cos 3t\ dt$ **(Nov'10)**

Solution .

$$\int_0^\infty e^{-2t}\cos 3t\ dt \;=\; L[\cos 3t]_{s=2} \;=\; \left[\frac{s}{s^2+9}\right]_{s=2} \;=\; \frac{2}{13}.$$

Question 3.11.11. Find $L[5^t]$. **(Jan.'16)**

Answer .

$$L[5^t] \;=\; L[e^{\log 5^t}] \;=\; L[e^{t\log 5}] \;=\; \frac{1}{s-\log 5}$$

Question 3.11.12. Find $L[2^t]$.

Answer . $L[2^t] \;=\; L[e^{\log 2^t}] \;=\; L[e^{t\log 2}] \;=\; \frac{1}{s-\log 2}$

Question 3.11.13. Find $L\left[t^2-2\right]$. **(Nov.'12)**

Solution .

$$L\left[t^2-2\right] = L\left[t^2\right]-L[2] = \frac{2!}{s^3}-\frac{2}{s}.$$

Question 3.11.14. Find $L\left[t^3-3t^2+2\right]$. **(Apr.'11)**

Solution .

$$L\left[t^3-3t^2+2\right] \;=\; L\left[t^3\right]-3L\left[t^2\right]+L[2] = \frac{6}{s^4}-\frac{6}{s^3}+\frac{2}{s}.$$

Question 3.11.15. Find $L\left[(e^t+e^{-t})^2\right]$.
Solution .

$$L\left[(e^t+e^{-t})^2\right] \;=\; L\left[e^{2t}+2+e^{-2t}\right] = \frac{1}{s-2}+\frac{2}{s}+\frac{1}{s+2}.$$

Question 3.11.16. Find $L\left[\frac{1+2t}{\sqrt{t}}\right]$.
Solution .

$$\begin{aligned}L\left[\frac{1+2t}{\sqrt{t}}\right] &= L\left[t^{-1/2}+2t^{1/2}\right] \;=\; \frac{\Gamma(1/2)}{s^{1/2}}+2\frac{\Gamma(3/2)}{s^{3/2}}\\ &= \frac{\sqrt{\pi}}{\sqrt{s}}+\frac{2.\frac{1}{2}\sqrt{\pi}}{\sqrt{s}} \quad \left[\Gamma\frac{1}{2}=\sqrt{\pi},\ \Gamma(n+1)=n\Gamma n\right]\\ &= \sqrt{\frac{\pi}{s}}\left[1+\frac{1}{s}\right].\end{aligned}$$

Question 3.11.17. Find Laplace transform of the function $\frac{1}{1-t}$.

Solution . W.k.t. $\frac{1}{1-t} \;=\; [1-t]^{-1} \;=\; 1+t+t^2+t^3+t^4+\ ...$

$$\begin{aligned}\therefore\ L\left[\frac{1}{1-t}\right] &= L[1]+L[t]+L[t^2]+L[t^3]+L[t^4]+\ ...\\ &= \frac{1}{s}+\frac{1!}{s^2}+\frac{2!}{s^3}+\frac{3!}{s^4}+\ ...\ = \sum_{n=0}^{\infty}\frac{n!}{s^{n+1}}.\end{aligned}$$

Question 3.11.18. Find $L[\sin^2 3t]$.
Solution .

$$\begin{aligned} L[\sin^2 3t] &= L\left[\frac{1-\cos 2(3t)}{2}\right] = \frac{1}{2}\left[L(1) - L(\cos 6t)\right] \\ &= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+36}\right] = \frac{18}{s(s^2+36)}. \end{aligned}$$

Question 3.11.19. Find $L[\cos^3 2t]$.
Solution .

$$\begin{aligned} L[\cos^3 2t] &= L\left[\frac{3\cos 2t + \cos 3(2t)}{4}\right] = \frac{1}{4}\left[3L(\cos 2t) + L(\cos 6t)\right] \\ &= \frac{1}{4}\left[\frac{3s}{s^2+4} + \frac{s}{s^2+36}\right] = \frac{s(s^2+28)}{(s^2+4)(s^2+36)}. \end{aligned}$$

Question 3.11.20. Find $L[\sin 3t \cos 4t]$. **(May'15)**

Solution .

$$\begin{aligned} L[\sin 3t \cos 4t] &= L\left[\frac{1}{2}\left(\sin 7t + \sin(-t)\right)\right] = \frac{1}{2}\left[L(\sin 7t) - L(\sin t)\right] \\ &= \frac{1}{2}\left[\frac{7}{s^2+49} - \frac{1}{s^2+1}\right] = \frac{3(s^2-7)}{(s^2+49)(s^2+1)}. \end{aligned}$$

Question 3.11.21. Find $L[\sin 3t \cos t]$. **(Apr.'13)**
Solution .

$$\begin{aligned} L[\sin 3t \cos t] &= L\left[\frac{1}{2}\left(\sin 4t + \sin 2t\right)\right] = \frac{1}{2}\left[L(\sin 4t) - L(\sin 2t)\right] \\ &= \frac{1}{2}\left[\frac{4}{s^2+16} - \frac{2}{s^2+4}\right] = \frac{s^2-8}{(s^2+16)(s^2+25)}. \end{aligned}$$

Question 3.11.22. Find $L[\sin t \sin 2t]$.
Solution .

$$\begin{aligned} L[\sin t \sin 2t] &= L\left[\frac{1}{2}\left(\cos t - \cos 3t\right)\right] = \frac{1}{2}\left[L(\cos t) - L(\cos 3t)\right] \\ &= \frac{1}{2}\left[\frac{s}{s^2+1} - \frac{s}{s^2+9}\right] = \frac{4s}{(s^2+1)(s^2+9)}. \end{aligned}$$

Question 3.11.23. Find $L[\cos(at+b)]$.

Solution .

$$\begin{aligned} L[\cos(at+b)] &= L\left[\cos at \cos b - \sin at \sin b\right] \\ &= \cos b\ L\left[\cos at\right] - \sin b\ L\left[\sin at\right] \\ &= \cos b \frac{s}{s^2+a^2} - \sin b \frac{s}{s^2+a^2}. \end{aligned}$$

Question 3.11.24. Find $L[t\cos at]$. **(Nov'10)**

Solution .

$$\begin{aligned} L[t\cos at] &= -\frac{d}{ds}L[\cos at] = -\frac{d}{ds}\left[\frac{s}{s^2+a^2}\right] \\ &= -\left[\frac{(s^2+a^2).1 - s.2s}{(s^2+a^2)^2}\right] = -\left[\frac{(s^2+a^2)-2s^2}{(s^2+a^2)^2}\right] \\ &= -\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right] = \left[\frac{s^2-a^2}{(s^2+a^2)^2}\right]. \end{aligned}$$

Question 3.11.25. Find $L[t\sin 2t]$. **(Nov'15)**

Solution .

$$\begin{aligned} L[t\sin 2t] &= -\frac{d}{ds}L[\sin 2t] = -\frac{d}{ds}\left[\frac{s}{s^2+4}\right] \\ &= -\left[\frac{(s^2+4).1 - s.2s}{(s^2+4)^2}\right] = -\left[\frac{(s^2+4)-2s^2}{(s^2+4)^2}\right] \\ &= -\left[\frac{4-s^2}{(s^2+4)^2}\right] = \left[\frac{s^2-4}{(s^2+4)^2}\right]. \end{aligned}$$

Question 3.11.26. Find $L[e^{-2t}t^2]$. **(Apr.'14)**

Solution .

$$L[e^{-2t}t^2] = L[t^2]_{s\to(s+2)} = \left[\frac{2}{s^3}\right]_{s\to(s+2)} = \frac{2}{(s+2)^3}.$$

Question 3.11.27. Find Laplace transform of Unit Step Function. **(Apr.'11, Apr.'14, Apr.'15)**

Solution . The Unit Step Function (or) Heavy Side's Function is defined as $U(t-a) = \begin{cases} 0, & \text{for } t < a; \\ 1, & \text{for } t \geq a. \end{cases}$ then

$$\begin{aligned} L\left[U(t-a)\right] &= \int_0^\infty e^{-st} U(t-a)\ dt \\ &= \int_0^a e^{-st}.0\ dt + \int_a^\infty e^{-st}.1\ dt \\ &= 0 + \left[\frac{e^{-st}}{-s}\right]_a^\infty \\ \therefore\ L\left[U(t-a)\right] &= \frac{e^{-as}}{s}. \end{aligned}$$

Question 3.11.28. Find Laplace transform of dirac delta function $\delta(t-a)$.

Solution .

$$L[\delta(t-a)] = \int_0^\infty e^{-st}\delta(t-a)\,dt = e^{-sa} \quad \left[\because \int_0^\infty f(t)\delta(t-a) = f(a)\right]$$

Question 3.11.29. Find $L\left[\frac{t}{e^t}\right]$. **(Nov.'13)**

Solution .

$$L\left[\frac{t}{e^t}\right] = L\left[te^{-t}\right] = L\left[t\right]_{s\to(s+1)} = \left[\frac{1}{s^2}\right]_{s\to(s+1)} = \frac{1}{(s+1)^2}.$$

Question 3.11.30. Find $L\left[e^{-t}\sin 3t\right]$.

Solution .

$$L\left[e^{-t}\sin 3t\right] = L\left[\sin 3t\right]_{s\to(s+1)} = \left[\frac{3}{s^2+9}\right]_{s\to(s+1)} = \frac{3}{s^2+2s+10}.$$

Question 3.11.31. Find $L\left[e^{-2t}\cos 3t\right]$.

Solution .

$$L\left[e^{-2t}\cos 3t\right] = L\left[\cos 3t\right]_{s\to(s+2)} = \left[\frac{s}{s^2+9}\right]_{s\to(s+2)} = \frac{s+2}{s^2+4s+13}.$$

Question 3.11.32. Find $L\left[t^2 2^t\right]$.

Solution .

$$L\left[t^2 2^t\right] = L\left[t^2 e^{t\log 2}\right] = L\left[t^2\right]_{s\to(s-\log 2)} = \frac{2}{(s-\log 2)^3}.$$

Question 3.11.33. Find $L\left[e^t \sin^2 t\right]$.

Solution .

$$\begin{aligned}L\left[e^t\sin^2 t\right] &= L\left[e^t\left(\frac{1-\cos 2t}{2}\right)\right] = \frac{1}{2}L\left[e^t(1-\cos 2t)\right]\\ &= \frac{1}{2}L\left[1-\cos 2t\right]_{s\to(s-1)} = \frac{1}{2}\left[\frac{1}{s}-\frac{s}{s^2+4}\right]_{s\to(s-1)}\\ &= \frac{1}{2}\left[\frac{1}{s-1}-\frac{s-1}{(s-1)^2+4}\right] = \frac{1}{2}\left[\frac{1}{s-1}-\frac{s-1}{s^2-2s+5}\right].\end{aligned}$$

Question 3.11.34. Find Laplace transform of $t\sin 2t$ **(Nov.'15)**

Solution .

$$\begin{aligned}L[t\sin 2t] &= (-1)\frac{d}{ds}\,L[\sin 2t] = (-1)\frac{d}{ds}\left[\frac{2}{s^2+4}\right]\\ &= (-1)\left[\frac{0-2.2s}{(s^2+4)^2}\right] = \frac{4s}{(s^2+4)^2}\end{aligned}$$

Question 3.11.35. Find the Laplace transform of $f(t) = te^{-2t}\sin 5t$.

Solution .

$$L\left[te^{-2t}\sin 5t\right] = L[t\sin 5t]_{s\to(s+2)} \qquad ...(1)$$

$$L[t\sin 5t] = (-1)\frac{d}{ds}\,L[\sin 5t] = (-1)\frac{d}{ds}\left[\frac{5}{s^2+25}\right] = \frac{10s}{(s^2+25)^2}...(2)$$

Using (2) in (1), we have

$$L\left[te^{-2t}\sin 5t\right] = L\left[\frac{10s}{(s^2+25)^2}\right]_{s\to(s+2)} = \frac{10(s+2)}{\left[s^2+4s+29\right]^2}.$$

Question 3.11.36. If $L[f(t)] = F(s)$, then prove that

$$L[f(t-a).u(t-a)] = e^{-as}F(s).$$

Proof:

$$\begin{aligned} L[f(t-a).u(t-a)] &= \int_0^\infty e^{-st} f(t-a)u(t-a)\ dt \\ &= \int_0^a e^{-st} f(t-a).(0)\ dt + \int_a^\infty e^{-st} f(t-a)\ dt \\ &= \int_0^\infty e^{-s(u+a)} f(u)\ du \quad [\text{Put } t-a=u] \\ &= e^{-as}\int_0^\infty e^{-su} f(u)\ du \\ &= e^{-as}F(s). \end{aligned}$$

Question 3.11.37. Find $L[(t-1)^3 u_1(t)]$.

Solution .

$$\begin{aligned} L[(t-1)^3 u_1(t)] &= L[(t-1)^3.u(t-1)], \quad \text{by second shifting} \\ &= e^{-as}L[t^3], \quad \text{where } a=1 \\ &= e^{-s}\frac{3!}{s^4} = \frac{6e^{-s}}{s^4}. \end{aligned}$$

Question 3.11.38. Find $L[4\sin(t-3)u(t-3)]$.

Solution .

$$\begin{aligned} L[4\sin(t-3)u(t-3)] &= 4L[\sin(t-3)u(t-3)] \quad \text{by second shifting} \\ &= 4e^{-as}L[\sin t], \quad \text{where } a=3 \\ &= \frac{4e^{-3s}}{s^2+1}. \end{aligned}$$

Question 3.11.39. Find $L[4u(t-\pi)\cos t]$.

Solution .

$$\begin{aligned} L[4u(t-\pi)\cos t] &= 4L\{u(t-\pi)\cos[(t-\pi)+\pi]\} \\ &\qquad \text{using second shifting property} \\ &= 4e^{-as}L[\cos(t+\pi)], \quad \text{where } a=\pi \end{aligned}$$

$$\begin{aligned} &= -4e^{-\pi s}L[\cos t] \\ &= -4e^{-\pi s}\left[\frac{s}{s^2+1}\right]. \end{aligned}$$

Question 3.11.40. If $L[f(t)] = F(s)$, prove $L\left[f(at)\right] = \frac{1}{a}\,F\left(\frac{s}{a}\right)$.
Proof:

$$\begin{aligned} L[f(at)] &= \int_0^\infty e^{-st}f(at)\ dt \ \ [\text{Put } at = u \ \ \therefore\ dt = du/a] \\ &= \int_0^\infty e^{-su/a}f(u)\ \frac{du}{a} \\ &= \frac{1}{a}\int_0^\infty e^{-(s/a)u}f(u)\ du \ = \ \frac{1}{a}F\left(\frac{s}{a}\right). \end{aligned}$$

Question 3.11.41. Find $L\left[\int_0^t \frac{\sin t}{t}\,dt\right]$ **(May'15)**

Answer .

$$\begin{aligned} L\left[\int_0^t \frac{\sin t}{t}\,dt\right] &= \frac{1}{s}L\left[\frac{\sin t}{t}\right] \\ &= \frac{1}{s}\int_s^\infty L(\sin t)\,ds \ = \ \frac{1}{s}\int_s^\infty \frac{ds}{s^2+1} \\ &= \frac{1}{s}\left[\tan^{-1}(s)\right]_s^\infty \\ &= \frac{1}{s}\left[\frac{\pi}{2} - \tan^{-1}s\right]. \end{aligned}$$

Question 3.11.42. State initial and final value theorems on Laplace transforms. **(Apr'11, Nov'11, Apr.'14, Jan.'15)**

Statement: If $L[f(t)]$, $L[f'(t)]$ are exist and $L[f(t)] = F(s)$ then

$$\begin{aligned} \text{(i)}\ \lim_{t\to\ 0} f(t) &= \lim_{s\to\ \infty} sF(s) \ \ [I.V.T] \\ \text{(ii)}\ \lim_{t\to\ \infty} f(t) &= \lim_{s\to\ 0} sF(s) \ \ [F.V.T] \end{aligned}$$

Question 3.11.43. Verify Initial value theorem for the following function $f(t) = 1 + e^{-t}[\sin t + \cos t]$. **(Nov'10, Apr'08)**

Solution . Given $f(t) = 1 + e^{-t}[\sin t + \cos t]$.

$$\begin{aligned} \therefore\ L[f(t)] &= L[1] + L\left\{e^{-t}[\sin t + \cos t]\right\} \\ \text{i.e., } F(s) &= \frac{1}{s} + \{L[\sin t] + L[\cos t]\}_{s\to s+1} \\ &= \frac{1}{s} + \left\{\frac{1}{s^2+1} + \frac{s}{s^2+1}\right\}_{s\to s+1} \\ &= \frac{1}{s} + \left\{\frac{1+s}{s^2+1}\right\}_{s\to s+1} \quad = \frac{1}{s} + \left\{\frac{s+2}{s^2+2s+2}\right\} \end{aligned}$$

$$\begin{aligned} \lim_{t\to 0} f(t) &= \lim_{t\to 0}\left\{1 + e^{-t}[\sin t + \cos t]\right\} \\ &= \left\{1 + e^{-0}[\sin 0 + \cos 0]\right\} \ = \ 1 + 1 = 2 \qquad \text{...(1)} \end{aligned}$$

$$\therefore\ \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} s\left[\frac{1}{s} + \frac{s+2}{s^2+2s+2}\right] = \lim_{s\to\infty}\left[1 + \frac{s^2+2s}{s^2+2s+2}\right]$$

$$\begin{aligned} &= \lim_{s\to\infty}\left\{1 + \left[\frac{s^2[1+(2/s)]}{s^2[1+(2/s)+(2/s^2)]}\right]\right\} \\ &= \lim_{s\to\infty}\left\{1 + \left[\frac{[1+(2/s)]}{[1+(2/s)+(2/s^2)]}\right]\right\} \\ &= 1 + \frac{1+0}{1+0+0} = 2\ . \qquad \text{...(2)} \end{aligned}$$

From (1) and (2) we have $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s) = 2$.
Hence Initial value theorem is verified.

Question 3.11.44. State Convolution theorem.
(Nov.'12, Apr.'14, Jan.'15, May'18)

Solution . Let $f(t)$ and $g(t)$ be two functions and

$$L^{-1}[F(s)] = f(t) \text{ and } L^{-1}[G(s)] = g(t)$$

then the Laplace transform of convolution of two functions is the product of their Laplace transforms.

$$\text{i.e., } \ L[f(t) * g(t)] = L[f(t)]L[g(t)] = F(s)G(s).$$

Question 3.11.45. Write the value of $1 * e^t$.
Solution .

$$1 * e^t \;=\; \int_0^t 1 * e^{t-u}\,du \;=\; \left[\frac{e^{t-u}}{-1}\right]_0^t \;=\; e^t - 1.$$

Question 3.11.46. Define periodic function $f(t)$. **(Apr.'14, Jan.'15)**

Answer . A function $f(t)$ is said to be periodic with period T, if for all T $f(t+T) = f(t)$; $T > 0$. The least value of T is called as period of $f(t)$.

Question 3.11.47. If $f(t)$ is a periodic function with period p, what is its Laplace transform?

Solution .

$$L[f(t)] \;=\; \frac{1}{1-e^{ps}}\int_0^p e^{-st}f(t)\,dt, \quad \text{where } p \text{ is the given period.}$$

Question 3.11.48. Evaluate: $\int_0^\infty te^{-3t}\sin 2t\;dt$.

Solution .

$$\int_0^\infty te^{-3t}\sin 2t\;dt \;=\; L[t\sin 2t]_{s=3} \;=\; \left[\frac{4s}{(s^2+4)^2}\right]_{s=3} \;=\; \frac{12}{169}.$$

Question 3.11.49. Using Laplace transform evaluate, $\int_0^\infty te^{-2t}\cos t\;dt$.

Solution .

$$\int_0^\infty te^{-2t}\cos t\;dt \;=\; L[t\cos t]_{s=2} \;=\; \left[\frac{s^2-1}{(s^2+1)^2}\right]_{s=2} \;=\; \frac{3}{25}.$$

Question 3.11.50. Find $L\left[\frac{\sin t}{t}\right]$.

Solution .

$$\begin{aligned} L\left[\frac{\sin t}{t}\right] &= \int_0^\infty L[\sin t]\;ds \;=\; \int_0^\infty \frac{1}{s^2+1}\;ds \\ &= \left[\tan^{-1} s\right]_0^\infty \;=\; \frac{\pi}{2} - \tan^{-1}(s) \;=\; \cot^{-1}(s). \end{aligned}$$

srivenkateshwaraa
College of Engineering & Technology
ASPIRE TO EXCEL
Ariyur, Puducherry-605 102.

ACCREDITED WITH GRADE A NAAC

ISO 21001

DEPARTMENT OF SCIENCE AND HUMANITIES

CIA-I EXAM

YEAR/SEM : I/ II (A-CSE) **MAX. MARKS** : 60 Marks
DATE : 27-04-2024 **DURATION** : 3Hrs

CSBS202 - MATHEMATICS - II

SECTION –A (10 Marks) - PART - I (10 x 2 = 20 Marks)

Answer the following

1. State Laplace transform. K1 CO2
2. Prove that $[e^{at}] = \frac{1}{s-a}; s > -a.$ K2 CO2
3. Find $L[t^3 - 3t^2 + 2]$. K2 CO2
4. Find $L[e^{-2t}t^2]$. K2 CO2
5. Define periodic function $f(t)$. K2 CO2
6. Find $L^{-1}\left[\frac{s^2-3s+4}{s^3}\right]$. K2 CO3
7. Find $L^{-1}\left[\frac{1}{(s+1)^2}\right]$. K2 CO3
8. Find $L^{-1}\left[\frac{s-1}{s^2-2s+5}\right]$. K2 CO3
9. Find $L^{-1}\left[\frac{1}{s(s-a)}\right]$. K1 CO3
10. Define convolution? K1 CO3

SECTION – B (40 Marks) – PART II (5 x 8 = 40 Marks)

Answer the Questions **Marks**

11. Find Laplace transforms of the following functions 8 K3 CO2
    i. $L[3e^{2t}]$
    ii. $L[e^{3t}\sin t \sin 2t]$
    iii. $t\,e^{-2t}\cos t$
    iv. $t^2e^{2t}\cos 3t$

(Or)

12. Evaluate 8 K3 CO2
    i. $L\left[\frac{e^{at}-\cos bt}{t}\right]$
    ii. $L\left[\frac{e^t-1}{te^{2t}}\right]$
    iii. $\int_0^\infty \left[\frac{\cos at-\cos bt}{t}\right] dt$
    iv. $L\left[e^{-2t}\int_0^t t\sin t\, dt\right]$

13. State and prove initial and final value theorems. 8 K3 CO2

(Or)

14. Verify initial and final value theorems for the function $f(t) = e^{-t}(t+2)^2$. 8 K3 CO2

8 K3 CO2

15. Find the Laplace transform of

$$f(t) = \begin{cases} \sin t, & \text{for } \ 0 \le t \le \pi \\ 0, & \text{for } \ \pi \le t \le 2\pi \end{cases} \text{ with period } 2\pi.$$

(Or)

16. Find the Laplace transform of 8 K3 CO2

$$f(t) = \begin{cases} 1, & \text{for } \ 0 \le t \le \frac{a}{2} \\ -1, & \text{for } \ \frac{a}{2} \le t \le a \end{cases} \text{ and } f(t+4) = f(t) \text{ for } t > 0.$$

17. Find the Inverse Laplace transform for the functions 8 K3 CO3

i. $\frac{1}{s^8}$ iii. $\frac{1}{(s+2)(s+3)}$

ii. $\frac{3s^2-4s+6}{s^4}$ iv. $\frac{1}{s^2(s^2+1)}$

(Or)

18. Find the Inverse Laplace transform for the functions 8 K3 CO3

i. $\frac{e^{-2s}}{s(s+1)}$ ii. $\frac{e^{-2s}}{s^2(s^2+1)}$ iii. $\frac{se^{-s}}{(s-3)^5}$

19. State and prove Convolution theorem? 8 K3 CO3

(Or)

20. Apply convolution theorem to evaluate 8 K3 CO3

i. $L^{-1}\left[\frac{2}{(s+1)(s^2+4)}\right]$ ii. $L^{-1}\left[\frac{s^2}{(s^2+9)(s^2+4)}\right]$

srivenkateshwaraa
College of Engineering & Technology
ASPIRE TO EXCEL
Ariyur, Puducherry-605 102.

NAAC A
ISO 21001

DEPARTMENT OF SCIENCE AND HUMANITIES

CIA-II EXAM

YEAR/SEM : I/ II (A-CSE)
DATE : 24-05-2024
MAX. MARKS : 60 Marks
DURATION : 3Hrs

CSBS202 - MATHEMATICS - II

SECTION –A (10 Marks) - PART - I (10 x 2 = 20 Marks)

Answer the following

1.	State the Fourier integral theorem.	K1	CO4
2.	Define Fourier transform.	K1	CO4
3.	State the convolution theorem for Fourier transform?	K1	CO4
4.	Find the Fourier sine transform of e^{-x}.	K2	CO4
5.	Find the Fourier sine transform of $f(x)$ if $f(x) = \begin{cases} x, & \text{if } 0 < x < \pi \\ 0, & \text{if } x \geq \pi \end{cases}$.	K2	CO4
6.	State Dirchlet's conditions for existence of Fourier series?	K1	CO5
7.	Defin Parseval's theorem on Fourier series?	K1	CO5
8.	What is harmonic analysis?	K1	CO5
9.	Define root mean square value?	K1	CO5
10.	Find the RMS value of $f(x) = x^2$ in $-\pi < x < \pi$.	K2	CO5

SECTION – B (40 Marks) – PART II (5 x 8 = 40 Marks)

Answer the Questions

		Marks		
11.	Find the Fourier Transform of $f(x) = \begin{cases} 1 - x^2, & \text{if } \lvert x \rvert < 1 \\ 0, & \text{if } \lvert x \rvert > 1 \end{cases}$ and hence evaluate $\int_0^\infty \left(\frac{\sin x - x\cos x}{x^3}\right) \cos\frac{x}{2}\, dx$.	8	K3	CO4
	(Or)			
12.	Find the Fourier Transform of $e^{-a^2x^2}$, $a > 0$ and hence deduce that $e^{-\frac{x^2}{2}}$ is self reciprocal under Fourier Transform.	8	K3	CO4
13.	Find the Fourier sine and cosine transform of $e^{-ax}, a > 0$ and deduce that. i. $\int_0^\infty \frac{\cos(sx)}{a^2+s^2} ds = \frac{\pi}{2a}\, e^{-ax}$ ii. $\int_0^\infty \frac{s\,\sin(sx)}{a^2+s^2} ds = \frac{\pi}{2}\, e^{-ax}$	8	K3	CO4
	(Or)			
14.	Find the Fourier sine transform, evaluate $\int_0^\infty \frac{dx}{(x^2+a^2)^2}$.	8	K3	CO4

15. Expand $f(x) = x^2$, when $-\pi \leq x \leq \pi$, in a Fourier series of periodicity 2π, hence deduce that 8 K3 CO5

i. $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$ ii. $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12}$

(Or)

16. Obtain the Fourier series expansion for the function $f(x) = (l - x)^2$ in the interval $0 < x < 2l$ and deduce that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$. 8 K3 CO5

17. Expand $f(x) = \sin x$, $0 < x < \pi$ in Fourier cosine series. 8 K3 CO5

(Or)

18. Expand $f(x) = x^2 - 2$, $0 < x < 2$ in a sine series. Hence deduce that $\pi^3 = 32\left[1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots\right]$. 8 K3 CO5

19. Find the Fourier series upto the third harmonic for the function $y = f(x)$ which is defined in $(0,2\pi)$ by means of the table of values given below. 8 K3 CO5

x	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	2π
y	1.0	1.4	1.9	1.7	1.5	1.2	1.0

(Or)

20. The value of x and the corresponding values of $f(x)$ over a period T are given below. 8 K3 CO5

x	0	$\frac{T}{6}$	$\frac{T}{3}$	$\frac{T}{2}$	$\frac{2T}{3}$	$\frac{5T}{6}$	T
y	1.98	1.30	1.05	1.30	-0.88	-0.25	1.98

Show that $f(x) = 0.75 + 0.37\, cos\theta + 1.004\, sin\theta$ where $\theta = \frac{2\pi x}{T}$.

srivenkateshwaraa
College of Engineering & Technology
ASPIRE TO EXCEL
Ariyur, Puducherry-605 102.
NAAC A
ISO 21001

DEPARTMENT OF SCIENCE AND HUMANITIES

MODEL EXAM

YEAR/SEM	**: I/ II (A-CSE)**	**MAX. MARKS**	**: 60 Marks**
DATE	**: 11-06-2024**	**DURATION**	**: 3Hrs**

CSBS202 - MATHEMATICS - II

SECTION –A (10 Marks) - PART - I (10 x 2 = 20 Marks)

Answer the following

1.	Form PDE by eliminating arbitrary constants $z = ax + by + a^2 + b^2$, where a, b are arbitrary constants.	K2	CO1
2.	Find the complementary function of $(D^3 - 2DD')z = \sin(x + 2y)$.	K2	CO1
3.	Prove that $[e^{-at}] = \frac{1}{s+a}; s > -a$.	K2	CO2
4.	Find $L[\sin 3t \cos t]$.	K2	CO2
5.	Find $L^{-1}\left[\frac{s^2-3s+4}{s^3}\right]$.	K2	CO3
6.	Find $L^{-1}\left[\frac{1}{(s+1)^2}\right]$.	K2	CO3
7.	Define Fourier transform pairs.	K1	CO4
8.	Find the Fourier sine transform of e^{-x}.	K2	CO4
9.	State Dirchlet's conditions for existence of Fourier series?	K1	CO5
10.	Find the RMS value of $f(x) = x^2$ in the interval $(0, \pi)$.	K2	CO5

SECTION – B (40 Marks) – PART II (5 x 8 = 40 Marks)

Answer the Questions

		Marks		
11.	i. Solve $xP + yQ = z$. ii. Solve $(mz - ny)P + (nx - lz)Q = ly - mx$.	8	K3	CO1
	(Or)			
12.	Solve $[D^3 - 7DD'^2 - 6D'^3]z = e^{2x+y} + \sin(x + 2y) + x^3y$.	8	K3	CO1
13.	Find Laplace transforms of the following functions i. $L[e^{-2t} cos^3 2t]$ ii. $L[\sinh t \ \cos 2t]$ iii. $t\, e^{-t} \sin t$ iv. $t^2 e^{2t} \cos 3t$	8	K3	CO2
	(Or)			
14.	Find the Laplace transform of $f(t) = \begin{cases} 1, & \text{for } 0 \le t \le \frac{a}{2} \\ -1, & \text{for } \frac{a}{2} \le t \le a \end{cases}$ and $f(t + a) = f(t)$ for $t > 0$.	8	K3	CO2

Find the Inverse Laplace transform for the functions

15. i. $\frac{1}{s^8}$ iii. $\frac{1}{(s+2)(s+3)}$ 8 K3 CO3

ii. $\frac{3s^2-4s+6}{s^4}$ iv. $\frac{1}{s^2(s^2+1)}$

(Or)

16. State and prove Convolution theorem? 8 K3 CO3

17. Find the Fourier Transform of $f(x) = \begin{cases} 1, & \text{for } |x| < a \\ 0, & \text{for } |x| > a \end{cases}$; $a > 0$ and hence evaluate $\int_0^\infty \frac{\sin x}{x} dx$. 8 K3 CO4

(Or)

18. Find the Fourier sine and cosine transform of $e^{-ax}, a > 0$ and deduce that. 8 K3 CO4

i. $\int_0^\infty \frac{\cos(sx)}{a^2+s^2} ds = \frac{\pi}{2a} e^{-ax}$ ii. $\int_0^\infty \frac{s \sin(sx)}{a^2+s^2} ds = \frac{\pi}{2} e^{-ax}$

19. Obtain the Fourier series expansion for the function $f(x) = (l - x)^2$ in the interval $0 < x < 2l$ and deduce that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$. 8 K3 CO5

(Or)

20. The value of x and the corresponding values of $f(x)$ over a period T are given below. 8 K3 CO5

x	0	$\frac{T}{6}$	$\frac{T}{3}$	$\frac{T}{2}$	$\frac{2T}{3}$	$\frac{5T}{6}$	T
y	1.98	1.30	1.05	1.30	-0.88	-0.25	1.98

Show that $f(x) = 0.75 + 0.37 \cos\theta + 1.004 \sin\theta$ where $\theta = \frac{2\pi x}{T}$.

srivenkateshwaraa
College of Engineering & Technology
ASPIRE TO EXCEL
Ariyur, Puducherry-605 102.
NAAC A
ISO 21001

DEPARTMENT OF SCIENCE AND HUMANITIES

CIA-I (A-CSE) ANSWER KEY

Section A

1. $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = F(s).$

2. $L[e^{at}] = \int_0^\infty e^{-(s-a)t}\,dt = \frac{1}{s-a}.$

3. $L[t^3 - 3t^2 + 2] = \frac{6}{s^4} - \frac{6}{s^3} + \frac{2}{s}.$

4. $L[e^{-2t} t^2] = L[t^2]_{s \to s+2}$

$= \left[\frac{2}{s^3}\right]_{s \to s+2} = \frac{2}{(s+2)^3}.$

5. $f(t+T) = f(t)\,;\; T > 0.$

6. $L^{-1}\left[\frac{s^2-3s+4}{s^3}\right] = L^{-1}\left[\frac{1}{s} - \frac{3}{s^2} + \frac{4}{s^3}\right]$

$= 1 - 3t - 2t^2.$

7. $L^{-1}\left[\frac{1}{(s+1)^2}\right] = e^{-t}\, L^{-1}\left[\frac{1}{s^2}\right] = e^{-t}\, t.$

8. $L^{-1}\left[\frac{s-1}{s^2-2s+5}\right] = L^{-1}\left[\frac{(s-1)}{(s-1)^2+4}\right] = e^{t}\, L^{-1}\left[\frac{s}{s^2+2^2}\right]$

$= e^{t} \cos 2t.$

9. $L^{-1}\left[\frac{1}{s(s-a)}\right] = \frac{1}{a} L^{-1}\left[\frac{1}{(s-a)} - \frac{1}{s}\right]$

$= \frac{1}{a}\left[e^{at} - 1\right].$

10. $L[f(t) * g(t)] = L[f(t)]\, L[g(t)]$

$= F(s)\, G(s).$

Section-B

11. (i) $L[3e^{2t}] = 3L[e^{2t}] = \frac{3}{s-2}.$

(ii) $L[e^{3t} \sin t \sin 2t] = L[\sin t \sin 2t]_{s \to s-3}$

$= \frac{1}{2} L[\cos t - \cos 3t]_{s \to s-3}$

$= \left[\frac{4s}{(s^2+1)(s^2+9)}\right]_{s \to s-3}.$

11-II

(iii) $L[t e^{-2t} \cos t] = \frac{(s+2)^2-1}{[(s+2)^2+1]^2} = \frac{s^2+4s+3}{[s^2+4s+5]^2}.$

$L[t \cos t] = \frac{s^2-1}{(s^2+1)^2}.$

(iv) $L[t^2 e^{2t} \cos 3t] = L[t^2 \cos 3t]_{s \to s-2}$

$L[t^2 \cos 3t] = \frac{2s^3 - 54s}{(s^2+9)^3}$

$L[t^2 e^{2t} \cos 3t] = \frac{2(s-2)^3 - 54(s-2)}{[s^2-4s+4+9]^3}$

$= \frac{2s^3 - 12s^2 - 30s + 92}{[s^2-4s+13]^3}.$

12. (i) $L\left[\frac{e^{at} - \cos bt}{t}\right] = \int_s^\infty L[e^{at} - \cos bt]\,ds$

$= \log\left(\frac{\sqrt{s^2+b^2}}{s-a}\right).$

(ii) $L\left[\frac{e^t - 1}{t e^{2t}}\right] = L\left[\frac{e^{-t} - e^{-2t}}{t}\right] = \log\left(\frac{s+2}{s+1}\right).$

(iii) $\int_0^\infty \left[\frac{\cos at - \cos bt}{t}\right] dt = L\left[\frac{\cos at - \cos bt}{t}\right]_{s=0}.$

$= \left(\log \sqrt{\frac{s^2+b^2}{s^2+a^2}}\right)_{s=0}.$

$= \log\left(\frac{b}{a}\right).$

(iv) $L\left[e^{-2t} \int_0^t t \sin t\,dt\right] = L\left[\int_0^t t \sin t\,dt\right]_{s \to s+2}$

$= \left\{\frac{1}{s} L[t \sin t]\right\}_{s \to s+2}$

$= \left\{\frac{1}{s}\left[\frac{2s}{(s^2+1)^2}\right]\right\}_{s \to s+2}$

$= \frac{2}{[s^2+4s+5]^2}.$

13. (i) $\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s)$ [I.V.T]

(ii) $\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$ [F.V.T].

$sF(s) = L[f'(t)] + f(0).$

14. $\lim_{t\to 0} f(t) = 4$ [IVT]

$\lim_{s\to\infty} sF(s) = 0+0+4 = 4.$

$\lim_{t\to\infty} f(t) = 0$, $\lim_{s\to 0} sF(s) = 0$ [FVT].

15. $T = 2\pi$

$$L\,f(t) = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt = \frac{1}{[1-e^{-s\pi}](s^2+1)}.$$

16. $T = a$

$$L[f(t)] = \frac{1}{s}\tanh\left[\frac{as}{4}\right].$$

17. (i) $L^{-1}\left[\frac{1}{s^8}\right] = \frac{1}{7!}L^{-1}\left[\frac{7!}{s^8}\right] = \frac{1}{7!}t^7.$

(ii) $L^{-1}\left[\frac{3s^2-4s+6}{s^4}\right] = 3L^{-1}\left[\frac{1}{s^2}\right] - 2L^{-1}\left[\frac{2!}{s^3}\right] + L^{-1}\left[\frac{3!}{s^4}\right]$

$= 3t - 2t^2 + t^3.$

(iii) $L^{-1}\left[\frac{1}{(s+2)(s+3)}\right] = L^{-1}\left[\frac{1}{s+2} - \frac{1}{s+3}\right]$

$= e^{-2t} - e^{-3t}.$

(iv) $L^{-1}\left[\frac{1}{s^2(s^2+1)}\right] = L^{-1}\left[\frac{1}{s^2} - \frac{1}{s^2+1}\right]$

$= t - \sin t.$

18. (i) $L^{-1}\left[\frac{e^{-2s}}{s(s+1)}\right] = L^{-1}\left[\frac{1}{s(s+1)}\right]_{t\to t-2}$

$= [1-e^{-t}]_{t\to t-2}$

$= (1-e^{-(t-2)}).u(t-2).$

(ii) $L^{-1}\left[\frac{e^{-2s}}{s^2(s^2+1)}\right] = L^{-1}\left[\frac{1}{s^2(s^2+1)}\right]_{t\to t-2}$

$= [t-\sin t]_{t\to t-2}$

$= [(t-2) - \sin(t-2)].u(t-2).$

(iii) $L^{-1}\left[\frac{s\,e^{-s}}{(s-3)^5}\right] = L^{-1}\left[\frac{s}{(s-3)^5}\right]_{t\to t-1}$

$= \left\{e^{3t}\left[\frac{t^3}{6}+\frac{t^4}{8}\right]\right\}_{t\to t-1}$

$= \left\{\frac{e^{3t}}{24}[4t^3 + 3t^4]\right\}_{t\to t-1}$

$= \left[\frac{e^{3(t-1)}}{24}\{4(t-1)^3 + 3(t-1)^4\}\right].u(t-1).$

19. $L[f(t)*g(t)] = L[f(t)].L[g(t)]$

$= F(s).G(s).$

$f(t)*g(t) = \int_0^t f(t)\,g(t-u)\,du.$

20. (i) $L^{-1}\left[\frac{2}{(s+1)(s^2+4)}\right] = L^{-1}\left[\frac{1}{(s+1)}\cdot\frac{2}{(s^2+4)}\right]$

$= e^{-t}\sin 2t = f(t).g(t).$

$\int_0^t f(u)\,g(t-u)\,du = \int_0^t e^{-u}\sin 2(t-u)\,du$

$= \frac{e^{-t}}{5}[2\sin^2 2t + 2\cos^2 2t] + \frac{\sin 2t - 2\cos 2t}{5}.$

$= \frac{\sin 2t - 2\cos 2t + 2e^{-t}}{5}.$

(ii) $L^{-1}\frac{s^2}{(s^2+9)(s^2+4)} = L^{-1}\left[\frac{s}{s^2+9}\cdot\frac{s}{s^2+4}\right]$

$= \cos 3t.\cos 2t = f(t).g(t).$

$\int_0^t f(u)\,g(t-u)\,du = \frac{3\sin 3t - 2\sin 2t}{5}.$

srivenkateshwaraa
College of Engineering & Technology
ASPIRE TO EXCEL
Ariyur, Puducherry-605 102.
NAAC A
ISO 21001

DEPARTMENT OF SCIENCE AND HUMANITIES

CIA-II (A-CSE) ANSWER KEY

1. f(x) – Piecewise continuously differentiable absolutely integrable on $(-\infty,\infty)$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, e^{is(x-t)}\,dt\,ds.$$

2. $F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\,dx$

$$F^{-1}\{F[f(x)]\} = f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F[f(x)]\, e^{-isx}\,ds$$

3. $F[f(x)] = F(s)$ & $F[g(x)] = G(s)$

$$F[f(x)*g(x)] = F(s)\,G(s)$$

$$f*g = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, g(x-t)\,dt.$$

4. $F_c[e^{-x}] = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} e^{-x}\cos sx\,dx$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{1}{s^2+1}\right].$$

5. $F_s\left[\frac{1}{x}\right] = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\frac{\sin sx}{x}\,dx = \sqrt{\frac{\pi}{2}}\ (s>0).$

$sx = t \Rightarrow x = t/s$. $dx = dt/s$.

6. $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos nx + b_n\sin nx]$

(i) Periodic, single valued and finite

(ii) Finite no. of discontinuities in any one period

(iii) at most a finite no. of maxima and minima

7. $y = f(x)$ in (a,b)

$$\frac{1}{b-a}\int_{a}^{b}[f(x)]^2\,dx = \bar{y}^2 = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2).$$

8. Process of finding the Fourier series for a function by numerical values.

9. $\bar{y} = \sqrt{\frac{1}{b-a}\int_{a}^{b}[f(x)]^2\,dx}$

10. $\bar{y}^2 = \frac{\pi^4}{5} \Rightarrow \bar{y} = \frac{\pi^2}{\sqrt{5}}.$

11. $F[f(x)] = 2\sqrt{\frac{2}{\pi}}\left[\frac{\sin s - s\cos s}{s^3}\right]$

$$f(x) = \frac{4}{\pi}\int_{0}^{\infty}\left[\frac{\sin s - s\cos s}{s^3}\right]\cos sx\,ds$$

$$x = \frac{1}{2},\quad \int_{0}^{\infty}\left[\frac{\sin x - x\cos x}{x^3}\right]\cos\frac{x}{2}\,dx = \frac{3\pi}{16}.$$

12. $F[f(x)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{isx}\,dx$

$$F[e^{-a^2x^2}] = \frac{1}{a\sqrt{2}}\, e^{-s^2/4a^2}.$$

$a^2 = 1/2$ (or) $a = \frac{1}{\sqrt{2}}$.

$$F[e^{-x^2/2}] = e^{-s^2/2}.$$

13. $F_c[f(x)] = \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+s^2}\right]$

$$\int_{0}^{\infty}\frac{\cos sx}{s^2+a^2}\,ds = \frac{\pi}{2a}\, e^{-ax}.$$

$F_s[f(x)] = \sqrt{\frac{2}{\pi}}\left[\frac{s}{a^2+s^2}\right]$

$$\int_{0}^{\infty}\frac{s\sin sx}{s^2+a^2}\,ds = \frac{\pi}{2}\, e^{-ax}.$$

14. $f(x) = e^{-ax}$.

$F_c[f(x)] = \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+s^2}\right]$

$$\int_{0}^{\infty}|f(x)|^2\,dx = \int_{0}^{\infty}|F_c(s)|^2\,ds$$

$$\Rightarrow \int_{0}^{\infty}\frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}.$$

15. $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2\pi^2}{3}$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{4(-1)^n}{n^2}.$$

$b_n = 0 \Rightarrow$ f(x) is even function

$$f(x) = \frac{1}{2}\cdot\frac{2\pi^2}{3} + \sum_{n=1}^{\infty}\left[4\frac{(-1)^n}{n^2}\cos nx + 0\right]$$

$$x^2 = \frac{\pi^2}{3} + 4\left[-\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \dots\right]$$

Put $x = \pi$,

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}.$$

$$x = 0 \Rightarrow \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots = \frac{\pi^2}{12}.$$

16. $a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx = \frac{2l^2}{3}$.

$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx = \frac{4l^2}{n^2\pi^2}$.

$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx = 0$.

$$f(x) = \frac{1}{2}\cdot\frac{2l^2}{3} + \sum_{n=1}^{\infty}\left[\frac{4l^2}{n^2\pi^2}\cos\frac{n\pi x}{l} + 0\cdot\sin\frac{n\pi x}{l}\right]$$

$$(l-x)^2 = \frac{l^2}{3} + \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos\frac{n\pi x}{l}}{n^2}.$$

$\underline{x=0}$ $\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}$.

17. $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{4}{\pi}$.

$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = -\frac{2}{\pi}\left[\frac{1+(-1)^n}{n^2-1}\right]$, $n \neq 1$.

$a_1 = \frac{2}{\pi}\int_0^{\pi} f(x)\cos x\,dx = 0$.

$$f(x) = \frac{1}{2}\cdot\frac{4}{\pi} + 0\cdot\cos x - \frac{2}{\pi}\sum_{n=2}^{\infty}\left[\frac{1+(-1)^n}{n^2-1}\cos nx\right]$$

$$\sin x = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos 2x}{2^2-1} + \frac{\cos 4x}{4^2-1} + \frac{\cos 6x}{6^2-1} + \dots\right]$$

18. $b_n = \frac{2}{l}\int_0^{l} f(x)\sin\frac{n\pi x}{l}\,dx$, $l=2$.

$$= \left[-4\frac{(-1)^n}{n\pi} + 16\frac{(-1)^n}{n^3\pi^3} - \frac{4}{n\pi} - \frac{16}{n^3\pi^3}\right]$$

$$x^2 - x = \sum_{n=1}^{\infty}\left[-4\frac{(-1)^n}{n\pi} + 16\frac{(-1)^n}{n^3\pi^3} - \frac{4}{n\pi} - \frac{16}{n^3\pi^3}\right]\sin\frac{n\pi x}{2}.$$

$\underline{\underline{x=1}}$ $1 - \frac{1}{3^3} + \frac{1}{5^3} - \dots = \frac{\pi^3}{32}$

19. $n=6$

$a_0 = 2.9$

$a_1 = -0.37$

$b_1 = 0.17$

$a_2 = -0.1$

$b_2 = -0.06$

$y = \frac{2.9}{2} + (-0.37\cos x + 0.17\sin x)$
$\quad + (-0.1\cos 2x - 0.06\sin 2x)$

$y = 1.45 + (-0.37\cos x + 0.17\sin x)$
$\quad - (0.1\cos 2x + 0.06\sin 2x)$!

20. $n=6$. $\theta = \frac{2\pi x}{T}$.

$a_0 = 1.5$

$a_1 = 0.37$

$b_1 = 1.004$.

$f(x) = \frac{1.5}{2} + (0.37\cos\theta + 1.004\sin\theta)$

$f(x) = 0.75 + \left[0.37\cos\frac{2\pi x}{T} + 1.004\sin\frac{2\pi x}{T}\right]$

srivenkateshwaraa
College of Engineering & Technology
ASPIRE TO EXCEL
Ariyur, Puducherry-605 102.

NAAC A
ISO 21001

DEPARTMENT OF SCIENCE AND HUMANITIES

MODEL (A-CSE) ANSWER KEY

Model Exam (CSE-A, BME, EEE, MECH)

Section - A

1. $z = ax + by + a^2 + b^2$

$\frac{dz}{dx} = p = a \qquad \frac{dz}{dy} = q = b$.

$z = px + qy + p^2 + q^2$.

2. $m^3 - 2m^2 = 0$.

$m^2(m-2) = 0$.

$m = 2,\ m = 0, 0$.

$y = f_1(y + 0x) + x f_2(y + 0x) + f_3(y + 2x)$.

$y = f_1(y) + x f_2(y) + f_3(y + 2x)$.

3. $L[e^{at}] = \int_0^\infty e^{-(s+a)t}\,dt$

$= \frac{1}{s+a}$.

4. $L[\sin 3t \cos t] = \frac{1}{2}\{L[\sin 4t] - L[\sin 2t]\}$

$= \frac{s^2 - 8}{(s^2+16)(s^2+25)}$

5. $L^{-1}\left[\frac{s^2 - 3s + 4}{s^3}\right] = L^{-1}\left[\frac{1}{s} - \frac{3}{s^2} + \frac{4}{s^3}\right)$

$= 1 - 3t + 2t^2$.

6. $L^{-1}\left[\frac{1}{(s+1)^2}\right] = e^{-t} L^{-1}\left[\frac{1}{s^2}\right]$

$= e^{-t} t$.

7. $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\,dx$

$F^{-1}\{F[f(x)]\} = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F[f(x)]\, e^{-isx}\,ds$.

8. $F_s[e^{-x}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \sin sx\,dx$

$= \sqrt{\frac{2}{\pi}} \left[\frac{s}{s^2+1}\right]$.

9. $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$

(i) periodic, single valued and finite

(ii) finite no. of discontinuities in any one period

(iii) must a finite no. of maxima and minima.

10. $\bar{y}^2 = \frac{1}{b-a} \int_a^b [f(x)]^2\,dx = \frac{\pi^4}{5}$.

$\bar{y} = \frac{\pi^2}{\sqrt{5}}$.

Section - B

11. (i) $xp + yq = z$.

$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$.

$\frac{dx}{x} = \frac{dy}{y} \Rightarrow \boxed{\frac{x}{y} = c_1}$

$\frac{dy}{y} = \frac{dz}{z} \Rightarrow \boxed{\frac{y}{z} = c_2}$

$\Rightarrow f\left(\frac{x}{y}, \frac{y}{z}\right) = 0$.

(ii) $(mz - ny)p + (nx - lz)q = ly - mx$.

$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}$.

$\Rightarrow l\,dx + m\,dy + n\,dz = 0$.

$\boxed{lx + my + nz = c_1}$

$\Rightarrow x\,dx + y\,dy + z\,dz = 0$.

$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c_2$.

$\Rightarrow f(lx + my + nz,\ x^2 + y^2 + z^2) = 0$.

12. $m = 3, -2, -1$.

$y = f(y + 3x) + f(y - 2x) + f(y - x)$.

$P.I_1 = \frac{-1}{12} e^{2x+y}$.

$P.I_2 = \frac{-7\cos(x+2y)}{525} = -\frac{\cos(x+2y)}{21}$.

$P.I_3 = \frac{x^6 y}{120}$

13. (i) $L[e^{-2t} \cos^3 2t] = L[\cos^3 2t]_{s \to s+2}$

$= L\left[\frac{3\cos 2t + \cos 6t}{4}\right]_{s \to s+2}$

$= \frac{1}{4}\left[\frac{3s}{s^2+4} + \frac{s}{s^2+36}\right]_{s \to s+2}$

$= \frac{(s+2)[(s+2)^2 + 28]}{[(s+2)^2 + 4][(s+2)^2 + 36]}$.

(ii) $L[\sinh t \cos 2t] = L\left[\left(\frac{e^t - e^{-t}}{2}\right) \cos 2t\right]$

$= \frac{1}{2}\left[\frac{(s-1)}{(s-1)^2 + 4} - \frac{(s+1)}{(s+1)^2 + 4}\right]$

(iii) $L[te^{-t}\sin t] = L[t\sin t]_{s\to s+1}$

$L[t\sin t] = \frac{2s}{(s^2+1)^2}$.

$\Rightarrow \frac{2(s+1)}{((s+1)^2+1)^2} = \frac{2(s+1)}{(s^2+2s+2)^2}$

$= \frac{2(s+1)}{(s^2+2s+2)^2}$.

(iv) $L[t^2e^{2t}\cos 3t] = L[t^2\cos 3t]_{s\to s-2}$.

$\Rightarrow \left[\frac{2s^3-54s}{(s^2+9)^3}\right]_{s\to s-2}$.

$\Rightarrow \frac{2s^3-12s^2-30s+92}{[s^2-4s+13]^3}$

14. $L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt$. $T=a$.

$L[f(t)] = \frac{1}{s}\tan h\left[\frac{as}{4}\right]$.

15. (i) $L^{-1}\left[\frac{1}{s^8}\right] = \frac{1}{7!}L^{-1}\left[\frac{7!}{s^8}\right] = \frac{1}{7!}t^7$

(ii) $L^{-1}\left[\frac{1}{(s+2)(s+3)}\right] = L^{-1}\left[\frac{1}{s+2} - \frac{1}{s+3}\right]$

$= e^{-2t} - e^{-3t}$.

(iii) $L^{-1}\left(\frac{3s^2-4s+6}{s^4}\right] = 3L^{-1}\left(\frac{1}{s^2}\right) - 2L^{-1}\left[\frac{2!}{s^3}\right) + L^{-1}\left[\frac{3!}{s^4}\right]$

$= 3t - 2t^2 + t^3$.

(iv) $L^{-1}\left[\frac{1}{s^2(s^2+1)}\right] = L^{-1}\left[\frac{1}{s^2} - \frac{1}{s^2+1}\right]$

$= t - \sin t$.

16. $L(f(t)*g(t)) = L[f(t)]L[g(t)] = F(s)G(s)$.

$L^{-1}[F(s)G(s)] = \int_0^t f(u)g(t-u)\,du$.

17. $F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\,dx = \sqrt{\frac{2}{\pi}}\left(\frac{\sin as}{s}\right]$

$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F[f(x)]\,e^{-isx}\,ds$.

$\Rightarrow \int_0^\infty \frac{\sin as}{s}\cos sx\,ds = \frac{\pi}{2}f(x)$.

$x=0$, $f(0)=1$.

$\int_0^\infty \frac{\sin t}{t/a}\frac{dt}{a} = \frac{\pi}{2} \Rightarrow \int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}$.

18. $F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$.

$F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+s^2}\right]$.

$\int_0^\infty \frac{\cos sx}{s^2+a^2}\,ds = \frac{\pi}{2a}e^{-ax}$

$F_s(f(x)) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx = \sqrt{\frac{2}{\pi}}\left[\frac{s}{a^2+s^2}\right]$.

$\int_0^\infty \frac{s\sin sx}{s^2+a^2}\,ds = \frac{\pi}{2}e^{-ax}$.

19. $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{l}\right) + b_n\sin\left(\frac{n\pi x}{l}\right)\right]$

$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx = \frac{2l^2}{3}$.

$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx = \frac{4l^2}{n^2\pi^2}$.

$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx = 0$.

$(l-x)^2 = \frac{l^2}{3} + \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos\frac{n\pi x}{l}}{n^2}$.

$x=0$, $f(0) = l^2$.

$\Rightarrow \frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}$.

20. $f(x) = \frac{a_0}{2} + (a_1\cos x + b_1\sin x)$

$a_0 = \frac{2}{n}\sum y = 1.5$

$a_1 = \frac{2}{n}\sum y\cos = 0.37$

$b_1 = \frac{2}{n}\sum y\sin\theta = 1.004$

$f(x) = 0.75 + \left[0.37\cos\frac{2\pi x}{T} + 1.004\sin\frac{2\pi x}{T}\right]$.

srivenkateshwaraa
COLLEGE OF ENGINEERING & TECHNOLOGY
Approved by AICTE, New Delhi & Affiliated to Pondicherry University, Puducherry
13-A, Pondy - Villupuram Main Road, Ariyur, Puducherry - 605 107.

ISO 21001

ASPIRE TO EXCEL

EXAM CELL

Part - I (to be filled by the candidate)

Name of the Exam :	CIA - I		
Date of the Exam :	27/04/2024	**Register No.:**	2 3 T D 0 6 8 1
Degree / Branch :	B.tech - CSE (A)	**Year / Semester :**	I/II
Subject Code :	CSBS202	**No. of Pages Written :**	
Subject Title :	Mathematics - II		
V. ANBUKARASI **Name of the Hall superintendent** **(With Designation)**		27/4/24 **Signature of the Hall Superintendent** **(With Date)**	

PART - II (to be filled by the Examiner)

Q.No.	Marks	Q.No.	Sub Division Marks (i)	(ii)	(iii)	(iv)	TOTAL MARKS
1.	2	1.	4				4
2.	1	2.					
3.	2	3.	8				8
4.	2	4.					
5.	1	5.	7				7
6.		6.					
7.		7.	3				3
8.		8.					
9.	1	9.	7				7
10.	1	10.					
TOTAL (A)	10	TOTAL (B)	29				29
Grand Total (A+B)	39	In Words :	three nine only				

Dr. C. John Sundar AP/MATHS **Name of the Hall superintendent** **(With Designation)**	**Signature of the Hall Superintendent** **(With Date)**

Janithaa

INSTRUCTIONS OF THE CANDIDATES

1. Check the Question paper first, its subject code / Title etc., before answering the questions.
2. Fill all the particulars in Part - I of this sheet.
3. Possession of any incrimination material, mobile phones / other electronic gadgets & mall practice of any nature are punishable as per rules.
4. Use both side of the paper for answering questions.
5. Answer must be legibly written in Ink (Blue, Black or Blue Black)
6. Writing or leaving any distinctive marks so as to identify your paper is strictly prohibited.
7. Strike all the blank pages left in the answer sheet at the end of the examination.
8. Rough work, if any must be done at the bottom of the page, reserving a fourth at the bottom of the page exclusively for this purpose.
9. Do write in the margin except the question number.
10. Graph or other sheets should be attached between pages and tied with white thread.

SECTION-A

2 1) $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = F(s)$

2) $[e^{at}] = \frac{1}{S-a}, \; S > -a$

$= \int_0^\infty e^{-st} e^{at} \, dt$

$= \int_0^\infty e^{-(s-a)t} \, dt$

$= \left[\frac{e^{-t}}{\mp(s-a)}\right]_0^\infty$

$e^0 = 1$
$e^\infty = 0$

$\left[\frac{e^{-\infty}}{-(s-a)} - \frac{e^{-0}}{-(s-a)}\right]$

$\therefore L[e^{at}] = \frac{1}{s-a}$

3) $L[t^3 - 3t^2 + 2] =$

$L[t^3] - 3L[t^2] + L[2]$

2 $\frac{3!}{s^4} - 3\left[\frac{2!}{s^3}\right] + \frac{2}{s}$

$\frac{6}{s^4} - \frac{6}{s^3} + \frac{2}{s}$

4) $L[e^{-2t}t^2]$

According to the first shifting property.

$$L[t^2]_{s\to s+2}$$

$$= \left[\frac{2!}{s^3}\right]_{s\to s+2}.$$

2

$$= \frac{2}{(s+2)^3}$$

5) Periodic function defines that the function f(t) is acting continous i,e with in period(T) of the given functional values.

$$= \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$$

1

The term period is denoted by (T)

and $f(t) = [t+T]$

6) $L^{-1}\left[\frac{s^2-3s+4}{s^3}\right]$

7) $L^{-1}\left[\frac{1}{(s+1)^2}\right] = \left[\frac{1}{s^2+2s+1}\right]$

$= L^{-1}\left[\frac{1}{(s+1)^2}\right]$

$L^{-1}\left[\frac{1}{(s+1)\ (s+1)}\right].$

$L^{-1}\left[\left(\frac{1}{s+1}\right)\left(\frac{1}{s+1}\right)\right]$

$L^{-1}\left[\frac{1}{s+1}\right] L^{-1}\left[\frac{1}{s+1}\right]$

$\therefore \left(e^{-at}\right)^2 . (u-t)$

$t \to u$

9) $L^{-1}\left[\frac{1}{s(s-a)}\right]$

$L^{-1}\left[\frac{1}{s}\right] L^{-1}\left[\frac{1}{s-a}\right]$

$$\frac{1}{s(s-a)} = \frac{A}{s} + \frac{B}{s-a}. \quad — ①$$

multiply eq ① with $s(s-a)$.

$s=0$.

$1 = A(s-a) + B(s).$ $As - Aa + Bs = 1.$

$A = 1/a$ $B = 1-Aa.$

$s=0$

$-Aa + B = 1$

~~Find $L^{-1}\left[\frac{1}{(s+1)^2}\right]$~~

~~Find $L^{-1}\left[\frac{s-1}{s^2-2s+5}\right]$~~

10. Convolution :

If there is an two function $f(t)$ and $g(t)$ of t ; then the product of the 2 main function tells the convolution : $[f(t) * g(t)]$

$L[f(t) * g(t)] = L[F(s)]\ L[g(s)]$.

SECTION-B

(9) Convolution theorem.

There are two functions $f(t)$ and $g(t)$ of function t. The convolution theoem states the product of two functions $[f(t) * g(t)]$

$$f(t) = L^{-1}[F(s)]$$

$$g(t) = L^{-1}[G(s)]$$

~~Proof~~ Therefore;

$$[f(t)^* g(t)] = L^{-1}[F(s)][G(s)].$$

$$= \int_0^T e^{-st} f(t)\, g(t)\, dt$$

Proof:

$$L[f(t) * g(t)] = \int_0^\infty e^{-st} f(t)\, dt.$$

$$= \int_0^\infty [f(t) : g(t)] e^{-st}\, dt.$$

$f(t) = u$ and $g(t) = t-u$

$$= \int_0^\infty e^{-st} u \int_0^\infty e^{-st} (t-u)\, du\, dt.$$

Changing; $t = u$ and $dt = du$

$t - u = x$ and $dt = dx$

limit $t = \infty$, $x = \infty$

$$\int_0^\infty e^{-s(x-u)} u \, du \int_0^\infty e^{-s(x-u)} (t-u) \, dt .$$

$$= \int_0^\infty e^{-s(x-u)} u \, du \int_0^\infty e^{-s(x-u)} x . dx .$$

$$. \int_0^\infty e^{-st} u \, du \int_0^\infty e^{-st} x . dx .$$

$$= \int_0^\infty e^{-st} f(t) \, dt \int_0^\infty e^{-st} g(t) \, dt .$$

Therefore

$$L[f(t) * g(t)] = L^{-1}[F(s)] \, L^{-1}[g(s)] =$$

$$L[f(t) * g(t)] = \int_0^T e^{-sT} f(t) \, g(t) \, dt = F(s) \, g(s)$$

∴ Convolution theorem is proved.

18) iv) $\left[\frac{e^{-2s}}{s(s+1)}\right] L^{-1}$

$$L^{-1}\left[\frac{e^{-2s}}{s}\right] \quad L^{-1}\left[\frac{e^{-2s}}{s+1}\right]$$

$$L^{-1}\left[\frac{1}{s}\right]_{s\to s+2} \quad L^{-1}\left[\frac{1}{s+1}\right]_{s\to s+2}.$$

$$\left[\frac{1}{s+2}\right] \left[\frac{1}{(s+2)s+1}\right]$$

$$\left[\frac{1}{s+2}\right] \left[\frac{1}{s^2+s+2s+2}\right]$$

$$\left[\frac{1}{s+2}\right] \left[\frac{1}{s^2+\cancel{2s}+s+2}\right]$$

$$\frac{1}{s+2(s^2+2s+s+2)} = \frac{A}{s+2} + \frac{Bx+C}{s^2+2s+s+2}$$ ①

multiply eq ① by $s+2(s^2+2s+s+2)$ oBS

$1 = A(s^2+2s+s+2) + (Bx+C)\,s+2.$

$1 = As^2 + 2As + As + 2A + Bxs + 2Bx + sC + 2C$

$0\ 1 = 2s^2 + 2A \quad 2Bx + 2C$

v) $\left[\frac{e^{-2s}}{s^2(s^2+1)}\right] L^{-1}$

$$L^{-1}\left[\frac{\cancel{e^{-2s}}1}{s^2(s^2+1)}\right]_{s\to s+2}$$

$$L^{-1}\left[\frac{1}{s^2}\right]_{s\to s+2} L^{-1}\left[\frac{1}{(s^2+1)}\right]_{s\to s+2}$$

$$L^{-1}\left[\frac{1}{(s+2)^2}\right] L^{-1}\left[\frac{1}{(s+2)^2+1}\right]$$

$$L^{-1}\left[\frac{1}{s^2+4s+4}\right] L^{-1}\left[\frac{1}{s^2\cancel{+4}s+5}\right]$$

$$\left(\frac{1}{s^2(s^2+1)} = \frac{Ax+B}{s^2+4s+4} + \frac{Cx+D}{s^2+4s+5}\right) \times s^2(s^2+1)$$

$$1 = \left[Ax+B(s^2+4s+5)\right] + Cx+D(s^2+4s+4)$$

$$1 = Axs^2+4Axs+5Ax+Bs^2+4Bs+5B+$$
$$Cxs^2+4Cxs+4Cx+Ds^2+4Ds+4D$$

$$\cancel{\frac{1}{1-e^{-ST}}\int_0^T e^{-st} f(t)\,dt.}$$

$T=4$ $\left[\cancel{T= a/2}\right]$

$$\cancel{\frac{1}{1-e^{-\cancel{a/2}\,4s}}\int_0^{a/2} e^{-st} f(t)\,dt + \int_{a/2}^{a} e^{-st} f(t)\,dt.}$$

$$\cancel{\frac{1}{1-e^{-s4}}\int_0^{a/2} e^{-st}\,dt - \int_{a/2}^{a} -e^{-s}\,dt.}$$

15) $$\frac{1}{1-e^{-s2\pi}}\int_0^{2\pi} e^{-st} f(t)\,dt$$ $[T= 2\pi]$

$$\frac{1}{1-e^{-s2\pi}}\int_0^{\pi} \sin t\; e^{-st}\,dt + \int_{\pi}^{2\pi} e^{-st}\,0\,.dt. \rightarrow 0$$

$u' = \cos t$

$u = \sin t$

$v = \frac{e^{-st}}{-s}$

$$\frac{1}{1-e^{-s2\pi}}\int_0^{\pi} \underbrace{e^{-st}}_{v}\,\underbrace{\sin t}_{u}\,dt.$$

$v_1\ \frac{e^{-st}}{-s}$

$$\frac{1}{1-e^{-s2\pi}}\int_0^{\pi} \sin t\; e^{-st} + \cos t\,\frac{e^{-st}}{-s}\,.\,dt.$$

$$\frac{1}{a^2+x^2}\left[\sin ax + \cos bx\right]$$

$$\frac{1}{1-e^{-s2\pi}}\left[\cos t\,\frac{e^{-st}}{-s} + \sin t\,\frac{e^{-st}}{s^2}\right]_0^{\pi}$$

$a =$ $b:0$

$$\cancel{= \frac{1}{1-e^{-2\pi s}} \left[\cos\pi \, \frac{e^{-s\pi}}{-s} \right.}$$

$$= \frac{1}{1-e^{2\pi s}} \left[\sin\pi \, e^{-s\pi} + \cos\pi \, \frac{e^{-s\pi}}{-s} \right] -$$

$$\left[\sin 0 \, e^{-s0} + \cos 0 \, \frac{e^{-s0}}{-s} \right]$$

$$= \frac{1}{1-e^{2\pi s}} \left[\frac{e^{-3\pi} - e^{-s\pi}}{-s} - \frac{1}{-s} \right]$$

$$= \frac{1}{1-e^{2\pi s}} \left[\frac{1-e^{\pi s} - 1}{-s} \right]$$

$$= \frac{1}{\cancel{(1-e^{\pi s})}(1+e^{\pi s})} \left[\frac{\cancel{1-e^{\pi s}}}{-s} \right]$$

$$= \frac{1}{-s(1-e^{\pi s})}$$

13) Statement: If $L[f(t)]$ and $L[f'(t)]$ exists then ;

$$L[f(t)] = sF(s)$$

Proof :

1] Initial value theorm :

$$\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s)$$

$$L[f'(t)] = sF(s) - f(0).$$

$$sF(s) = L[f'(t)] - f(0)$$

$$\left(sF(s) = \int_0^\infty e^{-st} f'(t) + f(0)\right)$$

sub $\lim_{s \to \infty}$ on both side

$$\lim_{s \to \infty} sF(s) = \lim_{s \to \infty} \int_0^\infty e^{-st} f'(t) \, dt + f(0)$$

$e^{\infty} = 0$.

$$= 0 + f(0).$$

$$\therefore \lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s).$$

∴ The initial value theorm is verified

ii) Final value theorem

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

$$\therefore L[f'(t)] = sF(s) - f(0).$$

$$sF(s) = L[f'(t)] + f(0).$$

$$sF(s) = \int_0^\infty e^{-st} f'(t)\,dt + f(0).$$

Sub $\lim_{s\to 0}$ in above equation.

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0} \int_0^\infty e^{-st} f'(t)\,dt + f(0).$$

$(e^0 = 1).$

$$= \left[f(t)\right]_0^\infty + f(0).$$

$$\lim_{s\to 0} sF(s) = \dots\dots f(0) - f(0).$$

6

$$\therefore \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s).$$

∴ Final value theorem is Verified.

12) i] $L\left[\frac{e^{at} - \cos bt}{t}\right]$

$$L\left[\frac{-\cos bt}{t}\right]_{s \to s-a}$$

$$\cancel{L\left[\sin bt\right]_{s \to s-a}}$$

$$\cancel{\left[\frac{a}{s^2 + a^2}\right]_{s \to s-a}} = \left[\frac{a}{s-a+a^2}\right]$$

$$= \boxed{\frac{a}{s+a}} \,//$$

ii] $L\left[\frac{e^{t}-1}{t\,e^{2t}}\right] =$

iii] $\int_0^\infty \left[\frac{\cos at - \cos bt}{t}\right] dt.$

iv] $\left[e^{-2t}\int_0^t t \sin t \, dt\right]L. = \cancel{e^{2t}\int_0^t \left[\frac{1}{s} \cdot \frac{1}{s^2+1}\right] dt}.$

11) i) $L[3e^{2t}]$

$$3L[e^{2t}] = 3\left[\frac{1}{s-2}\right] = \frac{3}{s-2}$$

iii) $L[t e^{-2t} \cos t] = (-1)\frac{d}{ds}[\cos t]$

$=$ [illegible]

$$= (-1)\frac{d}{ds}\left[\frac{s}{s^2+1}\right]_{s \to s+2}$$

$$\left[d\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}\right] \quad = -\left[\frac{(s^2+1)1 - s(2s)}{(s^2+1)^2}\right]_{s \to}$$

$$= \left[\frac{s^2+1-2s^2}{(s^2+1)^2}\right]_{s \to s+2}$$

iv) $L[t^2 e^{2t} \cos 3t] = (-1)^2 \frac{d^2}{ds^2}[\cos 3t]$

$$= \frac{d}{ds}\frac{d}{ds}[\cos 3t]$$

$$= \frac{d}{ds}\,\frac{d}{ds}\left[\frac{s}{s^2+9}\right]_{s \to s \mp 2}$$

$$= \frac{d}{ds}\left[\frac{(s^2+9) - s(2s)}{(s^2+9)^2}\right]_{s \to s-2} \qquad \frac{vu' - uv'}{v^2}$$

$$= \left[\frac{(s^2+9)^{\cancel{2}}\,2s - \cancel{(s^2+9)} - s(2s)\,2(s^2+9)}{(s^2+9)^4}\right]_{s \to s-2}$$

$$= \left[\frac{(s^2+9)\,2s + 2s^{\cancel{2}} - \cancel{2s}^{3} + 18}{(s^2+9)^4}\right]_{s \to s-2}$$

$$= \left[\frac{(s^2+9)\,2s + 18}{(s^2+9)^4}\right]_{s \to s-2}$$

4

ii) $L[e^{3t} \sin t \sin 2t] =$

$L[\sin t \sin 2t]_{s \to s-3}$.

$L[\sin t]\; L[\sin 2t]$

$$\left\{\left[\frac{1}{s^2+1}\right]\left[\frac{2}{s^2+4}\right]\right\}_{s\to s-3}$$

$$\frac{1}{(s-3)^2+1}\cdot\frac{2}{(s-3)^2+4} = \frac{2}{(s-3)^4+5} \text{ //}$$

~~$L\left[te^{-2t}\cos t\right] = (-1)\frac{d}{ds}\left[\cos t\right]$~~

17) v) $L^{-1}\left[\frac{1}{s^8}\right] = t^7$?

vii) $\left[\frac{1}{(s+2)(s-2)}\right]L^{-1} = L^{-1}\left[\frac{1}{s+2}\right]L^{-1}\left[\frac{1}{s-2}\right]$

$= \left(e^{-2t}\,e^{2t}\right).\, t-u$

viii) $L^{-1}\left[\frac{1}{s(s^2+1)}\right] = L^{-1}\left[\frac{1}{s}\right]\; L^{-1}\left[\frac{1}{s^2+1}\right]$

$= 1\left(\frac{a}{s^2+a^2}\right) = (\sin t)$

3

$=$ ~~$\sin t$~~ $(\sin t).\; t-u$

srivenkateshwaraa
COLLEGE OF ENGINEERING & TECHNOLOGY

Approved by AICTE, New Delhi & Affiliated to Pondicherry University, Puducherry
13-A, Pondy - Villupuram Main Road, Ariyur, Puducherry - 605 107.

ASPIRE TO EXCEL

ISO 21001

EXAM CELL

Part - I (to be filled by the candidate)

Name of the Exam :	CIA-II		
Date of the Exam :	24.05.2024	Register No.:	2 T D 0 0 5 7
Degree / Branch :	B.Tech, CB-A	Year / Semester :	I/II
Subject Code :	CBS202	No. of Pages Written :	
Subject Title :	MATHEMATICS-II		
Dr.C. John Sunder AP/MATHS Name of the Hall superintendent (With Designation)		24/5/24 Signature of the Hall Superintendent (With Date)	

PART - II (to be filled by the Examiner)

Q.No.	Marks	Q.No.	Sub Division Marks				TOTAL MARKS
			(i)	(ii)	(iii)	(iv)	
1.	2	11.	4				4
2.	2	12.					
3.	1	13.	5				5
4.	2	14.					
5.	2	15.					
6.	2	16.	5				5
7.	2	17.	8				8
8.	2	18.					
9.	2	19.	7				7
10.	1	210.					
TOTAL (A)	18	TOTAL (B)					29
Grand Total (A+B)	47	In Words :	FOUR SEVEN				

Dr.C. John Sunder AP/MATHS Name of the Hall superintendent (With Designation)	Signature of the Hall Superintendent (With Date)

INSTRUCTIONS OF THE CANDIDATES

1. Check the Question paper first, its subject code / Title etc., before answering the questions.
2. Fill all the particulars in Part - I of this sheet.
3. **Possession of any incrimination material, mobile phones / other electronic gadgets & mall practice of any nature are punishable as per rules.**
4. Use both side of the paper for answering questions.
5. Answer must be legibly written in Ink (Blue, Black or Blue Black)
6. Writing or leaving any distinctive marks so as to identify your paper is strictly prohibited.
7. Strike all the blank pages left in the answer sheet at the end of the examination.
8. Rough work, if any must be done at the bottom of the page, reserving a fourth at the bottom of the page exclusively for this purpose.
9. Do write in the margin except the question number.
10. Graph or other sheets should be attached between pages and tied with white thread.

Section - A

part - I

Answer the following :-

1. Fourier integral theorem:-

2 * If f(x) is peice wise continuously differentiated and absolutely integrated of $(-\infty, \infty)$.

$$f(x) = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^{\infty} f(t) \cos s (t-x)\, dt\, dx.$$

2. Fourier transform pairs.

$$* \quad f(x) = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx.$$

2 If f(x) is peice wise continuously differentiated and absolutely integrated of value by $(-\infty, \infty)$ by the fourier transform.

3. Convolution theorom for fourier transform.

* Convolution theorom state that the f(x) or g(x) which comes under the fourier transform, F(s) and G(s)

1

$$F[f*g] = F(s) \cdot G(s).$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)\, e^{isx}\, dx$$

mobile phones /

5. Fourier sine transform of $\frac{1}{x}$.

~~$F[\frac{1}{x}] = \frac{1}{2\pi}\int_0^\infty \frac{\sin sx}{x}\,dx$.~~

~~where, $sx = t$, $x = t/a$, $dx = \frac{dt}{a}$.~~

~~$= \frac{1}{2\pi}\sqrt{\frac{\pi}{2}}\int_0^\infty \frac{\sin t}{t/x}\,.dt/a$~~

~~$= \sqrt{\frac{\pi}{\pi}}$~~

soln:

$$F[\frac{1}{x}] = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin sx}{x}\,dt.$$

where, $sx = t$, $x = t/a$, $dx = dt/a$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin t}{t/a}\,.dt/a$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin t}{t}\,dt.$$

where, $\frac{\sin t}{t}\,dt = \cancel{2/\pi}\ \pi/2$

$$= \sqrt{\frac{2}{\pi}}\ \frac{\pi}{2}$$

$$= \sqrt{\frac{\pi}{2}}$$ //

s>0 provided,

6. Dirchlet's condition for existence of Fourier Series.

* Dirchlet's condition which statisfy the fouries series, of fourier transform, which is followed by the below conditions.

⇒ i) If f(x) is periodic, single valued and finite.

2

⇒ ii) If f(x) is more finite in any one of the period.

⇒ iii) If f(x) is the more finite of maxima and minima.

---

7. Parseval's theorem on Fourier series.

$$\Rightarrow f(x) = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left[a^2 n + b^2 n\right].$$

2

if the f(x) of the fourier series which explain the parseval's therom,

---

8. Harmonic analysis.

2

⇒ Harmonic analysis is the process of finding the fourier series of a function in given numerical value is called as harmonic analysis.

9. Root mean square value.

* Root mean square value (RMS), where $y = f(x)$

$$\Rightarrow \quad \bar{y}^2 = \frac{1}{b-a}\int_a^b y^2\,dx$$

(or)

$$\Rightarrow \quad \bar{y} = \sqrt{\frac{1}{b-a}\int_a^b y^2\,dx}$$

---

10. RMS value of $f(x) = x^2$; $-\pi < x < \pi$.

RMS $\Rightarrow \bar{y}^2 = \frac{1}{b-a}\int_a^b y^2\,dx$.

where, $a = -\pi$, $b = \pi$. $y = x^2$.

$$\bar{y}^2 = \frac{1}{\pi+\pi}\int_{-\pi}^{\cancel{b}\pi} x^2\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{x^3}{3}\right]$$

$$= \frac{1}{2\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\pi^3}{3} + \frac{\pi^3}{3}\right]$$

$$= \frac{1}{\cancel{2\pi}}\left[\frac{\cancel{2}\pi^{\cancel{3}}}{3}\right]$$

$$\bar{y}^2 = \frac{\pi^2}{3}$$

$$\bar{y} = \sqrt{\frac{\pi^2}{3}}$$

$$\bar{y} = \frac{\pi}{\sqrt{3}}$$

//.

Section -B.

Part -II.

Answer the Questions.

Fourier sine and cosine transform of e^{-ax}, $a > 0$

i) $\int_0^\infty \frac{\cos(sx)}{a^2+s^2}\, ds = \frac{\pi}{2a} e^{-ax}$

ii) $\int_0^\infty \frac{s\sin(sx)}{a^2+s^2}\, ds = \frac{\pi}{2} e^{-ax}$

soln:-

i) $\int_0^\infty \frac{\cos(sx)}{a^2+s^2}\, ds = \frac{\pi}{2a} e^{-ax}$.

Fourier cosine.

$F(x)$ $$F_c[f(x)] = \sqrt{\frac{\pi}{2}} \int_{+\infty}^{\infty} f(x) \cos e^{isx}\, dx$$

$$= \sqrt{\frac{\pi}{2}} \int_{+\infty}^{\infty} \cos e^{isx}\, dx$$

$$= \sqrt{\frac{\pi}{2}} \left[\frac{e^{isx}}{a^2+s^2} \right]_0^\infty dx$$

$$= \boxed{\sqrt{\pi/2}} \left[\frac{a}{a^2+s^2} \right] \text{—①}.$$

Fourier sine ,

$$F_s(f(x)] = \sqrt{\frac{\pi}{2}} \int_0^\infty f(x)\, e^{isx}\, dx$$

$$= \sqrt{\frac{\pi}{2}} \int_0^\infty \sin^{a}_{e} isx \; dx$$

$$= \sqrt{\frac{\pi}{2}} \left[\frac{e^{isx}}{a^2+s^2}\right]_0^\infty dx$$

$$= \sqrt{\frac{\pi}{2}} \left[\frac{s}{a^2+s^2}\right] \quad \text{—} ②.$$

The fourier cosine inverse form:

$$F_c[f(x)] = \sqrt{\frac{\pi}{2}} \int_0^\infty F_c(s)\, dx$$

$$= \sqrt{\frac{\pi}{2}} \int_0^\infty \cos sx\, dx$$

$$= \sqrt{\frac{\pi}{2}} \left[\frac{\sin \overset{a}{s}x - \cos \overset{a}{s}x}{a^2+s^2}\right]_0^\infty$$

$$= \sqrt{\frac{\pi}{2}} \left[\frac{\cos sx}{a^2+s^2}\right]$$

sub the $f(x)$

$$f(x) = \int_0^\infty \frac{\cos sx}{a^2+s^2} = \frac{\pi}{2a} e^{-ax}.$$

where;-

$$\boxed{\int_0^\infty \frac{\cos sx}{a^2+s^2} = \frac{\pi}{2a} e^{-ax}} \quad \text{———} ③.$$

The fourier sine inverse form.

$$F_s[f(x)] = \sqrt{\frac{\pi}{2}} \int_0^\infty F_s[s]\, dx$$

$$= \sqrt{\frac{\pi}{2}} \int_0^\infty s \sin sx\, dx.$$

$$= \sqrt{\frac{\pi}{2}} \int_0^{\infty} \left[\frac{\cos ax - s \sin sx}{a^2+s^2} \right]_0^{\infty}$$

$$= \sqrt{\frac{\pi}{2}} \left[\frac{s \sin sx}{a^2+s^2} \right].$$

$$f(x) = \int_0^{\infty} \frac{s \sin sx}{a^2+s^2} = \frac{\pi}{2} e^{-ax}$$

where,

$$f \cdot \boxed{\int_0^{\infty} \frac{s \cdot \sin sx}{a^2+s^2} = \frac{\pi}{2} e^{-ax}} \text{ — ④.}$$

where, (i) & (ii)

are proved,,

---

19. Fourier series of third harmonic, $y = f(x)$.

x	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
y	1.0	1.4	1.9	1.7	1.5	1.2	1.0

soln;-

$n = 6.$

$$y = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_1 \cos x + b_1 \sin x] + [a_2 \cos 2x + b_2 \sin 2x]$$

$$+ [a_3 \cos 3x + b_3 \sin 3x] + \ldots\ldots$$

$$a_0 = \frac{2}{n} \Sigma y$$

$$a_1 = \frac{2}{n} \Sigma y \cos x$$

$$b_1 = \frac{2}{n} \Sigma y \sin x$$

$$a_2 = \frac{2}{n} \Sigma\, y \cos 2x$$

$$b_2 = \frac{2}{n} \Sigma\, y \sin 2x$$

$$a_3 = \frac{2}{n} \Sigma\, y \cos 3x$$

$$b_3 = \frac{2}{n} \Sigma\, y \sin 3x .$$

	x	y	y x $x\cos y$	$y \sin x$	$y \cos 2x$	$y \sin 2x$	$y \cos 3x$	$y \sin 3x$
0°	0	1.0	1	0	1	0	1	0
60°	$\pi/3$	1.4	0.7	1.212	-0.7	1.1.212	-1.4	0
120°	$2\pi/3$	1.9	-0.95	1.645	-0.95	1.645	1.9	0
180°	π	1.7	-1.7	0	1.7	0	-1.7	0
240°	$\frac{4\pi}{3}$	1.5	-0.75	-1.299	-0.75	1.299	1.5	0
300°	$\frac{5\pi}{3}$	1.2	0.6	-1.039	-0.6	-1.039	-1.2	0
	Σ 2π	8.7	-1.1.	0.519.	-0.3	3.117.	0.1	0.

where

$$a_0 = \frac{2}{n} \Sigma y$$

$$= \frac{2}{6}\, 8.7$$

$$= \frac{17.4}{6}$$

$$= 2.9$$

$$a_1 = \frac{2}{n} \Sigma y \cos x$$

$$= \frac{2}{6} -1.1$$

$$= -0.366$$

$$b_2 = \frac{2}{n} \Sigma y \sin x$$

$$= \frac{2}{6} \quad 0.519.$$

$$= 0.173$$

$$a_2 = \frac{2}{n} \Sigma y \cos 2x$$

$$= \frac{2}{6} -0.3.$$

$$= -0.1$$

$$b_2 = \frac{2}{n} \Sigma y \sin 2x$$

$$= \frac{2}{6} \quad 3.117$$

$$= 1.039.$$

$$a_3 = \frac{2}{n} \Sigma y \cos 3x$$

$$= \frac{2}{6} \quad 0.1$$

$$= 0.033$$

$$b_3 = \frac{2}{n} \Sigma y \sin 3x$$

$$= \frac{2}{6} \quad 0$$

$$= 0.$$

$$y = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_1 \cos x + b_1 \sin x\right] + \left[a_2 \cos 2x +\right.$$

$$\left. b_2 \sin 2x\right] + \left[a_3 \cos 3x + b_3 \sin 3x\right] + \ldots$$

$$y = 1.45 + \left[-0.366 + 0.173\right] + \left[-0.1 + 1.039\right]$$

$$+ \left[0.333 + 0\right]$$

//.

---

16. $f(x) = (l-x)^2 \quad , 0 < x < 2l, \quad \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$ //.

Soln:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx\right]$$

where.

$$a_0 = \frac{l}{\pi} \int_0^{2l} f(x)\, dx$$

$$a_n = \frac{l}{\pi} \int_0^{2l} f(x) \cos \frac{n\pi x}{l}\, dx$$

$$b_n = \frac{l}{\pi} \int_0^{2l} f(x) \sin \frac{n\pi x}{l}\, dx.$$

To find a₀

$$a_0 = \frac{l}{\pi} \int_0^{2l} (l-x)^2\, dx$$

$$= \frac{l}{\pi} \int_0^{2l} \left[\frac{(l-x)^3}{3}\right]_0$$

$$= \frac{l}{\pi} \left[\frac{(l-x)^3}{3}\right]_0^{2l} \quad = \frac{l}{\pi} \left[\frac{[l-2l]^3}{3} - 0\right].$$

$$= \frac{l}{\pi} \frac{-l^3}{3} \qquad = \frac{l}{\pi} \frac{4\pi l^3}{3}$$

To find a_n.

$$a_n = \frac{l}{\pi}\int_0^{2l} \overset{(l-x)^2}{f(x)} \cdot \cos\frac{n\pi x}{l}\,dx$$

$$= \frac{l}{\pi}\int_0^{2l}\left[\frac{\sin n\pi x/l}{\frac{\pi^2 l^2}{n^2}} - \frac{\cos n\pi x/l}{\frac{\pi^2/l^2}{n^2}}\right]$$

$$= \frac{l}{\pi}\left[\frac{\sin n\pi x/l}{\frac{\pi^2 l^2}{n^2}} - \frac{\cos n\pi x/l}{\frac{\pi^2/l^2}{n^2}}\right]_0^{2l}$$

$$= \frac{l}{\pi}\left[\frac{\sin \pi x/l}{\pi^2 l^2} - \frac{\cos n\pi x/l}{n^2 l^2}\right]$$

$$= \boxed{\frac{l}{\pi}\frac{4\pi l^2}{n^2}}$$

To find b_n.

$$f(x) \to (l-x)^2$$

$$b_n = \frac{l}{\pi}\int_0^{2l} f(x)\ \sin\frac{n\pi x}{l}\,dx$$

$$= \frac{l}{\pi}\int_0^{2l}\left[\frac{\cos n\pi x/l}{\pi^2 n^2} - \frac{\sin n\pi x/l}{n^2\pi^2}\right]$$

$$= \frac{l}{\pi}\left[\frac{\cos n\pi x/l}{\pi^2/n^2} - \frac{\sin n\pi x/l}{n^2\pi^2}\right]_0^{2l}$$

$$= \frac{l}{\pi}\left[\frac{\cos n\pi x/l}{\frac{n^2\pi^2}{l^2}} - \frac{\sin n\pi x/l}{\frac{n^2\pi^2}{l^2}}\right]$$

$$= 0$$

Sub the value of a_0, a_n, b_n. in of equ ①.

$$f(x) = \frac{\cancel{2}}{\cancel{l}} \frac{l^3}{\pi} + \left[\frac{4\pi^2 l}{n^2} + 0 \right]$$

$$= \frac{l^3}{\pi} + \left[\frac{4\pi^2 l}{n^2} \right]$$

$$= \frac{\pi^2}{6}.$$

which is, deduce into the

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

$$\frac{\pi^2}{2+4} = \frac{1}{1^2} + 0 + \frac{1}{2^2} + 0 + \frac{1}{3^2} + \cdots$$

$$\frac{\pi}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

Hence It proved.

---

11. $$f(x) = \begin{cases} 1 - x^2, & \text{if } |x| < 1 \\ 0, & \text{if } |x| > 1. \end{cases}$$

$$\int_0^\infty \left(\frac{\sin x - x\cos x}{x^3} \right) \cos \frac{x}{2} \, dx.$$

Soln:

Fourier transform.

$$f(x) = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} \, dx.$$

$$= \sqrt{\frac{2}{\pi}} \int_{-1}^{1} (1 - x^2) e^{isx} \, dx.$$

Fourier transform of cosine.

$$F_c[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overset{f(x)}{\cos} e^{isx}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-x)^2 \cos e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(1-x)^2}{(1-x)} - \frac{\cos(1-x)^2}{(1-x)} \right]_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}} \left[0 - \frac{1}{(1-x)} + \frac{\cos(1-x)^2}{(1-x)} \right].$$

$$= \frac{1}{\sqrt{2\pi}} \left[-\frac{\cos(\cdot x)^2}{(1-x)} \right].$$

Fourisser transform of sine.

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-x)^2 \sin e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(ax)}{ax^3} - \frac{xa\cos ax}{ax^3} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(ax)}{ax^3} - \frac{xa\cos ax}{ax^3} \right]$$

put $a = 1$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin x}{x^3} - \frac{x\cos x}{x^3} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin x - x\cos x}{x^3} \right]$$

$$\therefore \int_0^{\infty} \left[\frac{\sin x - x\cos x}{x^3} \right] \cdot \frac{x}{2}\, dx$$

//.

17. $f(x) = \sin x,$

$0 < x < \pi.$

soln:

Fourier cosine series

$$F_c[f(x)] = \cancel{\frac{1}{\pi}} \int_0^{2\pi} f(x) \cos$$

$$= \sqrt{\frac{\pi}{2}} \int_{\cancel{0}}^{\infty} f(x) \cos \cdot e^{isx} \, dx.$$

$$= \sqrt{\frac{\pi}{2}} \int_0^{\overset{\pi}{\infty}} (\sin x) \cos e^{isx} \, dx$$

$$= \sqrt{\frac{\pi}{2}} \int_0^{\pi} \left[\frac{\sin \pi \overset{n}{x}}{n\pi} - \frac{\cos \pi n x}{n\pi} \right]$$

$$= \sqrt{\frac{\pi}{2}} \left[\frac{\sin \pi n x}{n\pi} - \frac{\cos \pi n x}{n\pi} \right]_0^{\pi}$$

$$= \sqrt{\frac{\pi}{2}} \left[\frac{\sin \pi^2 n}{n\pi^2} - \frac{\cos \pi^2 n}{n\pi^2} \right].$$

$$= \sqrt{\frac{\pi}{2}} \left[\frac{\pi}{2} \right]$$

$$= \sqrt{\frac{\pi^2}{2}}, \quad = \pi/2$$

$$= \pi/2$$

$$F_c[f(x)] = \sqrt{\pi/2} \int_0^{\pi} \sin x \cos e^{isx} \, dx$$

$$= \sqrt{\pi/2} \cdot (\sin x \, e^{isx}) + (\cos x \, e^{isx}) \, dx$$

$$= \sqrt{\pi/2} \left[\frac{\sin \pi n x}{n\pi} \right] + \left[\frac{\cos \pi n x}{n\pi} \right].$$

$$= \sqrt{\pi/2} \left\{ \frac{\sin \pi^2 n}{n\pi^2} - \frac{\cos n\pi}{n\pi^2} \right].$$

$$= \pi/2$$

Section A

part II.

4. Fourier cosine transform of e^{-x}.

Soln:-

$F(x) = e^{-x}.$

$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x}\, dx.$

$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-x}}{-x} .dt$ $\qquad x = t/a.$

$= \sqrt{2/\pi} \int_0^\infty \frac{e^{-t/a}}{t/\cancel{a}} .dt/\cancel{a}$

$= \sqrt{2/\pi} \int_0^\infty \frac{e^{-ta}}{t} dt$

$= \sqrt{2/\pi} \int_0^\infty \frac{\pi}{\cancel{2}}$

$\frac{1}{2}$ $= \sqrt{\frac{2}{\pi}}\, \pi$ //.

7

srivenkateshwaraa
COLLEGE OF ENGINEERING & TECHNOLOGY

Approved by AICTE, New Delhi & Affiliated to Pondicherry University, Puducherry
13-A, Pondy - Villupuram Main Road, Ariyur, Puducherry - 605 107.

ASPIRE TO EXCEL

ISO 21001

EXAM CELL

Part - I (to be filled by the candidate)

Name of the Exam :	MODEL EXAM		ABINAYA.V
Date of the Exam :	11/06/24	Register No.:	2 3 T D 0 6 5 4
Degree / Branch :	B. Tech - CSE - A	Year / Semester :	I/II
Subject Code :	CSBS 202	No. of Pages Written :	14
Subject Title :	MATHEMATICS - II.		

G. Palanivel.

Name of the Hall superintendent
(With Designation)

11/6/24

Signature of the Hall Superintendent
(With Date)

PART - II (to be filled by the Examiner)

Q.No.	Marks	Q.No.	Sub Division Marks				TOTAL MARKS
			(i)	(ii)	(iii)	(iv)	
1.	2	1.					
2.	1	2.	7				7
3.	2	3.					
4.	1	4.	6				6
5.		5.					
6.	1	6.	6				6
7.	1	7.					
8.		8.					
9.	2	9.	7				7
10.	1	10.	6				6
TOTAL (A)	11	TOTAL (B)	32				32
Grand Total (A+B)		43	In Words :	Four three only			

Dr. John Sundar AP/MATHS

Name of the Hall superintendent
(With Designation)

Signature of the Hall Superintendent
(With Date)

INSTRUCTIONS OF THE CANDIDATES

1. Check the Question paper first, its subject code / Title etc., before answering the questions.
2. Fill all the particulars in Part - I of this sheet.
3. Possession of any incrimination material, mobile phones / other electronic gadgets & mall practice of any nature are punishable as per rules.
4. Use both side of the paper for answering questions.
5. Answer must be legibly written in Ink (Blue, Black or Blue Black)
6. Writing or leaving any distinctive marks so as to identify your paper is strictly prohibited.
7. Strike all the blank pages left in the answer sheet at the end of the examination.
8. Rough work, if any must be done at the bottom of the page, reserving a fourth at the bottom of the page exclusively for this purpose.
9. Do write in the margin except the question number.
10. Graph or other sheets should be attached between pages and tied with white thread.

Section A

Answer the following :-

1. $z = ax + by + a^2 + b^2$, where a, b are arbitrary constants.

Soln: $z = ax + by + a^2 + b^2$.

$$\frac{\partial z}{\partial x} = a$$

2 $$\frac{\partial z}{\partial y} = b \qquad \therefore \; p + q + p^2 + q^2$$

---

3. Prove: $\left[e^{-at}\right] = \frac{1}{2\pi} \int_0^\infty e^{-st} f(t)\, dt$.

$$= \frac{1}{2\pi} \int_0^\infty e^{-at}\, d$$

$$= \left[e^{-at}\right]_0^\infty \Rightarrow \left[\frac{e^{-at}}{t+a}\right]_0^s$$

✓ $$= \frac{1}{s+a}$$

---

2. Find the complimentary function of:

$(D^3 - 2DD')z = \sin(x + 2y$.

$(D^3 - 2DD')z$

4. Find : $L[\sin 3t \cos 3t]$.

$\sin A \cos B = \sin A \cos B + \cos A \sin B$

$= 3t \cdot 3t + 3t \cdot 3t$

$= 18t.$

---

6. Find $L^{-1}\left[\frac{1}{(s+1)^2}\right]$.

$$L^{-1}\left[\frac{1}{(s+1)^2}\right] = L^{-1}\left[\frac{1}{s+1} \cdot \frac{1}{s+1}\right].$$

$$= L^{-1}\left[\frac{e^{-s}}{(s+1)(s+1)}\right]$$

---

7. Fourier Transform Pairs

$$= \cancel{\int_0^\infty} \int_{-\infty}^{\infty}$$

$$f(x) = 1/2\pi \int_0^\infty e^{-st} f(t)\, dt.$$

Conditions of Dirchlet's existence of Fourier series:

- * They are single - valued and finite
- * There are finite with discontinuity along any one period
- * The finite numbers are of maxima and minima,

Rms Value of $f(x)=x^2$ in interval $(0, \pi)$.

$$\bar{y}^2 = \frac{1}{b-a}\int_a^b y^2 \, dx \quad , \text{where } y^2 = f(x).$$

$$= \frac{1}{b-a}\int_a^b x^2 \, dx = \frac{1}{\pi}\int_0^\pi x^2 \, dx.$$

$$= 1/\pi \left[x\right]_0^\pi = 1/\pi \left[\pi - 0\right]$$

$$= 2\pi$$

5. Find $L^{-1}\left[\frac{s^2-3s+4}{s3}\right]$

$$\left[\frac{s^2-3s+4}{s3}\right]$$

Section - B

Answer the following:

14. Find Laplace transform of $f(t) = \begin{cases} 1, & \text{for } 0 \le t \le a/2 \\ -1, & \text{for } a/2 \le t \le a. \end{cases}$

& $f(t+a) = f(t)$ for $t > 0$

Soln:

$$f(t) = \frac{1}{1-e^{-st}} \int_0^T e^{-st}\,dt$$

$$= \frac{1}{1-e^{-st}} \int_0^T e^{-st}$$

$$= \frac{1}{1-e^{-st}} \int_0^{a/2} 1 \quad \int_{a/2}^{a}$$

6

$$= \frac{1}{1-e^{-st}} \left[a/2 - 0\right] \left[-a/2 - a\right]$$

$$= \frac{1}{1-e^{-st}} \left[a/2 \cdot -a/2\right]$$

$$= \tan h s/2$$

Convolution Theorum.

When the $L[f(t)] \times L[g(t)]$ is equal to the function of $f(s) : g(s)$.

$$L[f(t) * g(t)] = f(t) * g(t) = F(s) \cdot g(s).$$

Theorum:

$$L[f(t) * g(t)] = 1/2\pi \int_0^\infty \int_{-\infty}^\infty e^{-st} f(t)\, dt.$$

$$= 1/2\pi \int_0^\infty f(t) + \int_0^\infty g(t)\, e^{-st} f(t)\, dt.$$

$$= 1/2\pi \int_0^\infty f(t) + \int_0^\infty g(t)\, f(t)$$

$$= 1/2\pi \int_0^\infty f(t)\, e^{-st} + \int_0^\infty g(t)\, e^{-st}\, dt$$

$$= 1/2\pi \int_0^\infty \frac{e^{-st}}{s}\, dt + \int_0^\infty \frac{e^{-st}}{s}\, dt.$$

$$= \left[f(s) \cdot g(s) \right]$$

$$= f(s) \cdot g(s)$$

$$\Rightarrow L\left[f(t) \cdot g(t) \right] = f(s) \cdot g(s)$$

6

Hence Proved Convolution Theorem.

---

19. $f(x) = (l - x)^2$ in interval $0 < x < 2l$ & deduce that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots = \frac{\pi^2}{6}$.

Soln:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \{ a_n \cos nx + b_n \sin nx \}$$

Where,

$$a_0 = 1/\pi \int_0^{2l} a_n \cos \frac{2\pi x}{l} + b_n \sin \frac{2\pi x}{l}.$$

$$a_n = 1/\pi \int_0^{2l} a_n \cos \frac{2\pi x}{l} f(x).$$

12 Solve $[D^3 - 7DD'^2 - 6D'^3]z = e^{2x+y} + \sin(x+2y) + x^3y.$

Soln: $CF = f_1(y+3x) + f_2(y-2x) + f_3(y-x).$

$[D^3 - 7DD'^2 - 6D'^3]z = e^{2x+y} + \sin(x+2y) + x^3y.$

$$PI_1 = \frac{1}{[D^3 - 7DD'^2 - 6D'^3]} e^{2x+y}$$

$$= \frac{1}{D[D^2 - 7D' - 6D'^3]} e^{2x+y}$$

$$= -\frac{1}{12} e^{2x+y}.$$

$$PI_2 = \frac{1}{[D^3 - 7DD'^2 - 6D'^3]} \sin(x+2y)$$

$$= \frac{1}{[D^3 - 7DD'^2 - 6D'^3]} \cos(x+2y)$$

$$= \frac{1}{D[D^2 - 7D' - 6D'^3]} \cos(x+2y)$$

$$= \frac{\cos(x+2y)}{21}.$$

12 Solve: $[D^3 - 7DD'^2 - 6D'^3]z = e^{2x} + y + \sin(x+2y) + x^3y.$

Soln:

To Find: $PI_3 = \frac{1}{[D^3 - 7DD'^2 + 6D'^3]} x^3y.$

$$PI_3 = \frac{1}{\left[D^3 - \frac{7DD'^2}{D^3} + 6\frac{D'^3}{D^3}\right]} x^3y.$$

$$= \frac{1}{D^3}\left[1 - \frac{7D'^2}{D^2} + 6\frac{D'^3}{D^3}\right] x^3y.$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$$

$$= \frac{1}{D^3}\left[1 + \left(\frac{7D'^2}{D^2} - \frac{6D'}{D^2}\right) + \ldots\right] x^3y.$$

$$= \frac{1}{D^3}[1]\, x^3y.$$

$$= \frac{1}{D} \cdot \frac{1}{D}\left(\frac{1}{D} x^3y\right)$$

$$= \frac{1}{D} \cdot \frac{1}{D}\int x^3y\,dx$$

$$= \frac{1}{D} \cdot \frac{1}{D}\left[\frac{x^4y}{4}\right]$$

$$= \frac{1}{4}\,\frac{1}{D}\left(\frac{x^5}{5}y\right)$$

$$= \frac{1}{20}\left(\frac{x^6 y}{6}\right)$$

$$= \frac{x^6 y}{120}$$

Thus the general solution is

$$y = CF + PI_1 + PI_2 + PI_3$$

$$= f_1(y+3x) + f_2(y-2x) + f_3(y-x)$$

$$-\frac{1}{12}e^{2x+y} + \frac{\cos(x+2y)}{21} + \frac{x^6 y}{120}.$$

20.

x	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π
y	1.98	1.30	1.05	1.30	-0.88	-0.25	1.9

Soln:

Here 0 And π are the same value

So we can consider only 0 and its y value.

x	y	$\cos x$	$\sin x$	$\cos 2x$	$\sin 2x$	$y\cos x$	$y\sin x$	$y\cos 2x$	$y\sin 2x$
0	1.98	1	0	1	0	1.98	0	1.98	0
$\pi/6$	1.30	0.5	0.86	−0.5	0.86	0.65	1.11	−0.65	1.11
$\pi/3$	1.05	−0.5	0.86	−0.5	−0.86	−0.52	0.90	−0.52	−0.90
$\pi/2$	1.30	−1	0	1	0	−1.3	0	1.3	0
$2\pi/3$	−0.88	−0.5	−0.86	−0.5	0.86	0.44	0.75	0.44	−0.75
$5\pi/6$	−0.25	0.5	−0.86	−0.5	−0.86	−0.125	0.21	0.125	0.21
Σ	4.5					1.125	2.97	2.675	−0.33

$$f(x) = \frac{a_0}{2} + \left[a_1 \cos x + a_2 \sin x + a_1 \cos 2x + a_2 \sin 2x\right].$$

$$y = \frac{2}{n} \times 4.5$$

$$= 2/6 \times 4.5 = 1.5$$

$$y\cos x = 2/6 \times 1.125 = 0.375$$

6 $$y\sin x = 2/6 \times 2.97 = 0.99$$

$$y\cos 2x = 2/6 \times 2.675 = 0.891$$

$$y\sin 2x = 2/6 \times -0.33 = -0.11.$$

---

19. $f(x) = (l-x)^2$ in the interval $0 < x < 2l$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l}\right]$$

$$a_0 = 1/\pi \int_0^{2l} f(x)\,dx$$

$$a_n = 1/\pi \int_0^{2l} f(x) \cos \frac{n\pi x}{l}\,dx$$

$$b_n = 1/\pi \int_0^{2l} f(x) \sin \frac{n\pi x}{l}\, dx$$

$$a_0 = 1/\pi \int_0^{2l} f(x)\,dx$$

$$= 1/\pi \int_0^{2l} (l-x)^2\, dx$$

$$= 1/\pi \left[[l-x]^2 \right]_0^{2l}$$

$$= 0$$

$$a_n = 1/\pi \int_0^{2l} \cos \frac{n\pi x}{l}\, f(x)\,dx$$

$$= 1/\pi \int_0^{2l} \cos \frac{n\pi x}{l} (l-x)^2\, dx$$

$$\Rightarrow \left[uv - u'v_1 + u''v_2 - \ldots \right]$$

$$= (l-x)^2 \left(\sin \frac{n\pi x}{nl} \right)$$

$$= 1 + 0 \cdot 1/n^2 - 0 - 1/n^2.$$

$$= 0$$

$$b_n = 1/\pi \int_0^{2l} \sin \frac{n\pi x}{l} (l-x)^2\, dx$$

$= 1/n$

Sub a_0, a_n, b_n in eqn ①, we can get

$$\frac{1}{\sin x} + \frac{1}{\sin 2x} + \frac{1}{\sin 3x} + \cdots$$

$$\therefore \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \pi^2/6 \; //$$

---

17. $f(x) = \begin{cases} 1, & \text{for } |x| < a, \\ 0, & \text{for } |x| > a, \end{cases}$ $a > 0$ & deduce that

(i) $a > 0$ and hence evaluate $\int_0^\infty \frac{\sin x}{x}\, dx$.

srivenkateshwaraa
College of Engineering & Technology
ASPIRE TO EXCEL
Ariyur, Puducherry-605 102.

A NAAC

ISO 21001

Academic Year: 2023 – 2024
Year/Sem: I/II
Subject: Mathematics-II

S.NO	REG.NO	DATE OF THE EXAM	27.04.24	24.05.24	11.06.24	19.06.24
		NAME OF THE STUDENT	CIA I	CIA II	MODEL	UNIVERSITY RESULT
1	23TD0651	AARTHI .A	33	33	33	
2	23TD0654	ABINAYA . V	35	43	43	
3	23TD0657	ANISHA . V	40	47	45	
4	23TD0658	ARISHKUMAR . K	27	31	28	
5	23TD0659	ASHWIN . C	26	31	32	
6	23TD0660	ASWIN T	38	38	45	
7	23TD0666	DEVASRI . S	31	32	34	
8	23TD0667	DHANUSH . V	0	7	8	
9	23TD0668	DHARSHANI . S	33	41	53	
10	23TD0670	DULASI KRISHNA . P	1	29	31	
11	23TD0674	GOKUL . S	1	28	30	
12	23TD0680	HEMACHANDRAN . G	32	37	45	
13	23TD0681	JANITHAA K R	39	39	42	
14	23TD0683	JAYA SRINIVASAN . A	9	6	5	
15	23TD0684	JEEVARAAJAN S	28	34	36	
16	23TD0685	JEEVITHA . E	30	35	48	
17	23TD0686	KALAIVANAN . L	29	33	32	
18	23TD0687	KAMARAJ . M	13	32	AB	
19	23TD0688	KARTHIKEYAN . R	31	30	32	
20	23TD0690	KIROUBAKARAN . V	AB	31	31	
21	23TD0692	MADHAN . R	8	21	29	
22	23TD0694	MAHESH V	32	36	37	
23	23TD0696	MOHAMED ASIF . M	25	0	30	
24	23TD0697	MOHAMED IBRAHIM	26	30	32	
25	23TD0699	MOHAMMED AAQIL . M	30	30	30	
26	23TD0703	NIKILESHYOGAN . G	2	AB	AB	
27	23TD0706	PAVITHRA . K	34	41	42	
28	23TD0707	PRAKASHRAJ . S	31	20	32	
29	23TD0710	RAMA . S	32	35	31	
30	23TD0711	RANI .P	30	34	30	
31	23TD0712	RANJITH . S	1	24	10	
32	23TD0715	SABREEN . S	32	34	20	
33	23TD0716	SANIYASRI . J	37	34	37	
34	23TD0719	SATHYA J	32	25	30	
35	23TD0721	SHALINI . C	34	46	40	
36	23TD0722	SHARAN SHANTH . R	30	31	32	

37	23TD0723	SIVARANJANI K	36	40	43	
38	23TD0725	SOWKANTHINI . Y	37	44	46	
39	23TD0727	SRIVARDHINI .D	36	36	47	
40	23TD0731	VIJAYAVEL .R. S	30	32	30	
41	23TD0733	VITHYASAKAR S	AB	0	16	
42	23TD0735	YUVASREE . G	32	40	35	

Head of the Department
Sciences & Humanities
Sri Venkateshwaraa College of Engineering
& Technology,
Ariyur, Puducherry - 605 102.

MATHS ASSIGNMNET ①

NAME : ANISHA. V
CLASS : B.Tech CSE
SEC : 'A' - I st YEAR.
DATE : 22.04.2024.

$\frac{10}{10}$

1. Find the laplace transforms of the following function:-

i) $e^{-2t} \cos^3 2t$

soln;- $L[e^{-2t} \cos^3 2t] = L[\cos^3 2]._{s \to s+2}$

$$= L\left[\frac{3\cos 2t + \cos 6t}{4}\right]_{s \to s+2}$$

$$= \frac{1}{4}\left[\frac{3s}{s^2+4} + \frac{s}{s^2+36}\right]_{s \to s+2}$$

$$= \frac{1}{4}\left[\frac{3s[(s^2+36)+s(s^2+4)}{(s^2+4)(s^2+36)}\right]_{s \to s+2}$$

$$= \frac{1}{4}\left[\frac{3s^3+108s+s^3+4s}{(s^2+4)(s^2+36)}\right]_{s \to s+2}$$

$$= \frac{1}{4}\left[\frac{4s^3+112s}{(s^2+4)(s^2+36)}\right]_{s \to s+2}$$

$$= \frac{\cancel{4}}{\cancel{4}}\left[\frac{s(s^2+28)}{(s^2+4)(s^2+36)}\right]_{s \to s+2}$$

$$= \frac{(s+2)((s+2)^2) +28)}{((s+2)^2+4)((s+2)^2+36)} \;//.$$

ii) $e^{3t} \sin t \sin 2t$

solⁿ:-

$$L\left[e^{3t}\sin t \sin 2t\right] = L\left[\sin t \sin 2t\right]_{s\to s-3}$$

$$= \frac{1}{2}\cdot L\left[\cos t - \cos 3t\right]_{s\to s+3}$$

$$= \frac{1}{2}\left[\frac{s}{s^2+1} - \frac{s}{s^2+9}\right]_{s\to s+3} = \left[\frac{4s}{(s^2+1)(s^2+9)}\right]_{s\to s-3}$$

$$= \frac{4(s-3)}{((s-3)^2+4)((s-3)^2+9)}$$

---

iii) $\sinh t/2 \sin \sqrt{3}/2\, t$.

solⁿ:-

$$= \frac{1}{2}\left[e^{t/2} - e^{-t/2}\right]\sin \sqrt{3}/_2\, t$$

$$= \frac{1}{2}\left[e^{t/2}\sin \sqrt{3}/_2\, t - e^{-t/2}\sin \sqrt{3}/_2\, t\right]$$

$$L\left[\sinh \tfrac{1}{2} t \sin \sqrt{3}/_2\, t\right] = \frac{1}{2}\left\{L\left[e^{t/2}\sin \sqrt{3}/_2\, t - L\left[e^{-t/2}\sin \sqrt{3}/_2\, t\right]\right\}\right.$$

$$= \frac{1}{2}\left\{L\left[\sin \sqrt{3}/_2\, t\right]_{s\to s-1/2} - L\left[\sin \sqrt{3}/_2\, t\right]_{s\to s+1/2}\right\}$$

$$= \frac{1}{2}\left\{\left[\frac{\sqrt{3}/2}{s^2+3/4}\right]_{s\to s-1/2} - \left[\frac{\sqrt{3}/2}{s^2+3/4}\right]_{s\to s+1/2}\right\}$$

$$= \sqrt{3}/4\left[\frac{1}{(s-1/2)^2+3/4} - \frac{1}{(s+1/2)^2+3/4}\right]$$

$$= \sqrt{3}/4 \left[\frac{1}{s^2+\frac{1}{4}-2s/\cancel{2}+3/4} - \frac{1}{s^2+1/4+2s/\cancel{2}+3/4} \right]$$

$$= \sqrt{3}/4 \left[\frac{1}{s^2-s+1} - \frac{1}{s^2+s+1} \right]$$

$$= \sqrt{3}/4 \left[\frac{2s}{(s^2+s+1)\cdot(s^2-s+1)} \right] = \sqrt{3}/2 \left[\frac{s}{\cancel{s^4+}s^4+1} \right]$$

---

iv) $t \sinh^3 t$

soln:- $L[t \sinh^3 t] = (-1)\frac{d}{ds} L\left[\frac{e^t - e^{-t}}{2}\right]^3$

$$= \frac{-1}{8} \cdot \frac{d}{ds} L\left[e^{3t} - 3e^{t} + 3e^{-t} - e^{-3t}\right]$$

$$= -1/8 \cdot \frac{d}{ds}\left[\frac{1}{s-3} - \frac{3}{s-1} + \frac{3}{s+1} - \frac{1}{s+3}\right]$$

$$= 1/8 \left[\frac{1}{(s-3)^2} \cancel{- \frac{3}{(s-1)^2} + \frac{3}{(s+1)^2}} - \frac{1}{(s+3)^2}\right]$$

---

v) $(t \sin at)^2$

soln:- $L\left[t^2\left(\frac{1-\cos 2at}{2}\right)\right] = L[(t \sin at)^2]$

$$= \frac{(-1)^2}{2} \frac{d^2}{ds^2} L[1-\cos 2t] = \frac{1}{2} \cdot \frac{d^2}{ds^2}\left[\frac{1}{s} - \frac{s}{s^2+4a^2}\right]$$

$$= \frac{1}{2} \cdot \frac{d}{ds}\left[\frac{-1}{s^2} - \cancel{\left[\frac{s^2+4a^2-s-2s}{s^2+4a^2}\right]\}}\right.$$

$$= \frac{1}{2}\frac{d}{ds}\left\{-\frac{1}{s^2} + \left[\frac{s^2-4a^2}{(s^2+4a^2)^2}\right]\right\}$$

$$= \frac{1}{2}\left[\frac{2}{s^3} + \frac{2s(12a^2-s^2)}{(s^2+4a^2)^3}\right] = \frac{1}{s^3} + \frac{s(12a^2-a^2)}{(s^2+4a^2)^3}.$$

---

vi) $t^3 e^{-3t}$

soln:

$$L\left[t^3 e^{-3t}\right] = (-1)^3 \frac{d^3}{ds^3} L\left[e^{-3t}\right] = (-1)^3 \frac{d^3}{ds^3}\left[\frac{1}{s+3}\right]$$

$$= \frac{-(-1)^3 \cdot 3!}{(s+3)^{3+1}} = \frac{6}{(s+3)^4}$$

---

vii) $t e^{-2t} \sinh 3t$.

Soln:

$$L\left[t e^{-2t} \sinh 3t\right] = L\left[t \sinh 3t\right]_{s \to s+2} \longrightarrow ①$$

$$L\left[t \sinh 3t\right] = (-1)\frac{d}{ds}\left[\frac{3}{s^2-9}\right] = \frac{6s}{(s^2-9)^2} \quad — ②.$$

using ② in ① we get,

$$L\left[t e^{-2t} \sinh 3t\right] = \left[\frac{6s}{(s^2-9)^2}\right]_{s \to s+2}$$

$$= \frac{6(s+2)}{(s^2+4s-5)^2}$$

viii) $t \cosh t \cos t$.

soln:- $L[t \cosh \cos t] = L\left[t\left(\frac{e^t - e^{-t}}{2}\right) \cos t\right]$

$= \frac{1}{2}\left[L(t e^t \cos t) - L(t e^{-t} \cos t)\right]$

$= \frac{1}{2}\left[L[t \cos t)_{s\to s-1} - L(t \cos t)_{s\to s+1}\right]$

$= \frac{1}{2}\left\{\left[\frac{s^2-1}{(s^2+1)^2}\right]_{s\to s-1} - \left[\frac{s^2+2s}{(s^2+2s+2)^2}\right]\right\}$

$= \frac{1}{2}\left[\frac{s^2-2s}{(s^2+2s+2)^2} - \frac{s^2+2s}{(s^2\cancel{+2s}+2)^2}\right]$.

---

ix) $t^2 e^{2t} \cos 3t$.

soln:-

$L[t^2 e^{2t} \cos 3t] = L[t^2 \cos 3t]_{s\to s-3} \rightarrow (1)$

$= (-1)^2 \frac{d^2}{ds^2} L[\cos 3t]$

$= \frac{d}{ds} \frac{d}{ds}\left[\frac{s}{s^2+9}\right]$

$= \frac{d}{ds}\left[\frac{(s^2+9).1 - s.2s}{(s^2+9)^2}\right] = \frac{d}{ds}\left[\frac{(s^2+9)-2s^2}{(s^2+9)^2}\right]$

$= \frac{d}{ds}\left[\frac{9-s^2}{(s^2+9)^2}\right] = \frac{[s^2+9]^2.(-2s) - (9-s^2).[2(s^2+9).2s]}{(s^2+9)^4}$.

$$= \frac{-2s(s^2+9)-4s\cdot(9-s^2)}{(s^2+9)^3} = \frac{2s^3-54s}{(s^2+9)3} \to (2)$$

Using the equ (2) in (1).

$$L\left[t^2e^{2t}\cos 3t\right] = \left[\frac{2s^3-54s}{(s^2+9)^3}\right]_{s\to s-2} = \frac{2(s-2)^3-54(s-2)}{\left[(s-2)^2+9\right]^3}$$

$$= \frac{2\left[s^3-6s^2+12s+8\right]-54\left[s-2\right]}{\left[s^2-4s+4+9\right]^3}$$

$$= \frac{2s^2-12s^2-30s+92}{(s^2-4s+13]^3} = \frac{92-30s-10s^2}{[s^2-4s+13]^3}.$$

---

2. Evalute.

i) $L\left[\frac{e^t-1}{te^{2t}}\right]$

soln:-

$$L\left[\frac{e^t-1}{te^{2t}}\right] = \int_s^\infty t\left[e^{-t}-e^{-2t}\right]ds$$

$$= \int_s^\infty \left[\frac{1}{s+1}-\frac{1}{s+2}\right].ds$$

$$= \left[\log(s+1)-\log.(s+2)\right]_s^\infty = \left[\log\left(\frac{s+1}{s+2}\right)\right]_s^\infty$$

$$= \left[\log\left(\frac{1+1/s}{1+2/s}\right)\right]_{s\to\infty} - \log\left(\frac{s+1}{s+2}\right)$$

$$= \log(1)+\log\left(\frac{s+1}{s+2}\right)^{-1} = \log\left(\frac{s+2}{s+1}\right)$$

ii) $L\left[\frac{\sinh at}{t}\right]$

soln:- $L\left[\frac{\sinh at}{t}\right] = \int_s^\infty L[\sinh at]\,ds = \int_s^\infty \frac{a}{s^2-a^2}\,ds$

$= a\left[\frac{1}{2a}\log\left(\frac{s-a}{s+a}\right)\right]_s^\infty = \left[\frac{1}{2}\log\left(\frac{1-a/s}{1+a/s}\right)\right]_{s\to\infty}$

$-\frac{1}{2}\log\left(\frac{s-a}{s+a}\right)$

$= \frac{1}{2}\log\left(\frac{s+a}{s-a}\right)$

iii) $L\left[\frac{2\sin t\sin 2t}{t}\right]$

soln:- $= \int_s^\infty L[2\sin t\sin 2t]\,ds$

$= \int_s^\infty \left[\frac{s}{s^2+1} - \frac{s}{s^2+9}\right] ds = \left[\frac{1}{2}\log\left[\frac{s^2+1}{s^2+9}\right]\right]_s^\infty$

$= \left[\frac{1}{2}\log\left(\frac{1+1/s^2}{1+9/s^2}\right)\right]_{s\to\infty} - \left[\frac{1}{2}\log\left(\frac{s^2+1}{s^2+9}\right)\right]$

$= \frac{1}{2}\log\left(\frac{s^2+9}{s^2+1}\right)$

3. Verify I.V.T and F.V.T for the function.

i) $f(t) = e^{t} \cos^2 t$.

Soln:-

$$\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s)$$

$$F(s) = L[f(t) = L[e^{-t} \cos^2 t]$$

$$= L\left[e^{-t}\left(\frac{1+\cos 2t}{2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4}\right]_{s \to s+1}$$

$$= \frac{1}{2} \cdot \left[\frac{1}{s+1} + \frac{s+1}{s^2+2s+5}\right]$$

Thus, $sF(s) = \frac{1}{2}\left[\frac{s}{s+1} + \frac{s^2+s}{s^2+2s+5}\right]$

$$\lim_{t \to 0} f(t) = \lim_{t \to 0} [e^{-t} \cos^2 t] = 1 \text{ — ①}$$

$$\lim_{s \to 0} sF(s) = \lim_{s \to \infty} \frac{1}{2}\left[\frac{s}{s+1} + \frac{s^2+s}{s^2+2s+5}\right]$$

$$= \lim_{s \to \infty} \frac{1}{2}\left[\frac{1}{1+1/s} + \frac{1+1/s}{1+2/s+5/s^2}\right]$$

$$= \frac{1}{2}[1+1] = 1 \longrightarrow ②$$

From ① & ② we have;-

$$\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s) = 1.$$

Hence proved [I.V.T].

F.V.T,:

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} [e^{-t}\cos^2 t]$$

$$= 0 \rightarrow ③$$

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0} \cdot\frac{1}{2}\left[\frac{s}{s+1} + \frac{s^2+s}{s^2+2s+5}\right]$$

$$= 0 \longrightarrow ④.$$

From ③ & ④, we get,

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) = 0$$

Hence proved [F.V.T],,

---

ii) $f(t) = t^2 e^{-3t}$:

soln:- I.V.T,

The final value theorem of Laplace transform is

$$\lim_{t\to 0} f(t) = \lim_{s\to 0} sF(s).$$

$$F(s) = L[t^2 e^{-3t}] = L[t^2]_{s\to s+3}$$

$$= \left[\frac{2!}{s^3}\right]_{s\to s+3}$$

$$F(s) = \frac{2}{(s+3)^3} \; \& \; sF(s) = \frac{2s}{(s+3)^3}$$

$$\lim_{t\to 0} f(t) = \lim_{t\to 0} [t^2 e^{-3t}] . = 0 \longrightarrow ①$$

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0} \cdot \frac{2s}{(s+3)^3} = \lim_{s\to\infty} \cdot \frac{2}{s^2(1+3/s)^3}$$

$$= 0 \rightarrow ②.$$

from ① & ② we get,

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s) = 0$$

Hence it proved [I.V.T].

F.V.T.

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} t^2 e^{-3t}.$$

$$= 0 \rightarrow ③$$

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0} \frac{2s}{(s+3)^3} = 0 \rightarrow ④$$

Hence proved [F.V.T].

---

4. Evaluate the integrals.

i) $$\int_0^\infty \frac{\sin 2t}{te^t}\, dt$$

<u>soln</u>:-

$$\int_0^\infty \frac{\sin 2t}{te^t}\, dt = \left[\int_0^\infty e^{-t}\left[\frac{\sin 2t}{t}\right) dt\right] \rightarrow ①$$

$$L\left(\frac{\sin 2t}{t}\right) = \frac{1}{2} L\left[\frac{1-\cos^2 t}{t}\right]$$

$$= \frac{1}{2}\int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+4}\right] ds$$

$$= \frac{1}{2}\left[\log\left(\frac{s}{\sqrt{s^2+4}}\right)\right]_s^\infty$$

$$= \frac{1}{2}\log\left(\frac{\sqrt{s^2+4}}{s}\right) \quad -(2).$$

using (2) in (1) we get,

$$\int_0^\infty \frac{\sin^2 t}{te^t}\cdot dt = \left[\frac{1}{2}\log\left[\frac{\sqrt{s^2+4}}{s}\right]\right]_{s\to 1}.$$

$$= \frac{1}{4}\log(5) \;//.$$

(ii)] $L\left[t\int_0^t e^{4t}\sin 3t\, dt\right].$

soln:- $= (-1)\frac{d}{ds}L\left[\int_0^t e^{4t}\sin 3t\, dt\right] \quad -①$

Now, $L\left[\int_0^t e^{4t}\sin 3t\, dt\right] = \frac{1}{s}L\left[e^{4t}\sin 3t\right]$ by the theorem on LT of integrals.

$$= \frac{1}{s}L\left[\sin 3t\right]_{s\to(s-4)}$$

$$= \frac{1}{s}\left[\frac{3}{(s-4)^2+9}\right] = \frac{3}{s^3-8s^2+25s} \quad -②.$$

using ② in ①, we get,

$$L\left[t\int_0^t e^{4t}\sin 3t\, dt\right] = (-1)\frac{d}{ds}\left[\frac{3}{s^3-8s^2+25s}\right]$$

$$= \frac{9s^2-48s+75}{(s^2-8s^2+25s)^2} \;//.$$

$$L\left[\frac{1}{t}\int_0^t e^{-t}\sin t\,dt\right]$$

soln;-

$$= \int_s^\infty L\left[\int_0^t e^{-t}\sin t\,dt\right]ds \quad \text{—(1)}$$

$L\left[\int_0^t e^{-t}\sin t\,dt\right] = \frac{1}{s}L\left[e^{-t}\sin t\right]$ by the theorem on LT of Integral.

$$= \frac{1}{s}L\left[\sin t\right]_{s\to(s+1)} \qquad = \frac{1}{s}\left[\frac{1}{s^2+1}\right]_{s\to(s+1)}$$

$$= \frac{1}{s}\left[\frac{1}{(s+1)^2+1}\right] = \frac{1}{s.(s^2+2s+2)} \quad \text{—(2).}$$

on resolving the equation ② into partial fractions we get,.

$$\frac{1}{s(s^2+2s+2)} = \frac{A}{s} = \frac{Bs+C}{(s^2+2s+2)}$$

Multiplying both sides by $s(s^2+2s+2)$, we get,

$$1 = A(s^2+2s+2) + (Bs+C)s \quad \text{—(3).}$$

put $s=0$ in ③	comparing s^2 coefficient	Comparing s coeffinent.
$1=2A$	$0=A+B$	$0=2A+C$
$A=\frac{-1}{2}$	Sub A value we get,	Sub A value,
	$B=-1/2$	$C=-1$.

$$L\left[1/t\int_0^t e^{-t}\sin t\,dt\right] = \int_s^\infty\left[\frac{A}{s}+\frac{Bs+C}{(s^2+2s+2)}\right]ds$$

sub A, B, C Values,

$$= \int_s^\infty \left[\frac{1/2}{s} + \frac{-1/2 s - 1}{(s^2 + 2s + 2)} \right] ds$$

$$= \frac{1}{2} \int_s^\infty \left[\frac{1}{s} - \frac{s+2}{s^2+2s+2} \right] ds$$

On resolving the integral into partial,

$$= \frac{1}{2} \int_s^\infty \left[\frac{1}{s} - \frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1} \right] ds$$

$$= \frac{1}{2} \left[\log s - \frac{1}{2} \log\left[(s+1)^2+1\right] + \cot^{-1}(s+1) \right]_s^\infty$$

$$= \frac{1}{2} \left[\log\left(\frac{s}{\sqrt{s^2+2s+2}} \right) + \cot^{-1}(s+1) \right]_s^\infty$$

$$= \frac{1}{2} \left[\log\left(\frac{\sqrt{s^2+2s+2}}{s} \right) - \cot^{-1}(s+1) \right]$$

//.

Find the Laplace transform of. $f(t+4) = f(t)$.

i) $f(t) = \begin{cases} t & , \text{ for } 0 \leq t \leq 2 \\ (4-t) & , \text{ for } 2 \leq t \leq 4 \end{cases}$, $T = 4$.

soln:-

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-4s}} \cdot \int_0^2 e^{-st}\, t\,dt + \int_2^4 e^{-st}(4-t)\,dt$$

where,

$u = t$. $\quad V = e^{-st} \quad U = 4-t$

$u' = 1 \quad V_1 = e^{-st}/s^2 \quad U' = -1$

$u'' = 0 \quad U'' = 0$

$$= \frac{1}{1-e^{-4s}} \left[\left(\frac{te^{-st}}{-s} - (1)\frac{e^{-st}}{s^2} \right)_0^2 + \left[(4-1)\frac{e^{-st}}{-s} + \frac{e^{-st}}{s^2} \right]_2^4 \right]$$

$$= \frac{1}{1-e^{-4s}} \left[\left(\frac{2e^{-st}}{-s} - \frac{e^{-2s}}{s^2} - \left(0 - \frac{1}{s^2}\right) \right) \right] + \left[\left(0 + \frac{e^{-4s}}{s^2} - \left(\frac{2e^{-st}}{s} + \frac{e^{-2s}}{s^2} \right) \right) \right]$$

$$= \frac{4}{1-e^{-4s}} \left[\frac{2\cancel{e^{-st}}}{\cancel{-s}} - \frac{e^{-2s}}{s^2} + \frac{1}{s^2} + \frac{e^{-4s}}{s^2} - \frac{2\cancel{e^{-st}}}{\cancel{-s}} - \frac{e^{-2s}}{s^2} \right]$$

$$= \frac{1}{1-e^{-4s}} \left[\frac{-2e^{-2s} + e^{-4s} + 1}{s^2} \right] = \frac{1}{1-(e^{-2s})2} \left[\frac{(1-e^{-2s})^2}{s^2} \right]$$

$$= \frac{1}{(1+e^{-2s})\cancel{(1-e^{-2s})}} \cdot \frac{(1-e^{-2s})^{\cancel{2}}}{s^2} = \frac{1}{s^2} \cdot \frac{(1-e^{-2s})}{(1+e^{-2s})}$$

$$= \frac{1}{s^2}\left[\frac{(1-e^{-2s})\,(e^{+s})}{(1+e^{-2s})\,(e^{+s})}\right]$$

$$= \frac{1}{s^2}\left[\frac{e^{+s} - e^{-s}}{e + e^{-s}}\right]$$

$$= \frac{1}{s^2}\tanh(s).$$

---

ii) $f(t) = \begin{cases} \sin t & , \text{for } 0 \leq t \leq \pi \\ 0 & , \text{for } \pi \leq t \leq 2\pi \end{cases}$ with the period $= 2\pi$.

soln:-

$$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$$

$$L[f(t)] = \frac{1}{1-e^{-2\pi s}}\int_0^{\pi} e^{-st}\sin t\,dt + \int_{\pi}^{2\pi} e^{-st}(0)\,dt$$

$$= \frac{1}{1-e^{-2\pi s}}\int_0^{\pi} e^{-st}\sin t\,dt + 0$$

$$= \frac{1}{1-e^{-2\pi s}}\int_0^{\pi} \frac{e^{-st}}{s^2+1}\left[s\sin t - (1)\cos t\right]_0^{\pi}$$

$$= \frac{1}{1-e^{-2\pi s}}\left[\frac{e^{-\pi s}}{s^2+1}(0+1) - \frac{1}{s^2+1}(0-1)\right]$$

$$= \frac{1}{1-e^{(-\pi s)^2}}\left[\frac{1+e^{-\pi s}}{s^2+1}\right] = \frac{1}{(1-e^{-\pi s})(1+e^{-\pi s})}\frac{(1+e^{-\pi s})}{s^2+1}$$

$$= \frac{1}{(1-e^{-\pi s})(s^2+1)} \;//$$

srivenkateshwaraa
College of Engineering & Technology
ASPIRE TO EXCEL
Ariyur, Puducherry-605 102.

NAAC A

ISO 21001

DEPARTMENT OF SCIENCE AND HUMANITES

Subject Name with Code: CSBS202 - MATHEMATICS - II
Batch : 2023 - 2027

POs & PSOs Mapping with COs

Course Outcomes	Programme Outcomes (POs)												Programme Specific Outcomes (PSO's)	
	1	2	3	4	5	6	7	8	9	10	11	12	1	2
CO1	H	H	M	H	L	M	L	M	H	M	H	M	M	M
CO2	H	H	H	M	M	L	M	L	M	L	H	M	M	M
CO3	H	H	M	H	L	M	L	M	H	M	H	M	M	M
CO4	H	H	H	M	M	L	M	L	M	L	H	M	M	M
CO5	H	H	M	M	M	M	M	M	M	M	M	M	M	M

H - High contribution | M- Medium Contribution | L - Low Contribution

POs & PSOs Mapping with Correlation Value

Course Outcomes	Programme Outcomes (Pos)												Programme Specific Outcomes	
	1	2	3	4	5	6	7	8	9	10	11	12	1	2
CO1	3	3	2	3	1	2	1	2	3	2	3	2	2	2
CO2	3	3	3	2	2	1	2	1	2	1	3	2	2	2
CO3	3	3	2	3	1	2	1	2	3	2	3	2	2	2
CO4	3	3	3	2	2	1	2	1	2	1	3	2	2	2
CO5	3	3	2	2	2	2	2	2	2	2	2	2	2	2
Avg Value	**3**	**3**	**2.4**	**2.4**	**1.6**	**1.6**	**1.6**	**1.6**	**2.4**	**1.6**	**2.8**	**2**	**2**	**2**

Final CO Attainments	2.35
% CO Attainments	78.33

Attainment Status	✓

If CO Attainment is above Target percentage , then PO Attainment is full or else it is zero

Final PO & PSO Attainment Values	PO1	PO2	PO3	PO4	PO5	PO6	PO7	PO8	PO9	PO10	PO11	PO12	PSO1	PSO2
	2.35	2.35	1.88	1.88	1.253	1.253	1.253	1.253	1.88	1.253	2.193	1.567	1.567	1.567

Head of the Department
Sciences & Humanities
Sri Venkateshwaraa College of Engineering
& Technology,
Ariyur Puducherry - 605 102.